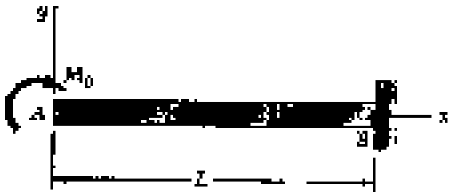
CHAPTER 9

PROBLEM 9.1

9.1 through 9.4 For the loading shown, determine (*a*) the equation of the elastic curve for the cantilever beam *AB*, (*b*) the deflection at the free end, (*c*) the slope at the free end.



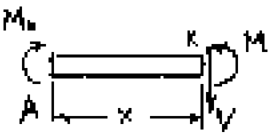
SOLUTION

$\sum M_K = 0 \qquad -M_0 + M = 0$

$M = M_0$

$[x = L, y = 0]$
$[x = L, \frac{dy}{dx} = 0]$

$$EI\frac{d^2y}{dx^2} = M = M_0$$



$$EI\frac{dy}{dx} = M_0 x + C_1$$

$[x = L, \frac{dy}{dx} = 0] \qquad 0 = M_0 L + C_1 \qquad C_1 = -M_0 L$

$$EI\,y = \frac{1}{2}M_0 x^2 + C_1 x + C_2$$

$[x = L, y = 0] \qquad 0 = \frac{1}{2}M_0 L^2 - M_0 L^2 + C_2$

$C_2 = \frac{1}{2}M_0 L^2$

(a) Elastic curve $\qquad y = \frac{M_0}{2EI}(x^2 - 2Lx + L^2)$

$$= \frac{M_0}{2EI}(L - x)^2 \quad ◀$$

(b) y @ $x = 0 \qquad y_A = \frac{M_0}{2EI}(L - 0)^2 = \frac{M_0 L^2}{2EI} \uparrow \quad ◀$

(c) $\frac{dy}{dx}$ @ $x = 0 \qquad \frac{dy}{dx} = -\frac{M_0}{EI}(L - x) = -\frac{M_0}{EI}(L - 0) = -\frac{M_0 L}{EI}$

$$\theta_A = \frac{M_0 L}{EI} \quad ◀$$

PROBLEM 9.2

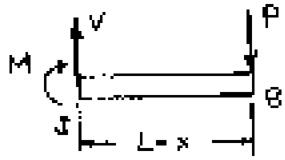
9.1 through 9.4 For the loading shown, determine (*a*) the equation of the elastic curve for the cantilever beam *AB*, (*b*) the deflection at the free end, (*c*) the slope at the free end.

SOLUTION

$[x = 0,\ y = 0]$

$[x = 0,\ \frac{dy}{dx} = 0]$

$\circlearrowleft \Sigma M_J = 0 \qquad -M - P(L - x) = 0$



$$M = -P(L - x)$$

$$EI\frac{d^2y}{dx^2} = -P(L - x) = -PL + Px$$

$$EI\frac{dy}{dx} = -PLx + \frac{1}{2}Px^2 + C_1$$

$[x = 0,\ \frac{dy}{dx} = 0] \qquad 0 = -0 + 0 + C_1$

$$C_1 = 0$$

$$EIy = -\frac{1}{2}PLx^2 + \frac{1}{6}Px^3 + C_1x + C_2$$

$[x = 0,\ y = 0] \qquad 0 = -0 + 0 + 0 + C_2 \qquad C_2 = 0$

(a) Elastic curve $\qquad y = -\frac{Px^2}{6EI}(3L - x)$ ◄

$$\frac{dy}{dx} = -\frac{Px}{2EI}(2L - x)$$

(b) y @ $x = L$ $\qquad y_B = -\frac{PL^2}{6EI}(3L - L) = -\frac{PL^3}{3EI}$

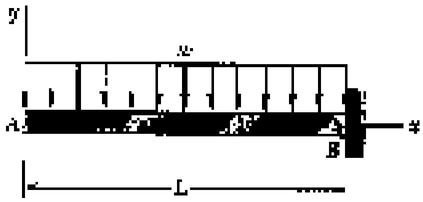
$$y_B = \frac{PL^3}{3EI} \downarrow$$ ◄

(c) $\frac{dy}{dx}$ @ $x = L$ $\qquad \left.\frac{dy}{dx}\right|_B = -\frac{PL}{2EI}(2L - L) = -\frac{PL^2}{2EI}$

$$\theta_B = \frac{PL^2}{2EI}$$ ◄

PROBLEM 9.3

9.1 through 9.4 For the loading shown, determine (*a*) the equation of the elastic curve for the cantilever beam *AB*, (*b*) the deflection at the free end, (*c*) the slope at the free end.



SOLUTION

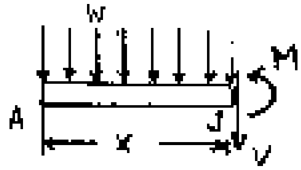
$+\circlearrowleft \Sigma M_J = 0 \qquad (wx)\frac{x}{2} + M = 0$

$$M = -\frac{1}{2}wx^2$$

[x = L, y = 0]

[x = L, $\frac{dy}{dx} = 0$]

$$EI\frac{d^2y}{dx^2} = M = -\frac{1}{2}wx^2$$



$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + C_1$$

[x = L, $\frac{dy}{dx} = 0$] $\qquad 0 = -\frac{1}{6}wL^3 + C_1$

$$C_1 = \frac{1}{6}wL^3$$

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{6}wL^3$$

$$EIy = -\frac{1}{24}wx^4 + \frac{1}{6}wL^3x + C_2$$

[x = L, y = 0] $\qquad 0 = -\frac{1}{24}wL^4 + \frac{1}{6}wL^4 + C_2 = 0$

$$C_2 = \left(\frac{1}{24} - \frac{1}{6}\right)wL^4 = -\frac{3}{24}wL^4$$

(a) Elastic curve $\qquad y = -\frac{w}{24EI}\left(x^4 - 4L^3x + 3L^4\right)$ ◄

(b) y @ x = 0 $\qquad y_A = -\frac{3wL^4}{24EI} = -\frac{wL^4}{8EI} \qquad y_A = \frac{wL^4}{8EI}\downarrow$ ◄

(c) $\frac{dy}{dx}$ @ x = 0 $\qquad \left.\frac{dy}{dx}\right|_A = \frac{wL^3}{6EI} \qquad \theta_A = \frac{wL^3}{6EI}$ ◄

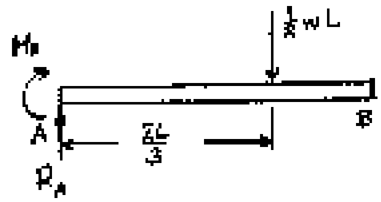
PROBLEM 9.4

9.1 through 9.4 For the loading shown, determine (*a*) the equation of the elastic curve for the cantilever beam *AB*, (*b*) the deflection at the free end, (*c*) the slope at the free end.

$[x = 0,\ y = 0]$

$[x = 0,\ \frac{dy}{dx} = 0]$

SOLUTION



$+\uparrow \Sigma F_y = 0$

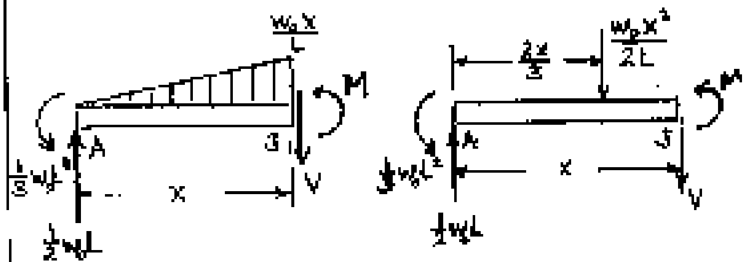
$R_A - \frac{1}{2}wL = 0$

$R_A = \frac{1}{2}w_0L$

$\circlearrowleft \Sigma M_A = 0$

$-M_A - \frac{2L}{3}\cdot\frac{w_0L}{2} = 0$

$M_A = -\frac{1}{3}w_0L^2$



$\circlearrowleft \Sigma M_J = 0 \qquad \frac{1}{3}w_0L^2 - \frac{1}{2}w_0Lx + \frac{w_0x^2}{2L}\cdot\frac{x}{3} + M = 0$

$M = -\frac{1}{3}w_0L^2 + \frac{1}{2}w_0Lx - \frac{w_0x^3}{6L}$

$EI\frac{d^2y}{dx^2} = -\frac{1}{3}w_0L^2 + \frac{1}{2}w_0Lx - \frac{w_0x^3}{6L}$

$EI\frac{dy}{dx} = -\frac{1}{3}w_0L^2x + \frac{1}{4}w_0Lx^2 - \frac{w_0x^4}{24L} + C_1$

$[x = 0,\ \frac{dy}{dx} = 0] \qquad 0 = -0 + 0 - 0 + C_1 \qquad C_1 = 0$

$EIy = -\frac{1}{6}w_0L^2x^2 + \frac{1}{12}w_0Lx^3 - \frac{w_0x^5}{120L} + C_2$

$[x = 0,\ y = 0] \qquad 0 = -0 + 0 - 0 + 0 + C_2 \qquad C_2 = 0$

(a) Elastic curve $\qquad y = -\frac{w_0}{EIL}\left(\frac{1}{6}L^3x^2 - \frac{1}{12}Lx^4 + \frac{1}{120}x^5\right)$ ◀

(b) y @ $x = L \qquad y_B = -\frac{w_0L^4}{EI}\left(\frac{1}{6} - \frac{1}{12} + \frac{1}{120}\right) = -\frac{11}{120}\frac{w_0L^4}{EI}$

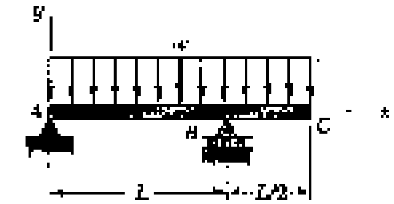
$y_B = \frac{11}{120}\frac{w_0L^4}{EI}\downarrow$ ◀

(c) $\frac{dy}{dx}$ @ $x = L \qquad \left.\frac{dy}{dx}\right|_B = -\frac{w_0L^3}{EI}\left(\frac{1}{3} - \frac{1}{4} + \frac{1}{24}\right) = -\frac{1}{8}\frac{w_0L^3}{EI}$

$\theta_B = \frac{1}{8}\frac{w_0L^3}{EI}$ ◀

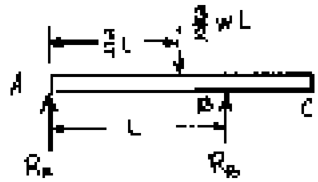
PROBLEM 9.5

9.5 For the beam and loading shown, determine (a) the equation of the elastic curve for portion AB of the beam, (b) the slope at A, (c) the slope at B.



$[x=0,\ y=0]$ $\quad$ $[x=L,\ y=0]$

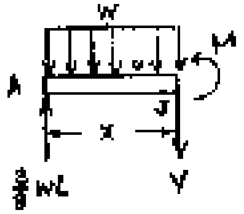
SOLUTION



$+\circlearrowleft \Sigma M_B = 0$

$-R_A L + \left(\frac{3}{2}wL\right)\left(\frac{1}{4}L\right) = 0$

$R_A = \frac{3}{8}wL$



For portion AB only $\quad (0 \le x < L)$

$+\circlearrowleft \Sigma M_J = 0 \qquad -\frac{3}{8}wLx + (wx)\frac{x}{2} + M = 0$

$M = \frac{3}{8}wLx - \frac{1}{2}wx^2$

$$EI\frac{d^2y}{dx^2} = \frac{3}{8}wLx - \frac{1}{2}wx^2$$

$$EI\frac{dy}{dx} = \frac{3}{16}wLx^2 - \frac{1}{6}wx^3 + C_1$$

$$EI\,y = \frac{1}{16}wLx^3 - \frac{1}{24}wx^4 + C_1x + C_2$$

$[x=0,\ y=0] \qquad 0 = 0 - 0 + 0 + C_2 \qquad C_2 = 0$

$[x=L,\ y=0] \qquad 0 = \frac{1}{16}wL^4 - \frac{1}{24}wL^4 + C_1L \qquad C_1 = -\frac{1}{48}wL^3$

(a) Elastic curve

$$y = \frac{w}{EI}\left(\frac{1}{16}Lx^3 - \frac{1}{24}x^4 - \frac{1}{48}L^3x\right)$$

$$\frac{dy}{dx} = \frac{w}{EI}\left(\frac{3}{16}Lx^2 - \frac{1}{6}x^3 - \frac{1}{48}L^3\right)$$

(b) $\frac{dy}{dx}$ @ $x=0$ $\qquad \left.\frac{dy}{dx}\right|_A = \frac{w}{EI}\left(0 - 0 - \frac{1}{48}L^3\right) = -\frac{wL^3}{48EI}$

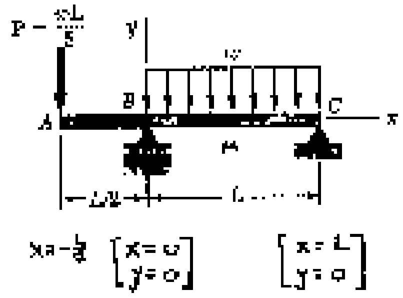
$$\theta_A = \frac{wL^3}{48EI}$$

(c) $\frac{dy}{dx}$ @ $x=L$ $\qquad \left.\frac{dy}{dx}\right|_B = \frac{w}{EI}\left(\frac{3}{16}L^3 - \frac{1}{6}L^3 - \frac{1}{48}L^3\right) = 0$

$$\theta_B = 0$$

PROBLEM 9.6

9.6 For the beam and loading shown, determine (a) the equation of the elastic curve for portion BC of the beam, (b) the deflection at midspan, (c) the slope at B.

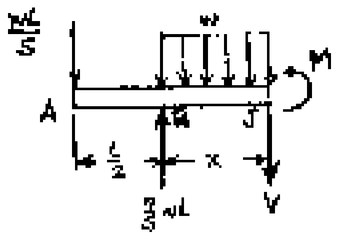


SOLUTION

Using ABC as a free body

$$\circlearrowleft \Sigma M_C = 0 \qquad \left(\frac{wL}{5}\right)\left(\frac{3L}{2}\right) - R_B L + (wL)\left(\frac{L}{2}\right) = 0$$

$$R_B = \frac{4}{5}wL$$



For portion BC only $\quad 0 < x < L$

$$\circlearrowleft \Sigma M_J = 0 \qquad \frac{wL}{5}\left(\frac{L}{2} + x\right) - \frac{4}{5}wLx + (wx)\frac{x}{2} + M = 0$$

$$M = \frac{3}{5}wLx - \frac{1}{2}wx^2 - \frac{1}{10}wL^2$$

$$EI\frac{d^2y}{dx^2} = \frac{3}{5}wLx - \frac{1}{2}wx^2 - \frac{1}{10}wL^2$$

$$EI\frac{dy}{dx} = \frac{3}{10}wLx^2 - \frac{1}{6}wx^3 - \frac{1}{10}wL^2x + C_1$$

$$EI\,y = \frac{1}{10}wLx^3 - \frac{1}{24}wx^4 - \frac{1}{20}wL^2x^2 + C_1x + C_2$$

$[x = 0,\ y = 0]$ $\qquad 0 - 0 - 0 - 0 + 0 + C_2 \qquad C_2 = 0$

$[x = L,\ y = 0]$ $\qquad 0 = \left(\frac{1}{10} - \frac{1}{24} - \frac{1}{20}\right)wL^4 + C_1L + 0 \qquad C_1 = -\frac{1}{120}wL^3$

(a) Elastic curve

$$y = \frac{w}{EI}\left(\frac{1}{10}Lx^3 - \frac{1}{24}x^4 - \frac{1}{20}L^2x^2 - \frac{1}{120}L^3x\right)$$

$$\frac{dy}{dx} = \frac{w}{EI}\left(\frac{3}{10}Lx^2 - \frac{1}{6}x^3 - \frac{1}{10}L^2x - \frac{1}{120}L^3\right)$$

(b) y @ $x = \frac{L}{2}$

$$y_M = \frac{w}{EI}\left\{\frac{1}{10}L\left(\frac{L}{2}\right)^3 - \frac{1}{24}\left(\frac{L}{2}\right)^4 - \frac{1}{20}L^2\left(\frac{L}{2}\right)^2 - \frac{1}{120}L^3\left(\frac{L}{2}\right)\right\}$$

$$= \frac{wL^4}{EI}\left\{\frac{1}{80} - \frac{1}{384} - \frac{1}{80} - \frac{1}{240}\right\} = -\frac{13\,wL^4}{1920\,EI}$$

$$y_M = \frac{13\,wL^4}{1920\,EI} \downarrow$$

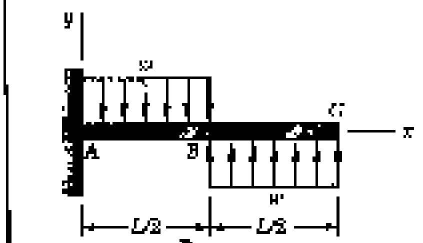
(c) $\frac{dy}{dx}$ @ $x = 0$

$$\left.\frac{dy}{dx}\right|_B = \frac{w}{EI}\left(0 - 0 - 0 - \frac{1}{120}L^3\right) = -\frac{wL^3}{120\,EI}$$

$$\theta_B = \frac{wL^3}{120\,EI}$$

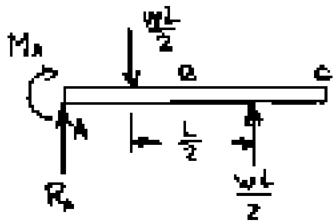
PROBLEM 9.7

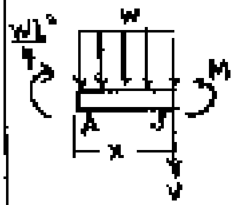
9.7 For the cantilever beam and loading shown, determine (*a*) the equation of the elastic curve for portion *AB* of the beam, (*b*) the deflection at *B*, (*c*) the slope at *B*.



$[x=0,\ y=0]$

$[x=0,\ \frac{dy}{dx}=0]$





SOLUTION

Using ABC as a free body

$+\uparrow \Sigma F_y = 0 \qquad R_A - \frac{wL}{2} + \frac{wL}{2} = 0 \qquad R_A = 0$

$+\circlearrowleft M_A = 0 \qquad -M_A + \left(\frac{wL}{2}\right)\left(\frac{L}{2}\right) = 0 \qquad M_A = \frac{wL^2}{4}$

Using AJ as a free body (Portion AB only)

$\circlearrowleft \Sigma M_J = 0 \qquad -\frac{wL^2}{4} + (wx)\frac{x}{2} + M = 0$

$M = \frac{1}{4}wL^2 - \frac{1}{2}wx^2$

$EI\frac{d^2y}{dx^2} = \frac{1}{4}wL^2 - \frac{1}{2}wx^2$

$EI\frac{dy}{dx} = \frac{1}{4}wL^2x - \frac{1}{6}wx^3 + C_1$

$[x=0,\ \frac{dy}{dx}=0] \qquad 0 = 0 - 0 + C_1 \qquad C_1 = 0$

$EIy = \frac{1}{8}wL^2x^2 - \frac{1}{24}wx^4 + C_1x + C_2$

$[x=0,\ y=0] \qquad 0 = 0 - 0 + 0 + C_2 \qquad C_2 = 0$

(a) Elastic curve $\qquad y = \frac{w}{EI}\left(\frac{1}{8}L^2x^2 - \frac{1}{24}x^4\right)$ ◄

$\frac{dy}{dx} = \frac{w}{EI}\left(\frac{1}{4}L^2x - \frac{1}{6}x^3\right)$

(b) y at $x = \frac{L}{2}$ $\qquad y_B = \frac{w}{EI}\left\{\frac{1}{8}L^2\left(\frac{L}{2}\right)^2 - \frac{1}{24}\left(\frac{L}{2}\right)^4\right\} = \frac{wL^4}{EI}\left\{\frac{1}{32} - \frac{1}{384}\right\}$

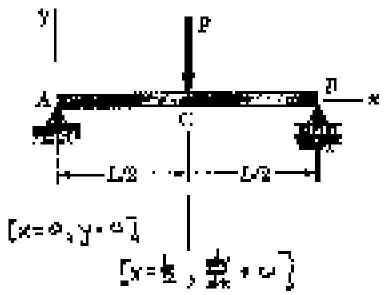
$= \frac{11\,wL^4}{384\,EI} \qquad y_B = \frac{11\,wL^4}{384\,EI}\ \uparrow$ ◄

(c) $\frac{dy}{dx}$ at $x = \frac{L}{2}$ $\qquad \theta_B = \frac{w}{EI}\left\{\frac{1}{4}L^2\left(\frac{L}{2}\right) - \frac{1}{6}\left(\frac{L}{2}\right)^3\right\} = \frac{wL^3}{EI}\left\{\frac{1}{8} - \frac{1}{48}\right\}$

$= \frac{5\,wL^3}{48\,EI} \qquad \theta_B = \frac{5\,wL^3}{48\,EI}$ ◄

PROBLEM 9.8

9.8 For the beam shown with load **P**, determine (*a*) the equation of the elastic curve for portion *AC* of the beam, (*b*) the slope at *A*, (*c*) the deflection at *C*.

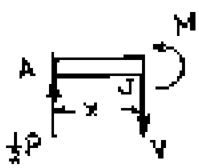


SOLUTION

Because of symmetry $\frac{dy}{dx} = 0$ at $x = \frac{L}{2}$.

Reaction at A $\quad R_A = \frac{1}{2}P$

For portion AC only: using free body AJ



$\circlearrowleft M_J = 0 \qquad -\frac{1}{2}Px + M = 0$

$M = \frac{1}{2}Px$

$EI\frac{d^2y}{dx^2} = \frac{1}{2}Px$

$EI\frac{dy}{dx} = \frac{1}{4}Px^2 + C_1$

$[x = \frac{L}{2}, \frac{dy}{dx} = 0] \qquad 0 = \frac{1}{4}P\left(\frac{L}{2}\right)^2 + C_1 \qquad C_1 = -\frac{1}{16}PL^2$

$EIy = \frac{1}{12}Px^3 + C_1x + C_2$

$[x = 0, y = 0] \qquad 0 = 0 + 0 + C_2 \qquad C_2 = 0$

(a) Elastic curve $\qquad y = \frac{P}{EI}\left(\frac{1}{12}x^3 - \frac{1}{16}L^2x\right)$ ◄

$\frac{dy}{dx} = \frac{P}{EI}\left(\frac{1}{4}x^2 - \frac{1}{16}L^2\right)$

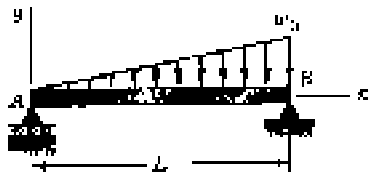
(b) $\frac{dy}{dx}$ at $x = 0 \qquad \theta_A = \frac{P}{EI}\left(0 - \frac{1}{16}L^2\right) = -\frac{PL^2}{16EI}$ or $\frac{PL^2}{16EI}$ ◄

(c) y at $x = \frac{L}{2} \qquad y_C = \frac{P}{EI}\left\{\frac{1}{12}\left(\frac{L}{2}\right)^3 - \frac{1}{16}L^2\left(\frac{L}{2}\right)\right\} = \frac{PL^3}{EI}\left\{\frac{1}{96} - \frac{1}{32}\right\}$

$= -\frac{PL^3}{48EI}$ or $\frac{PL^3}{48EI}\downarrow$ ◄

PROBLEM 9.9

9.9 and 9.10 For the beam and loading shown, (*a*) express the magnitude and location of the maximum deflection in terms of w_0, L, E, and I. (*b*) Calculate the value of the maximum deflection, assuming that beam AB is a W18 × 50 rolled shape and that w_0 = 4.5 kips/ft, L = 18 ft, and $E = 29 \times 10^6$ psi.



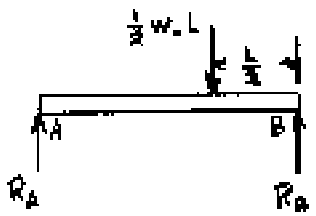
$[x=0, y=0]$ $\quad$ $[x=L, y=0]$

SOLUTION

Using entire beam as a free body

$$+\circlearrowleft \Sigma M_B = 0 \qquad -R_A L + \left(\tfrac{1}{2} w_0 L\right)\left(\tfrac{L}{3}\right) = 0$$

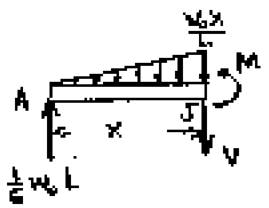
$$R_A = \tfrac{1}{6} w_0 L$$



Using AJ as a free body $\circlearrowleft \Sigma M_J = 0$

$$-\tfrac{1}{6} w_0 L x + \left(\tfrac{1}{2} \tfrac{w_0 x^2}{L}\right)\left(\tfrac{x}{3}\right) + M = 0$$

$$M = \tfrac{1}{6} w_0 L x - \tfrac{1}{6} \tfrac{w_0}{L} x^3$$



$$EI \frac{d^2y}{dx^2} = \tfrac{1}{6} w_0 L x - \tfrac{1}{6} \tfrac{w_0}{L} x^3$$

$$EI \frac{dy}{dx} = \tfrac{1}{12} w_0 L x^2 - \tfrac{1}{24} \tfrac{w_0}{L} x^4 + C_1$$

$$EI\, y = \tfrac{1}{36} w_0 L x^3 - \tfrac{1}{120} \tfrac{w_0}{L} x^5 + C_1 x + C_2$$

$[x=0, y=0]$ $\quad 0 = 0 - 0 + 0 + C_2 \qquad C_2 = 0$

$[x=L, y=0]$ $\quad 0 = \frac{1}{36} w_0 L^4 - \frac{1}{120} w_0 L^4 + C_1 L + 0 \qquad C_1 = -\frac{7 w_0 L^3}{360}$

Elastic curve $\quad y = \dfrac{w_0}{EI}\left\{ \dfrac{1}{36} L x^3 - \dfrac{1}{120} \dfrac{x^5}{L} - \dfrac{7}{360} L^3 x \right\}$

$$\frac{dy}{dx} = \frac{w_0}{EI}\left\{ \frac{1}{12} L x^2 - \frac{1}{24} \frac{x^4}{L} - \frac{7}{360} L^3 \right\}$$

To find location of maximum deflection set $\frac{dy}{dx} = 0$

$$15 x_m^4 - 30 L^2 x_m^2 + 7 L^4 = 0 \qquad x_m^2 = \frac{30 L^2 - \sqrt{900 L^4 - 420 L^4}}{30}$$

$$x_m^2 = \left(1 - \sqrt{\tfrac{8}{15}}\right) L^2 = 0.2697 L^2 \qquad x_m = 0.5193\, L$$

$$y_m = \frac{w_0}{EI}\left\{ \frac{1}{36} L (0.5193 L)^3 - \frac{1}{120} \frac{(0.5193 L)^5}{L} - \frac{7}{360} L^3 (0.5193 L) \right\}$$

$$= -0.00652 \frac{w_0 L^4}{EI} \quad \text{or} \quad 0.00652 \frac{w_0 L^4}{EI} \downarrow$$

Data: $w_0 = 4.5$ kips/ft $= \frac{4500}{12} = 375$ lb/in, $\quad L = 18$ ft $= 216$ in

$I = 800$ in^4 for W18×50

$$y_m = \frac{(0.00652)(375)(216)^4}{(29 \times 10^6)(800)} = 0.229 \text{ in.}$$

PROBLEM 9.10

9.9 and 9.10 For the beam and loading shown, (*a*) express the magnitude and location of the maximum deflection in terms of w_0, L, E, and I. (*b*) Calculate the value of the maximum deflection, assuming that beam AB is a W18 × 50 rolled shape and that $w_0 = 4.5$ kips/ft, $L = 18$ ft, and $E = 29 \times 10^6$ psi.

[x = 0, y = 0]   [x = L, y = 0]

SOLUTION

$$\frac{dV}{dx} = -w = -\frac{w_0}{L}(L-x)$$

$$V = -\frac{w_0}{L}\left(Lx - \frac{1}{2}x^2\right) + C_V = \frac{dM}{dx}$$

$$M = -\frac{w_0}{L}\left(\frac{1}{2}Lx^2 - \frac{1}{6}x^3\right) + C_V x + C_M$$

$[x = 0, M = 0]$ $0 = 0 + 0 + 0 + C_M$ $C_M = 0$

$[x = L, M = 0]$ $0 = -\frac{w_0}{L}\left(\frac{1}{2}L^3 - \frac{1}{6}L^3\right) + C_V L$ $C_V = \frac{1}{3}w_0 L$

$$EI\frac{d^2y}{dx^2} = M = \frac{w_0}{L}\left(\frac{1}{3}L^2x - \frac{1}{2}Lx^2 + \frac{1}{6}x^3\right)$$

$$EI\frac{dy}{dx} = \frac{w_0}{L}\left(\frac{1}{6}L^2x^2 - \frac{1}{6}Lx^3 + \frac{1}{24}x^4\right) + C_1$$

$$EI\,y = \frac{w_0}{L}\left(\frac{1}{18}L^2x^3 - \frac{1}{24}Lx^4 + \frac{1}{120}x^5\right) + C_1 x + C_2$$

$[x = 0, y = 0]$ $0 = 0 - 0 + 0 + 0 + C_2$ $C_2 = 0$

$[x = L, y = 0]$ $0 = \frac{w_0}{L}\left\{\frac{1}{18}L^5 - \frac{1}{24}L^5 + \frac{1}{120}L^5\right\} + C_1 L + 0$ $C_1 = -\frac{1}{45}w_0 L^3$

$$y = \frac{w_0}{EIL}\left(\frac{1}{18}L^2x^3 - \frac{1}{24}Lx^4 + \frac{1}{120}x^5 - \frac{1}{45}L^4x\right)$$

$$\frac{dy}{dx} = \frac{w_0}{EIL}\left(\frac{1}{6}L^2x^2 - \frac{1}{6}Lx^3 + \frac{1}{24}x^4 - \frac{1}{45}L^4\right)$$

To find location of maximum deflection set $\frac{dy}{dx} = 0$.

$$f = \frac{1}{6}L^2x_m^2 - \frac{1}{6}Lx_m^3 + \frac{1}{24}x_m^4 - \frac{1}{45}L^4 = 0$$

Let $z = \frac{x_m}{L}$ $f(z) = \frac{1}{6}z^2 - \frac{1}{6}z^3 + \frac{1}{24}z^4 - \frac{1}{45}$

$$\frac{df}{dz} = \frac{1}{3}z - \frac{1}{2}z^2 + \frac{1}{6}z^3$$

By Newton-Raphson method $z = z_0 - \frac{f(z)}{df/dz}$

$z = \cancel{0.5}, \cancel{0.4805}, 0.4807, 0.4807$ $x_m = 0.4807\,L$

$$y_m = \frac{w_0L^4}{EI}\left\{\frac{1}{18}(0.4807)^3 - \frac{1}{24}(0.4807)^4 + \frac{1}{120}(0.4807)^5 - \frac{1}{45}(0.4807)\right\}$$

$$= -0.00652\frac{w_0L^4}{EI}$$ $y_m = 0.00652\frac{w_0L^4}{EI}\downarrow$

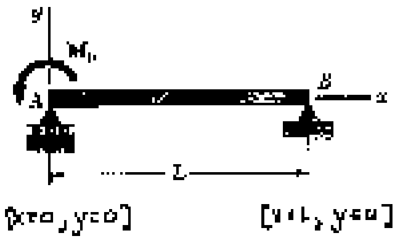
Data: $w_0 = 4.5$ kips/ft $= \frac{4500}{12} = 375$ lb/in, $L = 18$ ft $= 216$ in.

$I = 800$ in^4 for W 18×50

$$y_m = \frac{(0.00652)(375)(216)^4}{(29\times10^6)(800)} = 0.229 \text{ in.}$$

PROBLEM 9.11

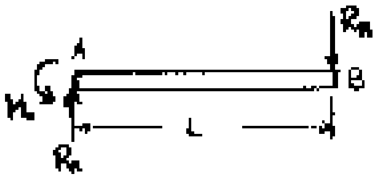
9.11 (*a*) Determine the location and magnitude of the maximum deflection of beam AB. (*b*) Assuming that beam AB is a W360 × 64, $L = 3.5$ m and $E = 200$ GPa, calculate the maximum allowable value of the applied moment M_0 if the maximum deflection is not to exceed 1 mm.



SOLUTION

Using entire beam as a free body

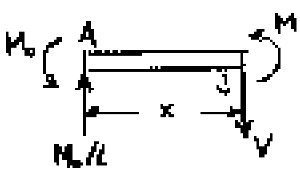
$+\circlearrowleft \Sigma M_B = 0 \qquad M_0 - R_A L = 0 \qquad R_A = \dfrac{M_0}{L}$



Using portion AJ

$+\circlearrowleft \Sigma M_J = 0 \qquad M_0 - \dfrac{M_0}{L}x + M = 0$

$M = \dfrac{M_0}{L}(x - L)$



$EI\dfrac{d^2y}{dx^2} = \dfrac{M_0}{L}(x - L)$

$EI\dfrac{dy}{dx} = \dfrac{M_0}{L}\left(\frac{1}{2}x^2 - Lx\right) + C_1$

$EIy = \dfrac{M_0}{L}\left(\frac{1}{6}x^3 - \frac{1}{2}Lx^2\right) + C_1x + C_2$

$[x = 0,\ y = 0] \qquad 0 = 0 - 0 + 0 + C_2 \qquad C_2 = 0$

$[x = L,\ y = 0] \qquad 0 = \dfrac{M_0}{L}\left(\frac{1}{6}L^3 - \frac{1}{2}L^3\right) + C_1L + 0 \qquad C_1 = \frac{1}{3}M_0L$

$y = \dfrac{M_0}{EIL}\left(\frac{1}{6}x^3 - \frac{1}{2}Lx^2 + \frac{1}{3}L^2x\right) \qquad \dfrac{dy}{dx} = \dfrac{M_0}{EIL}\left(\frac{1}{2}x^2 - Lx + \frac{1}{3}L^2\right)$

To find location of maximum deflection set $\dfrac{dy}{dx} = 0$

$\frac{1}{2}x_m^2 - Lx_m + \frac{1}{3}L^2 = 0 \qquad x_m = L - \sqrt{L^2 - (4)(\frac{1}{2})(\frac{1}{3}L^2)} = \left(1 - \sqrt{\tfrac{1}{3}}\right)L$

$= 0.42265\,L$ ◀

$y_m = \dfrac{M_0L^2}{EI}\left\{(\frac{1}{6})(0.42265)^3 - (\frac{1}{2})(0.42265)^2 + (\frac{1}{3})(0.42265)\right\} = 0.06415\dfrac{M_0L^2}{EI} \uparrow$ ◀

Solving for M_0 $\qquad M_0 = \dfrac{EI\,y_m}{0.06415\,L^2}$

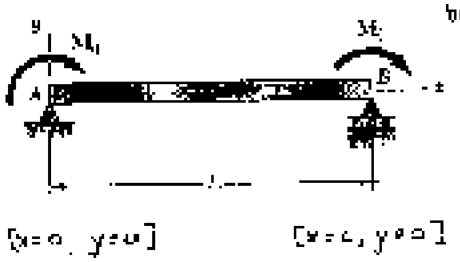
Data: $E = 200 \times 10^9$ Pa, $I = 178 \times 10^6$ mm^4 $= 178 \times 10^{-6}$ m^4

$L = 3.5$ m, $y_m = 1$ mm $= 10^{-3}$ m

$M_0 = \dfrac{(200\times10^9)(178\times10^{-6})(10^{-3})}{(0.06415)(3.5)^2} = 45.3\times10^3$ N·m $= 45.3$ kN·m ◀

PROBLEM 9.12

9.12 (a) Determine the location and magnitude of the maximum absolute deflection in AB between A and the center of the beam. (b) Assuming that beam AB is a W460 × 113, M_0 = 224 kN·m and E = 200 GPa, determine the maximum allowable length L of the beam if the maximum deflection is not to exceed 1.2 mm.

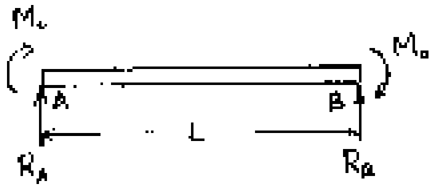


SOLUTION

Using AB as a free body

$$\Sigma M_B = 0 \qquad -2M_0 - R_A L = 0$$

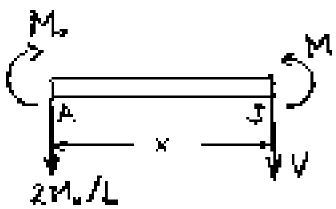
$$R_A = -\frac{2M_0}{L}$$



Using portion AJ as a free body

$$\Sigma M_J = 0 \qquad -M_0 + \frac{2M_0}{L}x - M = 0$$

$$M = \frac{M_0}{L}(L - 2x)$$



$$EI\frac{d^2y}{dx^2} = \frac{M_0}{L}(L - 2x)$$

$$EI\frac{dy}{dx} = \frac{M_0}{L}(Lx - x^2) + C_1$$

$$EI\,y = \frac{M_0}{L}\left(\tfrac{1}{2}Lx^2 - \tfrac{1}{3}x^3\right) + C_1 x + C_2$$

$$[x = 0,\ y = 0] \qquad 0 = 0 - 0 + 0 + C_2 \qquad C_2 = 0$$

$$[x = L,\ y = 0] \qquad 0 = \frac{M_0}{L}\left(\tfrac{1}{2}L^3 - \tfrac{1}{3}L^3\right) + C_1 L + 0 \qquad C_1 = -\tfrac{1}{6}M_0 L^2$$

$$y = \frac{M_0}{EIL}\left(\tfrac{1}{2}Lx^2 - \tfrac{1}{3}x^3 - \tfrac{1}{6}L^2 x\right) \qquad \frac{dy}{dx} = \frac{M_0}{EIL}\left(Lx - x^2 - \tfrac{1}{6}L^2\right)$$

To find location of maximum deflection set $\frac{dy}{dx} = 0$

$$x_m^2 - Lx_m - \tfrac{1}{6}L^2 = 0 \qquad x_m = \frac{L - \sqrt{L^2 - (4)(\frac{1}{6}L^2)}}{2} = \tfrac{1}{2}\left(1 - \tfrac{\sqrt{3}}{3}\right)L = 0.21132\ L$$

$$y_m = \frac{M_0 L^2}{EI}\left\{\left(\tfrac{1}{2}\right)(0.21132)^2 - \left(\tfrac{1}{3}\right)(0.21132)^3 - \left(\tfrac{1}{6}\right)(0.21132)\right\} = -0.0160375\frac{M_0 L^2}{EI}$$

$$|y_m| = 0.0160375\frac{M_0 L^2}{EI}$$

Solving for L
$$L = \left\{\frac{EI\,|y_m|}{0.0160375\,M_0}\right\}^{1/2}$$

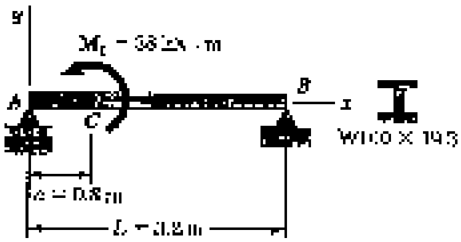
Data: $E = 200 \times 10^9$ Pa, $I = 556 \times 10^6\ \text{mm}^4 = 556 \times 10^{-6}\ \text{m}^4$

$|y_m| = 1.2\ \text{mm} = 1.2 \times 10^{-3}\ \text{m}$, $M_0 = 224 \times 10^3$ N·m

$$L = \left\{\frac{(200 \times 10^9)(556 \times 10^{-6})(1.2 \times 10^{-3})}{(0.0160375)(224 \times 10^3)}\right\}^{1/2} = 6.09\ \text{m}$$

PROBLEM 9.13

9.13 and 9.14 For the beam and loading shown, determine the deflection at point *C*. Use $E = 200$ GPa.



$[x=0, y=0]$ $\quad$ $[x=L, y=0]$

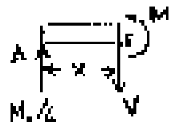
$[x=a, y=y]$

$[x=a, \frac{dy}{dx} = \frac{dy}{dx}]$

SOLUTION

Reactions: $R_A = M_0/L \uparrow$, $R_B = M_0/L \downarrow$

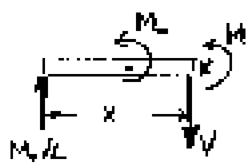
$0 < x < a$ $\qquad \circlearrowleft \Sigma M_J = 0$



$$-\frac{M_0}{L}x + M = 0$$

$$M = \frac{M_0}{L}x$$

$a < x < L$ $\qquad +\circlearrowleft M_J = 0$



$$-\frac{M_0}{L}x + M_0 + M = 0$$

$$M = \frac{M_0}{L}(x - L)$$

$\underline{0 < x < a}$

$$EI\frac{d^2y}{dx^2} = \frac{M_0}{L}x$$

$$EI\frac{dy}{dx} = \frac{M_0}{L}\left(\frac{1}{2}x^2\right) + C_1 \qquad (1)$$

$$EI\,y = \frac{M_0}{L}\left(\frac{1}{6}x^3\right) + C_1x + C_2 \qquad (2)$$

$\underline{a < x < L}$

$$EI\frac{d^2y}{dx^2} = \frac{M_0}{L}(x - L)$$

$$EI\frac{dy}{dx} = \frac{M_0}{L}\left(\frac{1}{2}x^2 - Lx\right) + C_3 \qquad (3)$$

$$EI\,y = \frac{M_0}{L}\left(\frac{1}{6}x^3 - \frac{1}{2}Lx^2\right) + C_3x + C_4 \qquad (4)$$

$[x=0, y=0]$ $\quad$ Eq. (2): $\quad 0 = 0 + 0 + C_2 \qquad C_2 = 0$

$[x=a, \frac{dy}{dx} = \frac{dy}{dx}]$ $\quad$ Eqs. (1) & (3): $\quad \frac{M_0}{L}\left(\frac{1}{2}a^2\right) + C_1 = \frac{M_0}{L}\left(\frac{1}{2}a^2 - La\right) + C_3$

$$C_3 = C_1 + M_0a$$

$[x=a, y=y]$ $\quad$ Eqs (2) & (4) $\quad \frac{M_0}{L}\left(\cancel{\frac{1}{6}a^3}\right) + \cancel{C_1a} = \frac{M_0}{L}\left(\cancel{\frac{1}{6}a^3} - \frac{1}{2}La^2\right) + (\cancel{C_1} + M_0a)a + C_4$

$$C_4 = -\frac{1}{2}M_0a^2$$

$[x=L, y=0]$ $\quad$ Eq(4) $\quad \frac{M_0}{L}\left(\frac{1}{6}L^3 - \frac{1}{2}L^3\right) + (C_1 + M_0a)L - \frac{1}{2}M_0a^2 = 0$

$$C_1 = \frac{M_0}{L}\left(\frac{1}{3}L^2 + \frac{1}{2}a^2 - aL\right)$$

Elastic curve for $0 < x < a$ $\qquad y = \frac{M_0}{EIL}\left[\frac{1}{6}x^3 + \left(\frac{1}{3}L^2 + \frac{1}{2}a^2 - aL\right)x\right]$

Make $x = a$ $\qquad y_C = \frac{M_0}{EIL}\left[\frac{1}{6}a^3 + \frac{1}{3}L^2a + \frac{1}{2}a^3 - a^2L\right] = \frac{M_0}{EIL}\left[\frac{2}{3}a^3 + \frac{1}{3}L^2a - La^2\right]$

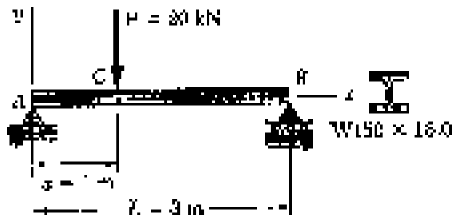
Data: $E = 200 \times 10^9$ Pa, $\quad I = 4.77 \times 10^6$ mm^4 $= 4.77 \times 10^{-6}$ m, $\quad M_0 = 38 \times 10^3$ N·m

$$y_C = \frac{(38 \times 10^3)\left[(2)(0.8)^3/3 + (3.2)^2(0.8)/3 - (3.2)(0.8)^2\right]}{(200 \times 10^9)(4.77 \times 10^{-6})(3.2)} = 12.75 \times 10^{-3} \text{ m}$$

$$= 12.75 \text{ mm} \longleftarrow$$

PROBLEM 9.14

9.13 and 9.14 For the beam and loading shown, determine the deflection at point C. Use $E = 200$ GPa.



$[x=0, y=0]$ $[x=L, y=0]$
$[x=a, y=y]$
$[x=a, \frac{dy}{dx}=\frac{dy}{dx}]$

SOLUTION

Let $b = L - a$

Reactions: $R_A = \frac{Pb}{L}\uparrow$, $R_B = \frac{Pa}{L}\uparrow$

Bending moments

$0 < x < a \qquad M = \frac{Pb}{L}x$

$a < x < L \qquad M = \frac{P}{L}[bx - L(x-a)]$

$0 < x < a$

$EI\frac{d^2y}{dx^2} = \frac{P}{L}(bx)$

$EI\frac{dy}{dx} = \frac{P}{L}(\frac{1}{2}bx^2) + C_1 \qquad (1)$

$EI\,y = \frac{P}{L}(\frac{1}{6}bx^3) + C_1x + C_2 \qquad (2)$

$a < x < L$

$EI\frac{d^2y}{dx^2} = \frac{P}{L}[bx - L(x-a)]$

$EI\frac{dy}{dx} = \frac{P}{L}[\frac{1}{2}bx^2 - \frac{1}{2}L(x-a)^2] + C_3 \qquad (3)$

$EIy = \frac{P}{L}[\frac{1}{6}bx^3 - \frac{1}{6}L(x-a)^3] + C_3x + C_4 \qquad (4)$

$[x=0, y=0]$ Eq (2) $\quad 0 = 0 + 0 + C_2 \qquad C_2 = 0$

$[x=a, \frac{dy}{dx}=\frac{dy}{dx}]$ Eqs. (1) and (3) $\quad \frac{P}{L}(\frac{1}{2}ba^2) + C_1 = \frac{P}{L}[\frac{1}{2}ba^2 + 0] + C_3 \quad \therefore \quad C_3 = C_1$

$[x=a, y=y]$ Eqs (2) and (4) $\quad \frac{P}{L}(\frac{1}{6}ba^3) + C_1a + C_2$
$= \frac{P}{L}[\frac{1}{6}ba^3 + 0] + C_1a + C_4 \qquad C_4 = C_2 = 0$

$[x=L, y=0]$ Eq (4) $\quad \frac{P}{L}[\frac{1}{6}bL^3 - \frac{1}{6}L(L-a)^3] + C_3L = 0$

$C_1 = C_3 = \frac{P}{L}[\frac{1}{6}(L-a)^3 - \frac{1}{6}bL^2] = \frac{P}{L}(\frac{1}{6}b^3 - \frac{1}{6}bL^2)$

Make $x = a$ in Eq (2)

$y_c = \frac{P}{EIL}[\frac{1}{6}ba^3 + \frac{1}{6}b^3a - \frac{1}{6}bL^2a] = \frac{P(ba^3 + b^3a - L^2ab)}{6EIL}$

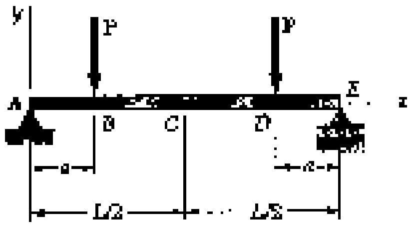
Data: $P = 20\times10^3$ N, $E = 200\times10^9$ Pa, $I = 9.17\times10^6$ mm^4 = 9.17×10^{-6} m^4
$L = 3$ m, $a = 1$ m, $b = 2$ m

$y_c = \frac{(20\times10^3)[(2)(1)^3 + (2)^3(1) - (3)^2(1)(2)]}{(6)(200\times10^9)(9.17\times10^{-6})(3)} = -4.85\times10^{-3}$ m

i.e. 4.85 mm $\downarrow$

PROBLEM 9.15

9.15 Knowing that beam AE is an S200 × 27.4 rolled shape and that $P = 17.5$ kN, $L = 2.5$ m, $a = 0.8$ m and $E = 200$ GPa, determine (a) the equation of the elastic curve for portion BD, (b) the deflection at the center C of the beam.



SOLUTION

Consider portion ABC only, and consider symmetry about C.

Reactions $R_A = R_E = P$

Boundary conditions: $[x=0,\ y=0]$, $[x=a,\ y=y]$, $[x=a,\ \frac{dy}{dx}=\frac{dy}{dx}]$, $[x=\frac{L}{2},\ \frac{dy}{dx}=0]$

$0 < x < a$

$$EI\frac{d^2y}{dx^2} = M = Px$$

$$EI\frac{dy}{dx} = \frac{1}{2}Px^2 + C_1 \qquad (1)$$

$$EIy = \frac{1}{6}Px^3 + C_1x + C_2 \qquad (2)$$

$$[x=0,\ y=0] \rightarrow C_2 = 0$$

$a < x < L-a$

$$EI\frac{d^2y}{dx^2} = M = Pa$$

$$EI\frac{dy}{dx} = Pax + C_3$$

$$EIy = \frac{1}{2}Pax^2 + C_3x + C_4$$

$$[x=\frac{L}{2},\ \frac{dy}{dx}=0] \rightarrow C_3 = -\frac{1}{2}PaL$$

$[x=a,\ \frac{dy}{dx}=\frac{dy}{dx}]$ $\qquad \frac{1}{2}Pa^2 + C_1 = Pa^2 - \frac{1}{2}PaL \qquad C_1 = \frac{1}{2}Pa^2 - \frac{1}{2}PaL$

$[x=a,\ y=y]$ $\qquad \frac{1}{6}Pa^3 + (\frac{1}{2}Pa^2 - \frac{1}{2}PaL)a = \frac{1}{2}Pa^3 - \frac{1}{2}Pa^2L + C_4$

$$C_4 = \frac{1}{6}Pa^3$$

(a) Elastic curve for portion BD

$$y = \frac{1}{EI}\left(\frac{1}{2}Pax^2 + C_3x + C_4\right)$$

$$= \frac{P}{EI}\left(\frac{1}{2}ax^2 - \frac{1}{2}aLx + \frac{1}{6}a^3\right)$$

For deflection at C set $x = \frac{L}{2}$

$$y_C = \frac{P}{EI}\left(\frac{1}{8}aL^2 - \frac{1}{4}aL^2 + \frac{1}{6}a^3\right) = -\frac{Pa}{EI}\left(\frac{1}{8}L^2 - \frac{1}{6}a^2\right)$$

Data: $I = 23.9\times10^6\ \text{mm}^4 = 23.9\times10^{-6}\ \text{m}^4$, $E = 200\times10^9$ Pa

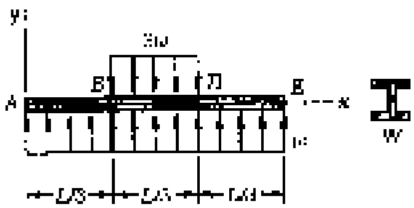
$P = 17.5\times10^3$ N, $L = 2.5$ m, $a = 0.8$ m

(b) $$y_C = -\frac{(17.5\times10^3)(0.8)}{(200\times10^9)(23.9\times10^{-6})}\left\{\frac{2.5^2}{8} - \frac{0.8^2}{6}\right\} = -1.976\times10^{-3}\ \text{m}$$

$$y_C = 1.976\ \text{mm} \downarrow$$

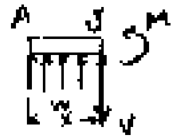
PROBLEM 9.16

9.16 Uniformly distributed loads are applied to beam *AE* as shown. (*a*) Selecting the *x* axis through the centers *A* and *E* of the end sections of the beam, determine the equation of the elastic curve for portion *AB* of the beam. (*b*) Knowing that the beam is a W200 × 35.9 rolled shape and that $L = 3$ m, $w = 5$ kN/m, and $E = 200$ GPa, determine the distance of the center of the beam from the *x* axis.



SOLUTION

$0 < x < \frac{L}{3}$ $\quad \circlearrowleft \Sigma M_J = 0$



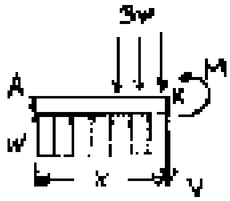
$-(wx)\left(\frac{x}{2}\right) + M = 0$

$M = \frac{1}{2}wx^2$

$EI\frac{d^2y}{dx^2} = \frac{1}{2}wx^2$

$EI\frac{dy}{dx} = \frac{1}{6}wx^3 + C_1$

$EIy = \frac{1}{24}wx^4 + C_1x + C_2$



$\frac{L}{3} \le x \le \frac{2L}{3}$ $\quad +\circlearrowleft \Sigma M_K = 0$ $\quad -(wx)\left(\frac{x}{2}\right) + 3w\left(x - \frac{L}{3}\right)\left(\frac{x - \frac{L}{3}}{2}\right) + M = 0$

$M = \frac{1}{2}wx^2 - \frac{3}{2}w\left(x - \frac{L}{3}\right)^2$

$EI\frac{d^2y}{dx^2} = \frac{1}{2}wx^2 - \frac{3}{2}w\left(x - \frac{L}{3}\right)^2$

$EI\frac{dy}{dx} = \frac{1}{6}wx^3 - \frac{1}{2}w\left(x - \frac{L}{3}\right)^3 + C_3$

$EIy = \frac{1}{24}wx^4 - \frac{1}{8}w\left(x - \frac{L}{3}\right)^4 + C_3x + C_4$

$[x = 0, y = 0]$ $\quad 0 = 0 + 0 + C_2$ $\quad C_2 = 0$

$\left[x = \frac{L}{3}, \frac{dy}{dx} = \frac{dy}{dx}\right]$ $\quad \frac{1}{6}w\left(\frac{L}{3}\right)^3 + C_1 = \frac{1}{6}w\left(\frac{L}{3}\right)^3 + 0 + C_3$ $\quad C_1 = C_3$

$\left[x = \frac{L}{3}, y = y\right]$ $\quad \cancel{\frac{1}{24}w\left(\frac{L}{3}\right)^4} + \cancel{C_1\frac{L}{3}} + C_2 = \cancel{\frac{1}{24}w\left(\frac{L}{3}\right)^4} + 0 + \cancel{C_3\frac{L}{3}} + C_4$

$C_4 = C_2 = 0$

Symmetry boundary condition $\quad \left[x = \frac{L}{2}, \frac{dy}{dx} = 0\right]$

$\frac{1}{6}w\left(\frac{L}{2}\right)^3 - \frac{1}{2}w\left(\frac{L}{2} - \frac{L}{3}\right)^3 + C_3 = 0$ $\quad C_3 = -\left(\frac{1}{48} - \frac{1}{432}\right)wL^3 = -\frac{1}{54}wL^3$

(a) Elastic curve for portion AB

$$y = \frac{1}{EI}\left\{\frac{1}{24}wx^4 + C_1x + C_2\right\} = \frac{w}{EI}\left(\frac{1}{24}x^4 - \frac{1}{54}L^3x\right)$$

(b) Deflection at center $\quad y_c = \frac{1}{EI}\left\{\frac{1}{24}w\left(\frac{L}{2}\right)^4 - \frac{1}{8}w\left(\frac{L}{2} - \frac{L}{3}\right)^4 - \frac{1}{54}wL^3\left(\frac{L}{2}\right) + 0\right\}$

$$= \frac{wL^4}{EI}\left\{\frac{1}{384} - \frac{1}{10368} - \frac{1}{108}\right\} = -\frac{35}{5184}\frac{wL^4}{EI}$$

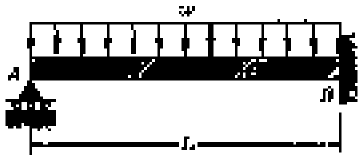
Data: $I = 34.4 \times 10^6\ \text{mm}^4 = 34.4 \times 10^{-6}\ \text{m}^4$, $E = 200 \times 10^9\ \text{Pa}$, $L = 3\ \text{m}$

$w = 5 \times 10^3\ \text{N/m}$

$$y_c = -\frac{35}{5184}\frac{(5 \times 10^3)(3)^4}{(200 \times 10^9)(34.4 \times 10^{-6})} = -397 \times 10^{-6}\ \text{m}$$, i.e. $0.397\ \text{mm} \downarrow$

PROBLEM 9.17

9.17 through 9.20 For the beam and loading shown, determine the reaction at the roller support.

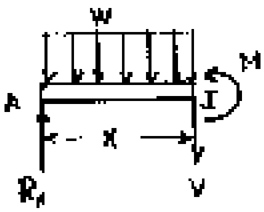


$[x=0,\ y=0]$ $[x=L,\ y=0]$ $[x=L,\ \frac{dy}{dx}=0]$

SOLUTION

Reactions are statically indeterminate.

Boundary conditions are shown at left.



$$\circlearrowleft \Sigma M_J = 0 \qquad -R_A x + (wx)\frac{x}{2} + M = 0$$

$$M = -\frac{1}{2}wx^2 + R_A x$$

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + R_A x$$

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{2}R_A x^2 + C_1$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{6}R_A x^3 + C_1 x + C_2$$

$[x=0,\ y=0]$ $\quad 0 = -0 + 0 + 0 + C_2 \qquad C_2 = 0$

$[x=L,\ \frac{dy}{dx}=0]$ $\quad 0 = -\frac{1}{6}wL^3 + \frac{1}{2}R_A L^2 + C_1 \qquad C_1 = \frac{1}{6}wL^3 - \frac{1}{2}R_A L^2$

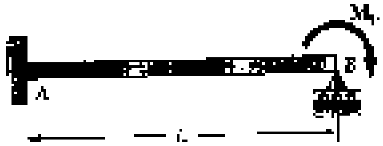
$[x=L,\ y=0]$ $\quad 0 = -\frac{1}{24}wL^4 + \frac{1}{6}R_A L^3 + \left(\frac{1}{6}wL^3 - \frac{1}{2}R_A L^2\right)L + 0$

$$\left(\frac{1}{2}-\frac{1}{6}\right)R_A = \left(\frac{1}{6}-\frac{1}{24}\right)wL \qquad \frac{1}{3}R_A = \frac{1}{8}wL$$

$$R_A = \frac{3}{8}wL \uparrow$$

PROBLEM 9.18

9.17 through 9.20 For the beam and loading shown, determine the reaction at the roller support.



$[x=0, y=0]$ $[x=L, y=0]$

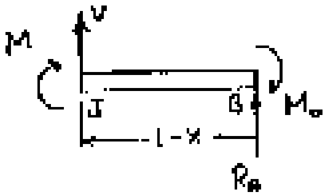
$[x=0, \frac{dy}{dx}=0]$

SOLUTION

Reactions are statically indeterminate.

Boundary condition are shown at left.

Using free body JB



$$\circlearrowleft \Sigma M_J = 0 \qquad -M + R_B(L-x) - M_0 = 0$$

$$M = -M_0 + R_B(L-x)$$

$$EI\frac{d^2y}{dx^2} = -M_0 + R_B(L-x)$$

$$EI\frac{dy}{dx} = -M_0 x + R_B\left(Lx - \tfrac{1}{2}x^2\right) + C_1$$

$$EI\,y = -\tfrac{1}{2}M_0 x^2 + R_B\left(\tfrac{1}{2}Lx^2 - \tfrac{1}{6}x^3\right) + C_1 x + C_2$$

$[x=0, y=0]$ $\quad 0 = -0 + 0 - 0 + 0 + C_2 \qquad C_2 = 0$

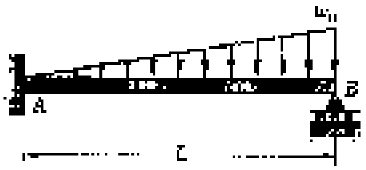
$[x=0, \frac{dy}{dx}=0]$ $\quad 0 = -0 + 0 - 0 + C_1 \qquad C_1 = 0$

$[x=L, y=0]$ $\quad 0 = -\frac{1}{2}M_0L^2 + R_B\left(\frac{1}{2}L^3 - \frac{1}{6}L^3\right)$

$$\frac{1}{3}R_B = \frac{1}{2}\frac{M_0}{L} \qquad R_B = \frac{3}{2}\frac{M_0}{L} \uparrow$$

PROBLEM 9.19

9.17 through 9.20 For the beam and loading shown, determine the reaction at the roller support.



$[x=0, y=0]$ $[x=L, y=0]$

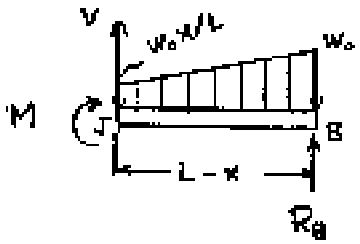
$[x=0, \frac{dy}{dx}=0]$

SOLUTION

Reactions are statically indeterminate.

Boundary conditions are shown at left.

Using free body JB $\circlearrowleft \Sigma M_J = 0$



$$-M + R_B(L-x) + \frac{1}{2}w_0(L-x)\frac{2}{3}(L-x) + \frac{1}{2}\frac{w_0 x}{L}(L-x)\frac{1}{3}(L-x) = 0$$

$$M = R_B(L-x) - \frac{w_0}{6L}\left[2L(L-x)^2 + x(L-x)^2\right]$$

$$= R_B(L-x) - \frac{w_0}{6L}\left[2L^3 - 4L^2x + 2Lx^2 + xL^2 - 2Lx^2 + x^3\right]$$

$$= R_B(L-x) - \frac{w_0}{6L}(x^3 - 3L^2x + 2L^3)$$

$$EI\frac{d^2y}{dx^2} = R_B(L-x) - \frac{w_0}{6L}(x^3 - 3L^2x + 2L^3)$$

$$EI\frac{dy}{dx} = R_B(Lx - \tfrac{1}{2}x^2) - \frac{w_0}{6L}\left(\tfrac{1}{4}x^4 - \tfrac{3}{2}L^2x^2 + 2L^3x\right) + C_1$$

$$EIy = R_B\left(\tfrac{1}{2}Lx^2 - \tfrac{1}{6}x^3\right) - \frac{w_0}{6L}\left(\tfrac{1}{20}x^5 - \tfrac{1}{2}L^2x^3 + L^3x^2\right) + C_1x + C_2$$

$[x=0, y=0]$ $\longrightarrow$ $C_2 = 0$

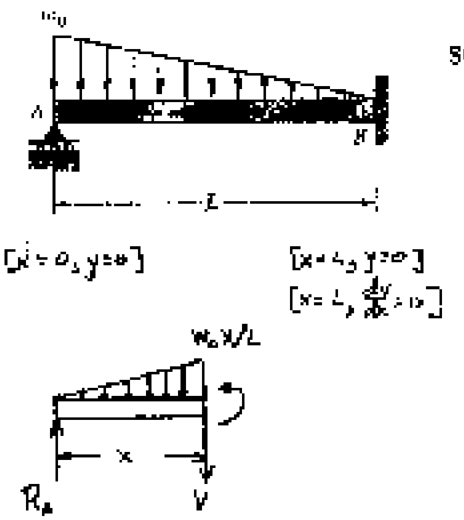
$[x=0, \frac{dy}{dx}=0]$ $\longrightarrow$ $C_1 = 0$

$[x=L, y=0]$ $\quad 0 = R_B L^3\left(\tfrac{1}{2} - \tfrac{1}{6}\right) - \frac{w_0 L^4}{6}\left(\tfrac{1}{20} - \tfrac{1}{2} + 1\right)$

$$\tfrac{1}{3}R_B = \left(\tfrac{1}{6}\right)\left(\tfrac{11}{20}\right)w_0 L \qquad R_B = \tfrac{11}{40}w_0 L \uparrow$$

PROBLEM 9.20

9.17 through 9.20 For the beam and loading shown, determine the reaction at the roller support.



SOLUTION

Reactions are statically indeterminate.

Boundary conditions are shown at left.

$$w = \frac{w_0}{L}(L-x)$$

$$\frac{dV}{dx} = -w = -\frac{w_0}{L}(L-x)$$

$$\frac{dM}{dx} = V = -\frac{w_0}{L}\left(Lx - \frac{1}{2}x^2\right) + R_A$$

$$M = -\frac{w_0}{L}\left(\frac{1}{2}Lx^2 - \frac{1}{6}x^3\right) + R_A x$$

$$EI\frac{d^2y}{dx^2} = -\frac{w_0}{L}\left(\frac{1}{2}Lx^2 - \frac{1}{6}x^3\right) + R_A x$$

$$EI\frac{dy}{dx} = -\frac{w_0}{L}\left(\frac{1}{6}Lx^3 - \frac{1}{24}x^4\right) + \frac{1}{2}R_A x^2 + C_1$$

$$EI\,y = -\frac{w_0}{L}\left(\frac{1}{24}Lx^4 - \frac{1}{120}x^5\right) + \frac{1}{6}R_A x^3 + C_1 x + C_2$$

$[x=0, y=0]$ $\quad 0 = 0 + 0 + 0 + C_2 \qquad C_2 = 0$

$[x=L, \frac{dy}{dx}=0]$ $\quad -\frac{w_0}{L}\left(\frac{1}{6}L^4 - \frac{1}{24}L^4\right) + \frac{1}{2}R_A L^2 + C_1 = 0$

$$C_1 = \frac{1}{8}w_0L^3 - \frac{1}{2}R_A L^2$$

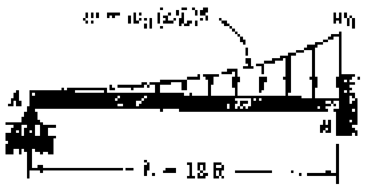
$[x=L, y=0]$ $\quad -\frac{w_0}{L}\left(\frac{1}{24}L^4 - \frac{1}{120}L^4\right) + \frac{1}{6}R_A L^3 + \left(\frac{1}{8}w_0L^3 - \frac{1}{2}R_A L^2\right)L = 0$

$$\left(\frac{1}{2} - \frac{1}{6}\right)R_A = \left(\frac{1}{8} - \frac{1}{24} + \frac{1}{120}\right)w_0 L$$

$$\frac{1}{3}R_A = \frac{11}{120}w_0 L \qquad R_A = \frac{11}{40}w_0 L \uparrow$$

PROBLEM 9.21

9.21 For the beam shown, determine the reaction at the roller support when $w_0 = 6$ kips/ft.



SOLUTION

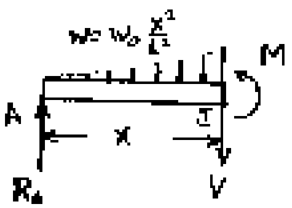
Reactions are statically indeterminate.

Boundary conditions are shown at left

$[x = 0,\ y = 0]$

$[x = L,\ y = 0]$

$[x = L,\ \frac{dy}{dx} = 0]$



$$w = w_0 \frac{x^2}{L^2}$$

$$\frac{dV}{dx} = -w = -\frac{w_0}{L^2} x^2$$

$$\frac{dM}{dx} = V = -\frac{w_0}{L^2}\frac{x^3}{3} + R_A$$

$$M = -\frac{w_0}{L^2}\frac{x^4}{12} + R_A x$$

$$EI\frac{d^2y}{dx^2} = -\frac{w_0}{L^2}\frac{x^4}{12} + R_A x$$

$$EI\frac{dy}{dx} = -\frac{w_0}{L^2}\frac{x^5}{60} + \frac{1}{2}R_A x^2 + C_1$$

$$EI\,y = -\frac{w_0}{L^2}\frac{x^6}{360} + \frac{1}{6}R_A x^3 + C_1 x + C_2$$

$[x = 0,\ y = 0]$ $\quad 0 = 0 + 0 + 0 + C_2 \qquad C_2 = 0$

$[x = L,\ \frac{dy}{dx} = 0]$ $\quad -\frac{1}{60}w_0 L^3 + \frac{1}{2}R_A L^2 + C_1 = 0 \qquad C_1 = \frac{1}{60}w_0 L^3 - \frac{1}{2}R_A L^2$

$[x = L,\ y = 0]$ $\quad -\frac{1}{360}w_0 L^4 + \frac{1}{6}R_A L^3 + \left(\frac{1}{60}w_0 L^3 - \frac{1}{2}R_A L^2\right)L = 0$

$$\left(\frac{1}{2} - \frac{1}{6}\right)R_A = \left(\frac{1}{60} - \frac{1}{360}\right)w_0 L$$

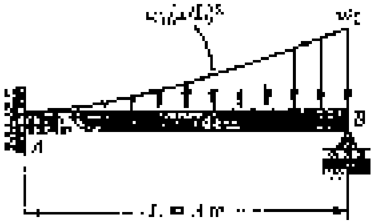
$$\frac{1}{3}R_A = \frac{1}{72}w_0 L \qquad R_A = \frac{1}{18}w_0 L$$

Data: $w_0 = 6$ kips/ft, $L = 12$ ft

$$R_A = \frac{1}{18}(6)(12) = 3 \text{ kips} \uparrow$$

PROBLEM 9.22

9.22 For the beam shown, determine the reaction at the roller support when $w_0 = 15$ kN/m.

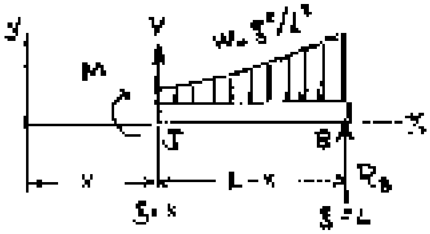


SOLUTION

Reactions are statically indeterminate.

Boundary conditions are shown at left.

$[x=0, y=0]$ $[x=L, y=0]$
$[x=0, \frac{dy}{dx}=0]$



Using free body JB $\circlearrowleft \Sigma M_J = 0$

$$-M + \int_x^L \frac{w_0}{L^2}\xi^2(\xi - x)\,d\xi + R_B(L-x) = 0$$

$$M = \frac{w_0}{L^2}\int_x^L \xi^2(\xi - x)\,d\xi - R_B(L-x)$$

$$= \frac{w_0}{L^2}\left(\frac{1}{4}\xi^4 - \frac{1}{3}x\xi^3\right)\Big|_x^L - R_B(L-x)$$

$$= \frac{w_0}{L^2}\left(\frac{1}{4}L^4 - \frac{1}{3}L^3x + \frac{1}{12}x^4\right) - R_B(L-x)$$

$$EI\frac{d^2y}{dx^2} = \frac{w_0}{L^2}\left(\frac{1}{4}L^4 - \frac{1}{3}L^3x + \frac{1}{12}x^4\right) - R_B(L-x)$$

$$EI\frac{dy}{dx} = \frac{w_0}{L^2}\left(\frac{1}{4}L^4x - \frac{1}{6}L^3x^2 + \frac{1}{60}x^5\right) - R_B\left(Lx - \frac{1}{2}x^2\right) + C_1$$

$$EI\,y = \frac{w_0}{L^2}\left(\frac{1}{8}L^4x^2 - \frac{1}{18}L^3x^3 + \frac{1}{360}x^6\right) - R_B\left(\frac{1}{2}Lx^2 - \frac{1}{6}x^3\right) + C_1x + C_2$$

$[x=0, \frac{dy}{dx}=0]$ $\quad 0 = 0 + 0 + C_1 \qquad C_1 = 0$

$[x=0, y=0]$ $\quad 0 = 0 + 0 + 0 + C_2 \qquad C_2 = 0$

$[x=L, y=0]$ $\quad \left(\frac{1}{8} - \frac{1}{18} + \frac{1}{360}\right)w_0L^4 - \left(\frac{1}{2} - \frac{1}{6}\right)R_BL^3 = 0$

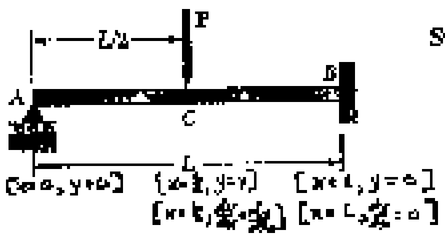
$$\frac{13}{180}w_0L^4 - \frac{1}{3}R_BL^3 = 0 \qquad R_B = \frac{13}{60}w_0L$$

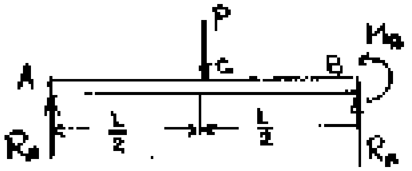
Data: $w_0 = 15\times10^3$ N/m $\qquad L = 3$ m

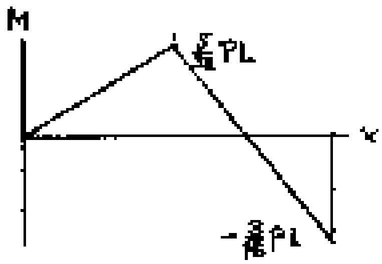
$$R_B = \frac{13}{60}(15\times10^3)(3) = 9.75\times10^3 \text{ N} \qquad 9.75 \text{ kN} \uparrow$$

PROBLEM 9.23

9.13 through 9.26 Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.







SOLUTION

Reactions are statically indeterminate

$0 < x < \frac{L}{2}$

$$EI\frac{d^2y}{dx^2} = M = R_A x \qquad (1)$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 + C_1 \qquad (2)$$

$$EI\,y = \frac{1}{6}R_A x^3 + C_1 x + C_2 \qquad (3)$$

$\frac{L}{2} < x < L$

$$EI\frac{d^2y}{dx^2} = M = R_A x - P\left(x - \frac{L}{2}\right) \qquad (4)$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 - \frac{1}{2}P\left(x - \frac{L}{2}\right)^2 + C_3 \qquad (5)$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{1}{6}P\left(x - \frac{L}{2}\right)^3 + C_3 x + C_4 \qquad (6)$$

$[x=0,\ y=0]$ $\quad 0 = 0 + 0 + C_2 \qquad C_2 = 0$

$[x=\frac{L}{2},\ \frac{dy}{dx}=\frac{dy}{dx}]$ $\quad \frac{1}{2}R_A\left(\frac{L}{2}\right)^2 + C_1 = \frac{1}{2}R_A\left(\frac{L}{2}\right)^2 + 0 + C_3 \qquad C_3 = C_1$

$[x=\frac{L}{2},\ y=y]$ $\quad \frac{1}{6}R_A\left(\frac{L}{2}\right)^3 + C_1\frac{L}{2} + \cancel{C_2} = \frac{1}{6}R_A\left(\frac{L}{2}\right)^3 + 0 + C_3\frac{L}{2} + C_4$

$C_4 = 0$

$[x=L,\ \frac{dy}{dx}=0]$ $\quad \frac{1}{2}R_A L^2 - \frac{1}{2}P\left(\frac{L}{2}\right)^2 + C_3 = 0 \qquad C_3 = \frac{1}{8}PL^2 - \frac{1}{2}R_A L^2$

$[x=L,\ y=0]$ $\quad \frac{1}{6}R_A L^3 - \frac{1}{6}P\left(\frac{L}{2}\right)^3 + \left(\frac{1}{8}PL^2 - \frac{1}{2}R_A L^2\right)L + 0 = 0$

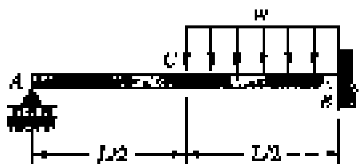
$$\left(\frac{1}{2} - \frac{1}{6}\right)R_A L^3 = \left(\frac{1}{8} - \frac{1}{48}\right)PL^3 \qquad \frac{1}{3}R_A = \frac{5}{48}P \qquad R_A = \frac{5}{16}P\uparrow$$

From (1), with $x = \frac{L}{2}$ $\qquad M_C = R_A\frac{L}{2} = \frac{5}{32}PL$

From (4), with $x = L$ $\qquad M_B = R_A L - \frac{1}{2}PL = -\frac{3}{16}PL$

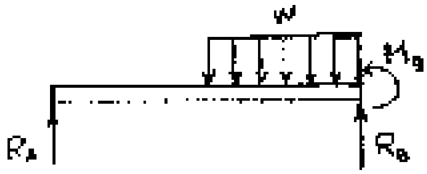
PROBLEM 9.24

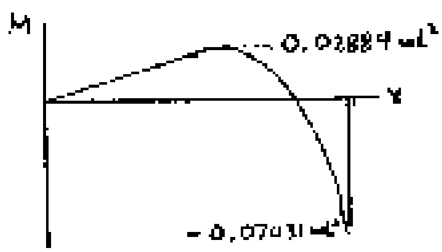
9.23 through 9.26 Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.



SOLUTION

Reactions are statically indeterminate.





$0 < x < \frac{L}{2}$

$$EI\frac{d^2y}{dx^2} = M = R_A x \qquad (1)$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 + C_1 \qquad (2)$$

$$EIy = \frac{1}{6}R_A x^3 + C_1 x + C_2 \qquad (3)$$

$\frac{L}{2} < x < L$

$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{1}{2}w\left(x-\frac{L}{2}\right)^2 \qquad (4)$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 - \frac{1}{6}w\left(x-\frac{L}{2}\right)^3 + C_3 \qquad (5)$$

$$EIy = \frac{1}{6}R_A x^3 - \frac{1}{24}w\left(x-\frac{L}{2}\right)^4 + C_3 x + C_4 \qquad (6)$$

$[x=0,\ y=0]$ $\quad 0 = 0 + 0 + C_2 \qquad C_2 = 0$

$[x=\frac{L}{2},\ \frac{dy}{dx}=\frac{dy}{dx}]$ $\quad \cancel{\frac{1}{2}R_A\left(\frac{L}{2}\right)^2} + C_1 = \cancel{\frac{1}{2}R_A\left(\frac{L}{2}\right)^2} - 0 + C_3 \qquad C_1 = C_3$

$[x=\frac{L}{2},\ y=y]$ $\quad \cancel{\frac{1}{6}R_A\left(\frac{L}{2}\right)^3} + \cancel{C_1\frac{L}{2}} + C_2 = \cancel{\frac{1}{6}R_A\left(\frac{L}{2}\right)^3} - 0 + \cancel{C_3\frac{L}{2}} + C_4 \qquad C_2 = C_4 = 0$

$[x=L,\ \frac{dy}{dx}=0]$ $\quad \frac{1}{2}R_A L^2 - \frac{1}{6}w\left(\frac{L}{2}\right)^3 + C_3 = 0 \qquad C_3 = \frac{1}{48}wL^3 - \frac{1}{2}R_A L^2$

$[x=L,\ y=0]$ $\quad \frac{1}{6}R_A L^3 - \frac{1}{24}w\left(\frac{L}{2}\right)^4 + \left(\frac{1}{48}wL^3 - \frac{1}{2}R_A L^2\right)L + 0 = 0$

$$\left(\frac{1}{2}-\frac{1}{6}\right)R_A L^3 = \left(\frac{1}{48}-\frac{1}{384}\right)wL^4 \qquad \frac{1}{3}R_A = \frac{7}{384}wL \qquad R_A = \frac{7}{128}wL\uparrow$$

From (1), with $x = \frac{L}{2}$ $\qquad M_C = R_A\left(\frac{L}{2}\right) = \frac{7}{256}wL^2 = 0.02734\ wL^2$

From (4), with $x = L$ $\qquad M_B = R_A L - \frac{1}{2}w\left(\frac{L}{2}\right)^2 = \left(\frac{7}{128}-\frac{1}{8}\right)wL^2 = -\frac{9}{128}wL^2$

$$= -0.07031\ wL$$

Location of maximum positive M

$\frac{L}{2} < x < L \qquad V_m = R_A - w\left(x_m - \frac{L}{2}\right) = 0 \qquad x_m - \frac{L}{2} = \frac{R_A}{w} = \frac{7}{128}L$

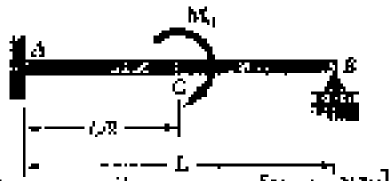
$$x_m = \frac{L}{2} + \frac{7}{128}L = \frac{71}{128}L$$

From (4), with $x = x_m$ $\qquad M_m = R_A x_m - \frac{1}{2}w\left(x_m - \frac{L}{2}\right)^2$

$$= \left(\frac{7}{128}wL\right)\left(\frac{71}{128}L\right) - \frac{1}{2}w\left(\frac{7}{128}L\right)^2 = 0.02884\ wL^2$$

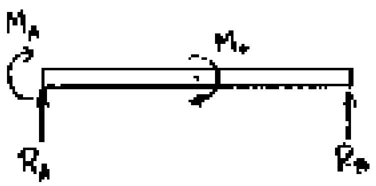
PROBLEM 9.26

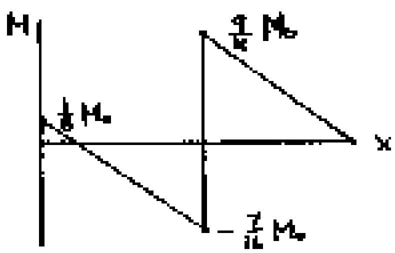
9.23 through 9.26 Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.



$[x=0,\ y=0]$ $[x=L,\ y=0]$

$[x=0,\ \frac{dy}{dx}=0]$





SOLUTION

Reactions are statically indeterminate.

$+\uparrow \Sigma F_y = 0 \qquad R_A + R_B = 0 \qquad R_A = -R_B$

$\circlearrowleft \Sigma M_A = 0 \qquad -M_A - M_0 + R_B L = 0$

$M_A = R_B L - M_0$

$0 < x < \frac{L}{2}$

$M = R_A x + M_A = -M_0 + R_B L - R_B x$

$EI\frac{d^2y}{dx^2} = -M_0 + R_B(L-x)$

$EI\frac{dy}{dx} = -M_0 x + R_B(Lx - \frac{1}{2}x^2) + C_1$

$EI\,y = -\frac{1}{2}M_0x^2 + R_B(\frac{1}{2}Lx^2 - \frac{1}{6}x^3) + C_1 x + C_2$

$\frac{L}{2} < x < L \qquad M = R_B(L-x)$

$EI\frac{d^2y}{dx^2} = R_B(L-x)$

$EI\frac{dy}{dx} = R_B(Lx - \frac{1}{2}x^2) + C_3$

$EI\,y = R_B(\frac{1}{2}Lx^2 - \frac{1}{6}x^3) + C_3 x + C_4$

$[x=0,\ \frac{dy}{dx}=0] \qquad 0 + 0 + C_1 = 0 \qquad C_1 = 0$

$[x=0,\ y=0] \qquad 0 + 0 + 0 + C_2 = 0 \qquad C_2 = 0$

$[x=\frac{L}{2},\ \frac{dy}{dx}=\frac{dy}{dx}] \qquad -M_0\frac{L}{2} + R_B(\frac{1}{2}L^2 - \frac{1}{8}L^2) = R_B(\frac{1}{2}L^2 - \frac{1}{8}L^2) + C_3 \qquad C_3 = -\frac{M_0L}{2}$

$[x=\frac{L}{2},\ y=y] \qquad -\frac{1}{2}M_0(\frac{L}{2})^2 + R_B(\frac{1}{8}L^3 - \frac{1}{48}L^3) = R_B(\frac{1}{8}L^3 - \frac{1}{48}L^3) + C_3\frac{L}{2} + C_4$

$C_4 = -\frac{1}{8}M_0L^2 - \frac{1}{2}C_3L = (-\frac{1}{8} + \frac{1}{4})M_0L^2 = \frac{1}{8}M_0L^2$

$[x=L,\ y=0] \qquad R_B(\frac{1}{2}L^3 - \frac{1}{6}L^3) - \frac{M_0L}{2}L + \frac{1}{8}M_0L^2 = 0$

$(\frac{1}{2} - \frac{1}{6})R_BL^3 = (\frac{1}{2} - \frac{1}{8})M_0L^2 \qquad \frac{1}{3}R_B = \frac{3}{8}\frac{M_0}{L}$

$R_B = \frac{9}{8}\frac{M_0}{L}\uparrow$ ◄

$M_A = \frac{9}{8}M_0 - M_0 = \frac{1}{8}M_0$ ◄

$M_{C^-} = -M_0 + \frac{9}{8}\frac{M_0}{L}\frac{L}{2} = -\frac{7}{16}M_0$ ◄

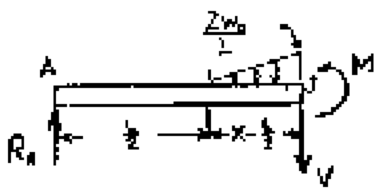
$M_{C^+} = R_B(L - \frac{L}{2}) = \frac{9}{8}\frac{M_0}{L}(\frac{L}{2}) = \frac{9}{16}M_0$ ◄

PROBLEM 9.25

9.23 through 9.26 Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.

SOLUTION

Reactions are statically indeterminate.



$0 \le x \le \frac{L}{2}$

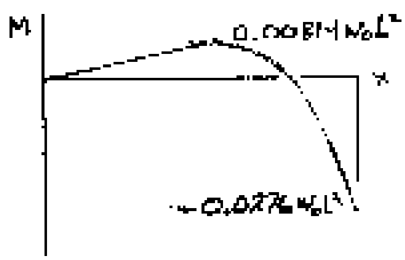
$$EI\frac{d^2y}{dx^2} = M = R_A x \qquad (1)$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 + C_1 \qquad (2)$$

$$EI\,y = \frac{1}{6}R_A x^3 + C_1 x + C_2 \qquad (3)$$

$\frac{L}{2} \le x \le L \qquad \circlearrowleft \Sigma M_J = 0$

$$-R_A x + \frac{1}{2}\frac{2w_0}{L}\left(x-\frac{L}{2}\right)\left(x-\frac{L}{2}\right)\frac{1}{3}\left(x-\frac{L}{2}\right) + M = 0$$



$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{1}{3}\frac{w_0}{L}\left(x-\frac{L}{2}\right)^3 \qquad (4)$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 - \frac{1}{12}\frac{w_0}{L}\left(x-\frac{L}{2}\right)^4 + C_3 \qquad (5)$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{1}{60}\frac{w_0}{L}\left(x-\frac{L}{2}\right)^5 + C_3 x + C_4 \qquad (6)$$

$[x=0,\ y=0] \qquad 0 = 0 + 0 + C_2 \qquad C_2 = 0$

$[x=\frac{L}{2},\ \frac{dy}{dx}=\frac{dy}{dx}] \qquad \frac{1}{2}R_A\left(\frac{L}{2}\right)^2 + C_1 = \frac{1}{2}R_A\left(\frac{L}{2}\right)^2 - 0 + C_3 \qquad C_1 = C_3$

$[x=\frac{L}{2},\ y=y] \qquad \frac{1}{6}R_A\left(\frac{L}{2}\right)^3 + C_1\frac{L}{2} + C_2 = \frac{1}{6}R_A\left(\frac{L}{2}\right)^3 - 0 + C_3\frac{L}{2} + C_4 \qquad C_4 = C_2 = 0$

$[x=L,\ \frac{dy}{dx}=0] \qquad \frac{1}{2}R_A L^2 - \frac{1}{12}\frac{w_0}{L}\left(\frac{L}{2}\right)^4 + C_3 = 0 \qquad C_3 = \frac{1}{192}w_0L^3 - \frac{1}{2}R_A L^2$

$[x=L,\ y=0] \qquad \frac{1}{6}R_A L^3 - \frac{1}{60}\frac{w_0}{L}\left(\frac{L}{2}\right)^5 + \frac{1}{192}w_0L^4 - \frac{1}{2}R_A L^3 + 0 = 0$

$$\left(\frac{1}{2}-\frac{1}{6}\right)R_A L^3 = \left(\frac{1}{192} - \frac{1}{1920}\right)w_0L^4 \qquad \frac{1}{3}R_A = \frac{3}{640}w_0L \qquad R_A = \frac{9}{640}w_0L \uparrow$$

From (1), with $x = \frac{L}{2}$ $\qquad M_C = R_A\frac{L}{2} = \frac{9}{1280}w_0L^2 = 0.007031\ w_0L^2$

From (4), with $x = L$ $\qquad M_B = \frac{9}{640}w_0L^2 - \frac{1}{3}\frac{w_0}{L}\left(\frac{L}{2}\right)^3 = -\frac{53}{1920}w_0L^2$

$$= -0.02761\ w_0L^2$$

Location of maximum positive M in portion CB.

$$\frac{dM}{dx} = R_A - \frac{w_0}{L}\left(x_m - \frac{L}{2}\right)^2 = 0 \qquad x_m - \frac{L}{2} = \sqrt{\frac{R_A L}{w_0}} = \sqrt{\frac{9}{640}}\,L = 0.1186\ L$$

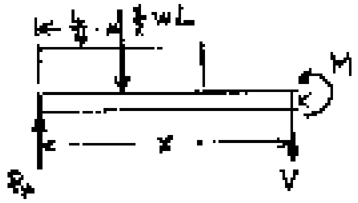
$$x_m = 0.5L + 0.1186\,L = 0.6186\ L$$

From (4), with $x = x_m$ $\qquad M_m = R_A(0.6186\,L) - \frac{1}{3}\frac{w_0}{L}(0.1186\,L)^3$

$$= 0.008143\ w_0L^2$$

PROBLEM 9.27

9.27 and 9.39 Determine the reaction at the roller support and the deflection at point C.



SOLUTION

Reactions are statically indeterminate.

$0 < x < \frac{L}{2}$

$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{1}{2}wx^2$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 - \frac{1}{6}wx^3 + C_1$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{1}{24}wx^4 + C_1 x + C_2$$

$\frac{L}{2} < x < L$ (See free body diagram)

$$+\circlearrowleft \Sigma M_K = 0 \qquad -R_A x + \frac{1}{2}wL\left(x - \frac{L}{4}\right) + M = 0$$

$$EI\frac{d^2y}{dx^2} = M = R_A x - \frac{1}{2}wL\left(x - \frac{1}{4}L\right)$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 - \frac{1}{4}wL\left(x - \frac{L}{4}\right)^2 + C_3$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{1}{12}wL\left(x - \frac{L}{4}\right)^3 + C_3 x + C_4$$

$[x = 0,\ y = 0]$ $\qquad 0 - 0 + 0 + C_2 = 0 \qquad C_2 = 0$

$[x = \frac{L}{2},\ \frac{dy}{dx} = \frac{dy}{dx}]$ $\qquad \frac{1}{2}R_A\left(\frac{L}{2}\right)^2 - \frac{1}{6}w\left(\frac{L}{2}\right)^3 + C_1 = \frac{1}{2}R_A\left(\frac{L}{2}\right)^2 - \frac{1}{4}wL\left(\frac{L}{4}\right)^2 + C_3$

$$C_1 = C_3 + \frac{1}{48}wL^3 - \frac{1}{64}wL^3 = C_3 + \frac{1}{192}wL^3$$

$[x = \frac{L}{2},\ y = y]$ $\qquad \frac{1}{6}R_A\left(\frac{L}{2}\right)^3 - \frac{1}{24}w\left(\frac{L}{2}\right)^4 + \left(\cancel{C_3} + \frac{1}{192}wL^3\right)\frac{L}{2} = \frac{1}{6}R_A\left(\frac{L}{2}\right)^3 - \frac{1}{12}wL\left(\frac{L}{4}\right)^3 + \cancel{C_3\frac{L}{2}} + C_4$

$$C_4 = -\frac{1}{384}wL^4 + \frac{1}{384}wL^4 + \frac{1}{768}wL^4 = \frac{1}{768}wL^4$$

$[x = L,\ \frac{dy}{dx} = 0]$ $\qquad \frac{1}{2}R_A L^2 - \frac{1}{4}wL\left(\frac{3L}{4}\right)^2 + C_3 = 0 \qquad C_3 = \frac{9}{64}wL^3 - \frac{1}{2}R_A L^2$

$[x = L,\ y = 0]$ $\qquad \frac{1}{6}R_A L^3 - \frac{1}{12}wL\left(\frac{3L}{4}\right)^3 + \left(\frac{9}{64}wL^3 - \frac{1}{2}R_A L^2\right)L + \frac{1}{768}wL^4 = 0$

$$\left(\frac{1}{2} - \frac{1}{6}\right)R_A L^3 = \left(\frac{9}{64} - \frac{27}{768} + \frac{1}{768}\right)wL^4 \qquad \frac{1}{3}R_A = \frac{41}{384}wL \qquad R_A = \frac{41}{128}wL \uparrow \blacktriangleleft$$

$$C_3 = \frac{9}{64}wL^3 - \frac{1}{2}\cdot\frac{41}{128}wL^3 = -\frac{5}{256}wL^3$$

$$C_1 = -\frac{5}{256}wL^3 + \frac{1}{192}wL^3 = -\frac{11}{768}wL^3$$

Deflection at C (y at $x = \frac{L}{2}$)

$$y_C = \frac{wL^4}{EI}\left\{\frac{1}{6}\cdot\frac{41}{128}\cdot\left(\frac{1}{2}\right)^3 - \frac{1}{24}\cdot\left(\frac{1}{2}\right)^4 - \frac{11}{768}\cdot\frac{1}{2} + 0\right\}$$

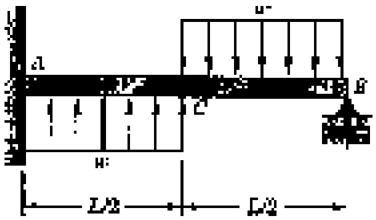
$$= \left(\frac{41}{6144} - \frac{1}{384} - \frac{11}{1536}\right)\frac{wL^4}{EI} = -\frac{19}{6144}\frac{wL^4}{EI} \qquad y_C = \frac{19}{6144}\frac{wL^4}{EI} \downarrow \blacktriangleleft$$

or $$y_C = \frac{wL^4}{EI}\left\{\frac{1}{6}\cdot\frac{41}{128}\left(\frac{1}{2}\right)^3 - \frac{1}{12}\cdot\left(\frac{1}{4}\right)^3 - \frac{5}{256}\cdot\frac{1}{2} + \frac{1}{768}\right\}$$

$$= \left(\frac{41}{6144} - \cancel{\frac{1}{768}} - \frac{5}{512} + \cancel{\frac{1}{768}}\right)\frac{wL^4}{EI} = -\frac{19}{6144}\frac{wL^4}{EI}$$

PROBLEM 9.28

9.27 and 9.28 Determine the reaction at the roller support and the deflection at point C.

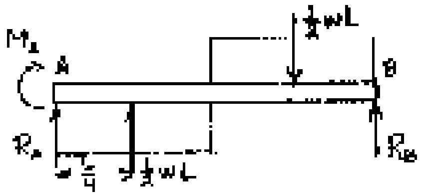


$[x=0,\ y=0]$ $\quad [x=L,\ y=0]$

$[x=0,\ \frac{dy}{dx}=0]$

$[x=\frac{L}{2},\ y=y]$

$[x=\frac{L}{2},\ \frac{dy}{dx}=\frac{dy}{dx}]$



SOLUTION

Reactions are statically indeterminate.

$+\uparrow \Sigma F_y = 0 \qquad R_A + \frac{1}{2}wL - \frac{1}{2}wL + R_B = 0 \qquad R_A = -R_B$

$\circlearrowleft \Sigma M_A = 0 \qquad -M_A - (\frac{1}{2}wL)\frac{L}{2} + R_B L = 0$

$M_A = R_B L - \frac{1}{4}wL^2$

From A to C $\qquad 0 < x \le \frac{L}{2}$

$EI\frac{d^2y}{dx^2} = M = M_A + R_A x + \frac{1}{2}wx^2$

$EI\frac{dy}{dx} = M_A x + \frac{1}{2}R_A x^2 + \frac{1}{6}wx^3 + C_1$

$EI\,y = \frac{1}{2}M_A x^2 + \frac{1}{6}R_A x^3 + \frac{1}{24}wx^4 + C_1 x + C_2$

From C to B $\qquad \frac{L}{2} \le x < L$

$EI\frac{d^2y}{dx^2} = M = M_A + R_A x + \frac{1}{2}wL(x - \frac{L}{4}) - \frac{1}{2}w(x - \frac{L}{2})^2$

$EI\frac{dy}{dx} = M_A x + \frac{1}{2}R_A x^2 + \frac{1}{4}wL(x - \frac{L}{4})^2 - \frac{1}{6}w(x - \frac{L}{2})^3 + C_3$

$EI\,y = \frac{1}{2}M_A x^2 + \frac{1}{6}R_A x^3 + \frac{1}{12}wL(x - \frac{L}{4})^3 - \frac{1}{24}(x - \frac{L}{2})^4 + C_3 x + C_4$

$[x=0,\ \frac{dy}{dx}=0] \qquad 0 + 0 + 0 + C_1 = 0 \qquad C_1 = 0$

$[x=0,\ y=0] \qquad 0 + 0 + 0 + 0 + C_2 = 0 \qquad C_2 = 0$

$[x=\frac{L}{2},\ \frac{dy}{dx}=\frac{dy}{dx}] \qquad \cancel{M_A\frac{L}{2}} + \cancel{\frac{1}{2}R_A(\frac{L}{2})^2} + \frac{1}{6}w(\frac{L}{2})^3 = \cancel{M_A\frac{L}{2}} + \cancel{\frac{1}{2}R_A(\frac{L}{2})^2} + \frac{1}{4}wL(\frac{L}{4})^2 - 0 + C_3$

$C_3 = (\frac{1}{48} - \frac{1}{64})wL^3 = \frac{1}{192}wL^3$

$[x=\frac{L}{2},\ y=y] \qquad \cancel{\frac{1}{2}M_A(\frac{L}{2})^2} + \cancel{\frac{1}{6}R_A(\frac{L}{2})^3} + \frac{1}{24}w(\frac{L}{2})^4$

$= \cancel{\frac{1}{2}M_A(\frac{L}{2})^2} + \cancel{\frac{1}{6}R_A(\frac{L}{2})^3} + \frac{1}{12}wL(\frac{L}{4})^3 - 0 + \frac{1}{192}wL^3(\frac{L}{2}) + C_4$

$C_4 = (\frac{1}{384} - \frac{1}{768} - \frac{1}{384})wL^4 = -\frac{1}{768}wL^4$

$[x=L,\ y=0] \qquad \frac{1}{2}M_A L^2 + \frac{1}{6}R_A L^3 + \frac{1}{12}wL(\frac{3L}{4})^3 - \frac{1}{24}w(\frac{L}{2})^4 + \frac{1}{192}wL^3(L) - \frac{1}{768}wL^4 = 0$

$\frac{1}{2}(R_B L - \frac{1}{4}wL^2)L^2 + \frac{1}{6}(-R_B)L^3 + (\frac{27}{768} - \frac{1}{384} + \frac{1}{192} - \frac{1}{768})wL^4 = 0$

$(\frac{1}{2} - \frac{1}{6})R_B L^3 = (\frac{1}{8} - \frac{7}{192})wL^4 \qquad \frac{1}{3}R_B = \frac{17}{192}wL \qquad ; R_B = \frac{17}{64}wL \uparrow$

$R_A = -R_B = -\frac{17}{64}wL$

$M_A = R_B L - \frac{1}{4}wL^2 = (\frac{17}{64} - \frac{1}{4})wL^2 = \frac{1}{64}wL^2$

(b) Deflection at C $\qquad$ (y at $x = \frac{L}{2}$)

$EI\,y_C = \frac{1}{2}M_A(\frac{L}{2})^2 + \frac{1}{6}R_A(\frac{L}{2})^3 + \frac{1}{24}w(\frac{L}{2})^4 = \frac{1}{2}(\frac{1}{64}wL^2)(\frac{L}{2})^2 + \frac{1}{6}(-\frac{17}{64}wL)(\frac{L}{2})^3 + \frac{1}{24}w(\frac{L}{2})^4$

$= (\frac{1}{512} - \frac{17}{3072} + \frac{1}{384})wL^4 = -\frac{1}{1024}wL^4$

$y_C = \frac{1}{1024}\frac{wL^4}{EI} \downarrow$

PROBLEM 9.29

9.29 and 9.30 Determine the reaction at the roller support and the deflection at point D, knowing that a is equal to $L/3$.

$[x=0, y=0]$ $\quad [x=L, y=0]$ $\quad [x=L, \frac{dy}{dx}=0]$

$[x=a, \frac{dy}{dx}=\frac{dy}{dx}]$ $\quad [x=a, y=y]$

SOLUTION

Reactions are statically indeterminate.

$0 < x < a \qquad M = R_A x$

$$EI\frac{d^2y}{dx^2} = M = R_A x$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 + C_1$$

$$EI\,y = \frac{1}{6}R_A x^3 + C_1 x + C_2$$

$a < x < L \qquad M = R_A x - M_0$

$$EI\frac{d^2y}{dx^2} = R_A x - M_0$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 - M_0(x-a) + C_3$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{1}{2}M_0(x-a)^2 + C_3 x + C_4$$

$[x=0, y=0] \qquad 0 + 0 + C_2 = 0 \qquad C_2 = 0$

$[x=a, \frac{dy}{dx}=\frac{dy}{dx}] \qquad \frac{1}{2}R_A a^2 + C_1 = \frac{1}{2}R_A a^2 - 0 + C_3 \qquad C_1 = C_3$

$[x=a, y=y] \qquad \frac{1}{6}R_A a^3 + C_1 a + C_2 = \frac{1}{6}R_A a^3 + 0 + C_3 a + C_4 \qquad C_2 = C_4 = 0$

$[x=L, \frac{dy}{dx}=0] \qquad \frac{1}{2}R_A L^2 - M_0(L-a) + C_3 = 0 \qquad C_3 = M_0(L-a) - \frac{1}{2}R_A L^2$

$[x=L, y=0] \qquad \frac{1}{6}R_A L^3 - \frac{1}{2}M_0(L-a)^2 + [M_0(L-a) - \frac{1}{2}R_A L^2]L + 0 = 0$

$$\left(\frac{1}{2}-\frac{1}{6}\right)R_A L^3 = M_0\left[(L-a)L - \frac{1}{2}(L-a)^2\right]$$

$$\frac{1}{3}R_A L^3 = M_0\left[L^2 - \cancel{aL} - \frac{1}{2}L^2 + \cancel{aL} - \frac{1}{2}a^2\right] = \frac{1}{2}M_0(L^2-a^2)$$

$$R_A = \frac{3}{2}\frac{M_0}{L^3}(L^2-a^2) = \frac{3}{2}\frac{M_0}{L^3}\left[L^2 - \left(\frac{L}{3}\right)^2\right] = \frac{4}{3}\frac{M_0}{L}\uparrow$$

Deflection at D $\quad$ (y at $x = a = \frac{L}{3}$)

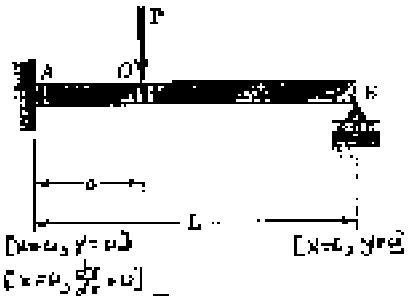
$$y_D = \frac{1}{EI}\left\{\frac{1}{6}R_A\left(\frac{L}{3}\right)^3 + C_1\left(\frac{L}{3}\right)\right\} = \frac{1}{EI}\left\{\frac{1}{6}\left(\frac{4}{3}\frac{M_0}{L}\right)\left(\frac{L}{3}\right)^3 + C_3\left(\frac{L}{3}\right)\right\}$$

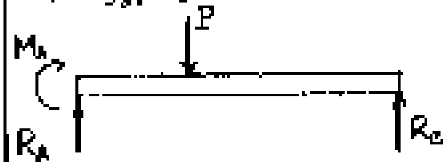
$$= \frac{1}{EI}\left\{\frac{4}{486}M_0L^2 + \left[M_0\left(L-\frac{L}{3}\right) - \frac{1}{2}\cdot\frac{4}{3}\frac{M_0}{L}L^2\right]\frac{L}{3}\right\}$$

$$= \frac{M_0L^2}{EI}\left(\frac{4}{486} + \frac{2}{9} - \frac{4}{18}\right) = \frac{2}{243}\frac{M_0L^2}{EI}\uparrow$$

PROBLEM 9.30

9.29 and 9.30 Determine the reaction at the roller support and the deflection of point D, knowing that a is equal to $L/3$.





SOLUTION

Reactions are statically indeterminate.

$+\uparrow \Sigma F_y = 0 \qquad R_A + R_B - P = 0 \qquad R_A = P - R_B$

$+\circlearrowleft \Sigma M_A = 0 \qquad -M_A - Pa - R_B L = 0$

$M_A = R_B L - Pa$

$0 < x < a \qquad M = M_A + R_A x$

$EI \frac{d^2y}{dx^2} = M = M_A + R_A x$

$EI \frac{dy}{dx} = M_A x + \frac{1}{2} R_A x^2 + C_1$

$EI\, y = \frac{1}{2} M_A x^2 + \frac{1}{6} R_A x^3 + C_1 x + C_2$

$a < x < L \qquad M = M_A + R_A x - P(x-a)$

$EI \frac{d^2y}{dx^2} = M = M_A + R_A x - P(x-a)$

$EI \frac{dy}{dx} = M_A x + \frac{1}{2} R_A x^2 - \frac{1}{2} P (x-a)^2 + C_3$

$EI\, y = \frac{1}{2} M_A x^2 + \frac{1}{6} R_A x^3 - \frac{1}{6} P (x-a)^3 + C_3 x + C_4$

$[x=0, \frac{dy}{dx}=0] \qquad 0 + 0 + C_1 = 0 \qquad C_1 = 0$

$[x=0, y=0] \qquad 0 + 0 + 0 + C_2 = 0 \qquad C_2 = 0$

$[x=a, \frac{dy}{dx} = \frac{dy}{dx}] \qquad \cancel{M_A a} + \cancel{\frac{1}{2} R_A a^2} + C_1 = \cancel{M_A a} + \cancel{\frac{1}{2} R_A a^2} - 0 + C_3 \qquad C_3 = C_1 = 0$

$[x=a, y=y] \qquad \cancel{\frac{1}{2} M_A a^2} + \cancel{\frac{1}{6} R_A a^3} + \cancel{C_1 a} + C_2$

$= \cancel{\frac{1}{2} M_A a^2} + \cancel{\frac{1}{6} R_A a^3} - 0 + \cancel{C_3 a} + C_4 \qquad C_4 = C_2 = 0$

$[x=L, y=0] \qquad \frac{1}{2} M_A L^2 + \frac{1}{6} R_A L^3 - \frac{1}{6} P (L-a)^3 + 0 + 0 = 0$

$\frac{1}{2}(R_B L - Pa) L^2 + \frac{1}{6}(P - R_B) L^3 - \frac{1}{6} P (L-a)^3 = 0$

$(\frac{1}{2} - \frac{1}{6}) R_B L^3 = P \left[\frac{1}{2} a L^2 - \frac{1}{6} L^3 + \frac{1}{6} (L-a)^3 \right]$

$\frac{1}{3} R_B L^3 = P \left[\cancel{\frac{1}{2} a L^2} - \cancel{\frac{1}{6} L^3} + \cancel{\frac{1}{6} L^3} - \cancel{\frac{1}{2} L^2 a} + \frac{1}{2} L a^2 - \frac{1}{6} a^3 \right]$

$= P a^2 (\frac{1}{2} L - \frac{1}{6} a)$

$R_B = \frac{Pa^2}{2L^3}(3L - a) = \frac{P(L/3)^2}{2L^3}(3L - \frac{L}{3}) = \frac{4}{27} P \uparrow$ ◀

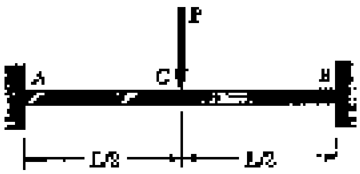
Deflection at D $\quad (y \text{ at } x = a = \frac{L}{3})$

$y_D = \frac{1}{EI}\left\{ \frac{1}{2} M_A (\frac{L}{3})^2 + \frac{1}{6} R_A (\frac{L}{3})^3 \right\} = \frac{1}{EI}\left\{ \frac{1}{18}(R_B L - P\frac{L}{3}) L^2 + \frac{1}{162}(P - R_B) L^3 \right\}$

$= \frac{PL^3}{EI}\left\{ \frac{1}{18}(\frac{4}{27} - \frac{1}{3}) + \frac{1}{162}(1 - \frac{4}{27}) \right\} = -\frac{11}{2187} \frac{PL^3}{EI}, \qquad y_D = \frac{11}{2187} \frac{PL^3}{EI} \downarrow$ ◀

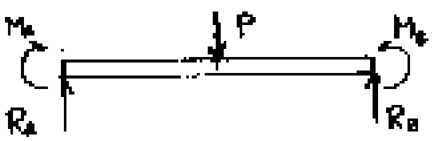
PROBLEM 9.31

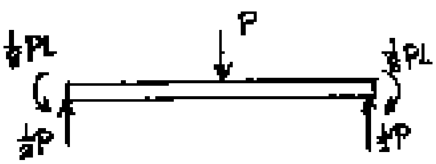
9.31 and 9.32 Determine the reaction at *A* and draw the bending moment diagram for the beam and loading shown.

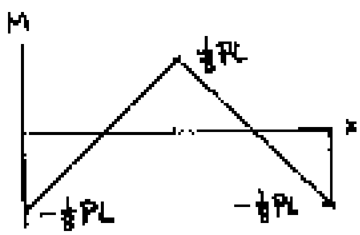


$[x=0,\ y=0]$
$[x=0,\ \frac{dy}{dx}=0]$

$[x=\frac{L}{2},\ \frac{dy}{dx}=0]$







SOLUTION

By symmetry, $R_A = R_B$ and $\frac{dy}{dx} = 0$ at $x = \frac{L}{2}$.

$+\uparrow \Sigma F_y = 0 \quad R_A + R_B - P = 0 \qquad R_A = R_B = \frac{1}{2}P$ ◄

Moment reaction is statically indeterminate.

$0 < x < \frac{L}{2} \qquad M = M_A + R_A x = M_A + \frac{1}{2}Px$

$EI\frac{d^2y}{dx^2} = M_A + \frac{1}{2}Px$

$EI\frac{dy}{dx} = M_A x + \frac{1}{4}Px^2 + C_1$

$[x=0,\ \frac{dy}{dx}=0] \qquad 0 - 0 + C_1 = 0 \qquad C_1 = 0$

$[x=\frac{L}{2},\ \frac{dy}{dx}=0] \qquad M_A\frac{L}{2} + \frac{1}{4}P(\frac{L}{2})^2 + 0 = 0$

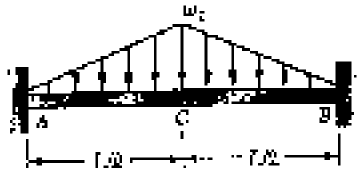
$M_A = -\frac{1}{8}PL \qquad M_A = \frac{1}{8}PL$ ↺ ◄

By symmetry $\quad M_B = M_A = \frac{1}{8}PL$ ↻ ◄

$M_C = M_A + \frac{1}{2}P\frac{L}{2} = -\frac{1}{8}PL + \frac{1}{4}PL = \frac{1}{8}PL$ ◄

PROBLEM 9.32

9.31 and 9.32 Determine the reaction at A and draw the bending moment diagram for the beam and loading shown.

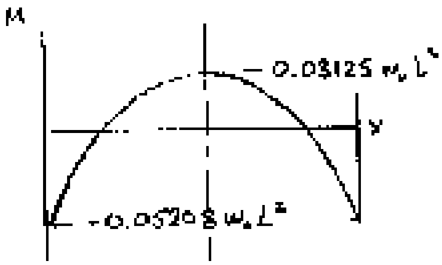


$[x=0, y=0]$ $[x=L, y=0]$

$[x=0, \frac{dy}{dx}=0]$ $[x=L, \frac{dy}{dx}=0]$

$[x=\frac{L}{2}, V=0]$

$[x=\frac{L}{2}, \frac{dy}{dx}=0]$



SOLUTION

Reactions are statically indeterminate.

Because of symmetry $\frac{dy}{dx} = 0$ and $V = 0$ at $x = \frac{L}{2}$.

Use portion AC of beam $(0 < x \le \frac{L}{2})$

$$\frac{dV}{dx} = -w = -2\frac{w_0}{L}x$$

$$\frac{dM}{dx} = V = -\frac{w_0}{L}x^2 + R_A \qquad (1)$$

$$EI\frac{d^2y}{dx^2} = M = -\frac{1}{3}\frac{w_0}{L}x^3 + R_A x + M_A \qquad (2)$$

$$EI\frac{dy}{dx} = -\frac{1}{12}\frac{w_0}{L}x^4 + \frac{1}{2}R_A x^2 + M_A x + C_1 \qquad (3)$$

$$EIy = -\frac{1}{60}\frac{w_0}{L}x^5 + \frac{1}{6}R_A x^3 + \frac{1}{2}M_A x^2 + C_1 x + C_2 \qquad (4)$$

$[x=0, \frac{dy}{dx}=0]$ $\quad 0 = 0 + 0 + 0 + C_1 \qquad C_1 = 0$

$[x=0, y=0]$ $\quad 0 = 0 + 0 + 0 + 0 + C_2 \qquad C_2 = 0$

$[x=\frac{L}{2}, V=0]$ $\quad -\frac{w_0}{L}\left(\frac{L}{2}\right)^2 + R_A = 0 \qquad R_A = \frac{1}{4}w_0 L \uparrow$ ◄

$[x=\frac{L}{2}, \frac{dy}{dx}=0]$ $\quad -\frac{1}{12}\frac{w_0}{L}\left(\frac{L}{2}\right)^4 + \frac{1}{2}\left(\frac{1}{4}w_0 L\right)\left(\frac{L}{2}\right)^2 + M_A\frac{L}{2} + 0 = 0$

$$M_A = -2\left(\frac{1}{32} - \frac{1}{192}\right)w_0 L^2 = -\frac{5}{96}w_0 L^2 = -0.05208\, w_0 L^2$$ ◄

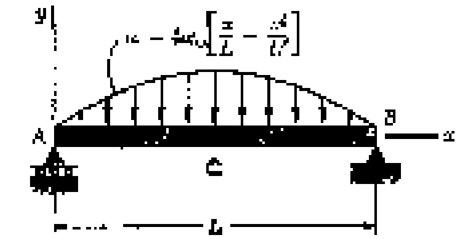
From (2), with $x = \frac{L}{2}$

$$M_C = -\frac{1}{3}\frac{w_0}{L}\left(\frac{L}{2}\right)^3 + \left(\frac{1}{4}w_0 L\right)\left(\frac{L}{2}\right) - \frac{5}{96}w_0 L^2$$

$$= \left(-\frac{1}{24} + \frac{1}{8} - \frac{5}{96}\right)w_0 L^2 = \frac{1}{32}w_0 L^2 = 0.03125\, w_0 L^2$$ ◄

PROBLEM 9.33

9.33 For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the slope at end *A*, (*c*) the deflection at the midpoint of the span.



$[x=0,\ M=0]$ $\quad$ $[x=L,\ M=0]$

$[x=0,\ y=0]$ $\quad$ $[x=L,\ y=0]$

SOLUTION

Boundary conditions at A and B are noted.

$$w = \frac{w_0}{L^2}(4Lx - 4x^2)$$

$$\frac{dV}{dx} = -w = \frac{w_0}{L^2}(4x^2 - 4Lx)$$

$$\frac{dM}{dx} = V = \frac{w_0}{L^2}\left(\frac{4}{3}x^3 - 2Lx^2\right) + C_1$$

$$M = \frac{w_0}{L^2}\left(\frac{1}{3}x^4 - \frac{2}{3}Lx^3\right) + C_1 x + C_2$$

$[x=0,\ M=0]$ $\quad 0 = 0 + 0 + 0 + C_2 \quad\quad C_2 = 0$

$[x=L,\ M=0]$ $\quad 0 = \frac{w_0}{L^2}\left(\frac{1}{3}L^4 - \frac{2}{3}L^4\right) + C_1 L + 0 \quad\quad C_1 = \frac{1}{3}w_0 L^2$

$$EI\frac{d^2y}{dx^2} = M = \frac{w_0}{L^2}\left(\frac{1}{3}x^4 - \frac{2}{3}Lx^3 + \frac{1}{3}L^3 x\right)$$

$$EI\frac{dy}{dx} = \frac{w_0}{L^2}\left(\frac{1}{15}x^5 - \frac{1}{6}Lx^4 + \frac{1}{6}L^3x^2\right) + C_3$$

$$EI\,y = \frac{w_0}{L^2}\left(\frac{1}{90}x^6 - \frac{1}{30}Lx^5 + \frac{1}{18}L^3x^3\right) + C_3 x + C_4$$

$[x=0,\ y=0]$ $\quad 0 = 0 + 0 + 0 + 0 + C_4 \quad\quad C_4 = 0$

$[x=L,\ y=0]$ $\quad 0 = \frac{w_0}{L^2}\left(\frac{1}{90}L^6 - \frac{1}{30}L^6 + \frac{1}{18}L^6\right) + C_3 L + 0 \quad\quad C_3 = -\frac{1}{30}w_0 L^3$

(a) Elastic curve: $\quad y = \frac{w_0}{EIL^2}\left(\frac{1}{90}x^6 - \frac{1}{30}Lx^5 + \frac{1}{18}L^3x^3 - \frac{1}{30}L^5 x\right)$ ◄

$$\frac{dy}{dx} = \frac{w_0}{EIL^2}\left(\frac{1}{15}x^5 - \frac{1}{6}Lx^4 + \frac{1}{6}L^3x^2 - \frac{1}{30}L^5\right)$$

(b) Slope at end A. $\quad$ Set $x = 0$ in $\frac{dy}{dx}$ $\quad\quad \left.\frac{dy}{dx}\right|_A = -\frac{1}{30}\frac{w_0L^3}{EI}$

$$\theta_A = \frac{1}{30}\frac{w_0 L^3}{EI}$$ ◄

(c) Deflection at midpoint. $\quad$ Set $x = \frac{L}{2}$ in y

$$y_C = \frac{w_0L^4}{EI}\left\{\left(\frac{1}{90}\right)\left(\frac{1}{2}\right)^6 - \left(\frac{1}{30}\right)\left(\frac{1}{2}\right)^5 + \frac{1}{18}\left(\frac{1}{2}\right)^3 - \frac{1}{30}\left(\frac{1}{2}\right)\right\}$$

$$= \frac{w_0L^4}{EI}\left\{\frac{1}{5760} - \frac{1}{960} + \frac{1}{144} - \frac{1}{60}\right\} = -\frac{61}{5760}\frac{w_0L^4}{EI}$$

$$y_C = \frac{61}{5760}\frac{w_0L^4}{EI} \downarrow$$ ◄

PROBLEM 9.34

9.34 For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the slope at the free end, (*c*) the deflection at the free end.

$[x=0, V=0]$ $[x=0, M=0]$ $[x=L, \frac{dy}{dx}=0]$ $[x=L, y=0]$

SOLUTION

$$\frac{dV}{dx} = -w = -w_0 \cos\frac{\pi x}{2L}$$

$$V = -\frac{2w_0 L}{\pi}\sin\frac{\pi x}{2L} + C_1$$

$[x=0, V=0]$ $\quad 0 = 0 + C_1 \quad C_1 = 0$

$$\frac{dM}{dx} = V = -\frac{2w_0 L}{\pi}\sin\frac{\pi x}{2L}$$

$$M = \frac{4w_0 L^2}{\pi^2}\cos\frac{\pi x}{2L} + C_2$$

$[x=0, M=0]$ $\quad C_2 = -\frac{4w_0 L^2}{\pi^2}$

$$EI\frac{d^2y}{dx^2} = M = \frac{4w_0 L^2}{\pi^2}\left(\cos\frac{\pi x}{2L} - 1\right)$$

$$EI\frac{dy}{dx} = \frac{4w_0 L^2}{\pi^2}\left(\frac{2L}{\pi}\sin\frac{\pi x}{2L} - x\right) + C_3$$

$[x=L, \frac{dy}{dx}=0]$ $\quad \frac{4w_0 L^2}{\pi^2}\left(\frac{2L}{\pi} - L\right) + C_3 = 0 \qquad C_3 = \frac{4w_0 L^3}{\pi^3}(\pi - 2)$

$$EIy = \frac{4w_0 L^2}{\pi^2}\left[-\frac{4L^2}{\pi^2}\cos\frac{\pi x}{2L} - \frac{1}{2}x^2\right] + C_3 x + C_4$$

$[x=L, y=0]$ $\quad \frac{4w_0 L^2}{\pi^2}\left(-\frac{1}{2}L^2\right) + C_3 L + C_4 = 0$

$$C_4 = \frac{2w_0 L^4}{\pi^2} - C_3 L$$

(a) Elastic curve $\quad y = \frac{w_0}{EI}\left\{-\frac{16L^4}{\pi^4}\cos\frac{\pi x}{2L} - \frac{2L^2x^2}{\pi^2} + \frac{4L^3}{\pi^3}(\pi-2)(x-L) + \frac{2L^4}{\pi^2}\right\}$

$$y = \frac{2w_0 L^4}{\pi^4 EI}\left\{-8\cos\frac{\pi x}{2L} - \pi^2\frac{x^2}{L^2} + 2\pi(\pi-2)\frac{x}{L} + \pi(4-\pi)\right\}$$

(b) Slope at free end. $(x=0)$

$$EI\frac{dy}{dx}\Big|_{x=0} = C_3 = \frac{4(\pi-2)}{\pi^3}w_0 L^3$$

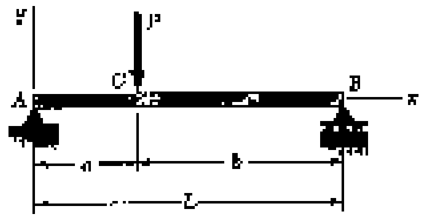
$$\frac{dy}{dx}\Big|_A = \frac{4(\pi-2)}{\pi^3}\frac{w_0 L^3}{EI} = 0.14727\frac{w_0 L^3}{EI}$$

(c) Deflection at free end $(x=0)$

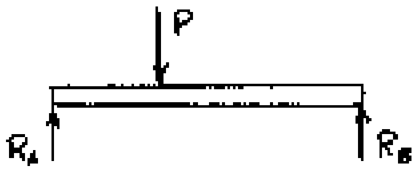
$$y_A = \frac{2w_0 L^4}{\pi^4 EI}\{-8 + \pi(4-\pi)\} = -0.10889\frac{w_0 L^4}{EI}$$

PROBLEM 9.35

9.35 through 9.38 For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the slope at end *A*, (*c*) the deflection of point *C*.



$[x=0,\ M=0]$ $[x=L,\ M=0]$
$[x=0,\ y=0]$ $[x=L,\ y=0]$



SOLUTION

$$+\circlearrowleft \Sigma M_B = 0 \qquad -R_A L + Pb = 0 \qquad R_A = \frac{Pb}{L}$$

$$\frac{dM}{dx} = V = R_A - P\langle x-a\rangle^0 = \frac{Pb}{L} - P\langle x-a\rangle^0$$

$$M = \frac{Pb}{L}x - P\langle x-a\rangle^1 + \cancel{M_A}$$

$$EI\frac{d^2y}{dx^2} = \frac{Pb}{L}x - P\langle x-a\rangle$$

$$EI\frac{dy}{dx} = \frac{Pb}{2L}x^2 - \frac{1}{2}P\langle x-a\rangle^2 + C_1$$

$$EI\,y = \frac{Pb}{6L}x^3 - \frac{1}{6}P\langle x-a\rangle^3 + C_1x + C_2$$

$[x=0,\ y=0]$ $\qquad C_2 = 0$

$[x=L,\ y=0]$ $\qquad \dfrac{Pb}{6L}L^3 - \dfrac{1}{6}P(L-a)^3 + C_1L = 0$

$$C_1 = -\frac{1}{6}\frac{P}{L}(bL^2 - b^3) = -\frac{1}{6}\frac{Pb}{L}(L^2-b^2)$$

(a) Elastic curve

$$y = \frac{P}{EI}\left\{\frac{b}{6L}x^3 - \frac{1}{6}\langle x-a\rangle^3 - \frac{1}{6}\frac{b}{L}(L^2-b^2)x\right\}$$

$$y = \frac{P}{6EIL}\left\{bx^3 - L\langle x-a\rangle^3 - b(L^2-b^2)\right\}$$

(b) Slope at end A.

$$EI\left.\frac{dy}{dx}\right|_{x=0} = C_1 = -\frac{Pb}{6L}(L^2-b^2)$$

$$\theta_A = -\frac{Pb}{6EIL}(L^2-b^2)$$

(c) Deflection at C

$$EI\,y_C = \frac{Pb}{6L}a^3 + C_1a = \frac{Pba^3}{6L} - \frac{Pb(L^2-b^2)a}{6L}$$

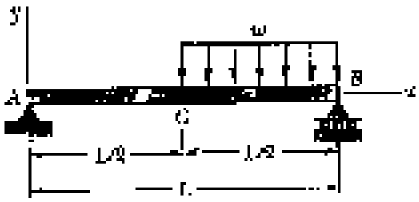
$$= \frac{Pba}{6L}(a^2 - L^2 - b^2)$$

$$y_C = -\frac{Pab}{6EIL}(L^2-a^2-b^2) = -\frac{Pab}{6EIL}\{\cancel{a^2}+2ab+\cancel{b^2}-\cancel{a^2}-\cancel{b^2}\}$$

$$= -\frac{Pa^2b^2}{3EIL}$$

PROBLEM 9.36

9.35 through 9.38 For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the slope at end *A*, (*c*) the deflection of point *C*.



[x=0, M=0] [x=L, M=0]
[x=0, y=0] [x=L, y=0]

SOLUTION

$$\frac{dV}{dx} = -w\langle x-\tfrac{L}{2}\rangle^0$$

$$\frac{dM}{dx} = V = R_A - w\langle x-\tfrac{L}{2}\rangle^1$$

$$M = \cancel{M_A} + R_A x - \tfrac{1}{2}w\langle x-\tfrac{L}{2}\rangle^2$$

$[x=L, M=0]$ $\quad R_A L - \tfrac{1}{2}w\left(\tfrac{L}{2}\right)^2 = 0$

$$R_A = \tfrac{1}{8}wL$$

$$EI\frac{d^2y}{dx^2} = \tfrac{1}{8}wLx - \tfrac{1}{2}w\langle x-\tfrac{L}{2}\rangle^2$$

$$EI\frac{dy}{dx} = \tfrac{1}{16}wLx^2 - \tfrac{1}{6}w\langle x-\tfrac{L}{2}\rangle^3 + C_1$$

$$EIy = \tfrac{1}{48}wLx^3 - \tfrac{1}{24}w\langle x-\tfrac{L}{2}\rangle^4 + C_1x + C_2$$

$[x=0, y=0]$ $\quad 0 = 0 + 0 + 0 + C_2 \qquad C_2 = 0$

$[x=L, y=0]$ $\quad \tfrac{1}{48}wL^4 - \tfrac{1}{24}w\left(\tfrac{L}{2}\right)^4 + C_1L + 0 = 0$

$$C_1 = -\left(\tfrac{1}{48} - \tfrac{1}{24}\cdot\tfrac{1}{16}\right)wL^3 = -\tfrac{7}{384}wL^3$$

(a) Elastic curve

$$EIy = \tfrac{1}{48}wLx^3 - \tfrac{1}{24}w\langle x-\tfrac{L}{2}\rangle^4 - \tfrac{7}{384}wL^3x$$

$$y = \frac{w}{EI}\left\{\tfrac{1}{48}Lx^3 - \tfrac{1}{24}\langle x-\tfrac{L}{2}\rangle^4 - \tfrac{7}{384}L^3x\right\}$$

$$\frac{dy}{dx} = \frac{w}{EI}\left\{\tfrac{1}{16}Lx^2 - \tfrac{1}{6}\langle x-\tfrac{L}{2}\rangle^3 - \tfrac{7}{384}L^3\right\}$$

(b) Slope at A $\quad$ ($x=0$ in $\frac{dy}{dx}$)

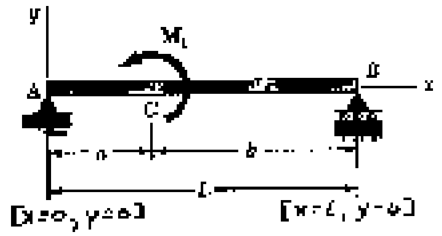
$$\theta_A = -\frac{7}{384}\frac{wL^3}{EI}$$

(c) Deflection at C $\quad$ ($x = \frac{L}{2}$ in y)

$$y_C = \frac{wL^4}{EI}\left\{\tfrac{1}{48}\cdot\tfrac{1}{8} - \tfrac{7}{384}\cdot\tfrac{1}{2}\right\} = \left(\tfrac{1}{384} - \tfrac{7}{768}\right)\frac{wL^4}{EI} = -\frac{5}{768}\frac{wL^4}{EI}$$

PROBLEM 9.37

9.35 through 9.38 For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the slope at end *A*, (*c*) the deflection of point *C*.

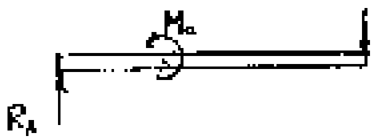


SOLUTION

Reactions $R_A = \frac{M_0}{L} \uparrow$, $R_B = \frac{M_0}{L} \downarrow$

$0 < x < a \qquad M = R_A x$

$a < x < L \qquad M = R_A x - M_0$



Using singularity functions

$$EI \frac{d^2y}{dx^2} = M = R_A x - M_0 \langle x-a \rangle^0$$

$$EI \frac{dy}{dx} = \frac{1}{2} R_A x^2 - M_0 \langle x-a \rangle^1 + C_1$$

$$EI\, y = \frac{1}{6} R_A x^3 - \frac{1}{2} M_0 \langle x-a \rangle^2 + C_1 x + C_2$$

$[x=0, y=0] \qquad 0 = 0 - 0 + 0 + C_2 \qquad C_2 = 0$

$[x=L, y=0] \qquad \frac{1}{6} R_A L^3 - \frac{1}{2} M_0 (L-a)^2 + C_1 L + 0 = 0$

$C_1 L = -\frac{1}{6} \frac{M_0}{L} L^3 + \frac{1}{2} M_0 b^2 \qquad C_1 = \frac{M_0}{6L}(3b^2 - L^2)$

(a) Elastic curve $\quad y = \frac{1}{EI}\left\{ \frac{1}{6} \frac{M_0}{L} x^3 - \frac{1}{2} M_0 \langle x-a \rangle^2 + \frac{M_0}{6L}(3b^2 - L^2)x \right\}$

$$= \frac{M_0}{6EIL}\left\{ x^3 - 3L\langle x-a \rangle^2 + (3b^2 - L^2)x \right\}$$

$$\frac{dy}{dx} = \frac{M_0}{6EIL}\left\{ 3x^2 - 6L\langle x-a \rangle^1 + (3b^2 - L^2) \right\}$$

(b) Slope at A $\quad \left(\frac{dy}{dx} \text{ at } x = 0 \right)$

$$\theta_A = \frac{M_0}{6EIL}\left\{ 0 - 0 + 3b^2 - L^2 \right\} = \frac{M_0}{6EIL}(3b^2 - L^2)$$

(c) Deflection at C $\quad (y \text{ at } x = a)$

$$y_C = \frac{M_0}{6EIL}\left\{ a^3 - 0 + (3b^2 - L^2)a \right\} = \frac{M_0 a}{6EIL}\left\{ a^2 + 3b^2 - (a+b)^2 \right\}$$

$$= \frac{M_0 a}{6EIL}\left\{ \cancel{a^2} + 3b^2 - \cancel{a^2} - 2ab - b^2 \right\} = \frac{M_0 a}{6EIL}\left\{ 2b^2 - 2ab \right\}$$

$$= \frac{M_0 ab}{3EIL}(b-a) \uparrow$$

PROBLEM 9.38

9.35 through 9.38 For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the slope at end *A*, (*c*) the deflection of point *C*.

[x = 0, y = 0]
[x = 0, M = 0]

[x = L, y = 0]
[x = L, M = 0]

SOLUTION

$$w = \frac{2w_0}{L}\langle x - \tfrac{L}{2}\rangle^1$$

$$\frac{dV}{dx} = -w = -\frac{2w_0}{L}\langle x - \tfrac{L}{2}\rangle^1$$

$$\frac{dM}{dx} = V = -\frac{w_0}{L}\langle x - \tfrac{L}{2}\rangle^2 + R_A$$

$$M = -\frac{1}{3}\frac{w_0}{L}\langle x - \tfrac{L}{2}\rangle^3 + R_A x + M_A$$

[x = 0, M = 0] $\quad M_A = 0$

[x = L, M = 0] $\quad -\frac{1}{3}\frac{w_0}{L}\left(\frac{L}{2}\right)^3 + R_A L + 0 = 0 \qquad R_A = \frac{1}{24}w_0 L$

$$EI\frac{d^2y}{dx^2} = M = -\frac{1}{3}\frac{w_0}{L}\langle x - \tfrac{L}{2}\rangle^3 + \frac{1}{24}w_0 L x$$

$$EI\frac{dy}{dx} = -\frac{1}{12}\frac{w_0}{L}\langle x - \tfrac{L}{2}\rangle^4 + \frac{1}{48}w_0 L x^2 + C_1$$

$$EI\,y = -\frac{1}{60}\frac{w_0}{L}\langle x - \tfrac{L}{2}\rangle^5 + \frac{1}{144}w_0 L x^3 + C_1 x + C_2$$

[x = 0, y = 0] $\quad 0 = 0 + 0 + 0 + C_2 \qquad C_2 = 0$

[x = L, y = 0] $\quad -\frac{1}{60}\frac{w_0}{L}\left(\frac{L}{2}\right)^5 + \frac{1}{144}w_0 L^4 + C_1 L + 0 = 0$

$$C_1 = -\left(\frac{1}{144} - \frac{1}{1920}\right)w_0 L^3 = -\frac{37}{5760}w_0 L^3$$

(a) Elastic curve $\quad y = \frac{w_0}{EIL}\left\{-\frac{1}{60}\langle x - \tfrac{L}{2}\rangle^5 + \frac{1}{144}L^2x^3 - \frac{37}{5760}L^4 x\right\}$

$$\frac{dy}{dx} = \frac{w_0}{EIL}\left\{-\frac{1}{12}\langle x - \tfrac{L}{2}\rangle^4 + \frac{1}{48}L^2x^2 - \frac{37}{5760}L^4\right\}$$

(b) Slope at A $\quad \left(\frac{dy}{dx} \text{ at } x = 0\right)$

$$\theta_A = \frac{w_0}{EIL}\left\{0 + 0 - \frac{37}{5760}L^4\right\} = -\frac{37}{5760}\frac{w_0L^3}{EI}, \qquad \theta_A = \frac{37}{5760}\frac{w_0L^3}{EI}$$

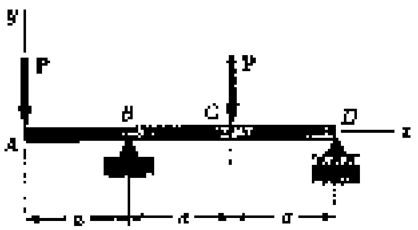
(c) Deflection at C $\quad \left(y \text{ at } x = \frac{L}{2}\right)$

$$y_C = \frac{w_0}{EIL}\left\{-0 + \frac{1}{144}L^2\left(\frac{L}{2}\right)^3 - \frac{37}{5760}L^4\left(\frac{L}{2}\right)\right\}$$

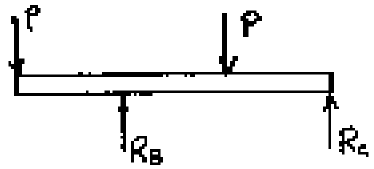
$$= \left(\frac{1}{1152} - \frac{37}{11520}\right)\frac{w_0L^4}{EI} = -\frac{3}{1280}\frac{w_0L^4}{EI}$$

PROBLEM 9.39

9.39 and 9.40 For the beam and loading shown, determine (*a*) the deflection at end *A*. (*b*) the deflection point *C*, (*c*) the slope at end *D*.



$[x=a,\ y=0]$ $[x=3a,\ y=0]$



SOLUTION

Reactions: $R_A = 2P \uparrow$, $R_D = 0$

$0 < x < a \qquad V = -P$

$a < x < 2a \qquad V = -P + 2P$

$2a < x < 3a \qquad V = -P + 2P - P$

Using singularity functions

$$\frac{dM}{dx} = V = -P + 2P\langle x-a\rangle^0 - P\langle x-2a\rangle^0$$

$$M = -Px + 2P\langle x-a\rangle^1 - P\langle x-2a\rangle^1 + M_A$$

But $M = 0$ at $x=0$ $\qquad M_A = 0$

$$EI\frac{d^2y}{dx^2} = M = -Px + 2P\langle x-a\rangle^1 - P\langle x-2a\rangle^1 \qquad (1)$$

$$EI\frac{dy}{dx} = -\frac{1}{2}Px^2 + P\langle x-a\rangle^2 - \frac{1}{2}P\langle x-2a\rangle^2 + C_1 \qquad (2)$$

$$EI\,y = -\frac{1}{6}Px^3 + \frac{1}{3}P\langle x-a\rangle^3 - \frac{1}{6}P\langle x-2a\rangle^3 + C_1x + C_2 \qquad (3)$$

$[x=a,\ y=0]$ $\quad -\frac{1}{6}Pa^3 + 0 - 0 + C_1a + C_2 = 0 \qquad aC_1 + C_2 = \frac{1}{6}Pa^3 \quad (4)$

$[x=3a,\ y=0]$ $\quad -\frac{1}{6}P(3a)^3 + \frac{1}{3}P(2a)^3 - \frac{1}{6}Pa^3 + C_1(3a) + C_2 = 0 \qquad 3aC_1 + C_2 = 2Pa^3 \quad (5)$

Eq(5) - Eq(4) $\qquad 2C_1a = \frac{11}{6}Pa^3 \qquad C_1 = \frac{11}{12}Pa^2$

$$C_2 = \frac{1}{6}Pa^3 - aC_1 = -\frac{3}{4}Pa^3$$

$$y = \frac{P}{EI}\left\{-\frac{1}{6}x^3 + \frac{1}{3}\langle x-a\rangle^3 - \frac{1}{6}\langle x-2a\rangle^3 + \frac{11}{12}a^2x - \frac{3}{4}a^3\right\}$$

$$\frac{dy}{dx} = \frac{P}{EI}\left\{-\frac{1}{2}x^2 + \langle x-a\rangle^2 - \frac{1}{2}\langle x-2a\rangle^2 + \frac{11}{12}a^2\right\}$$

(a) Deflection at A $\quad$ (y at $x=0$)

$$y_A = \frac{Pa^3}{EI}\left\{0 + 0 - 0 + 0 - \frac{3}{4}\right\} = -\frac{3}{4}\frac{Pa^3}{EI} \qquad y_A = \frac{3}{4}\frac{Pa^3}{EI}\downarrow$$

(b) Deflection at C $\quad$ (y at $x=2a$)

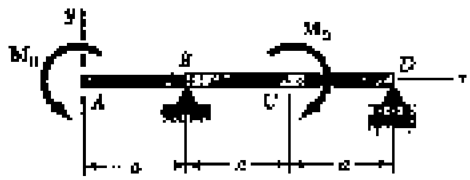
$$y_C = \frac{Pa^3}{EI}\left\{-\frac{1}{6}(2)^3 + \frac{1}{3}(1)^3 - 0 + \frac{11}{12}(2) - \frac{3}{4}\right\} = \frac{1}{12}\frac{Pa^3}{EI}\uparrow$$

(c) Slope at D $\quad$ ($\frac{dy}{dx}$ at $x=3a$)

$$\theta_D = \frac{Pa^2}{EI}\left\{-\frac{1}{2}(3)^2 + (2)^2 - \frac{1}{2}(1)^2 + \frac{11}{12}\right\} = -\frac{1}{12}\frac{Pa^2}{EI} \qquad \theta_D = \frac{1}{12}\frac{Pa^2}{EI}$$

PROBLEM 9.40

9.39 and 9.40 For the beam and loading shown, determine (a) the deflection at end A, (b) the deflection point C, (c) the slope at end D.



$[x=a, y=0]$ $[x=3a, y=0]$

SOLUTION

Since loads self equilibrate

$R_B = 0$ $R_D = 0$

$0 < x < 2a$ $M = -M_0$

$2a < x < 3a$ $M = -M_0 + M_0 = 0$

Using singularity functions

$$EI\frac{d^2y}{dx^2} = M = -M_0 + M_0\langle x-2a\rangle^0$$

$$EI\frac{dy}{dx} = -M_0 x + M_0\langle x-2a\rangle^1 + C_1$$

$$EIy = -\tfrac{1}{2}M_0x^2 + \tfrac{1}{2}M_0\langle x-2a\rangle^2 + C_1x + C_2$$

$[x=3a, y=0]$ $-\tfrac{1}{2}M_0(3a)^2 + \tfrac{1}{2}M_0a^2 + C_1(3a) + C_2 = 0$ $3aC_1 + C_2 = 4M_0a^2$

$[x=a, y=0]$ $-\tfrac{1}{2}M_0a^2 + 0 + C_1a + C_2 = 0$ $aC_1 + C_2 = \tfrac{1}{2}M_0a^2$

Subtracting $2aC_1 = \tfrac{7}{2}M_0a^2$ $C_1 = \tfrac{7}{4}M_0a$

$$C_2 = \tfrac{1}{2}M_0a^2 - aC_1 = -\tfrac{5}{4}M_0a^2$$

$$y = \frac{M_0}{EI}\left\{-\tfrac{1}{2}x^2 + \tfrac{1}{2}\langle x-2a\rangle^2 + \tfrac{7}{4}ax - \tfrac{5}{4}a^2\right\}$$

$$\frac{dy}{dx} = \frac{M_0}{EI}\left\{-x + \langle x-a\rangle^1 + \tfrac{7}{4}a\right\}$$

(a) Deflection at A $(y \text{ at } x=0)$

$$y_A = \frac{M_0a^2}{EI}\left\{-0 + 0 + 0 - \tfrac{5}{4}\right\} = -\tfrac{5}{4}\frac{M_0a^2}{EI}, \quad y_A = \tfrac{5}{4}\frac{M_0a^2}{EI}\downarrow$$

(b) Deflection at C $(y \text{ at } x=2a)$

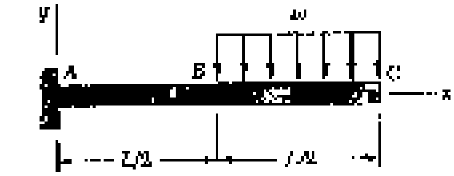
$$y_C = \frac{M_0a^2}{EI}\left\{-\tfrac{1}{2}(2)^2 + 0 + \tfrac{7}{4}(2) - \tfrac{5}{4}\right\} = \tfrac{1}{4}\frac{M_0a^2}{EI}\uparrow$$

(c) Slope at D $\left(\frac{dy}{dx} \text{ at } x=3a\right)$

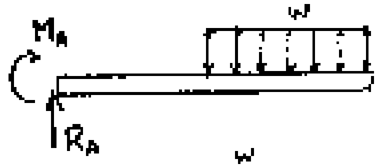
$$\theta_D = \frac{M_0a}{EI}\left\{-3 + 1 + \tfrac{7}{4}\right\} = -\tfrac{1}{4}\frac{M_0a}{EI}, \quad \theta_D = \tfrac{1}{4}\frac{M_0a}{EI}$$

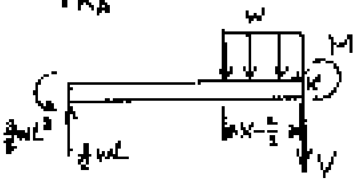
PROBLEM 9.41

9.41 For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the deflection at point *B*, (*c*) the deflection at point *C*.



$[x=0,\ \frac{dy}{dx}=0]$
$[x=0,\ y=0]$





SOLUTION

$+\uparrow \Sigma F_y = 0 \qquad R_A - \frac{1}{2}wL = 0 \qquad R_A = \frac{1}{2}wL$

$\circlearrowleft \Sigma M_A = 0 \qquad -M_A - \left(\frac{1}{2}wL\right)\left(\frac{3}{4}L\right) = 0$

$M_A = -\frac{3}{8}wL^2$

$0 < x < \frac{L}{2} \qquad M = -\frac{3}{8}wL^2 + \frac{1}{2}wLx$

$\frac{L}{2} < x < L$ (See free body diagram.)

$\circlearrowleft \Sigma M_K = 0$

$\frac{3}{8}wL^2 - \frac{1}{2}wLx + \frac{1}{2}w\left(x-\frac{L}{2}\right)^2 + M = 0$

$M = -\frac{3}{8}wL^2 + \frac{1}{2}wLx - \frac{1}{2}w\left(x-\frac{L}{2}\right)^2$

Using singularity functions

$EI\frac{d^2y}{dx^2} = M = -\frac{3}{8}wL^2 + \frac{1}{2}wLx - \frac{1}{2}w\left\langle x-\frac{L}{2}\right\rangle^2$

$EI\frac{dy}{dx} = -\frac{3}{8}wL^2x + \frac{1}{4}wLx^2 - \frac{1}{6}w\left\langle x-\frac{L}{2}\right\rangle^3 + C_1$

$[x=0,\ \frac{dy}{dx}=0] \qquad -0+0-0+C_1 = 0 \qquad C_1 = 0$

$EIy = -\frac{3}{16}wL^2x^2 + \frac{1}{12}wLx^3 - \frac{1}{24}w\left\langle x-\frac{L}{2}\right\rangle^4 + C_1x + C_2$

$[x=0,\ y=0] \qquad -0+0-0+0+C_2 = 0 \qquad C_2 = 0$

(a) Elastic curve $\qquad y = \frac{w}{EI}\left\{-\frac{3}{16}L^2x^2 + \frac{1}{12}Lx^3 - \frac{1}{24}\left\langle x-\frac{L}{2}\right\rangle^4\right\}$

$\frac{dy}{dx} = \frac{w}{EI}\left\{-\frac{3}{8}L^2x + \frac{1}{4}Lx^2 - \frac{1}{6}\left\langle x-\frac{L}{2}\right\rangle^3\right\}$

(b) Deflection at B $\quad$ (y at $x = \frac{L}{2}$)

$y_B = \frac{wL^4}{EI}\left\{-\frac{3}{16}\left(\frac{1}{2}\right)^2 + \frac{1}{12}\left(\frac{1}{2}\right)^3 - 0\right\} = -\frac{7}{192}\frac{wL^4}{EI}, \qquad y_B = \frac{7}{192}\frac{wL^4}{EI}\downarrow$

(c) Deflection at C $\quad$ (y at $x = L$)

$y_C = \frac{wL^4}{EI}\left\{-\frac{3}{16}(1)^2 + \frac{1}{12}(1)^3 - \frac{1}{24}\left(\frac{1}{2}\right)^4\right\} = -\frac{41}{384}\frac{wL^4}{EI}, \qquad y_C = \frac{41}{384}\frac{wL^4}{EI}\downarrow$

PROBLEM 9.42

9.42 and 9.43 For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the deflection at the midpoint *C*.

[x = 0, y = 0]
[x = 0, M = 0]
[x = 4a, y = 0]
[x = 4a, M = 0]

SOLUTION

By symmetry $R_A = R_B$

$+\uparrow \Sigma F_y = 0 \qquad R_A + R_B - 2wa = 0 \qquad R_A = wa$

$$w(x) = w\langle x-a\rangle^0 - w\langle x-3a\rangle^0$$

$$\frac{dV}{dx} = -w(x) = -w\langle x-a\rangle^0 + w\langle x-3a\rangle^0$$

$$\frac{dM}{dx} = V = R_A - w\langle x-a\rangle^1 + w\langle x-3a\rangle^1$$

$$M = M_A + R_A x - \tfrac{1}{2}w\langle x-a\rangle^2 + \tfrac{1}{2}\langle x-3a\rangle^2 \qquad \text{with } M_A = 0$$

$$EI\frac{d^2y}{dx^2} = M = wax - \tfrac{1}{2}w\langle x-a\rangle^2 + \tfrac{1}{2}w\langle x-3a\rangle^2$$

$$EI\frac{dy}{dx} = \tfrac{1}{2}wax^2 - \tfrac{1}{6}w\langle x-a\rangle^3 + \tfrac{1}{6}w\langle x-3a\rangle^3 + C_1$$

$$EI\,y = \tfrac{1}{6}wax^3 - \tfrac{1}{24}w\langle x-a\rangle^4 + \tfrac{1}{24}w\langle x-3a\rangle^4 + C_1x + C_2$$

$[x = 0,\ y = 0] \qquad 0 - 0 + 0 + 0 + C_2 = 0 \qquad C_2 = 0$

$[x = 4a,\ y = 0] \qquad \tfrac{1}{6}wa(4a)^3 - \tfrac{1}{24}w(3a)^4 + \tfrac{1}{24}(a)^4 + C_1(4a) = 0$

$$4C_1 = wa^3\left\{-\frac{64}{6} + \frac{81}{24} - \frac{1}{24}\right\} = -\frac{22}{3}wa^3 \qquad C_1 = -\frac{11}{6}wa^3$$

(a) Equation of elastic curve

$$y = \frac{w}{EI}\left\{\frac{1}{6}ax^3 - \frac{1}{24}\langle x-a\rangle^4 + \frac{1}{24}\langle x-3a\rangle^4 - \frac{11}{6}a^3x\right\}$$

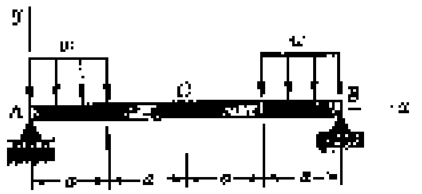
(b) Deflection at C $\quad (y \text{ at } x = 2a)$

$$y_C = \frac{wa^4}{EI}\left\{\frac{1}{6}(2)^3 - \frac{1}{24}(1)^4 + 0 - \frac{11}{6}(2)\right\} = -\frac{19}{8}\frac{wa^4}{EI}$$

$$y_C = \frac{19}{8}\frac{wa^4}{EI}\downarrow$$

PROBLEM 9.43

9.42 and 9.43 For the beam and loading shown, determine (a) the equation of the elastic curve, (b) the deflection at the midpoint C.



$[x=0, y=0]$ $[x=4a, y=0]$

$[x=0, M=0]$ $[x=4a, M=0]$

SOLUTION

By symmetry $R_A = R_B$

$+\uparrow \Sigma F_y = 0 \qquad R_A + R_B - 2wa = 0 \qquad R_A = wa$

$$w(x) = w - w\langle x-a\rangle^0 + w\langle x-3a\rangle^0$$

$$\frac{dV}{dx} = -w(x) = -w + w\langle x-a\rangle^0 - w\langle x-3a\rangle^0$$

$$\frac{dM}{dx} = V = R_A - wx + w\langle x-a\rangle^1 - w\langle x-3a\rangle^1$$

$$M = M_A + R_A x - \frac{1}{2}wx^2 + \frac{1}{2}w\langle x-a\rangle^2 - \frac{1}{2}w\langle x-3a\rangle^2 \qquad \text{with } M_A = 0$$

$$EI\frac{d^2y}{dx^2} = M = wax - \frac{1}{2}wx^2 + \frac{1}{2}w\langle x-a\rangle^2 - \frac{1}{2}w\langle x-3a\rangle^2$$

$$EI\frac{dy}{dx} = \frac{1}{2}wax^2 - \frac{1}{6}wx^3 + \frac{1}{6}w\langle x-a\rangle^3 - \frac{1}{6}w\langle x-3a\rangle^3 + C_1$$

$$EI\,y = \frac{1}{6}wax^3 - \frac{1}{24}wx^4 + \frac{1}{24}w\langle x-a\rangle^4 - \frac{1}{24}w\langle x-3a\rangle^4 + C_1x + C_2$$

$[x=0, y=0] \qquad 0 - 0 + 0 - 0 + 0 + C_2 = 0 \qquad C_2 = 0$

$[x=4a, y=0] \qquad \frac{1}{6}wa(4a)^3 - \frac{1}{24}w(4a)^4 + \frac{1}{24}w(3a)^4 - \frac{1}{24}wa^4 + C_1(4a) = 0$

$$4C_1 = wa^3\left\{-\frac{64}{6} + \frac{256}{24} - \frac{81}{24} + \frac{1}{24}\right\} = -\frac{10}{3}wa^3 \qquad C_1 = -\frac{5}{6}wa^3$$

(a) Equation of elastic curve

$$y = \frac{w}{EI}\left\{\frac{1}{6}ax^3 - \frac{1}{24}x^4 + \frac{1}{24}\langle x-a\rangle^4 - \frac{1}{24}\langle x-3a\rangle^4 - \frac{5}{6}a^3x\right\}$$

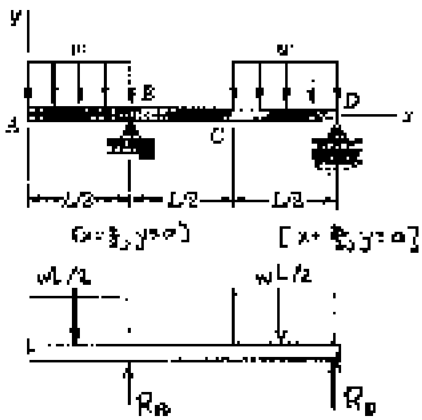
(b) Deflection at C $\quad$ (y at $x = 2a$)

$$y_C = \frac{wa^4}{EI}\left\{\frac{1}{6}(2)^3 - \frac{1}{24}(2)^4 + \frac{1}{24}(1)^4 + 0 - \frac{5}{6}(2)\right\} = -\frac{23}{24}\frac{wa^4}{EI}$$

$$y_C = \frac{23}{24}\frac{wa^4}{EI}\downarrow$$

PROBLEM 9.44

9.44 For the beam and loading shown, determine (*a*) the equation of the elastic curve, (*b*) the deflection at point *A*, (*c*) the deflection at point *C*.



SOLUTION

$$\circlearrowleft M_D = 0 \qquad \frac{wL}{2}\cdot\frac{5L}{4} - R_B L + \frac{wL}{2}\cdot\frac{L}{4} = 0$$

$$R_B = \frac{3}{4}wL$$

$$w(x) = w - w\langle x-\tfrac{L}{2}\rangle^0 + w\langle x-L\rangle^0$$

$$\frac{dV}{dx} = -w(x) = -w + w\langle x-\tfrac{L}{2}\rangle^0 - w\langle x-L\rangle^0$$

$$\frac{dM}{dx} = V = -wx + R_B\langle x-\tfrac{L}{2}\rangle^0 + w\langle x-\tfrac{L}{2}\rangle^1 - w\langle x-L\rangle^1$$

$$EI\frac{d^2y}{dx^2} = M = -\tfrac{1}{2}wx^2 + \tfrac{3}{4}wL\langle x-\tfrac{L}{2}\rangle^1 + \tfrac{1}{2}w\langle x-\tfrac{L}{2}\rangle^2 - \tfrac{1}{2}w\langle x-L\rangle^2$$

$$EI\frac{dy}{dx} = -\tfrac{1}{6}wx^3 + \tfrac{3}{8}wL\langle x-\tfrac{L}{2}\rangle^2 + \tfrac{1}{6}w\langle x-\tfrac{L}{2}\rangle^3 - \tfrac{1}{6}w\langle x-L\rangle^3 + C_1$$

$$EI\,y = -\tfrac{1}{24}wx^4 + \tfrac{1}{8}wL\langle x-\tfrac{L}{2}\rangle^3 + \tfrac{1}{24}w\langle x-\tfrac{L}{2}\rangle^4 - \tfrac{1}{24}w\langle x-L\rangle^4 + C_1x + C_2$$

$$[x=\tfrac{L}{2},\ y=0] \quad -\tfrac{1}{24}w(\tfrac{L}{2})^4 + 0 + 0 - 0 + C_1\tfrac{L}{2} + C_2 = 0$$

$$C_2 = \tfrac{1}{384}wL^4 - C_1\tfrac{L}{2}$$

$$[x=\tfrac{3L}{2},\ y=0] \quad -\tfrac{1}{24}w(\tfrac{3L}{2})^4 + \tfrac{1}{8}wL(L)^3 + \tfrac{1}{24}wL^4 - \tfrac{1}{24}w(\tfrac{L}{2})^4 + C_1\tfrac{3L}{2} + (\tfrac{1}{384}wL^4 - C_1\tfrac{L}{2}) = 0$$

$$(\tfrac{3}{2}-\tfrac{1}{2})C_1L = (\tfrac{1}{24}\cdot\tfrac{81}{16} - \tfrac{1}{8} - \tfrac{1}{24} + \tfrac{1}{24}\cdot\tfrac{1}{16} - \tfrac{1}{384})wL^4 \qquad C_1 = \tfrac{17}{384}wL^3$$

$$C_2 = (\tfrac{1}{384} - \tfrac{17}{768})wL^4 = -\tfrac{5}{256}wL^4$$

(a) $$y = \frac{w}{EI}\left\{-\tfrac{1}{24}x^4 + \tfrac{1}{8}L\langle x-\tfrac{L}{2}\rangle^3 + \tfrac{1}{24}\langle x-\tfrac{L}{2}\rangle^4 - \tfrac{1}{24}\langle x-L\rangle^4 + \tfrac{17}{384}L^3x - \tfrac{5}{256}L^4\right\}$$

(b) Deflection at A (y at x = 0)

$$y_A = \frac{w}{EI}\left\{0+0+0+0+0-\tfrac{5}{256}L^4\right\} = -\frac{5}{256}\frac{wL^4}{EI}$$

$$y_A = \frac{5}{256}\frac{wL^4}{EI}\downarrow$$

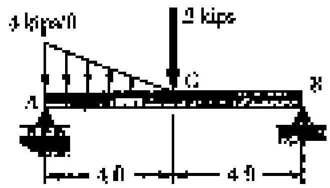
(c) Deflection at C (y at x = L)

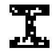
$$y_C = \frac{w}{EI}\left\{-\tfrac{1}{24}L^4 + \tfrac{1}{8}L\tfrac{L^3}{8} + \tfrac{1}{24}\tfrac{L^4}{16} - 0 + \tfrac{17}{384}L^3L - \tfrac{5}{256}L^4\right\}$$

$$= \frac{1}{768}\frac{wL^4}{EI}\uparrow$$

PROBLEM 9.45

9.45 For the beam and loading shown, determine (*a*) the slope at end *A*, (*b*) the deflection at the midpoint *C*. Use $E = 29 \times 10^6$ psi.





S8 × 18.5

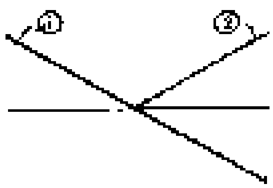
SOLUTION

Distributed loads: ① $w_1(x) = w_0 - kx$

② $w_2(x) = kx$

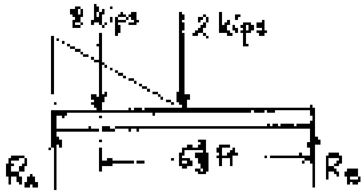
Data: $a = 4$ ft, $w_0 = 4$ kips/ft, $k = 1$ kip/ft^2

$P = 2$ kips.



$$+\circlearrowleft \Sigma M_B = 0 \qquad -8R_A + (8)(6\tfrac{2}{3}) + (2)(4) = 0 \qquad R_A = \tfrac{23}{3} \text{ kips}$$

$$w(x) = w_0 - kx + k\langle x-4\rangle^1$$
$$= 4 - x + \langle x-4\rangle^1$$



$$\frac{dV}{dx} = -w = -4 + x - \langle x-4\rangle^1$$

$$\frac{dM}{dx} = V = \frac{23}{3} - 4x + \frac{1}{2}x^2 - \frac{1}{2}\langle x-4\rangle^2 - 2\langle x-4\rangle^0$$

$$EI\frac{d^2y}{dx^2} = M = \frac{23}{3}x - 2x^2 + \frac{1}{6}x^3 - \frac{1}{6}\langle x-4\rangle^3 - 2\langle x-4\rangle^1 \quad \text{kip}\cdot\text{ft}$$

$$EI\frac{dy}{dx} = \frac{23}{6}x^2 - \frac{2}{3}x^3 + \frac{1}{24}x^4 - \frac{1}{24}\langle x-4\rangle^4 - \langle x-4\rangle^2 + C_1 \quad \text{kip}\cdot\text{ft}^2$$

$$EIy = \frac{23}{18}x^3 - \frac{1}{6}x^4 + \frac{1}{120}x^5 - \frac{1}{120}\langle x-4\rangle^5 - \frac{1}{3}\langle x-4\rangle^3 + C_1x + C_2 \quad \text{kip}\cdot\text{ft}^3$$

$[x = 0, y = 0]$ $\quad 0 - 0 + 0 + 0 + 0 + 0 + 0 + C_2 = 0 \qquad C_2 = 0$

$[x = 8, y = 0]$ $\quad \left(\frac{23}{18}\right)(8)^3 - \frac{1}{6}(8)^4 + \frac{1}{120}(8)^5 - \frac{1}{120}(4)^5 - \frac{1}{3}(4)^3 + C_1(8) = 0$

$C_1 = -26.844$ kip·ft^2

Data: $E = 29 \times 10^6$ psi $= 29 \times 10^3$ ksi $\qquad I = 22.1$ in^4

$EI = (29 \times 10^3)(22.1) = 640.9 \times 10^3$ kip·in^2 $= 4451$ kip·ft^2

(a) Slope at A $\quad \left(\frac{dy}{dx} \text{ at } x = 0\right)$

$$EI\theta_A = 0 + 0 + 0 + 0 + 0 - 26.844 \quad \text{kip}\cdot\text{ft}^2$$

$$\theta_A = -\frac{26.844}{4451} = -6.03 \times 10^{-3} \text{ rad}$$

(b) Deflection at C $\quad$ (y at $x = 4$ ft.)

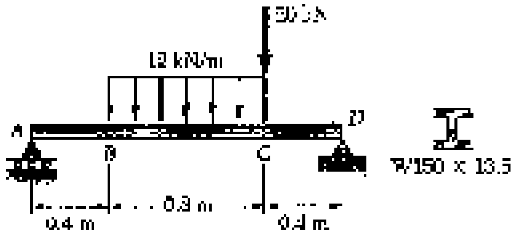
$$EIy_C = \frac{23}{18}(4)^3 - \frac{1}{6}(4)^4 + \frac{1}{120}(4)^5 - 0 - 0 - (26.844)(4) + 0$$
$$= -59.73 \quad \text{kip}\cdot\text{ft}^3$$

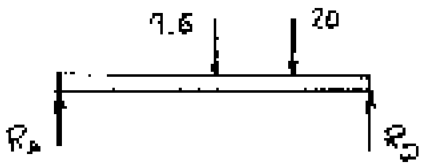
$$y_C = -\frac{59.73}{4451} = -13.42 \times 10^{-3} \text{ ft}$$

$$= 0.1610 \text{ in.} \downarrow$$

PROBLEM 9.46

9.46 For the beam and loading shown, determine (a) the slope at end A, (b) the deflection at point C. Use E = 200 GPa.





SOLUTION

Units: Forces in kN, lengths in m

$+\circlearrowleft M_D = 0$

$-1.6 R_A + (9.6)(0.8) + (20)(0.4) = 0$

$R_A = 9.8$ kN

$w(x) = 12\langle x-0.4\rangle^0 - 12\langle x-1.2\rangle^0$ kN/m

$\frac{dV}{dx} = -w(x) = -12\langle x-0.4\rangle^0 + 12\langle x-1.2\rangle^0$ kN/m

$\frac{dM}{dx} = V = 9.8 - 12\langle x-0.4\rangle^1 + 12\langle x-1.2\rangle^1 - 20\langle x-1.2\rangle^0$ kN

$EI\frac{d^2y}{dx^2} = M = 9.8x - 6\langle x-0.4\rangle^2 + 6\langle x-1.2\rangle^2 - 20\langle x-1.2\rangle^1$ kN·m

$EI\frac{dy}{dx} = 4.9x^2 - 2\langle x-0.4\rangle^3 + 2\langle x-1.2\rangle^3 - 10\langle x-1.2\rangle^2 + C_1$ kN·m²

$EIy = 1.63333x^3 - \frac{1}{2}\langle x-0.4\rangle^4 + \frac{1}{2}\langle x-1.2\rangle^4 - \frac{10}{3}\langle x-1.2\rangle^3 + C_1x + C_2$ kN·m³

$[x=0, y=0]$ $\quad 0 - 0 + 0 - 0 + 0 + C_2 = 0 \qquad C_2 = 0$

$[x=1.6, y=0]$ $\quad (1.63333)(1.6)^3 - \frac{1}{2}(1.2)^4 + \frac{1}{2}(0.4)^4 - \frac{10}{3}(0.4)^3 + C_1(1.6) + 0 = 0$

$C_1 = -3.4080$ kN·m²

Data: $E = 200\times10^9$ Pa $\quad I = 6.87\times10^6$ mm⁴ $= 6.87\times10^{-6}$ m⁴

$EI = (200\times10^9)(6.87\times10^{-6}) = 1.374\times10^6$ N·m² $= 1374$ kN·m²

(a) Slope at A $\left(\frac{dy}{dx}\text{ at } x=0\right)$

$EI\frac{dy}{dx} = 0 - 0 + 0 - 0 - 3.4080$ kN·m²

$\theta_A = -\frac{3.4080}{1374} = -2.48\times10^{-3}$ rad $= 2.48\times10^{-3}$ rad ⟍

(b) Deflection at C (y at $x = 1.2$ m)

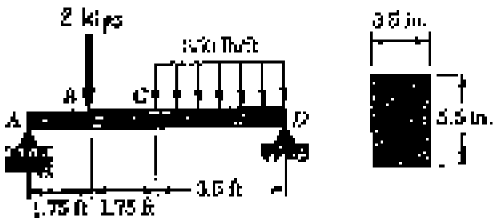
$EIy_C = (1.63333)(1.2)^3 - \frac{1}{2}(0.8)^4 + 0 - 0 - (3.4080)(1.2) + 0$

$= -1.4720$ kN·m³

$y_C = -\frac{1.4720}{1374} = -1.071\times10^{-3}$ m $= 1.071$ mm ↓

PROBLEM 9.47

9.47 For the timber beam and loading shown, determine (*a*) the slope at end *A*, (*b*) the deflection at the midpoint *C*. Use $E = 1.6 \times 10^6$ psi.



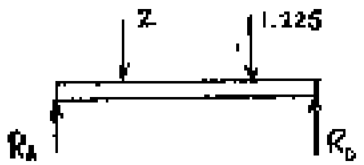
SOLUTION

Units: Forces in kips, lengths in ft.

$+\circlearrowleft \Sigma M_D = 0$

$-7R_A + (2)(5.25) + (1.225)(1.75) = 0$

$R_A = 1.80625$ kips



$w(x) = 0.350\langle x-3.5\rangle^0$

$\frac{dV}{dx} = -w = -0.35\langle x-3.5\rangle^0$

$\frac{dM}{dx} = V = 1.80625 - 2\langle x-1.75\rangle^0 - 0.35\langle x-3.5\rangle^1$

$EI\frac{d^2y}{dx^2} = M = 1.80625x - 2\langle x-1.75\rangle^1 - 0.175\langle x-3.5\rangle^2$ kip·ft

$EI\frac{dy}{dx} = 0.903125x^2 - 1\langle x-1.75\rangle^2 - 0.05833\langle x-3.5\rangle^3 + C_1$ kip·ft²

$EI\,y = 0.301042x^3 - \frac{1}{3}\langle x-1.75\rangle^3 - 0.014583\langle x-3.5\rangle^4 + C_1x + C_2$ kip·ft³

$[x=0,\ y=0]$ $C_2 = 0$

$[x=7,\ y=0]$ $(0.301042)(7)^3 - \frac{1}{3}(5.25)^3 - 0.014583(3.5)^4 + C_1(7) + 0 = 0$

$C_1 = -7.54779$ kip·ft²

Data: $E = 1.6\times10^6$ psi $= 1.6\times10^3$ ksi

$I = \frac{1}{12}(3.5)(5.5)^3 = 48.526$ in⁴

$EI = (1.6\times10^3)(48.526) = 77.6417$ kip·in² $= 539.18$ kip·ft²

(a) Slope at A $\left(\frac{dy}{dx}\text{ at } x = 0\right)$

$EI\frac{dy}{dx} = 0 - 0 - 0 - 7.54779$ kip·ft²

$\theta_A = -\frac{7.54779}{539.18} = -14.00\times10^{-3}$ rad $= 14.00\times10^{-3}$ rad

(b) Deflection at C (y at $x = 3.5$ ft)

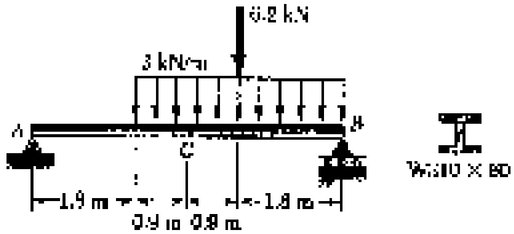
$EI\,y_C = (0.301042)(3.5)^3 - \frac{1}{3}(1.75)^3 - 0 - (7.54779)(3.5) + 0$

$= -15.297$ kip·ft³

$y_C = -\frac{15.297}{539.18} = -28.37\times10^{-3}$ ft $= 0.340$ in ↓

PROBLEM 9.48

9.48 For the beam and loading shown, determine (*a*) the slope at end *A*, (*b*) the deflection at the midpoint *C*. Use *E* = 200 GPa.



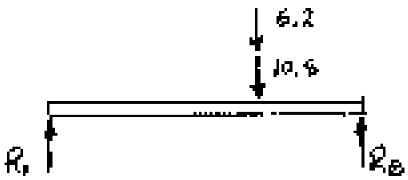
SOLUTION

Units: Forces in kN, lengths in meters.

$+\circlearrowleft \Sigma M_B = 0$

$-5.4 R_A - (1.8)(6.2 + 10.8) = 0$

$R_A = 5.6667 \text{ kN}$



$w(x) = 3\langle x - 1.8\rangle^0$

$\frac{dV}{dx} = -w(x) = -3\langle x - 1.8\rangle^0$

$\frac{dM}{dx} = V = 5.6667 - 3\langle x - 1.8\rangle^1 - 6.2\langle x - 3.6\rangle^0$

$EI\frac{d^2y}{dx^2} = M = 5.6667x - \frac{3}{2}\langle x - 1.8\rangle^2 - 6.2\langle x - 3.6\rangle^1$ kN·m

$EI\frac{dy}{dx} = 2.8333x^2 - \frac{1}{2}\langle x - 1.8\rangle^3 - 3.1\langle x - 3.6\rangle^2 + C_1$ kN·m²

$EIy = 0.9444x^3 - \frac{1}{8}\langle x - 1.8\rangle^4 - 1.0333\langle x - 3.6\rangle^3 + C_1 x + C_2$ kN·m³

$[x = 0, y = 0]$ $0 - 0 - 0 + 0 + C_2 = 0$ $C_2 = 0$

$[x = 5.4, y = 0]$ $(0.9444)(5.4)^3 - \frac{1}{8}(3.6)^4 - 1.0333(1.8)^3 + C_1(5.4) + 0 = 0$

$C_1 = -22.535 \text{ kN·m}^2$

Data: $E = 200 \times 10^9$ Pa, $I = 129 \times 10^6 \text{ mm}^4 = 129 \times 10^{-6} \text{ m}^4$

$EI = (200 \times 10^9)(129 \times 10^{-6}) = 25.8 \times 10^6 \text{ N·m}^2 = 25.8 \times 10^3 \text{ kN·m}^2$

(a) Slope at A   ($\frac{dy}{dx}$ at $x = 0$)

$EI\frac{dy}{dx} = 0 - 0 - 0 - 22.535$ kN·m²

$\theta_A = -\frac{22.535}{25.8 \times 10^3} = -873 \times 10^{-6} = 0.873 \times 10^{-3} \text{ rad}$ ⟶

(b) Deflection at C   (y at $x = 2.7$ m)

$EIy_C = (0.9444)(2.7)^3 - \frac{1}{8}(0.9)^4 - 0 - (22.535)(2.7) + 0$

$= -42.337 \text{ kN·m}^3$

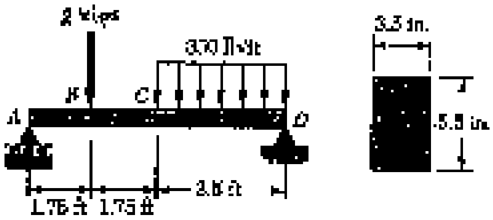
$y_C = -\frac{42.337}{25.8 \times 10^3} = -1.641 \times 10^{-3} \text{ m}$

$y_C = 1.641 \text{ mm} \downarrow$ ⟶

PROBLEM 9.49

9.49 and 9.50 For the beam and loading indicated, write a computer program and use it to calculate the slope and deflection of the beam at intervals ΔL, starting at point A and ending at the right-hand support.

9.49 Beam and loading of Prob. 9.47 with $\Delta L = 3.0$ in.



SOLUTION

See solution to Prob. 9.47 for the derivation of the equations used in the following.

$EI = 539.18 \ \text{kip}\cdot\text{ft}^2$

$$EI\frac{dy}{dx} = 0.903125x^2 - 1\langle x - 1.75\rangle^2 - 0.05833\langle x - 3.5\rangle^3 - 7.54779 \quad \text{kip}\cdot\text{ft}^2$$

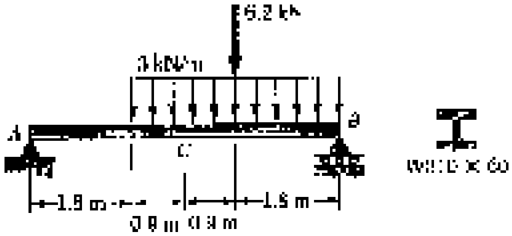
$$EIy = 0.301042x^3 - \tfrac{1}{3}\langle x - 1.75\rangle^3 - 0.014583\langle x - 3.5\rangle^4 - 7.54779x \quad \text{kip}\cdot\text{ft}^3$$

x (in)	x (ft)	θ (10^{-3} rad)	y (10^{-3} ft)	y (in)
0	0	−14.00	0	0
3	0.25	−13.89	−3.49	−0.042
6	0.5	−13.58	−6.93	−0.083
9	0.75	−13.06	−10.26	−0.123
12	1.0	−12.32	−13.44	−0.161
15	1.25	−11.38	−16.41	−0.197
18	1.5	−10.23	−19.11	−0.229
21	1.75	−8.87	−21.51	−0.258
24	2.0	−7.41	−23.54	−0.282
27	2.25	−5.98	−25.21	−0.303
30	2.5	−4.57	−26.53	−0.318
33	2.75	−3.19	−27.50	−0.330
36	3.0	−1.82	−28.13	−0.338
39	3.25	−0.48	−28.42	−0.341
42	3.5	0.84	−28.37	−0.340
45	3.75	2.14	−28.00	−0.336
48	4.0	3.40	−27.30	−0.328
51	4.25	4.62	−26.30	−0.316
54	4.5	5.79	−25.00	−0.300
57	4.75	6.89	−23.41	−0.281
60	5.0	7.92	−21.56	−0.259
63	5.25	8.87	−19.46	−0.234
66	5.5	9.72	−17.13	−0.206
69	5.75	10.47	−14.61	−0.175
72	6.0	11.11	−11.91	−0.143
75	6.25	11.62	−9.06	−0.109
78	6.5	12.00	−6.11	−0.073
81	6.75	12.24	−3.07	−0.037
84	7.0	12.32	0	0

PROBLEM 9.50

9.49 and 9.50 For the beam and loading indicated, write a computer program and use it to calculate the slope and deflection of the beam at intervals ΔL, starting at point A and ending at the right-hand support.

9.50 Beam and loading of Prob. 9.48 with $\Delta L = 0.3$ m.



SOLUTION

See solution to Prob. 9.48 for the derivation of the equations used in the following.

$$EI = 25.8 \times 10^3 \quad \text{kN}\cdot\text{m}^2$$

$$EI\,\frac{dy}{dx} = 2.8333\,x^2 - \tfrac{1}{2}\langle x-1.8\rangle^3 - 3.1\langle x-3.6\rangle^2 - 22.535 \qquad \text{kN}\cdot\text{m}^2$$

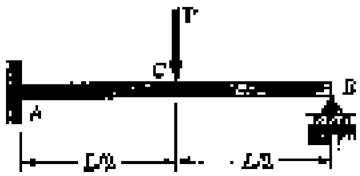
$$EI\,y = 0.9444\,x^3 - \tfrac{1}{8}\langle x-1.8\rangle^4 - 1.03333\langle x-3.6\rangle^3 - 22.535\,x \qquad \text{kN}\cdot\text{m}^3$$

x (m)	θ (10^{-6} rad)	y (mm)
0	−873	0
0.3	−864	−0.261
0.6	−834	−0.516
0.9	−784	−0.759
1.2	−715	−0.985
1.5	−626	−1.187
→ 1.8	−518	−1.359
2.1	−390	−1.495
2.4	−245	−1.591
2.7	−87	−1.641
3.0	81	−1.642
3.3	257	−1.591
→ 3.6	437	−1.487
3.9	606	−1.330
4.2	753	−1.126
4.5	872	−0.882
4.8	960	−0.606
5.1	1016	−0.309
5.4	1035	0

(b) Deflection at C

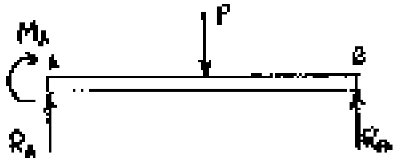
PROBLEM 9.51

9.51 through 9.54 For the beam and loading shown, determine (*a*) the reaction at the roller support, (*b*) the deflection at point C.



$[x=0,\ y=0]$ $\quad$ $[x=L,\ y=0]$

$[x=0,\ \frac{dy}{dx}=0]$



SOLUTION

$+\uparrow \Sigma F_y = 0 \qquad R_A + R_B - P = 0 \qquad R_A = P - R_B$

$\circlearrowleft \Sigma M_A = 0 \qquad -M_A - P\frac{L}{2} + R_B L = 0 \qquad M_A = R_B L - \frac{1}{2}PL$

Reactions are statically indeterminate.

$$\frac{dM}{dx} = V = R_A - P\langle x - \tfrac{L}{2}\rangle^0$$

$$EI\frac{d^2y}{dx^2} = M = M_A + R_A x - P\langle x - \tfrac{L}{2}\rangle^1$$

$$EI\frac{dy}{dx} = M_A x + \tfrac{1}{2}R_A x^2 - \tfrac{1}{2}P\langle x - \tfrac{L}{2}\rangle^2 + C_1$$

$$EIy = \tfrac{1}{2}M_A x^2 + \tfrac{1}{6}R_A x^3 - \tfrac{1}{6}P\langle x - \tfrac{L}{2}\rangle^3 + C_1 x + C_2$$

$[x=0,\ \frac{dy}{dx}=0] \qquad 0 + 0 + 0 + C_1 = 0 \qquad C_1 = 0$

$[x=0,\ y=0] \qquad 0+0+0+0+C_2 = 0 \qquad C_2 = 0$

$[x=L,\ y=0] \qquad \tfrac{1}{2}M_A L^2 + \tfrac{1}{6}R_A L^3 - \tfrac{1}{6}P(\tfrac{L}{2})^3 + 0 + 0 = 0$

$$\tfrac{1}{2}(R_B L - \tfrac{1}{2}PL)L^2 + \tfrac{1}{6}(P - R_B)L^3 - \tfrac{1}{48}PL^3 = 0$$

$(\tfrac{1}{2} - \tfrac{1}{6})R_B L^3 = (\tfrac{1}{4} - \tfrac{1}{6} + \tfrac{1}{48})PL^3 \qquad \tfrac{1}{3}R_B = \tfrac{5}{48}P \qquad R_B = \tfrac{5}{16}P \uparrow$ ◄

$$R_A = P - \tfrac{5}{16}P = \tfrac{11}{16}P$$

$$M_A = \tfrac{5}{16}PL - \tfrac{1}{2}PL = -\tfrac{3}{16}PL$$

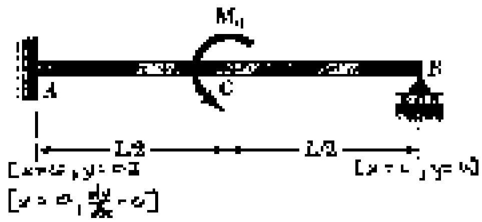
(b) Deflection at C $\qquad$ (y at $x = \frac{L}{2}$)

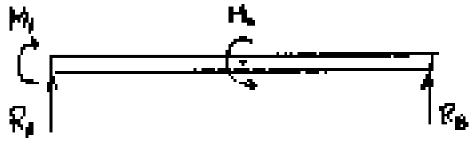
$$y_C = \frac{1}{EI}\left\{\tfrac{1}{2}M_A(\tfrac{L}{2})^2 + \tfrac{1}{6}R_A(\tfrac{L}{2})^3 + 0 + 0 + 0\right\}$$

$= \frac{PL^3}{EI}\left\{(\tfrac{1}{2})(-\tfrac{3}{16})(\tfrac{1}{4}) + (\tfrac{1}{6})(\tfrac{11}{16})(\tfrac{1}{8})\right\} = -\frac{7}{768}\frac{PL^3}{EI} \qquad y_C = \frac{7}{768}\frac{PL^3}{EI} \downarrow$ ◄

PROBLEM 9.52

9.51 through 9.54 For the beam and loading shown, determine (*a*) the reaction at the roller support, (*b*) the deflection at point C.





SOLUTION

$+\uparrow \Sigma F_y = 0 \qquad R_A + R_B = 0 \qquad R_A = -R_B$

$+\circlearrowleft \Sigma M_A = 0 \qquad -M_A + M_0 - R_B L = 0 \qquad M_A = M_0 + R_B L$

Reactions are statically indeterminate.

$$EI\frac{d^2y}{dx^2} = M = M_A + R_A x - M_0\langle x - \tfrac{L}{2}\rangle^0$$

$$EI\frac{dy}{dx} = M_A x + \tfrac{1}{2}R_A x^2 \quad M_0\langle x - \tfrac{L}{2}\rangle^1 + C_1$$

$$EIy = \tfrac{1}{2}M_A x^2 + \tfrac{1}{6}R_A x^3 - \tfrac{1}{2}M_0\langle x - \tfrac{L}{2}\rangle^2 + C_1 x + C_2$$

$[x = 0, \frac{dy}{dx} = 0] \qquad 0 + 0 + 0 + C_1 = 0 \qquad C_1 = 0$

$[x = 0, y = 0] \qquad 0 + 0 + 0 + 0 + C_2 = 0 \qquad C_2 = 0$

$[x = L, y = 0] \qquad \tfrac{1}{2}M_A L^2 + \tfrac{1}{6}R_A L^3 - \tfrac{1}{2}M_0(\tfrac{L}{2})^2 + 0 + 0 = 0$

$\qquad \tfrac{1}{2}(M_0 + R_B L)L^2 - \tfrac{1}{6}R_B L^3 + \tfrac{1}{8}M_0 L^2 = 0$

$(\tfrac{1}{2} - \tfrac{1}{6})R_B L^3 = (\tfrac{1}{8} - \tfrac{1}{2})M_0 L^2 \qquad \tfrac{1}{3}R_B = -\tfrac{3}{8}\frac{M_0}{L} \qquad R_B = -\tfrac{9}{8}\frac{M_0}{L}$

$R_B = \tfrac{9}{8}\frac{M_0}{L}\downarrow$ ◄

$R_A = \tfrac{9}{8}\frac{M_0}{L}$

$M_A = M_0 - \tfrac{9}{8}\frac{M_0}{L}\cdot L = -\tfrac{1}{8}M_0$

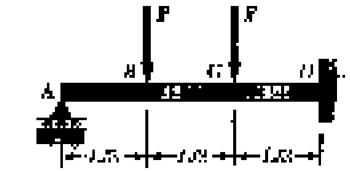
(b) Deflection at C $\quad$ (y at $x = \frac{L}{2}$)

$$y_C = \frac{1}{EI}\left\{\tfrac{1}{2}M_A\left(\tfrac{L}{2}\right)^2 + \tfrac{1}{6}R_A\left(\tfrac{L}{2}\right)^3\right\} = \frac{M_0 L^2}{EI}\left\{(\tfrac{1}{2})(-\tfrac{1}{8})(\tfrac{1}{4}) + (\tfrac{1}{6})(\tfrac{9}{8})(\tfrac{1}{8})\right\}$$

$$= \frac{1}{128}\frac{M_0 L^2}{EI}\uparrow$$ ◄

PROBLEM 9.53

9.51 through 9.54 For the beam and loading shown, determine (*a*) the reaction at the roller support, (*b*) the deflection at point C.



$[x=0,\ y=0]$ $\quad$ $[x=L,\ y=0]$

$[x=0,\ M=0]$ $\quad$ $[x=L,\ \frac{dy}{dx}=0]$

SOLUTION

$$\frac{dM}{dx} = V = R_A - P\langle x-\tfrac{L}{3}\rangle^0 - P\langle x-\tfrac{2L}{3}\rangle^0$$

$$EI\frac{d^2y}{dx^2} = M = R_A x - P\langle x-\tfrac{L}{3}\rangle^1 - P\langle x-\tfrac{2L}{3}\rangle^1$$

$$EI\frac{dy}{dx} = \tfrac{1}{2}R_A x^2 - \tfrac{1}{2}P\langle x-\tfrac{L}{3}\rangle^2 - \tfrac{1}{2}P\langle x-\tfrac{2L}{3}\rangle^2 + C_1$$

$$EIy = \tfrac{1}{6}R_A x^3 - \tfrac{1}{6}P\langle x-\tfrac{L}{3}\rangle^3 - \tfrac{1}{6}P\langle x-\tfrac{2L}{3}\rangle^3 + C_1 x + C_2$$

$[x=0,\ y=0]$ $\qquad 0+0+0+0+C_2 = 0 \qquad C_2 = 0$

$[x=L,\ \frac{dy}{dx}=0]$ $\qquad \tfrac{1}{2}R_A L^2 - \tfrac{1}{2}P(\tfrac{2L}{3})^2 - \tfrac{1}{2}P(\tfrac{L}{3})^2 + C_1 + 0 = 0$

$$C_1 = \tfrac{1}{2}\left[\left(\tfrac{4}{9}+\tfrac{1}{9}\right)P - R_A\right]L^2 = \tfrac{1}{2}\left(\tfrac{5}{9}P - R_A\right)L^2$$

$[x=L,\ y=0]$ $\qquad \tfrac{1}{6}R_A L^3 - \tfrac{1}{6}P(\tfrac{2L}{3})^3 - \tfrac{1}{6}P(\tfrac{L}{3})^3 + \tfrac{1}{2}(\tfrac{5}{9}P - R_A)L^2 L + 0 = 0$

$$\left(\tfrac{1}{2}-\tfrac{1}{6}\right)R_A L^3 = \left[\left(\tfrac{1}{2}\right)\left(\tfrac{5}{9}\right) - \left(\tfrac{1}{6}\right)\left(\tfrac{8}{27}\right) - \left(\tfrac{1}{6}\right)\left(\tfrac{1}{27}\right)\right]PL^3, \qquad \tfrac{1}{3}R_A = \tfrac{2}{9}P, \qquad R_A = \tfrac{2}{3}P \uparrow$$

$$C_1 = \tfrac{1}{2}\left(\tfrac{5}{9}P - \tfrac{2}{3}P\right)L^2 = -\tfrac{1}{18}PL^2$$

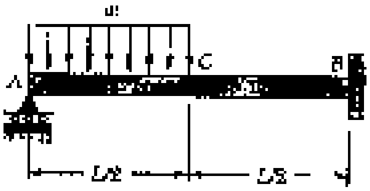
(b) Deflection at C $\quad$ (y at $x = \frac{2L}{3}$)

$$y_C = \frac{1}{EI}\left\{\tfrac{1}{6}\left(\tfrac{2}{3}P\right)\left(\tfrac{2L}{3}\right)^3 - \tfrac{1}{6}P\left(\tfrac{L}{3}\right)^3 - 0 - \tfrac{1}{18}PL^2\left(\tfrac{2L}{3}\right)\right\}$$

$$= \frac{PL^3}{EI}\left(\frac{16}{486} - \frac{1}{162} - \frac{2}{54}\right) = -\frac{5}{486}\frac{PL^3}{EI} \qquad y_C = \frac{5}{486}\frac{PL^3}{EI} \downarrow$$

PROBLEM 9.54

9.51 through 9.54 For the beam and loading shown, determine (*a*) the reaction at the roller support, (*b*) the deflection at point *C*.



SOLUTION

$$w(x) = w - w\langle x-\tfrac{L}{2}\rangle^0$$

$$\frac{dV}{dx} = -w(x) = -w + w\langle x-\tfrac{L}{2}\rangle^0$$

$$\frac{dM}{dx} = V = R_A - wx + w\langle x-\tfrac{L}{2}\rangle^1$$

$$EI\frac{d^2y}{dx^2} = M = R_A x - \tfrac{1}{2}wx^2 + \tfrac{1}{2}w\langle x-\tfrac{L}{2}\rangle^2$$

$$EI\frac{dy}{dx} = \tfrac{1}{2}R_A x^2 - \tfrac{1}{6}wx^3 + \tfrac{1}{6}w\langle x-\tfrac{L}{2}\rangle^3 + C_1$$

$$EIy = \tfrac{1}{6}R_A x^3 - \tfrac{1}{24}wx^4 + \tfrac{1}{24}\langle x-\tfrac{L}{2}\rangle^4 + C_1 x + C_2$$

$[x=0,\ y=0]$ $\quad 0+0+0+0+C_2=0 \quad\quad C_2=0$

$[x=L,\ \frac{dy}{dx}=0]$ $\quad \tfrac{1}{2}R_A L^2 - \tfrac{1}{6}wL^3 + \tfrac{1}{6}w(\tfrac{L}{2})^3 + C_1 = 0 \quad\quad C_1 = \tfrac{7}{48}wL^3 - \tfrac{1}{2}R_A L^2$

$[x=L,\ y=0]$ $\quad \tfrac{1}{6}R_A L^3 - \tfrac{1}{24}wL^4 + \tfrac{1}{24}w(\tfrac{L}{2})^4 + (\tfrac{7}{48}wL^3 - \tfrac{1}{2}R_A L^2)L + 0 = 0$

$$(\tfrac{1}{2}-\tfrac{1}{6})R_A L^3 = (-\tfrac{1}{24} + \tfrac{1}{24}(\tfrac{1}{16}) + \tfrac{7}{48})wL^4 \quad\quad \tfrac{1}{3}R_A = \tfrac{41}{384}wL \quad\quad R_A = \tfrac{41}{128}wL\uparrow$$

$$C_1 = \tfrac{7}{48}wL^3 - \tfrac{1}{2}(\tfrac{41}{128}wL)L^2 = -\tfrac{11}{768}wL^3$$

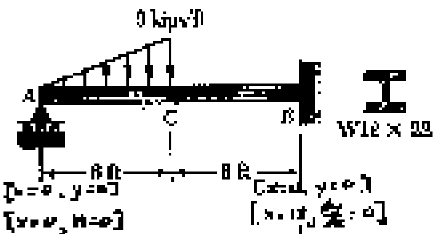
(b) Deflection at C $\quad$ (y at $x=\frac{L}{2}$)

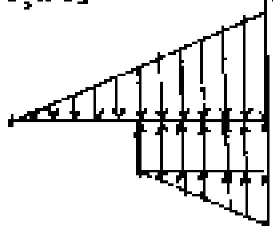
$$y_C = \frac{1}{EI}\left\{(\tfrac{1}{6})(\tfrac{41}{128}wL)(\tfrac{L}{2})^3 - \tfrac{1}{24}w(\tfrac{L}{2})^4 + 0 - \tfrac{11}{768}wL^3\tfrac{L}{2} + 0\right\}$$

$$= \frac{wL^4}{EI}\left(\frac{41}{6144} - \frac{1}{384} - \frac{11}{1536}\right) = -\frac{19}{6144}\frac{wL^4}{EI} \quad\quad y_C = \frac{19}{6144}\frac{wL^4}{EI}\downarrow$$

PROBLEM 9.55

9.55 and 9.56 For the beam and loading shown, determine (a) the reaction at A, (b) the deflection at C. Use $E = 29 \times 10^6$ psi.





SOLUTION

Units: Forces in kips, lengths in ft.

$$k = \frac{9 \text{ kips/ft}}{6 \text{ ft}} = 1.5 \text{ kips/ft}^2$$

$$w(x) = 1.5x - 9\langle x-6\rangle^0 - 1.5\langle x-6\rangle^1$$

$$\frac{dV}{dx} = -w(x) = -1.5x + 9\langle x-6\rangle^0 + 1.5\langle x-6\rangle^1$$

$$\frac{dM}{dx} = V = R_A - 0.75x^2 + 9\langle x-6\rangle^1 + 0.75\langle x-6\rangle^2$$

$$EI\frac{d^2y}{dx^2} = M = R_A x - 0.25x^3 + 4.5\langle x-6\rangle^2 + 0.25\langle x-6\rangle^3$$

$$EI\frac{dy}{dx} = \tfrac{1}{2}R_A x^2 - 0.0625x^4 + 1.5\langle x-6\rangle^3 + 0.0625\langle x-6\rangle^4 + C_1 \qquad \text{kip}\cdot\text{ft}^2$$

$$EIy = \tfrac{1}{6}R_A x^3 - 0.0125x^5 + 0.375\langle x-6\rangle^4 + 0.0125\langle x-6\rangle^5 + C_1 x + C_2 \qquad \text{kip}\cdot\text{ft}^3$$

$[x = 0, y = 0]$ $\qquad 0 + 0 + 0 + 0 + 0 + C_2 = 0 \qquad C_2 = 0$

$[x = 12, \frac{dy}{dx} = 0]$ $\qquad \tfrac{1}{2}(R_A)(12)^2 - (0.0625)(12)^4 + (1.5)(6)^3 + (0.0625)(6)^4 + C_1 = 0$

$$C_1 = 891 - 72R_A = 0 \qquad \text{kip-ft}^2$$

$[x = 12, y = 0]$ $\qquad \tfrac{1}{6}R_A(12)^3 - (0.0125)(12)^5 + (0.375)(6)^4 + (0.0125)(6)^5 + (891 - 72R_A)(12) + 0 = 0$

$(864 - 288)R_A = 8164.8 \qquad R_A = 14.175$ kips ↑

$$C_1 = 891 - (72)(14.175) = -129.6 \text{ kip}\cdot\text{ft}^2$$

Data: $E = 29 \times 10^6$ psi $= 29 \times 10^3$ ksi $\qquad I = 156$ in^4

$EI = (29 \times 10^3)(156) = 4.524 \times 10^6$ kip·in^2 $= 31417$ kip·ft^2

(b) Deflection at C $\qquad$ (y at x = 6)

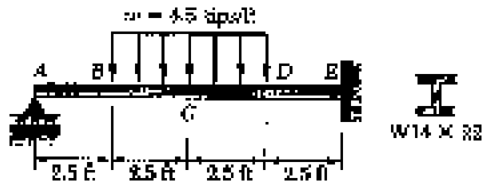
$$EIy_C = \tfrac{1}{6}(14.175)(6)^3 - (0.0125)(6)^5 + 0 + 0 - (129.6)(6) + 0$$
$$= -364.5 \text{ kip}\cdot\text{ft}^3$$

$$y_C = -\frac{364.5}{31417} = -11.60 \times 10^{-3} \text{ ft}$$

$y_C = 0.1392$ in ↓

PROBLEM 9.56

9.55 and 9.56 For the beam and loading shown, determine (*a*) the reaction at *A*, (*b*) the deflection at *C*. Use $E = 29 \times 10^6$ psi.



$[x=0, y=0]$
$[x=0, M=0]$
$[x=10, y=0]$
$[x=10, \frac{dy}{dx}=0]$

SOLUTION

Units: Forces in kips, lengths in ft.

$$w(x) = 4.5\langle x-2.5\rangle^0 - 4.5\langle x-7.5\rangle^0$$

$$\frac{dV}{dx} = -w(x) = -4.5\langle x-2.5\rangle^0 + 4.5\langle x-7.5\rangle^0 \quad \text{kips/ft}$$

$$\frac{dM}{dx} = V = R_A - 4.5\langle x-2.5\rangle^1 + 4.5\langle x-7.5\rangle^1 \quad \text{kips}$$

$$EI\frac{d^2y}{dx^2} = M = R_A x - 2.25\langle x-2.5\rangle^2 + 2.25\langle x-7.5\rangle^2 \quad \text{kip}\cdot\text{ft}$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 - \frac{2.25}{3}\langle x-2.5\rangle^3 + \frac{2.25}{3}\langle x-7.5\rangle^3 + C_1 \quad \text{kip}\cdot\text{ft}^2$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{2.25}{12}\langle x-2.5\rangle^4 + \frac{2.25}{12}\langle x-2.5\rangle^4 + C_1 x + C_2 \quad \text{kip}\cdot\text{ft}^3$$

$[x=0, y=0]$ $\quad 0+0+0+0+C_2 = 0 \qquad C_2 = 0$

$[x=10, \frac{dy}{dx}=0]$ $\quad \frac{1}{2}R_A(10)^2 - \frac{2.25}{3}(7.5)^3 + \frac{2.25}{3}\langle 2.5\rangle^3 + C_1 = 0$

$$C_1 = 304.69 - 50R_A \quad \text{kip}\cdot\text{ft}^2$$

$[x=10, y=0]$ $\quad \frac{1}{6}R_A(10)^3 - \frac{2.25}{12}(7.5)^4 + \frac{2.25}{12}(2.5)^4 + (304.69 - 50R_A)(10) + 0 = 0$

$$\left(500 - \frac{1000}{6}\right)R_A = 2460.9 \qquad R_A = 7.3833 \text{ kips} \uparrow$$

$$C_1 = 304.69 - (50)(7.3833) = -64.45 \quad \text{kip}\cdot\text{ft}^2$$

Data: $E = 29\times10^6$ psi $= 29\times10^3$ ksi, $\quad I = 199$ in^4

$EI = (29\times10^3)(199) = 5.771\times10^6$ kip$\cdot$in^2 $= 40076$ kip$\cdot$ft^2

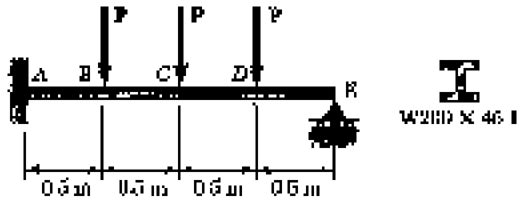
(b) Deflection at C $\quad$ (y at $x = 5$ ft)

$$EI\,y_C = \frac{1}{6}(7.3833)(5)^3 - \frac{2.25}{12}(2.5)^4 + 0 - (64.45)(5) + 0 = -175.76 \quad \text{kip}\cdot\text{ft}^3$$

$$y_C = -\frac{175.76}{40076} = -4.3856\times10^{-3} \text{ ft} \qquad y_C = -0.0526 \text{ in.}$$

PROBLEM 9.57

9.57 For the beam shown and knowing that $P = 40$ kN, determine (*a*) the reaction at E, (*b*) the deflection at C. Use $E = 200$ GPa.



$[x=0, y=0]$ $\quad$ $[x=2, y=0]$

$[x=0, \frac{dy}{dx}=0]$

SOLUTION

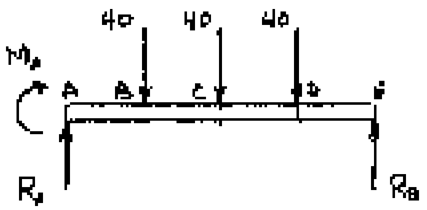
Units: Forces in kN; lengths in m.

$+\uparrow \Sigma F_y = 0 \qquad R_A - 40 - 40 - 40 + R_E = 0$

$R_A = 120 - R_E \quad$ kN

$\circlearrowleft \Sigma M_A = 0 \qquad -M_A - 20 - 40 - 60 + 2R_E = 0$

$M_A = 2R_E - 120 \quad$ kN·m



Reactions are statically indeterminate.

$$\frac{dM}{dx} = V = R_A - 40\langle x-0.5\rangle^0 - 40\langle x-1\rangle^0 - 40\langle x-1.5\rangle^0$$

$$EI\frac{d^2y}{dx^2} = M = M_A + R_A x - 40\langle x-0.5\rangle^1 - 40\langle x-1\rangle^1 - 40\langle x-1.5\rangle^1$$

$$EI\frac{dy}{dx} = M_A x + \tfrac{1}{2}R_A x^2 - 20\langle x-0.5\rangle^2 - 20\langle x-1\rangle^2 - 20\langle x-1.5\rangle^2 + C_1$$

$$EI\,y = \tfrac{1}{2}M_A x^2 + \tfrac{1}{6}R_A x^3 - \tfrac{20}{3}\langle x-0.5\rangle^3 - \tfrac{20}{3}\langle x-1\rangle^3 - \tfrac{20}{3}\langle x-1.5\rangle^3 + C_1 x + C_2$$

$[x=0, \frac{dy}{dx}=0] \qquad 0+0+0+0+0+C_1 = 0 \qquad C_1 = 0$

$[x=0, y=0] \qquad 0+0+0+0+0+0+C_2 = 0 \qquad C_2 = 0$

$[x=2, y=0] \qquad \tfrac{1}{2}M_A(2)^2 + \tfrac{1}{6}R_A(2)^3 - \tfrac{20}{3}(1.5)^3 - \tfrac{20}{3}(1)^3 - \tfrac{20}{3}(0.5)^3 + 0 + 0 = 0$

$\tfrac{1}{2}(2R_E - 120)(2)^2 + \tfrac{1}{6}(120 - R_E)(2)^3 = 30$

$2.66667\,R_E = 30 + 240 - 160 = 110 \qquad R_E = 41.25$ kN $\uparrow$

$M_A = (2)(41.25) - 120 = -37.5$ kN·m

$R_A = 120 - 41.25 = 78.75$ kN

Data: $E = 200\times10^9$ Pa, $I = 45.5\times10^6$ mm^4 $= 45.5\times10^{-6}$ m^4

$EI = (200\times10^9)(45.5\times10^{-6}) = 9.10\times10^6$ N·m^2 $= 9100$ kN·m^2

(b) Deflection at C $\quad$ (y at $x = 1$ m)

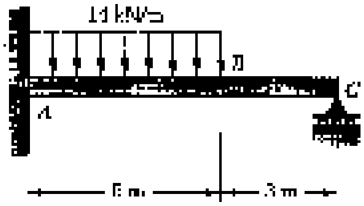
$EI\,y_C = \tfrac{1}{2}(-37.5)(1)^2 + \tfrac{1}{6}(78.75)(1)^3 - \tfrac{20}{3}(0.5)^3 - 0 - 0 + 0 + 0$

$= -6.4583$ kN·m^3

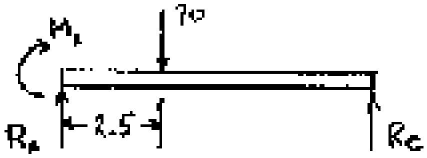
$y_C = -\dfrac{6.4583}{9100} = -0.710\times10^{-3}$ m $\qquad y_C = 0.710$ mm $\downarrow$

PROBLEM 9.58

9.58 For the beam and loading shown, determine (a) the reaction at C, (b) the deflection at B. Use $E = 200$ GPa.



$[x=0, y=0]$ $[x=0, \frac{dy}{dx}=0]$ $[x=8, y=0]$



SOLUTION

Units: Forces in kN, lengths in m.

$+\uparrow \Sigma F_y = 0 \quad R_A - 70 + R_C = 0$

$R_A = 70 - R_C \quad$ kN

$+\circlearrowleft M_A = 0 \quad -M_A - (70)(2.5) + 8R_C = 0$

$M_A = 8R_C - 175 \quad$ kN·m

Reactions are statically indeterminate.

$w(x) = 14 - 14\langle x-5\rangle^0 \quad$ kN/m

$\frac{dV}{dx} = -w = -14 + 14\langle x-5\rangle^0 \quad$ kN/m

$\frac{dM}{dx} = V = R_A - 14x + 14\langle x-5\rangle^1 \quad$ kN

$EI\frac{d^2y}{dx^2} = M = M_A + R_A x - 7x^2 + 7\langle x-5\rangle^2 \quad$ kN·m

$EI\frac{dy}{dx} = M_A x + \frac{1}{2}R_A x^2 - \frac{7}{3}x^3 + \frac{7}{3}\langle x-5\rangle^3 + C_1 \quad$ kN·m²

$EI\,y = \frac{1}{2}M_A x^2 + \frac{1}{6}R_A x^3 - \frac{7}{12}x^4 + \frac{7}{12}\langle x-5\rangle^4 + C_1 x + C_2 \quad$ kN·m³

$[x=0, \frac{dy}{dx}=0] \quad 0+0+0+0+C_1=0 \quad C_1 = 0$

$[x=0, y=0] \quad 0+0+0+0+0+C_2=0 \quad C_2=0$

$[x=8, y=0] \quad \frac{1}{2}M_A(8)^2 + \frac{1}{6}R_A(8)^3 - \frac{7}{12}(8)^4 + \frac{7}{12}(3)^4 + 0 + 0 = 0$

$32(8R_C - 175) + \frac{512}{6}(70 - R_C) - \frac{28105}{12} = 0$

$170.667R_C = 5600 - \frac{35840}{6} + \frac{28105}{12} = 1968.75 \qquad R_C = 11.536$ kN$\uparrow$ ◄

$M_A = (8)(11.536) - 175 = -82.715$ kN·m

$R_A = 70 - 11.536 = 58.464$ kN

Data: $E = 200\times10^9$ Pa $\quad I = 216\times10^6$ mm⁴ $= 216\times10^{-6}$ m⁴

$EI = (200\times10^9)(216\times10^{-6}) = 43.2\times10^6$ N·m² $= 43200$ kN·m²

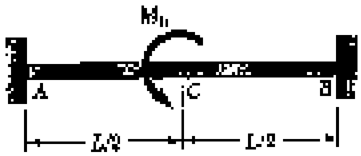
(b) Deflection at B $\quad$ (y at $x = 5$ m)

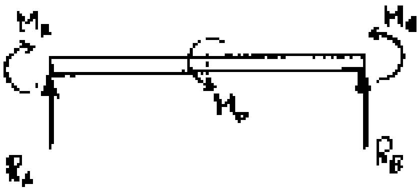
$EI\,y_B = \frac{1}{2}(-82.715)(5)^2 + \frac{1}{6}(58.464)(5)^3 - \frac{7}{12}(5)^4 = -180.52$ kN·m³

$y_B = -\frac{180.52}{43200} = -4.18\times10^{-3}$ m $\qquad y_B = 4.18$ mm$\downarrow$ ◄

PROBLEM 9.59

9.59 For the beam and loading shown, determine (*a*) the reaction at *A*, (*b*) the slope at *C*.





SOLUTION

Reactions are statically indeterminate

$$EI\frac{d^2y}{dx^2} = M = M_A + R_A x - M_0\langle x - \tfrac{L}{2}\rangle^0$$

$$EI\frac{dy}{dx} = M_A x + \tfrac{1}{2}R_A x^2 - M_0\langle x - \tfrac{L}{2}\rangle^1 + C_1$$

$$EI\,y = \tfrac{1}{2}M_A x^2 + \tfrac{1}{6}R_A x^3 - \tfrac{1}{2}M_0\langle x - \tfrac{L}{2}\rangle^2 + C_1 x + C_2$$

$[x = 0,\ \frac{dy}{dx} = 0]$ $\qquad C_1 = 0$

$[x = 0,\ y = 0]$ $\qquad C_2 = 0$

$[x = L,\ \frac{dy}{dx} = 0]$ $\qquad M_A L + \tfrac{1}{2}R_A L^2 - M_0\frac{L}{2} = 0$ $\qquad M_A = \tfrac{1}{2}M_0 - \tfrac{1}{2}R_A L$

$[x = L,\ y = 0]$ $\qquad \tfrac{1}{2}M_A L^2 + \tfrac{1}{6}R_A L^3 - \tfrac{1}{2}M_0(\tfrac{L}{2})^2 = 0$

$$\tfrac{1}{2}(\tfrac{1}{2}M_0 - \tfrac{1}{2}R_A L)L^2 + \tfrac{1}{6}R_A L^3 - \tfrac{1}{8}M_0 L^2 = 0$$

$$(\tfrac{1}{4} - \tfrac{1}{6})R_A L^3 = (\tfrac{1}{4} - \tfrac{1}{8})M_0 L^2 \qquad R_A = \tfrac{3}{2}\frac{M_0}{L}\uparrow$$

$$M_A = \tfrac{1}{2}M_0 - \tfrac{1}{2}\cdot\tfrac{3}{2}\frac{M_0}{L}L = -\tfrac{1}{4}M_0$$

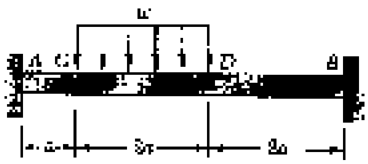
(b) Slope at C $\qquad \frac{dy}{dx}$ at $x = \frac{L}{2}$

$$\theta_C = \frac{1}{EI}\left\{(-\tfrac{1}{4}M_0)\tfrac{L}{2} + \tfrac{1}{2}(\tfrac{3}{2}\tfrac{M_0}{L})(\tfrac{L}{2})^2 + 0 + 0\right\} = -\frac{1}{16}\frac{M_0 L}{EI}$$

$$\theta_C = \frac{1}{16}\frac{M_0 L}{EI}$$

PROBLEM 9.60

9.60 For the beam and loading shown, determine (*a*) the reaction at *A*, (*b*) the deflection at *D*.



[x=0, y=0]
[x=0, $\frac{dy}{dx}=0$]
[x=5a, y=0]
[x=5a, $\frac{dy}{dx}=0$]

SOLUTION

$$w(x) = w\langle x-a\rangle^0 - w\langle x-3a\rangle^0$$

$$\frac{dV}{dx} = -w(x) = -w\langle x-a\rangle^0 + w\langle x-3a\rangle^0$$

$$\frac{dM}{dx} = R_A - w\langle x-a\rangle^1 + w\langle x-3a\rangle^1$$

$$EI\frac{d^2y}{dx^2} = M = M_A + R_A x - \frac{1}{2}w\langle x-a\rangle^2 + \frac{1}{2}w\langle x-3a\rangle^2$$

$$EI\frac{dy}{dx} = M_A x + \frac{1}{2}R_A x^2 - \frac{1}{6}w\langle x-a\rangle^3 + \frac{1}{6}w\langle x-3a\rangle^3 + C_1$$

$$EIy = \frac{1}{2}M_A x^2 + \frac{1}{6}R_A x^3 - \frac{1}{24}w\langle x-a\rangle^4 + \frac{1}{24}\langle x-3a\rangle^4 + C_1 x + C_2$$

$\left[x=0,\ \frac{dy}{dx}=0\right]$ $\quad 0+0+0+0+C_1=0 \qquad C_1=0$

$[x=0,\ y=0]$ $\quad 0+0+0+0+0+C_2=0 \qquad C_2=0$

$\left[x=5a,\ \frac{dy}{dx}=0\right]$ $\quad M_A(5a)+\frac{1}{2}R_A(5a)^2-\frac{1}{6}w(4a)^3+\frac{1}{6}w(2a)^3+0=0$

$$5M_A a + 12.5R_A a^2 = 9.3333\,wa^3 \qquad (1)$$

$[x=5a,\ y=0]$ $\quad \frac{1}{2}M_A(5a)^2+\frac{1}{6}R_A(5a)^3-\frac{1}{24}w(4a)^4+\frac{1}{24}(2a)^4+0+0=0$

$$12.5M_A a^2 + 20.8333R_A a^3 = 10\,wa^4 \qquad (2)$$

Solving (1) and (2) simultaneously $\qquad M_A = -1.3333\,wa^2$ ◄

$$R_A = 1.280\,wa \uparrow \quad ◄$$

(b) Deflection at D $\quad$ (y at x = 3a)

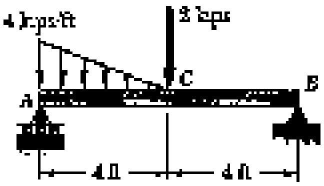
$$y_D = \frac{1}{EI}\left\{\frac{1}{2}M_A(3a)^2+\frac{1}{6}R_A(3a)^3-\frac{1}{24}w(2a)^4+0+0+0\right\}$$

$$= \left[\frac{9}{2}(-1.3333)+\frac{1}{6}(1.28)(27)-\frac{1}{24}(16)\right]\frac{wa^4}{EI} = -0.907\frac{wa^4}{EI}$$

$$y_D = 0.907\frac{wa^4}{EI}\downarrow \quad ◄$$

PROBLEM 9.61

9.61 through 9.64 For the beam and loading indicated, determine the magnitude and location of the largest downward deflection.
9.61 Beam and loading of Prob. 9.45.



SOLUTION

See solution to Prob. 9.45 for the derivation of the equations used in the following.

$EI = 4451 \text{ kip}\cdot\text{ft}^2$

$$EI\frac{dy}{dx} = \frac{23}{6}x^2 - \frac{2}{3}x^3 + \frac{1}{24}x^4 - \frac{1}{24}\langle x-4\rangle^4 - \langle x-4\rangle^2 - 26.844 \qquad \text{kip}\cdot\text{ft}^2$$

$$EIy = \frac{23}{18}x^3 - \frac{1}{6}x^4 + \frac{1}{120}x^5 - \frac{1}{120}\langle x-4\rangle^5 - \frac{1}{3}\langle x-4\rangle^3 - 26.844x \qquad \text{kip}\cdot\text{ft}^3$$

To find location of maximum $|y|$, set $\frac{dy}{dx} = 0$. Assume $0 < x < 4$ ft.

$$EI\frac{dy}{dx} = \frac{23}{6}x_m^2 - \frac{2}{3}x_m^3 + \frac{1}{24}x_m^4 - 26.844 = 0 \qquad f.$$

Solve by iteration: $x_m =$ 4.0 3.73 3.735 $\qquad x_m = 3.735$ ft. ◄
$df/dx =$ 9.33 9.42

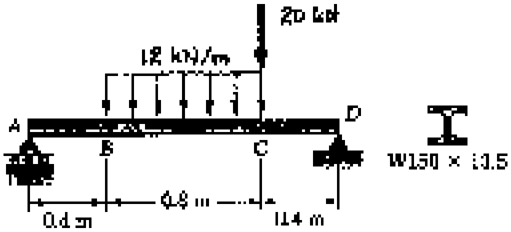
$$EIy_m = \frac{23}{18}(3.735)^3 - \frac{1}{6}(3.735)^4 + \frac{1}{120}(3.735)^5 - (26.844)(3.735) = -60.06 \text{ kip}\cdot\text{ft}^3$$

$$y_m = -\frac{60.06}{4451} = -13.49\times10^{-3} \text{ ft.} \qquad y_m = 0.1619 \text{ in.}$$ ◄

PROBLEM 9.62

9.61 through 9.64 For the beam and loading indicated, determine the magnitude and location of the largest downward deflection.

9.62 Beam and loading of Prob. 9.46.



SOLUTION

See solution to Prob. 9.46 for the derivation of the equations used in the following.

$EI = EI = 1374 \text{ kN}\cdot\text{m}^2$

$EI\frac{dy}{dx} = 4.9x^2 - 2\langle x-0.4\rangle^3 + 2\langle x-1.2\rangle^3 - 10\langle x-1.2\rangle^2 - 3.4080 \quad \text{kN}\cdot\text{m}^2$

$EIy = 1.63333x^3 - \frac{1}{2}\langle x-0.4\rangle^4 + \frac{1}{2}\langle x-1.2\rangle^4 - \frac{10}{3}\langle x-1.2\rangle^3 - 3.4080x \quad \text{kN}\cdot\text{m}^3$

To find the location of maximum $|y|$, set $\frac{dy}{dx} = 0$. Assume $0.4 < x_m < 1.2$

$4.9x_m^2 - 2(x_m - 0.4)^3 - 3.4080 = f(x_m) = 0$

Solve by iteration $\quad x_m = \quad 0.8 \quad 0.858 \quad 0.857 \quad 0.8570 \qquad x_m = 0.8570 \text{ m}$ ◄
$\qquad df/dx = \quad 6.98 \quad 7.128 \quad 7.145$

$EIy = (1.63333)(0.8570)^3 - \frac{1}{2}(0.8570-0.4)^4 - (3.4080)(0.8570)$

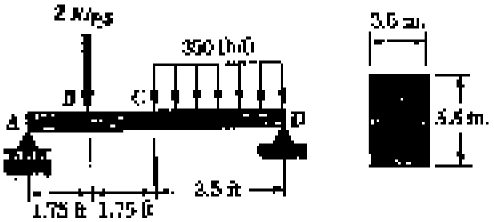
$\quad = -1.9144 \quad \text{kN}\cdot\text{m}$

$y = -\frac{1.9144}{1374} = -1.393\times10^{-3} \text{ m} = 1.393 \text{ mm} \downarrow$ ◄

PROBLEM 9.63

9.61 through 9.64 For the beam and loading indicated, determine the magnitude and location of the largest downward deflection.

9.63 Beam and loading of Prob. 9.47.



SOLUTION

See solution to Prob. 9.47 for the derivation of the equations used in the following.

$EI = 539.18 \text{ kip}\cdot\text{ft}^2$

$EI\dfrac{dy}{dx} = 0.903125\,x^2 - 1\langle x-1.75\rangle^2 - 0.05833\langle x-3.5\rangle^3 - 7.54779 \qquad \text{kip}\cdot\text{ft}^2$

$EI\,y = 0.301042\,x^3 - \frac{1}{3}\langle x-1.75\rangle^3 - 0.014583\langle x-3.5\rangle^4 - 7.54779\,x \qquad \text{kip}\cdot\text{ft}^3$

To find the location of maximum $|y|$, set $\dfrac{dy}{dx} = 0$. Assume $1.75 < x_m < 3.5$

$0.903125\,x_m^2 - 1\,(x_m - 1.75)^2 - 7.54779 = 0$

$0.096875\,x_m^2 - 3.5\,x_m + 10.61029 = 0$

$$x_m = \frac{3.5 - \sqrt{(3.5)^2 - (4)(0.096875)(10.61029)}}{(2)(0.096875)} = 3.340 \text{ ft}$$

$EI\,y = (0.301042)(3.340)^3 - \frac{1}{3}(3.340-1.75)^3 - (7.54779)(3.340)$

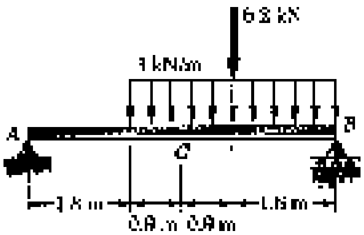
$= -15.3328 \quad \text{kip ft}^3$

$$y = -\frac{15.3328}{539.18} = -28.44\times10^{-3} \text{ ft} = 0.341 \text{ in.}\downarrow$$

PROBLEM 9.64

9.61 through 9.64 For the beam and loading indicated, determine the magnitude and location of the largest downward deflection.

9.64 Beam and loading of Prob. 9.48.



SOLUTION

See solution to Prob. 9.48 for the derivation of the equations used in the following.

$EI = 25.8 \times 10^3$ kN-m

$EI\frac{dy}{dx} = 2.8333x^2 - \frac{1}{2}\langle x-1.8\rangle^3 - 3.1\langle x-3.6\rangle^2 - 22.535$

$EI\,y = 0.9444x^3 - \frac{1}{8}\langle x-1.8\rangle^4 - 1.03333\langle x-3.6\rangle^3 - 22.535x$

To find location of maximum $|y|$, set $\frac{dy}{dx} = 0$. Assume $1.8 < x_m < 3.6$

$EI\frac{dy}{dx} = 2.8333\,x_m^2 - \frac{1}{2}(x_m - 1.8)^3 - 22.535 = 0 \qquad F_1$

Solving by iteration: $x_m = 3$, 2.86, 2.855 $\qquad x_m = 2.855$ m ◄

$dF/dx = 15.8$, 15.15

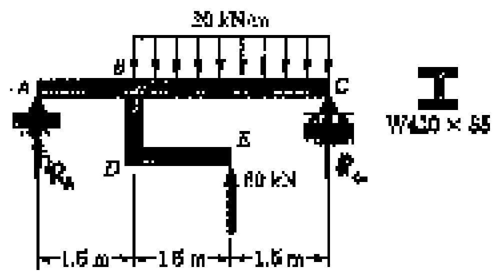
$EI\,y_m = 0.9444\,x_m^3 - \frac{1}{8}(x_m - 1.8)^4 - 22.535\,x_m$

$= (0.9444)(2.855)^3 - \frac{1}{8}(2.855 - 1.8)^4 - (22.535)(2.855) = -42.507$ kN-m³

$y_m = -\frac{42.507}{25.8 \times 10^3} = -1.648 \times 10^{-3}$ m $\qquad y_m = 1.648$ mm ↓ ◄

PROBLEM 9.65

9.65 The rigid bar *BDE* is welded at point *B* to the rolled steel beam *AC*. For the loading shown, determine (*a*) the slope at point *A*, (*b*) the deflection at point *B*. Use $E = 200$ GPa.



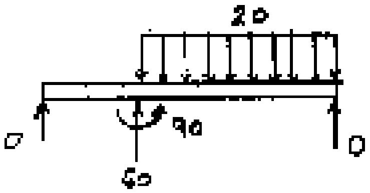
SOLUTION

$$\circlearrowleft \Sigma M_C = 0$$

$$-4.5 R_A + (20)(3)(1.5) - (60)(1.5) = 0$$

$$R_A = 0$$

Units: Forces in kN, lengths in m



$$EI \frac{d^2y}{dx^2} = M = 60\langle x-1.5\rangle^1 - 90\langle x-1.5\rangle^0 - \tfrac{1}{2}(20)\langle x-1.5\rangle^2$$

$$EI \frac{dy}{dx} = 30\langle x-1.5\rangle^2 - 90\langle x-1.5\rangle^1 - (\tfrac{1}{6})(20)\langle x-1.5\rangle^3 + C_1$$

$$EI\, y = 10\langle x-1.5\rangle^3 - 45\langle x-1.5\rangle^2 - \tfrac{1}{24}(20)\langle x-1.5\rangle^4 + C_1 x + C_2$$

$[x=0,\ y=0]$ $\quad 0 + 0 + 0 + 0 + C_2 = 0 \qquad C_2 = 0$

$[x=4.5,\ y=0]$ $\quad (10)(3)^3 - (45)(3)^2 - \tfrac{1}{24}(20)(3)^4 + 4.5C_1 + 0 = 0$

$$C_1 = 45 \text{ kN}\cdot\text{m}^2$$

Data: $E = 200\times10^9$ Pa, $I = 315\times10^6 \text{ mm}^4 = 315\times10^{-6} \text{ m}^4$

$EI = (200\times10^9)(315\times10^{-6}) = 63\times10^6 \text{ N}\cdot\text{m}^2 = 63000 \text{ kN}\cdot\text{m}^2$

(a) Slope at A $\quad (\frac{dy}{dx}$ at $x = 0)$

$$EI\,\theta_A = C_1 = 45 \text{ kN}\cdot\text{m}^2$$

$$\theta_A = \frac{45}{63000} = 0.714\times10^{-3} \text{ rad}$$

$\theta_A = 0.714\times10^{-3}$ rad ◄

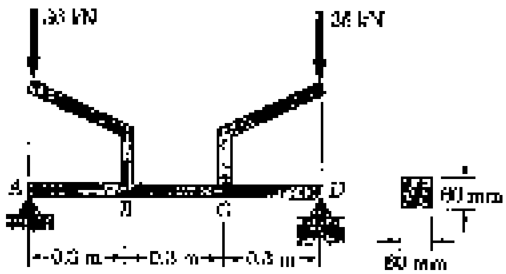
(b) Deflection at B $\quad (y$ at $x = 1.5)$

$$EI\,y_B = (C_1)(1.5) = (45)(1.5) = 67.5 \text{ kN}\cdot\text{m}^3$$

$$y_B = \frac{67.5}{63000} = 1.071\times10^{-3} \text{ m} = 1.071 \text{ mm} \uparrow$$ ◄

PROBLEM 9.66

9.66 Rigid bars are welded to the steel rod AD as shown. For the loading shown, determine (a) the deflection at point B, (b) the slope at end A. Use $E = 200$ GPa.



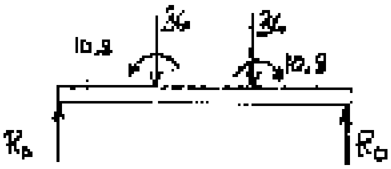
SOLUTION

Units: Use kN for forces, m for lengths.

By symmetry $R_A = R_D$

$+\uparrow \Sigma F_y = 0 \quad R_A + R_A - 36 - 36 = 0 \qquad R_A = 36$ kN

$$EI \frac{d^2y}{dx^2} = M = 36x - 36\langle x-0.3\rangle^1 - 36\langle x-0.6\rangle^1 - 10.8\langle x-0.3\rangle^0 + 10.8\langle x-0.6\rangle^0$$



$$(\text{kN}\cdot\text{m}^2)\ EI\frac{dy}{dx} = 18x^2 - 18\langle x-0.3\rangle^2 - 18\langle x-0.6\rangle^2 - 10.8\langle x-0.3\rangle^1 + 10.8\langle x-0.6\rangle^1 + C_1$$

$$(\text{kN}\cdot\text{m}^3)\ EIy = 6x^3 - 6\langle x-0.3\rangle^3 - 6\langle x-0.6\rangle^3 - 5.4\langle x-0.3\rangle^2 + 5.4\langle x-0.6\rangle^2 + C_1x + C_2$$

$[x=0,\ y=0]$ $\quad 0+0+0+0+0+0+C_2 = 0 \qquad C_2 = 0$

$[x=0.9,\ y=0]$ $\quad (6)(0.9)^3 - (6)(0.6)^3 - (6)(0.3)^3 - (5.4)(0.6)^2 + (5.4)(0.3)^2 + 0.9C_1 + 0 = 0$

$C_1 = -1.62 \text{ kN}\cdot\text{m}^2$

Data: $E = 200\times10^9$ Pa, $I = bh^3 = \frac{1}{12}(60)(60)^3 = 1.08\times10^6 \text{ mm}^4 = 1.08\times10^{-6} \text{ m}^4$

$EI = (200\times10^9)(1.08\times10^{-6}) = 216\times10^3 \text{ N}\cdot\text{m}^2 = 216 \text{ kN}\cdot\text{m}^2$

(a) Deflection at B $\quad$ (y at $x = 0.3$)

$EIy_B = (6)(0.3)^3 - 0 - 0 - 0 + 0 + (1.62)(0.3) = -0.324 \text{ kN}\cdot\text{m}^3$

$y_B = -\frac{0.324}{216} = -1.500\times10^{-3}$ m $\qquad y_B = 1.500$ mm $\downarrow$

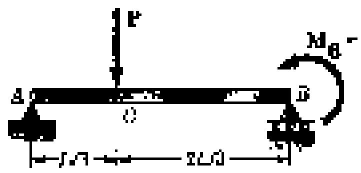
(b) Slope at A $\quad$ ($\frac{dy}{dx}$ at $x=0$)

$EI\theta_A = C_1 = -1.62 \text{ kN}\cdot\text{m}^2$

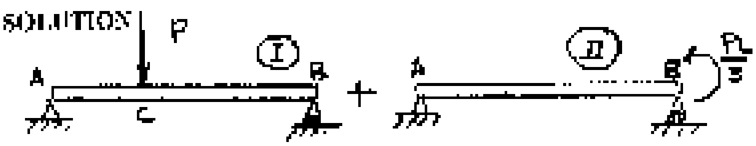
$\theta_A = -\frac{1.62}{216} = -7.50\times10^{-3}$ rad $\qquad \theta_A = 7.50\times10^{-3}$ rad

PROBLEM 9.67

9.67 and 9.68 For the beam and loading shown, determine (*a*) the deflection at point *C*, (*b*) the slope at end *A*.



SOLUTION



Loading I: Case 5 $\quad a = \frac{L}{3}, \quad b = \frac{2L}{3}, \quad P = P, \quad x = a$

$$y_C = -\frac{Pa^2b^2}{6EIL} = -\frac{P}{6EIL}\left(\frac{L}{3}\right)^2\left(\frac{2L}{3}\right)^2 = -\frac{4}{243}\frac{PL^3}{EI}$$

$$\theta_A = -\frac{Pb(L^2-b^2)}{6EIL} = -\frac{P}{6EIL}\left(\frac{2L}{3}\right)\left[L^2-\left(\frac{2L}{3}\right)^2\right] = -\frac{5}{81}\frac{PL^2}{EI}$$

Loading II: Case 7 $\quad M = -\frac{PL}{3} \quad x = \frac{L}{3}$

$$y_C = -\frac{M}{6EIL}(x^3 - L^2x) = +\frac{PL/3}{6EIL}\left\{\left(\frac{L}{3}\right)^3 - L^2\left(\frac{L}{3}\right)\right\} = -\frac{4}{243}\frac{PL^3}{EI}$$

$$\theta_A = +\frac{ML}{6EI} = \frac{-(PL/3)L}{6EI} = -\frac{1}{18}\frac{PL^2}{EI}$$

(a) Deflection at C: $\quad y_C = -\frac{4}{243}\frac{PL^3}{EI} - \frac{4}{243}\frac{PL^3}{EI} = -\frac{8}{243}\frac{PL^3}{EI}$

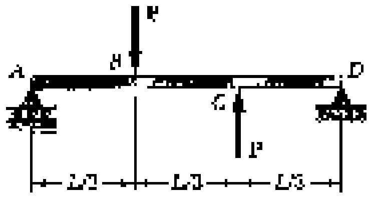
$$y_C = \frac{8}{243}\frac{PL^3}{EI} \downarrow$$

(b) Slope at A: $\quad \theta_A = -\frac{5}{81}\frac{PL^2}{EI} - \frac{1}{18}\frac{PL^2}{EI} = -\frac{19}{162}\frac{PL^2}{EI}$

$$\theta_A = \frac{19}{162}\frac{PL^2}{EI}$$

PROBLEM 9.68

9.67 and 9.68 For the beam and loading shown, determine (a) the deflection at point C, (b) the slope at end A.



SOLUTION

Loading I: Downward load P at B

Use Case 5 of Appendix D with

$P = P, \quad a = \frac{L}{3}, \quad b = \frac{2L}{3}, \quad L = L, \quad x = \frac{2L}{3}$

For $x < a$, given elastic curve is $y = \frac{Pb}{6EIL}[x^3 - (L^2 - b^2)x]$

To obtain elastic curve for $x > a$ replace x by $L - x$ and interchange a and b to get

$$y = \frac{Pa}{6EIL}\left[(L-x)^3 - (L^2 - a^2)(L-x)\right] \quad \text{with } x = \frac{2L}{3} \text{ at point C}$$

$$y_C = \frac{P(L/3)}{6EIL}\left[\left(\frac{L}{3}\right)^3 - \left(L^2 - \left(\frac{L}{3}\right)^2\right)\left(\frac{L}{3}\right)\right] = -\frac{7}{486}\frac{PL^3}{EI}$$

$$\theta_A = -\frac{Pb(L^2 - b^2)}{6EIL} = -\frac{P(2L/3)[L^2 - (2L/3)^2]}{6EIL} = -\frac{5}{81}\frac{PL^2}{EI}$$

Loading II: Upward load at C. Use Case 5 of Appendix D with

$P = -P, \quad a = \frac{2L}{3}, \quad b = \frac{L}{3}, \quad L = L, \quad x = a = \frac{2L}{3}$

$$y_C = -\frac{(-P)(2L/3)^2(L/3)^2}{3EIL} = \frac{4}{243}\frac{PL^3}{EI}$$

$$\theta_A = -\frac{(-P)(L/3)(L^2 - (L/3)^2)}{6EIL} = \frac{4}{81}\frac{PL^2}{EI}$$

(a) Deflection at C $\quad y_C = -\frac{7}{486}\frac{PL^3}{EI} + \frac{4}{243}\frac{PL^3}{EI} = \frac{1}{486}\frac{PL^3}{EI} \uparrow$

(b) Slope at A $\quad \theta_A = -\frac{5}{81}\frac{PL^2}{EI} + \frac{4}{81}\frac{PL^2}{EI} = \frac{1}{81}\frac{PL^2}{EI}$

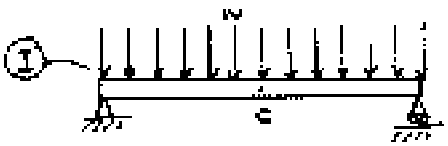
PROBLEM 9.69

9.69 and 9.70 For the beam and loading shown, determine (*a*) the deflection at the midpoint *C*, (*b*) the slope at end *A*.

SOLUTION

Loading I: Case 6 in Appendix D

$$y_C = -\frac{5}{384}\frac{wL^4}{EI} \; ; \quad \theta_A = -\frac{1}{24}\frac{wL^3}{EI}$$

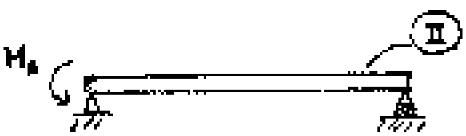


Loading II: Case 7 of Appendix D.

Note that center deflection is

$$y_C = -\frac{M_A}{6EIL}\left[\left(\frac{L}{2}\right)^3 - L^2\left(\frac{L}{2}\right)\right]$$

$$= \frac{1}{16}\frac{M_A L}{EI}$$



$$\theta_A = \frac{M_A L}{3EI}$$

with $M_A = \frac{wL^2}{12}$, $\quad y_C = \frac{1}{192}\frac{wL^4}{EI}$, $\quad \theta_A = \frac{1}{36}\frac{wL^3}{EI}$

(a) Deflection at C. $\quad y_C = -\frac{5}{384}\frac{wL^4}{EI} + \frac{1}{192}\frac{wL^4}{EI} = -\frac{1}{128}\frac{wL^4}{EI}$

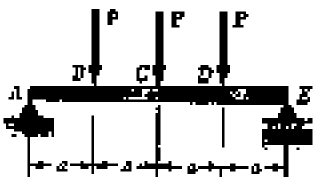
$$y_C = \frac{1}{128}\frac{wL^4}{EI} \downarrow$$

(b) Slope at A. $\quad \theta_A = -\frac{1}{24}\frac{wL^3}{EI} + \frac{1}{36}\frac{wL^3}{EI} = -\frac{1}{72}\frac{wL^3}{EI}$

$$\theta_A = \frac{1}{72}\frac{wL^3}{EI}$$

PROBLEM 9.70

9.69 and 9.70 For the beam and loading shown, determine (a) the deflection at the midpoint C, (b) the slope at end A.



SOLUTION

Loading I : Load at B

Case 5 in Appendix D.

$L = 4a, \quad a = a, \quad b = 3a, \quad x = 2a$

For $x > a$, replace x by $L - x$ and interchange a and b in expression for elastic curve given.

$$y = \frac{Pa}{6EIL}[(L-x)^3 - (L^2 - a^2)(L-x)]$$

$$y_C = \frac{Pa}{6EI(4a)}\left[(2a)^3 - (16a^2 - a^2)(2a)\right] = -\frac{11}{12}\cdot\frac{Pa^3}{EI}$$

$$\theta_A = -\frac{Pb(L^2 - b^2)}{6EIL} = -\frac{P(3a)(16a^2 - 9a^2)}{6EI(4a)} = -\frac{7}{8}\cdot\frac{Pa^2}{EI}$$

Loading II Load at C Case 4 of Appendix D with $L = 4a$

$$y_C = -\frac{PL^3}{48EI} = -\frac{P(4a)^3}{48EI} = -\frac{4}{3}\frac{Pa^3}{EI}$$

$$\theta_A = -\frac{PL^2}{16EI} = -\frac{P(4a)^2}{16EI} = -\frac{Pa^2}{EI}$$

Loading III Load at D Case 5 of Appendix D

$L = 4a, \quad a = 3a, \quad b = a, \quad x = 2a$ at point C

$$y_C = \frac{Pb}{6EIL}[x^3 - (L^2 - b^2)x] = \frac{Pa}{6EI(4a)}\left[(2a)^3 - (16a^2 - a^2)(2a)\right]$$

$$= -\frac{11}{12}\frac{Pa^3}{EI}$$

$$\theta_A = -\frac{Pb(L^2 - b^2)}{6EIL} = -\frac{Pa(16a^2 - a^2)}{6EI(4a)} = -\frac{5}{8}\frac{Pa^2}{EI}$$

(a) Deflection at C :

$$y_C = -\frac{11}{12}\frac{Pa^3}{EI} - \frac{4}{3}\frac{Pa^3}{EI} - \frac{11}{12}\frac{Pa^3}{EI} = -\frac{19}{6}\frac{Pa^3}{EI}$$

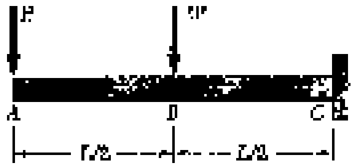
$$y_C = \frac{19}{6}\frac{Pa^3}{EI} \downarrow$$

(b) Slope at A :

$$\theta_A = -\frac{7}{8}\frac{Pa^2}{EI} - \frac{Pa^2}{EI} - \frac{5}{8}\frac{Pa^2}{EI} = -\frac{5}{2}\frac{Pa^2}{EI}$$

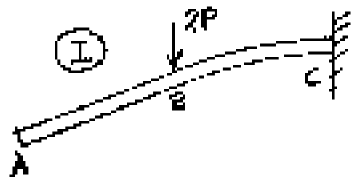
$$\theta_A = \frac{5}{2}\frac{Pa^2}{EI}$$

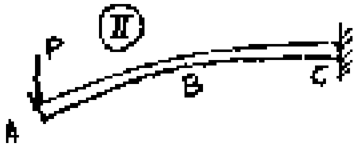
PROBLEM 9.71



9.71 and 9.72 For the cantilever beam and loading shown, determine the slope and deflection at the free end.

SOLUTION





Loading I: 2P downward at B.

Case 1 of Appendix D applied to portion BC.

$$\theta_B' = \frac{(2P)(L/2)^2}{2EI} = \frac{1}{4}\frac{PL^2}{EI}$$

$$y_B' = \frac{(2P)(L/2)^3}{3EI} = \frac{1}{12}\frac{PL^3}{EI}$$

AB remains straight.

$$\theta_A' = \theta_B' = \frac{1}{4}\frac{PL^2}{EI}$$

$$y_A' = y_B' - \left(\frac{L}{2}\right)\theta_B' = -\frac{1}{12}\frac{PL^3}{EI} - \frac{1}{8}\frac{PL^3}{EI} = -\frac{5}{24}\frac{PL^3}{EI}$$

Loading II P downward at A. Case 1 of Appendix D.

$$\theta_A'' = \frac{PL^2}{2EI}, \quad y_A'' = -\frac{PL^3}{3EI}$$

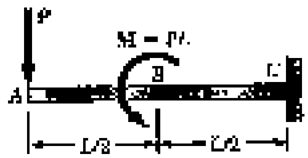
By superposition

$$\theta_A = \theta_A' + \theta_A'' = \frac{1}{4}\frac{PL^2}{EI} + \frac{1}{2}\frac{PL^2}{EI} = \frac{3}{4}\frac{PL^2}{EI}$$

$$y_A = y_A' + y_A'' = -\frac{5}{24}\frac{PL^3}{EI} - \frac{1}{3}\frac{PL^3}{EI} = -\frac{13}{24}\frac{PL^3}{EI}$$

PROBLEM 9.72

9.71 and 9.72 For the cantilever beam and loading shown, determine the slope and deflection at the free end.



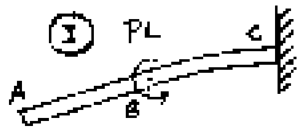
SOLUTION

Loading I: Counterclockwise couple PL at B.

Case 3 of Appendix D applied to portion BC.

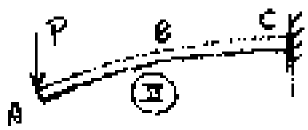
$$\theta_B' = \frac{(PL)(L/2)}{EI} = \frac{1}{2}\frac{PL^2}{EI}$$

$$y_B' = \frac{(PL)(L/2)^2}{2EI} = \frac{1}{8}\frac{PL^3}{EI}$$



AB remains straight.

$$\theta_A' = \theta_B' = \frac{1}{2}\frac{PL^2}{EI}$$



$$y_A' = y_B - \left(\frac{L}{2}\right)\theta_B = -\frac{1}{8}\frac{PL^3}{EI} - \frac{1}{4}\frac{PL^3}{EI} = -\frac{3}{8}\frac{PL^3}{EI}$$

Loading II Case 1 of Appendix D.

$$\theta_A'' = \frac{PL^2}{2EI}, \qquad y_A'' = -\frac{PL^3}{3EI}$$

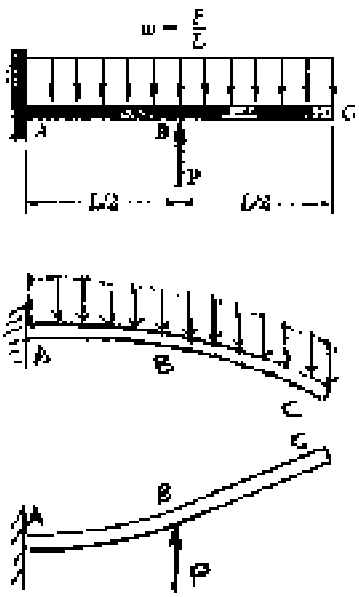
By superposition

$$\theta_A = \theta_A' + \theta_A'' = \frac{1}{2}\frac{PL^2}{EI} + \frac{1}{2}\frac{PL^2}{EI} = \frac{PL^2}{EI} = \frac{PL^2}{EI} \measuredangle$$

$$y_A = y_A' + y_A'' = -\frac{3}{8}\frac{PL^3}{EI} - \frac{1}{3}\frac{PL^3}{EI} = -\frac{17}{24}\frac{PL^3}{EI} = \frac{17}{24}\frac{PL^3}{3EI} \downarrow$$

PROBLEM 9.73

9.73 and 9.74 For the cantilever beam and loading shown, determine the slope and deflection at point C.



SOLUTION

Loading I Uniformly distributed downward loading with $w = P/L$.

Case 2 of Appendix D.

$$\theta_C' = -\frac{(P/L)L^3}{6EI} = -\frac{1}{6}\frac{PL^2}{EI}$$

$$y_C' = -\frac{(P/L)L^4}{8EI} = -\frac{1}{8}\frac{PL^3}{EI}$$

Loading II Upward concentrated load at P.

Case 1 of Appendix D applied to portion AB.

$$\theta_B'' = \frac{P(L/2)^2}{2EI} = \frac{1}{8}\frac{PL^2}{EI}$$

$$y_B'' = \frac{P(L/2)^3}{3EI} = \frac{1}{24}\frac{PL^3}{EI}$$

Portion BC remains straight.

$$\theta_C'' = \theta_B'' = \frac{1}{8}\frac{PL^2}{EI}$$

$$y_C'' = y_B'' + \frac{L}{2}\theta_B'' = \frac{1}{24}\frac{PL^3}{EI} + \frac{1}{16}\frac{PL^3}{EI} = \frac{5}{48}\frac{PL^3}{EI}$$

By superposition

$$\theta_C = \theta_C' + \theta_C'' = -\frac{1}{6}\frac{PL^2}{EI} + \frac{1}{8}\frac{PL^2}{EI} = -\frac{1}{24}\frac{PL^2}{EI} = \frac{PL^2}{24EI} \quad \blacktriangleleft$$

$$y_C = y_C' + y_C'' = -\frac{1}{8}\frac{PL^3}{EI} + \frac{5}{48}\frac{PL^3}{EI} = -\frac{1}{48}\frac{PL^3}{EI} = \frac{PL^3}{48EI}\downarrow \quad \blacktriangleleft$$

PROBLEM 9.74

9.73 and 9.74 For the cantilever beam and loading shown, determine the slope and deflection at point C.

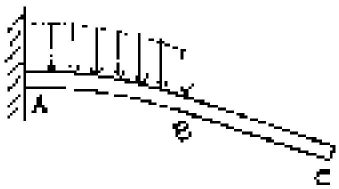
SOLUTION

Loading I: Downward distributed load w applied to portion AB.

Case 2 of Appendix D applied to portion AB.

$$\theta_B' = -\frac{w(L/2)^3}{6EI} = -\frac{1}{48}\frac{wL^3}{EI}$$

$$y_B' = -\frac{w(L/2)^4}{8EI} = -\frac{1}{128}\frac{wL^4}{EI}$$



Portion BC remains straight.

$$\theta_C' = \theta_B' = -\frac{1}{48}\frac{wL^3}{EI}$$

$$y_C' = y_B' + \left(\frac{L}{2}\right)\theta_B' = -\frac{1}{128}\frac{wL^4}{EI} - \frac{1}{96}\frac{wL^4}{EI} = -\frac{7}{384}\frac{wL^4}{EI}$$

Loading II: Counterclockwise couple $\frac{wL^2}{24}$ applied at C

Case 3 of Appendix D

$$\theta_C'' = \frac{(wL^2/24)L}{EI} = \frac{1}{24}\frac{wL^3}{EI}$$

$$y_C'' = \frac{(wL^2/24)L^2}{2EI} = \frac{1}{48}\frac{wL^4}{EI}$$

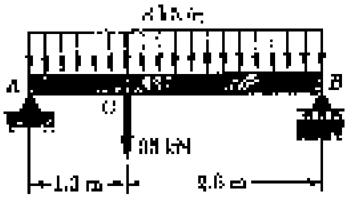
By superposition

$$\theta_C = \theta_C' + \theta_C'' = -\frac{1}{48}\frac{wL^3}{EI} + \frac{1}{24}\frac{wL^3}{EI} = \frac{1}{48}\frac{wL^3}{EI} \ \measuredangle$$

$$y_C = y_C' + y_C'' = -\frac{7}{384}\frac{wL^4}{EI} + \frac{1}{48}\frac{wL^4}{EI} = \frac{1}{384}\frac{wL^4}{EI} \uparrow$$

PROBLEM 9.75

9.75 For the W360 × 39 beam and loading shown, determine (a) the slope at end A, (b) the deflection at point C. Use $E = 200$ GPa.



SOLUTION

Units: Forces in kN, lengths in m.

Loading I: 8 kN/m uniformly distributed.

Case 6: $w = 8$ kN/m, $L = 3.9$ m, $x = 1.3$ m

$$\theta_A = -\frac{wL^3}{24EI} = -\frac{(8)(3.9)^3}{24\,EI} = -\frac{19.773}{EI}$$

$$y_C = -\frac{w}{24EI}\left[x^4 - 2Lx^3 + L^3x\right] = -\frac{8}{24\,EI}\left[(1.3)^4 - (2)(3.9)(1.3)^3 + (3.9)^3(1.3)\right]$$

$$= -\frac{20.945}{EI}$$

Loading II 35 kN concentrated load at C. Case 5 of Appendix D

$P = 35$ kN, $L = 3.9$ m, $a = 1.3$ m, $b = 2.6$ m, $x = a = 1.3$ m

$$\theta_A = -\frac{Pb(L^2-b^2)}{6EIL} = -\frac{(35)(2.6)(3.9^2 - 2.6^2)}{6\,EI\,(3.9)} = -\frac{32.861}{EI}$$

$$y_C = -\frac{Pa^2b^2}{3EIL} = -\frac{(35)(1.3)^2(2.6)^2}{3\,EI\,(3.9)} = -\frac{34.176}{EI}$$

Data: $E = 200 \times 10^9$, $I = 102.0 \times 10^6\ \text{mm}^4 = 102.0 \times 10^{-6}\ \text{m}^4$

$EI = (200\times10^9)(102.0\times10^{-6}) = 20.4\times10^6\ \text{N}\cdot\text{m}^2 = 20400\ \text{kN}\cdot\text{m}^2$

(a) Slope at A $\qquad \theta_A = -\dfrac{19.773 + 32.861}{20400} = -2.58\times10^{-3}$ rad

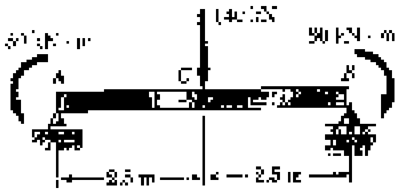
$\theta_A = 2.58\times10^{-3}$ rad ◄

(b) Deflection at C $\qquad y_C = -\dfrac{20.934 + 34.176}{20400} = -2.70\times10^{-3}$ m

$y_C = 2.70$ mm ↓ ◄

PROBLEM 9.76

9.76 For the W410 × 46.1 beam and loading shown, determine (*a*) the slope at end *A*, (*b*) the deflection at point *C*. Use *E* = 200 GPa.



SOLUTION

Units: Forces in kN, lengths in m.

Loading I: Moment at B

Case 7 of Appendix D $M = 80$ kN·m, $L = 5.0$ m, $x = 2.5$ m

$$\theta_A = \frac{ML}{6EI} = \frac{(80)(5.0)}{6EI} = \frac{66.667}{EI}$$

$$y_c = -\frac{M}{6EIL}(x^3 - L^2 x) = -\frac{80}{6EI(5.0)}\left[2.5^3 - (5.0)^2(2.5)\right] = \frac{125}{EI}$$

Loading II Moment at A Case 7 of Appendix D

$M = 80$ kN·m, $L = 5.0$ m, $x = 2.5$ m

$$\theta_A = \frac{ML}{3EI} = \frac{(80)(5.0)}{3EI} = \frac{133.333}{EI}$$

$$y_c = \frac{125}{EI} \quad \text{(Same as loading I)}$$

Loading III 140 kN concentrated load at C $P = 140$ kN

$$\theta_A = -\frac{PL^2}{16EI} = -\frac{(140)(5.0)^2}{16EI} = -\frac{218.75}{EI}$$

$$y_c = -\frac{PL^3}{48EI} = -\frac{(140)(5.0)^3}{48EI} = -\frac{364.583}{EI}$$

Data: $E = 200 \times 10^9$ Pa, $I = 156 \times 10^6$ mm^4 $= 156 \times 10^{-6}$ m^4

$$EI = (200 \times 10^9)(156 \times 10^{-6}) = 31.2 \times 10^6 \text{ N·m}^2 = 31200 \text{ kN·m}^2$$

(a) Slope at A $\theta_A = \dfrac{66.667 + 133.333 - 218.75}{31200} = -0.601 \times 10^{-3}$ rad

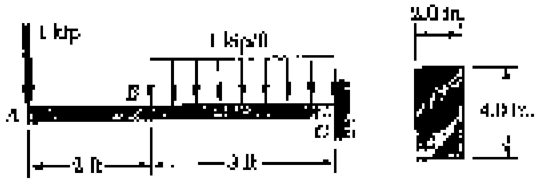
$\theta_A = 0.601 \times 10^{-3}$ ◄

(b) Deflection at C $y_c = \dfrac{125 + 125 - 364.583}{31200} = -3.67 \times 10^{-3}$ m

$y_c = 3.67$ mm ↓ ◄

PROBLEM 9.77

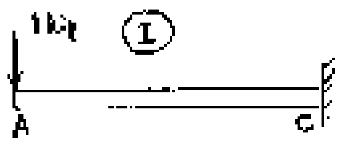
9.77 For the cantilever beam shown, determine the slope and deflection at end A. Use $E = 29 \times 10^6$ psi.



SOLUTION

Units: Forces in kips, lengths in ft.

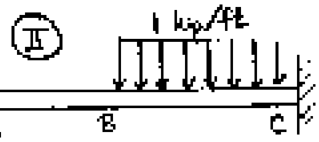
Loading I: Concentrated load at A



Case 1 of Appendix D.

$$\theta_A' = \frac{PL^2}{2EI} = \frac{(1)(5)^2}{2EI} = \frac{12.5}{EI}$$

$$y_A' = -\frac{PL^3}{3EI} = -\frac{(1)(5)^3}{3EI} = -\frac{41.667}{EI}$$



Loading II: Uniformly distributed load over portion BC.

Case 2 of Appendix D applied to portion BC

$$\theta_B'' = \frac{wL^3}{6EI} = \frac{(1)(3)^3}{6EI} = \frac{4.5}{EI}$$

$$y_B'' = -\frac{wL^4}{8EI} = -\frac{(1)(3)^4}{8EI} = -\frac{10.125}{EI}$$

Portion AB remains straight. $\quad \theta_A'' = \theta_B'' = \frac{4.5}{EI}$

$$y_A'' = y_B'' - a\theta_B'' = -\frac{10.125}{EI} - (2)\left(\frac{4.5}{EI}\right) = -\frac{19.125}{EI}$$

By superposition

$$\theta_A = \theta_A' + \theta_A'' = \frac{12.5}{EI} + \frac{4.5}{EI} = \frac{17}{EI}$$

$$y_A = y_A' + y_A'' = -\frac{41.667}{EI} - \frac{19.125}{EI} = -\frac{60.792}{EI}$$

Data: $E = 29 \times 10^6$ psi $= 29 \times 10^3$ ksi

$I = \frac{1}{12}(2.0)(4.0)^3 = 10.667$ in^4

$EI = (29 \times 10^3)(10.667) = 309.33 \times 10^3$ kip·in^2 $= 2148$ kip·ft^2

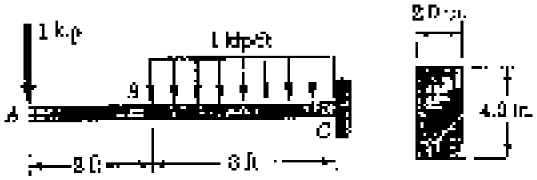
Slope at A $\quad \theta_A = \frac{17}{2148} = 7.91 \times 10^{-3}$ rad

Deflection at A $\quad y_A = -\frac{60.792}{2148} = -28.30 \times 10^{-3}$ ft

$= 0.340$ in. ↓

PROBLEM 9.78

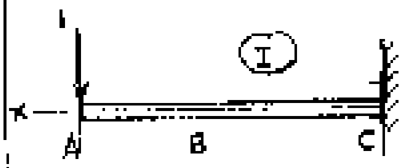
9.78 For the cantilever beam shown, determine the slope and deflection at point B. Use $E = 29 \times 10^6$ psi.



SOLUTION

Units: Forces in kips, lengths in ft.

Loading I: Concentrated load at A.



Case 1 of Appendix D.

$$y = \frac{P}{6EI}\left[x^3 - 3Lx^2\right]$$

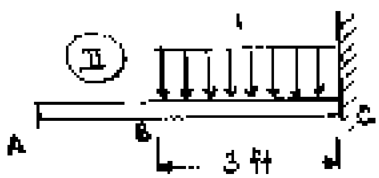
$$\frac{dy}{dx} = \frac{P}{6EI}\left[3x^2 - 6Lx\right]$$

with $P = 1$ kip, $L = 5$ ft, $x = 3$ ft.

$$y_B' = \frac{1}{6EI}\left[(3)^3 - (3)(5)(3)^2\right] = -\frac{18}{EI}$$

$$\left.\frac{dy}{dx}\right|_B' = \frac{1}{6EI}\left[(3)(3)^2 - (6)(5)(3)\right] = -\frac{10.5}{EI}$$

Adjusting the sign $\quad \theta_B' = \dfrac{10.5}{EI}$



Loading II Uniformly distributed load over portion BC.

Case 2 of Appendix D applied to portion BC

$$y_B'' = -\frac{wL^4}{8EI} = -\frac{(1)(3)^4}{8EI} = -\frac{10.125}{EI}$$

$$\theta_B'' = \frac{wL^3}{6EI} = \frac{(1)(3)^3}{6EI} = \frac{4.5}{EI}$$

By superposition

$$\theta_B = \theta_B' + \theta_B'' = \frac{10.5}{EI} + \frac{4.5}{EI} = \frac{15}{EI}$$

$$y_B = y_B' + y_B'' = -\frac{18}{EI} - \frac{10.125}{EI} = -\frac{28.125}{EI}$$

Data: $E = 29 \times 10^6$ psi $= 29 \times 10^3$ ksi

$I = \frac{1}{12}(2.0)(4.0)^3 = 10.667$ in^4

$EI = (29 \times 10^3)(10.667) = 309.33 \times 10^3$ kip·in^2 $= 2148$ kip·ft^2

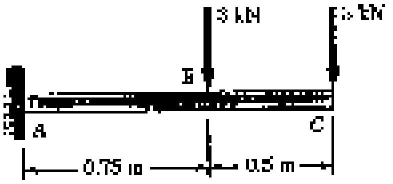
Slope at B $\qquad \theta_B = \dfrac{15}{2148} = 6.98 \times 10^{-3}$ rad. ↺ ◀

Deflection at B $\qquad y_B = -\dfrac{28.125}{2148} = -13.09 \times 10^{-3}$ ft

$= 0.1571$ in. ↓ ◀

PROBLEM 9.79

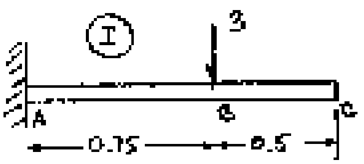
9.79 For the cantilever beam shown, determine the slope and deflection at end C. Use $E = 200$ GPa.



SOLUTION

Units: Forces in kN, lengths in m.

Loading I: Concentrated load at B



Case 1 of Appendix D applied to portion AB.

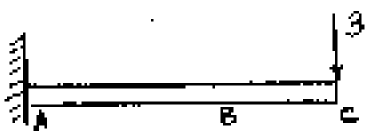
$$\theta_B' = -\frac{PL^2}{2EI} = -\frac{(3)(0.75)^2}{2EI} = -\frac{0.84375}{EI}$$

$$y_B' = -\frac{PL^3}{3EI} = -\frac{(3)(0.75)^3}{3EI} = -\frac{0.421875}{EI}$$

Portion BC remains straight

$$\theta_C' = \theta_B' = -\frac{0.84375}{EI}$$

$$y_C' = y_B' - (0.5)\theta_B' = -\frac{0.84375}{EI}$$



Loading II: Concentrated load at C. Case 1 of Appendix D.

$$\theta_A'' = -\frac{PL^2}{2EI} = -\frac{(3)(1.25)^2}{2EI} = -\frac{2.34375}{EI}$$

$$y_A'' = -\frac{PL^3}{3EI} = -\frac{(3)(1.25)^3}{3EI} = -\frac{1.953125}{EI}$$

By superposition: $\theta_A = \theta_A' + \theta_A'' = -\frac{3.1875}{EI}$

$$y_A = y_A' + y_A'' = -\frac{2.796875}{EI}$$

Data: $E = 200 \times 10^9$ Pa, $I = 2.53 \times 10^6$ mm^4 = 2.53×10^{-6} m^4

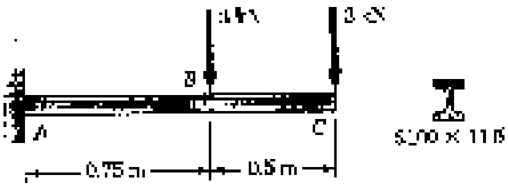
$$EI = (200 \times 10^9)(2.53 \times 10^{-6}) = 506 \times 10^3 \text{ N}\cdot\text{m}^2 = 506 \text{ kN}\cdot\text{m}^2$$

Slope at C $\quad \theta_C = -\frac{3.1875}{506} = -6.30 \times 10^{-3}$ rad $= 6.30 \times 10^{-3}$ rad ◄

Deflection at C $\quad y_C = -\frac{2.796875}{506} = -5.53 \times 10^{-3}$ m $= 5.53$ mm ↓ ◄

PROBLEM 9.80

9.80 For the cantilever beam shown, determine the slope and deflection at point *B*. Use $E = 200$ GPa.

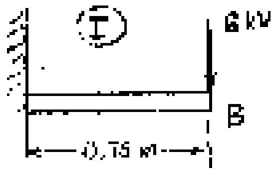


SOLUTION

Units: Forces in kN, lengths in m.

The slope and deflection at B depend only on the deformation of portion AB.

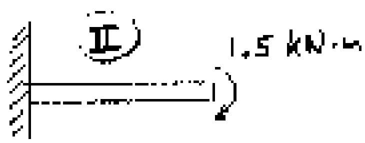
Reducing the force at C to an equivalent force-couple system at B and adding the force already at B gives the loadings I and II shown.



Loading I: Case 1 of Appendix D

$$\theta_B' = -\frac{PL^2}{2EI} = -\frac{(6)(0.75)^2}{2EI} = -\frac{1.6875}{EI}$$

$$y_B' = -\frac{PL^3}{3EI} = -\frac{(6)(0.75)^3}{3EI} = -\frac{0.84375}{EI}$$



Loading II: Case 3 of Appendix D

$$\theta_B'' = -\frac{ML}{EI} = -\frac{(1.5)(0.75)}{EI} = -\frac{1.125}{EI}$$

$$y_B'' = -\frac{ML^2}{2EI} = -\frac{(1.5)(0.75)^2}{EI} = -\frac{0.421875}{EI}$$

By superposition

$$\theta_B = \theta_B' + \theta_B'' = -\frac{2.8125}{EI}$$

$$y_B = y_B' + y_B'' = -\frac{1.265625}{EI}$$

Data: $E = 200 \times 10^9$ Pa, $I = 2.53 \times 10^6\ \text{mm}^4 = 2.53 \times 10^{-6}\ \text{m}^4$

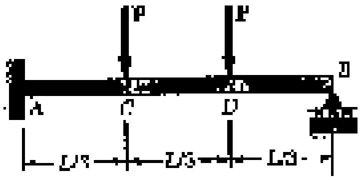
$$EI = (200 \times 10^9)(2.53 \times 10^{-6}) = 506 \times 10^3\ \text{N}\cdot\text{m}^2 = 506\ \text{kN}\cdot\text{m}^2$$

Slope at B $\quad \theta_B = -\frac{2.8125}{506} = -5.56 \times 10^{-3}\ \text{rad} = 5.56\ \text{rad}$ ◄

Deflection at B $\quad y_B = -\frac{1.265625}{506} = -2.50 \times 10^{-3}\ \text{m} = 2.50\ \text{mm} \downarrow$ ◄

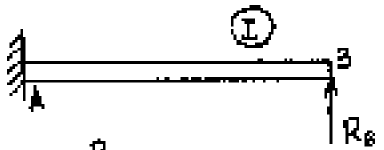
PROBLEM 9.81

9.81 and 9.82 For the uniform beam shown, determine (*a*) the reaction at *A*, (*b*) the reaction at *B*.



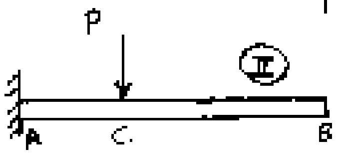
SOLUTION

Consider R_B as redundant and replace loading system by I, II, and III



Loading I   Case 1 of Appendix D applied to AB.

$$(y_B)_I = \frac{R_B L^3}{3EI}$$



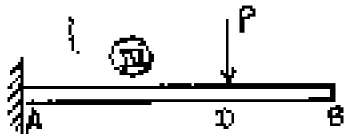
Loading II   Case 1 applied to portion AC.

$$(\theta_C)_{II} = -\frac{P(L/3)^2}{2EI} = -\frac{1}{18}\frac{PL^2}{EI}$$

$$(y_C)_{II} = -\frac{P(L/3)^3}{3EI} = -\frac{1}{81}\frac{PL^3}{EI}$$

Portion CB remains straight

$$(y_B)_{II} = (y_C)_{II} + \frac{2L}{3}(\theta_C)_{II} = -\frac{4}{81}\frac{PL^3}{EI}$$



Loading III   Case 1 applied to portion AD

$$(\theta_D)_{III} = \frac{P(2L/3)^2}{2EI} = -\frac{2}{9}\frac{PL^2}{EI}$$

$$(y_D)_{III} = \frac{P(2L/3)^3}{3EI} = -\frac{8}{81}\frac{PL^3}{EI}$$

Portion DB remains straight

$$(y_B)_{III} = (y_D)_{III} + \frac{L}{3}(\theta_D)_{III} = -\frac{14}{81}\frac{PL^3}{EI}$$

Superposition and constraint. $y_B = (y_B)_I + (y_B)_{II} + (y_B)_{III} = 0$

$$\frac{1}{3}R_B L^3 - \frac{4}{81}\frac{PL^3}{EI} - \frac{14}{81}\frac{PL^3}{EI} = \frac{1}{3}\frac{R_B L^3}{EI} - \frac{2}{9}\frac{PL^3}{EI} = 0 \qquad R_B = \frac{2}{3}P\uparrow$$

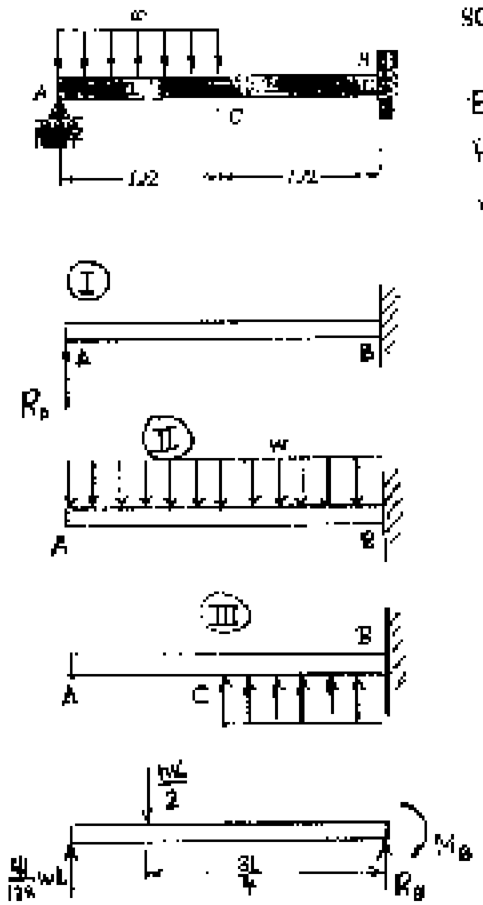
Statics

$+\uparrow \Sigma F_y = 0 \qquad R_A - P - P + \frac{2}{3}P = 0 \qquad R_A = \frac{4}{3}P\uparrow$

$\circlearrowleft \Sigma M_A = 0 \qquad M_A - P\left(\frac{L}{3}\right) - P\left(\frac{2L}{3}\right) + \left(\frac{2}{3}P\right)(L) = 0 \qquad M_A = \frac{1}{3}PL \circlearrowleft$

PROBLEM 9.82

9.81 and 9.82 For the uniform beam shown, determine (a) the reaction at A, (b) the reaction at B.



SOLUTION

Beam is indeterminate to first degree. Consider R_A as redundant and replace the given loading by loadings I, II, and III.

Loading I: Case 1 of Appendix D

$$(y_A)_I = \frac{R_A L^3}{3EI}$$

Loading II: Case 2 of Appendix D

$$(y_A)_{II} = -\frac{wL^4}{8EI}$$

Loading III Case 2 of Appendix D (Portion CB)

$$(\theta_C)_{III} = -\frac{w(L/2)^3}{6EI} = -\frac{1}{48}\frac{wL^3}{EI}$$

$$(y_C)_{III} = \frac{w(L/2)^4}{8EI} = \frac{1}{128}\frac{wL^4}{EI}$$

Portion AC remains straight

$$(y_A)_{III} = (y_C)_{III} + \frac{L}{2}(\theta_C)_{III} = \frac{7}{384}\frac{wL^4}{EI}$$

Superposition and constraint $\quad y_A = (y_A)_I + (y_A)_{II} + (y_A)_{III} = 0$

$$\frac{1}{3}\frac{R_A L^3}{EI} - \frac{1}{8}\frac{wL^4}{EI} + \frac{7}{384}\frac{wL^4}{EI} = \frac{1}{3}\frac{R_A L^3}{EI} - \frac{41}{384}\frac{wL^4}{EI} = 0 \qquad R_A = \frac{41}{128}wL \uparrow$$

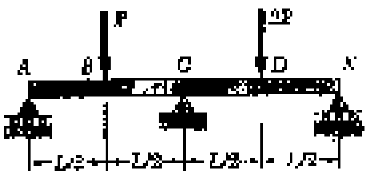
Statics

$$+\uparrow \Sigma F_y = 0 \qquad \frac{41}{128}wL - \frac{1}{2}wL + R_B = 0 \qquad R_B = \frac{23}{128}wL \uparrow$$

$$+\circlearrowleft \Sigma M_B = 0 \qquad -\left(\frac{41}{128}wL\right)L + \left(\frac{1}{2}wL\right)\left(\frac{3L}{4}\right) - M_B = 0 \qquad M_B = \frac{7}{128}wL^2$$

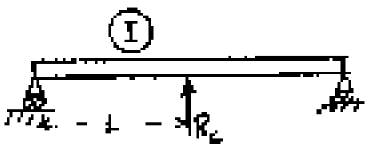
PROBLEM 9.83

9.83 and 9.84 For the uniform beam shown, determine the reaction at each of the three supports.



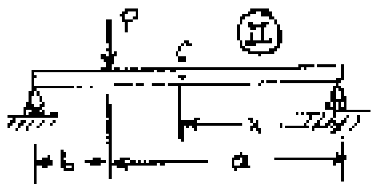
SOLUTION

Beam is indeterminate to first degree. Consider R_C to be the redundant reaction, and replace the loading by loadings I, II, and III.



Loading I   Case 4 of Appendix D.

$$(y_C)_I = \frac{R_C(2L)^3}{48EI} = \frac{1}{6}\frac{R_C L^3}{EI}$$

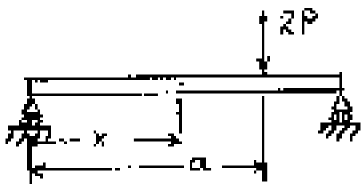


Loading II   Case 5 of Appendix D.

$$(y_C)_{II} = \frac{Pb}{6EI(2L)}\left[x^3 - \{(2L)^2 - b^2\}x\right]$$

$$= \frac{P(L/2)}{12EIL}\left[L^3 - \{4L^2 - (\tfrac{L}{2})^2\}L\right]$$

$$= -\frac{11}{48}\frac{PL^3}{EI}$$



Loading III   Case 5 of Appendix D.

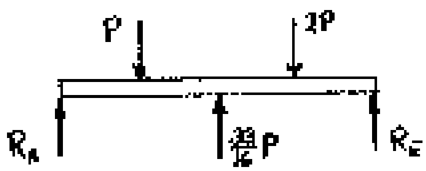
$(y_C)_{III}$ = twice that of loading II

$$(y_C)_{III} = -\frac{11}{24}\frac{PL^3}{EI}$$

Superposition and constraint $y_C = (y_C)_I + (y_C)_{II} + (y_C)_{III} = 0$

$$\frac{1}{6}\frac{R_C L^3}{EI} - \frac{11}{48}\frac{PL^3}{EI} - \frac{11}{24}\frac{PL^3}{EI} = \frac{1}{6}\frac{R_C L^3}{EI} - \frac{11}{16}\frac{PL^3}{EI} = 0 \qquad R_C = \frac{33}{16}P \uparrow$$

Statics



$+\circlearrowleft \Sigma M_E = 0$

$$-R_A(2L) + P\left(\frac{3L}{2}\right) - \left(\frac{33}{16}P\right)L + (2P)\left(\frac{L}{2}\right) = 0$$

$$R_A = \frac{7}{32}P \uparrow$$

$+\uparrow \Sigma F_y = 0$

$$\frac{7}{32}P - P + \frac{33}{16}P - 2P + R_E = 0$$

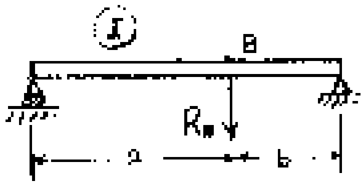
$$R_E = \frac{23}{32}P \uparrow$$

PROBLEM 9.84

9.83 and 9.84 For the uniform beam shown, determine the reaction at each of the three supports.

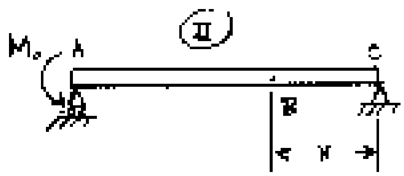
SOLUTION

Beam is statically indeterminate to first degree. Consider R_B to be the redundant reaction, and replace the loading by loadings I and II.



Loading I: Case 5 of Appendix D.

$$(y_B)_I = -\frac{R_B a^2 b^2}{3EIL} = -\frac{R_B (2L/3)^2 (L/3)^2}{3EIL} = -\frac{4}{243}\frac{R_B L^3}{EI}$$



Loading II: Case 7 of Appendix D.

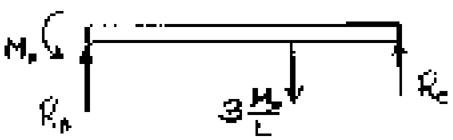
$$(y_B)_{II} = -\frac{M_0}{6EIL}(x^3 - L^2 x) = -\frac{M_0}{6EIL}\left[\left(\frac{L}{3}\right)^3 - L^2\left(\frac{L}{3}\right)\right]$$

$$= \frac{4}{81}\frac{M_0 L^2}{EI}$$

Superposition and constraint.

$$y_B = (y_B)_I + (y_B)_{II} = 0$$

$$-\frac{4}{243}\frac{R_B L^3}{EI} + \frac{4}{81}\frac{M_0 L^2}{EI} = 0 \qquad R_B = 3\frac{M_0}{L}\downarrow$$



Statics

$\circlearrowleft \Sigma M_C = 0$

$$-R_A L + M_0 + 3\frac{M_0}{L}\cdot\frac{L}{3} = 0 \qquad R_A = 2\frac{M_0}{L}\uparrow$$

$+\uparrow \Sigma F_y = 0$

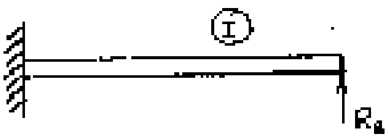
$$2\frac{M_0}{L} - 3\frac{M_0}{L} + R_C = 0 \qquad R_C = \frac{M_0}{L}\uparrow$$

PROBLEM 9.85

9.85 and 9.86 For the beam shown, determine the reaction at *B*.

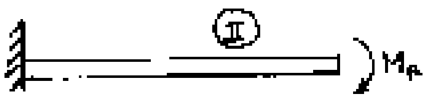
SOLUTION

Beam is second degree indeterminate. Choose R_B and M_B as redundant reactions.



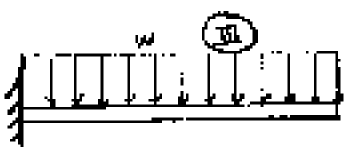
Loading I: Case 1 of Appendix D.

$$(y_B)_I = \frac{R_B L^3}{3EI} \qquad (\theta_B)_I = \frac{R_B L^2}{2EI}$$



Loading II: Case 3 of Appendix D

$$(y_B)_{II} = -\frac{M_B L^2}{2EI} \qquad (\theta_B)_{II} = -\frac{M_B L}{EI}$$



Loading III: Case 2 of Appendix D

$$(y_B)_{III} = -\frac{wL^4}{8EI} \qquad (\theta_B)_{III} = -\frac{wL^3}{6EI}$$

Superposition and constraint

$$y_B = (y_B)_I + (y_B)_{II} + (y_B)_{III} = 0$$

$$\frac{L^3}{3EI} R_B - \frac{L^2}{2EI} M_B - \frac{wL^4}{8EI} = 0 \qquad (1)$$

$$\theta_B = (\theta_B)_I + (\theta_B)_{II} + (\theta_B)_{III} = 0$$

$$\frac{L^2}{2EI} R_B - \frac{L}{EI} M_B - \frac{wL^3}{6EI} = 0 \qquad (2)$$

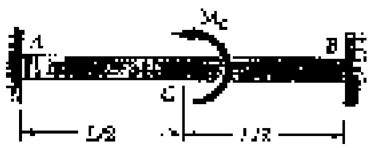
Solving (1) and (2) simultaneously

$$R_B = \tfrac{1}{2} wL \uparrow$$

$$M_B = \tfrac{1}{12} wL^2 \circlearrowright$$

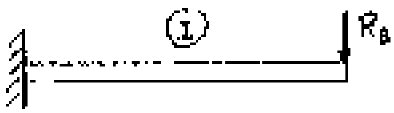
PROBLEM 9.86

9.85 and 9.86 For the beam shown, determine the reaction at B.



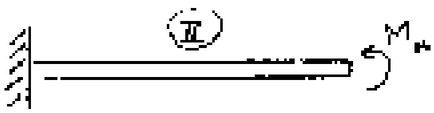
SOLUTION

Beam is second degree indeterminate. Choose R_B and M_B as redundant reactions.



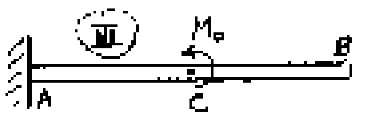
Loading I: Case 1 of Appendix D.

$$(y_B)_I = -\frac{R_B L^3}{3EI}, \quad (\theta_B)_I = -\frac{R_B L^2}{2EI}$$



Loading II: Case 3 of Appendix D.

$$(y_B)_{II} = \frac{M_B L^2}{2EI}, \quad (\theta_B)_{II} = \frac{M_B L}{EI}$$



Loading III: Case 3 applied to portion AC.

$$(y_C)_{III} = \frac{M_0 (L/2)^2}{2EI} = \frac{M_0 L^2}{8EI}$$

$$(\theta_C)_{III} = \frac{M_0 (L/2)}{EI} = \frac{M_0 L}{2EI}$$

Portion CB remains straight

$$(y_B)_{III} = (y_C)_{III} + \frac{L}{2}(\theta_C)_{III} = \frac{3}{8}\frac{M_0 L^2}{EI}$$

$$(\theta_B)_{III} = (\theta_C)_{III} = \frac{1}{2}\frac{M_0 L}{EI}$$

Superposition and constraint

$$y_B = (y_B)_I + (y_B)_{II} + (y_B)_{III} = 0$$

$$-\frac{L^3}{3EI} R_B + \frac{L^2}{2EI} M_B + \frac{3}{8}\frac{M_0 L^2}{EI} = 0 \qquad (1)$$

$$\theta_B = (\theta_B)_I + (\theta_B)_{II} + (\theta_B)_{III} = 0$$

$$-\frac{L^2}{2EI} R_B + \frac{L}{EI} M_B + \frac{1}{2}\frac{M_0 L}{EI} = 0 \qquad (2)$$

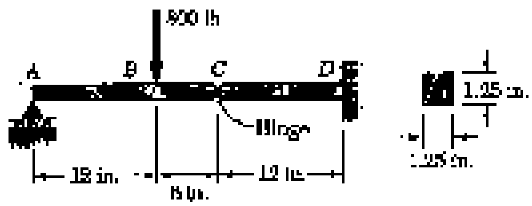
Solving (1) and (2) simultaneously

$$R_B = \frac{3}{2}\frac{M_0}{L} \downarrow$$ ◄

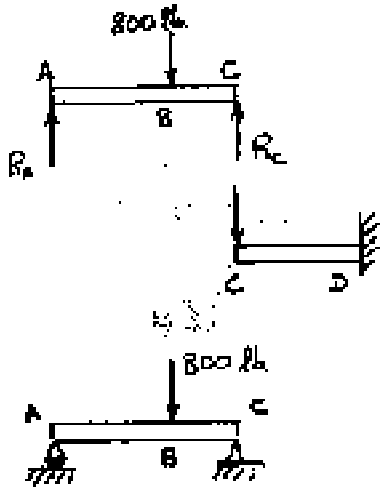
$$M_B = \frac{1}{4} M_0 \circlearrowleft$$ ◄

PROBLEM 9.87

9.87 The two beams shown have the same cross section and are joined by a hinge at C. For the loading shown, determine (a) the slope at point A, (b) the deflection at point B. Use $E = 29 \times 10^6$ psi.



SOLUTION



Using free body ABC

$\circlearrowright \Sigma M_A = 0 \qquad 18 R_C - (12)(800) = 0$

$R_C = 533.33$ lb.

$E = 29 \times 10^6$ psi

$I = \frac{1}{12} bh^3 = \frac{1}{12}(1.25)(1.25)^3 = 0.20345$ in^4

$EI = (29 \times 10^6)(0.20345) = 5.900 \times 10^6$ lb·in^2

Using cantilever beam CD with load R_C

Case 1 of Appendix D

$$y_C = -\frac{R_C L_{CD}^3}{3EI} = -\frac{(533.33)(12)^3}{(3)(5.900 \times 10^6)} = -52.067 \times 10^{-3} \text{ in.}$$

Calculation of θ_A' and y_B' assuming that point C does not move.

Case 5 of Appendix D $\qquad P = 800$ lb, $L = 18$ in., $a = 12$ in., $b = 6$ in.

$$\theta_A' = -\frac{Pb(L^2 - b^2)}{6EIL} = -\frac{(800)(6)(18^2 - 6^2)}{(6)(5.900 \times 10^6)(18)} = -2.1695 \times 10^{-3} \text{ rad.}$$

$$y_B' = -\frac{Pb^2a^2}{3EIL} = -\frac{(800)(6)^2(12)^2}{(3)(5.900 \times 10^6)(18)} = -13.017 \times 10^{-3} \text{ in}$$

Addition slope and deflection due to movement of point C

$$\theta_A'' = \frac{y_C}{L_{AC}} = -\frac{52.067 \times 10^{-3}}{18} = -2.8926 \times 10^{-3} \text{ rad.}$$

$$y_B'' = \frac{a}{L} y_C = -\frac{(12)(52.067 \times 10^{-3})}{18} = -34.711 \times 10^{-3} \text{ in.}$$

(a) Slope at A $\qquad \theta_A = \theta_A' + \theta_A'' = -2.1695 \times 10^{-3} - 2.8926 \times 10^{-3}$

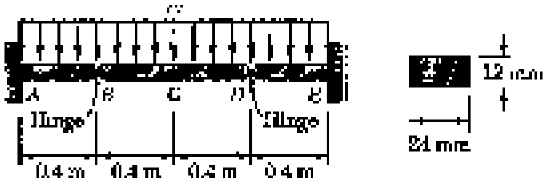
$= -5.06 \times 10^{-3}$ rad $= 5.06 \times 10^{-3}$ rad ◄

(b) Deflection at B $\qquad y_B = y_B' + y_B'' = -13.017 \times 10^{-3} - 34.711 \times 10^{-3}$

$= -47.7 \times 10^{-3}$ in. $= 47.7 \times 10^{-3}$ in. ↓ ◄

PROBLEM 9.88

9.88 A central beam BD is joined at hinges to two cantilever beams AB and DE. All beams have the cross section shown. For the loading shown, determine the largest allowable value of w if the deflection at C is not to exceed 3 mm. Use $E = 200$ GPa.



SOLUTION

Let $a = 0.4$ m

Cantilever beams AB and CD.

Cases 1 and 2 of Appendix D $\quad y_C = -\frac{(wa)a^3}{3EI} - \frac{wa^4}{8EI} = \frac{11}{24}\frac{wa^4}{EI}$

Beam BCD, with $L = 0.8$ m, assuming that points B and D do not move.

Case 6 of Appendix

$$y_C' = -\frac{5wL^4}{384EI}$$

Additional deflection due to movement of points B and D.

$$y_C'' = y_B = y_D = -\frac{11}{24}\frac{wa^4}{EI}$$

Total deflection at C $\quad y_C = y_C' + y_C''$

$$y_C = -\frac{w}{EI}\left\{\frac{5L^4}{384} + \frac{11a^4}{24}\right\}$$

Data: $E = 200 \times 10^9$ Pa, $\quad I = \frac{1}{12}(24)(12)^3 = 3.456 \times 10^3\ \text{mm}^4 = 3.456 \times 10^{-9}\ \text{m}^4$

$EI = (200 \times 10^9)(3.456 \times 10^{-9}) = 691.2\ \text{N}\cdot\text{m}^2$

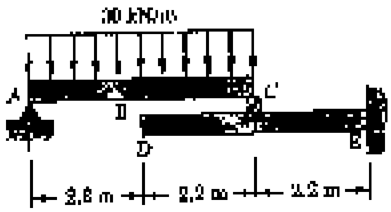
$y_C = -3 \times 10^{-3}$ m

$$-3 \times 10^{-3} = -\frac{w}{691.2}\left\{\frac{(5)(0.8)^4}{384} + \frac{(11)(0.4)^4}{24}\right\} = -24.69 \times 10^{-6}\, w$$

$w = 121.5$ N/m ◄

PROBLEM 9.89

9.89 Beam AC rests on the cantilever beam DE, as shown. Knowing that a W410 × 38.8 rolled-steel shape is used for each beam, determine for the loading shown (*a*) the deflection at point B, (*b*) the deflection at point D. Use $E = 200$ GPa.

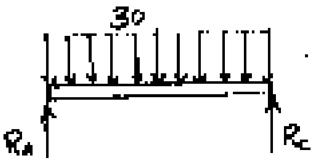


SOLUTION

Units: Forces in kN, lengths in m.

Using free body ABC, $\circlearrowright \Sigma M_A = 0$

$4.4\,R_C - (4.4)(30)(2.2) = 0 \qquad R_C = 66.0$ kN

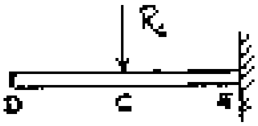


$E = 200\times10^9$ Pa

$I = 127\times10^6\ \text{mm}^4 = 127\times10^{-6}\ \text{m}^4$

$EI = (200\times10^9)(127\times10^{-6}) = 25.4\times10^6\ \text{N}\cdot\text{m}^2$

$= 25400\ \text{kN}\cdot\text{m}^2$



For slope and deflection at C, use Case 1, Appendix D applied to portion CE of beam DCE.

$$\theta_C = \frac{R_C L^2}{2EI} = \frac{(66.0)(2.2)^2}{(2)(25400)} = 6.2882\times10^{-3}\ \text{rad}$$

$$y_C = -\frac{R_C L^3}{3EI} = \frac{(66.0)(2.2)^3}{(3)(25400)} = -9.2227\times10^{-3}\ \text{m}$$

Deflection at B assuming that point C does not move.

Use Case 6 of Appendix D. $\quad (y_B)_1 = -\dfrac{5wL^4}{384\,EI} = -\dfrac{(5)(30)(4.4)^4}{(384)(25400)}$

$= -5.7642\times10^{-3}$

Additional deflection at B due to movement of point C

$(y_B)_2 = \frac{1}{2}y_C = -4.6113\times10^{-3}$ m

Total deflection at B $\quad y_B = (y_B)_1 + (y_B)_2 = -10.38\times10^{-3}$ m $= 10.38$ mm $\downarrow$

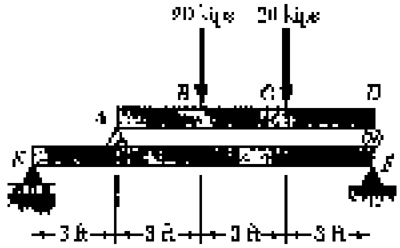
Portion DC of beam DCB remains straight.

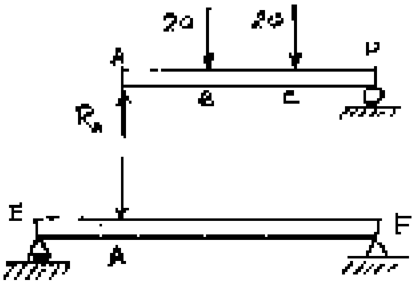
$$y_D = y_C - a\theta_C = -9.2227\times10^{-3} - (2.2)(6.2882\times10^{-3})$$

$= -23.1\times10^{-3}$ m $= 23.1$ mm $\downarrow$

PROBLEM 9.90

9.90 Beam AD rests on beam EF as shown. Knowing that a W12 × 26 rolled-steel shape is used for each beam, determine for the loading shown the deflection at points B and C. Use $E = 29 \times 10^6$ psi.





SOLUTION

$E = 29 \times 10^3$ ksi $\qquad I = 204$ in^4

$EI = (29\times10^3)(204) = 5.916\times10^6$ kip·in^2 = 41083 kip·ft^2

For equilibrium of beam ABCD $\quad R_A = 20$ kips.

Deflection at point A is due to bending of beam EAF. Using Case 5 of

$$y_A = -\frac{Pa^2b^2}{3EIL} = -\frac{(20)(3)^2(9)^2}{(3)(EI)(12)} = -\frac{405}{EI}\ \text{ft}$$

Assuming that beam ABCD is rigid

$$y_B' = \frac{6}{9}y_A = -\frac{270}{EI}\ \text{ft}, \qquad y_C' = \frac{3}{9}y_A = -\frac{135}{EI}\ \text{ft}$$

Additional deflection at B due to bending of beam ABCD. Using Case 5

$$y_B'' = -\frac{P_B a^2 b^2}{3EIL} + \frac{P_C b}{6EIL}\left[x^3 - (L^2 - b^2)x\right]$$

$$= -\frac{(20)(3)^2(6)^2}{(3)(EI)(9)} + \frac{(20)(3)\left[(3)^3 - (9^2 - 3^2)(3)\right]}{(6)(EI)(9)} = -\frac{240}{EI} - \frac{210}{EI} = -\frac{450}{EI}\ \text{ft}$$

Additional deflection at C due to bending of beam ABCD.

By symmetry $\quad y_C'' = y_B'' = -\frac{450}{EI}$ ft

Total deflection at B

$$y_B = y_B' + y_B'' = -\frac{270}{EI} - \frac{450}{EI} = -\frac{720}{EI} = -\frac{720}{41083} = -17.525\times10^{-3}\ \text{ft}$$

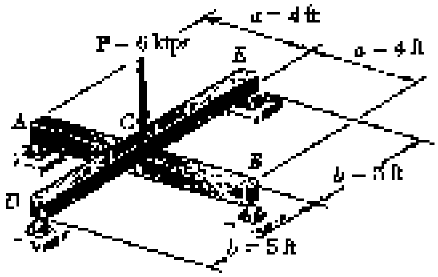
$= 0.210$ in. ↓

$$y_C = y_C' + y_C'' = -\frac{135}{EI} - \frac{450}{EI} = -\frac{585}{EI} = -\frac{585}{41083} = -14.239\times10^{-3}\ \text{ft}$$

$= 0.171$ in. ↓

PROBLEM 9.91

9.91 For the loading shown, and knowing that beams *AB* and *DE* have the same flexural rigidity, determine the reaction (*a*) at *B*, (*b*) at *E*.

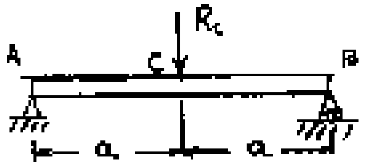


SOLUTION

Units: Forces in kips, lengths in ft.

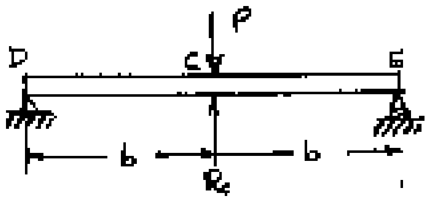
For beam ACB, using Case 4 of Appendix D.

$$(y_C)_1 = -\frac{R_C(2a)^3}{48EI}$$



For beam DCE, using Case 4 of Appendix D.

$$(y_C)_2 = \frac{(R_C - P)(2b)^3}{48EI}$$



Matching deflections at C

$$-\frac{R_C(2a)^3}{48EI} = \frac{(R_C - P)(2b)^3}{48EI}$$

$$R_C = \frac{Pb^3}{a^3 + b^3} = \frac{(6)(5)^3}{4^3 + 5^3} = 3.968 \text{ kips}$$

$$P - R_C = 6 - 3.968 = 2.032 \text{ kips}$$

Using free body ACB $\circlearrowleft M_A = 0$ $\quad 2aR_B - aR_C = 0$

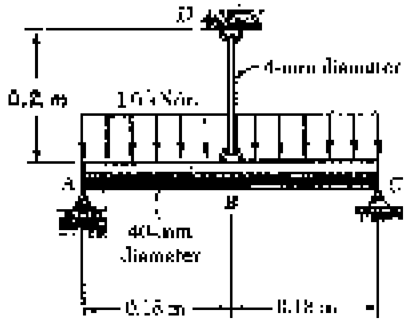
$$R_B = \tfrac{1}{2}R_C = 1.984 \text{ kips}$$

Using free body DCE $\circlearrowleft M_D = 0$ $\quad 2bR_E - b(P - R_C) = 0$

$$R_E = \tfrac{1}{2}(P - R_C) = 1.016 \text{ kips}$$

PROBLEM 9.92

9.92 Knowing that the rod ABC and the wire BD are both made of steel, determine (*a*) the deflection at B, (*b*) the reaction at A. Use $E = 200$ GPa.



SOLUTION

Let F_{BD} be the tension in wire BD. The elongation of the wire is

$$\delta_{BD} = \frac{F_{BD}\, l}{EA}$$

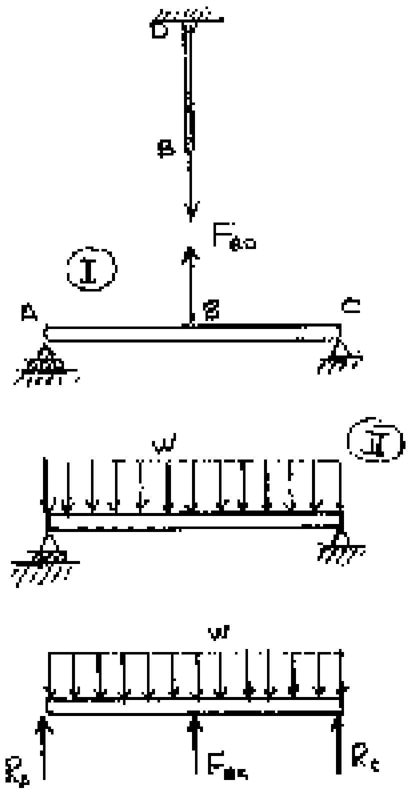
Beam ABC is subjected to loads F_{BD} (I) and w (II).

Loading I: Case 4 of Appendix D.

$$(y_B)_I = \frac{F_{BD} L^3}{48\, EI}$$

Loading II: Case 6 of Appendix D.

$$(y_B)_{II} = -\frac{5}{384}\frac{wL^4}{EI}$$



Deflection at B

$$-\delta_{BD} = y_B = (y_B)_I + (y_B)_{II}$$

$$-\frac{F_{BD}\, l}{EA} = \frac{F_{BD} L^3}{48\, EI} - \frac{5}{384}\frac{wL^4}{EI}$$

$$\left(\frac{l}{EA} + \frac{L^3}{48\, EI}\right) F_{BD} = \frac{5}{384}\frac{wL^4}{EI}$$

Data: $l = 0.2$ m

$A = \frac{\pi}{4} d^2 = \frac{\pi}{4}(4)^2 = 12.566\ \text{mm}^2 = 12.566\times10^{-6}\ \text{m}^2$

$E = 200\times10^9$ Pa

$\frac{l}{EA} = 79.58\times10^{-9}$ m/N

$L = 0.36$ m $\qquad w = 1.6\times10^3$ N/m

$I = \frac{\pi}{4}c^4 = \frac{\pi}{4}\left(\frac{40}{2}\right)^4 = 125.66\times10^3\ \text{mm}^4 = 125.66\times10^{-9}\ \text{m}^4$

$EI = (200\times10^9)(125.66\times10^{-9}) = 25.132\times10^3\ \text{N}\cdot\text{m}^2$

$$\left[79.58\times10^{-9} + \frac{(0.36)^3}{(48)(25.132\times10^3)}\right] F_{BD} = \frac{(5)(1.6\times10^3)(0.36)^4}{(384)(25.132\times10^3)}$$

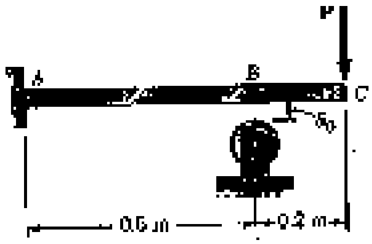
$118.256\times10^{-9}\, F_{BD} = 13.923\times10^{-6} \qquad F_{BD} = 117.74$ N

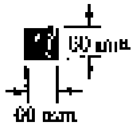
(a) Deflection at B $\quad \delta_B = \frac{F_{BD}\, l}{EA} = (117.74)(79.58\times10^{-9}) = 9.37\times10^{-6}$ m $= 0.00937$ mm ↓

(b) $R_A = R_C = \frac{1}{2}\left[wL - F_{BD}\right] = \frac{1}{2}\left[(1600)(0.36) - 117.74\right] = 229$ N ↑

PROBLEM 9.93

9.93 Before the load P was applied, a gap $\delta_0 = 0.5$ mm existed between the cantilever beam AC and the support at B. Knowing that $E = 200$ GPa, determine the magnitude of P for which the deflection at C is 1 mm.

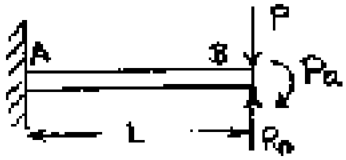




SOLUTION

Let length AB = L = 0.5 m
length BC = a = 0.2 m

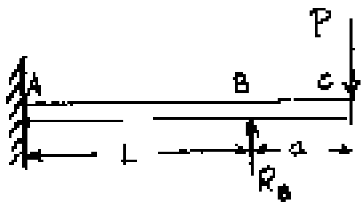
Consider portion AB of beam ABC.



The loading becomes forces P and R_B at B plus the couple Pa. The deflection at B is δ_0. Using Cases 1 and 3 of Appendix D.

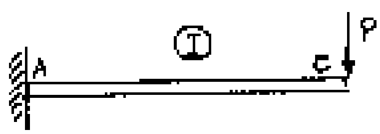
$$\delta_0 = \frac{(P - R_B)L^3}{3EI} + \frac{PaL^2}{2EI}$$

$$\left(\frac{L^3}{3} + \frac{L^2 a}{2}\right)P - \frac{L^3}{3}R_B = EI\delta_0 \qquad (1)$$



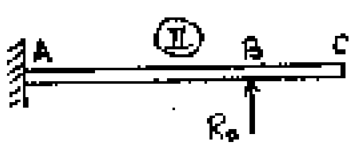
The deflection at C depends on the deformation of beam ABC subjected to loads P and R_B.

For loading I, using Case 1 of Appendix D



$$(\delta_c)_1 = \frac{P(L+a)^3}{3EI}$$

For loading II, using Case 1 of Appendix D



$$y_B = \frac{R_B L^3}{3EI} \qquad \theta_B = \frac{R_B L^2}{2EI}$$

Portion BC remains straight

$$y_c = y_B + a\theta_B = \left(\frac{L^3}{3} + \frac{L^2 a}{2}\right)\frac{R_B}{EI}$$

By superposition the downward deflection at C is

$$\delta_c = \frac{P(L+a)^3}{3EI} - \left(\frac{L^3}{3} + \frac{L^2 a}{2}\right)\frac{R_B}{EI}$$

$$\frac{(L+a)^3}{3}P - \left(\frac{L^3}{3} + \frac{L^2 a}{2}\right)R_B = EI\delta_c \qquad (2)$$

Data: $E = 200 \times 10^9$ Pa

$I = \frac{1}{12}(60)(60)^3 = 1.08 \times 10^6$ mm^4
$= 1.08 \times 10^{-6}$ m^4

$EI = 216 \times 10^3$ N·m^2

$\delta_0 = 0.5 \times 10^{-3}$ m

$\delta_c = 1.0 \times 10^{-3}$ m

Using the data, eqs (1) and (2) become

$$0.06667\,P - 0.04167\,R_B = 108 \qquad (1)'$$

$$0.11433\,P - 0.06667\,R_B = 216 \qquad (2)'$$

Solving simultaneously

$P = 5.63 \times 10^3$ N $= 5.63$ kN $\downarrow$

$R_B = 6.42 \times 10^3$ N

PROBLEM 9.94

9.94 Before the 60-kip·ft couple was applied, a gap, $\delta_0 = 0.05$ in., existed between the W16 × 26 beam and the support at C. Knowing that $E = 29 \times 10^6$ psi, determine the reaction at each support after the couple is applied.

SOLUTION

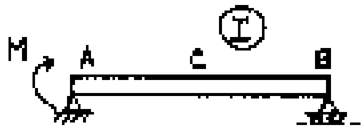
Units: Forces in kips, lengths in ft.

$\delta_0 = 0.05$ in $= 4.1667 \times 10^{-3}$ ft

$E = 29 \times 10^6$ psi $= 29 \times 10^3$ ksi

$I = 301$ in^4

$EI = 8.729 \times 10^6$ kip·in^2 $= 60618$ kip·ft^2

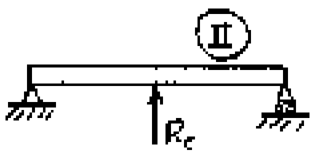


Loading I: Case 7 of Appendix D

$$y = -\frac{M}{6EIL}\left(x^3 - L^2 x\right)$$

with $M = 60$ kip·ft, $L = 13$ ft, $x = 6.5$ ft

$$(y_C)_1 = -\frac{(60)\left[6.5^3 - (13)^2(6.5)\right]}{(6)(60618)(13)} = 10.454 \times 10^{-3} \text{ ft}$$



Loading II: Case 4 of Appendix D

$$(y_C)_2 = \frac{R_C L^3}{48EI} = \frac{(13)^3 R_C}{(48)(60618)}$$

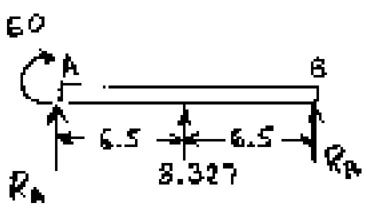
$$= 755.07 \times 10^{-6} R_C$$

Deflection at C

$$y_C = (y_C)_1 + (y_C)_2 = -\delta_0$$

$$-10.454 \times 10^{-3} + 755.07 \times 10^{-6} R_C = -4.1667 \times 10^{-3}$$

$$R_C = 8.327 \text{ kips} \uparrow$$



Statics:

$\curvearrowleft \Sigma M_B = 0$

$$-13 R_A - 60 - (6.5)(8.327) = 0$$

$R_A = -8.779$ kips $\qquad R_A = 8.779$ kips $\downarrow$

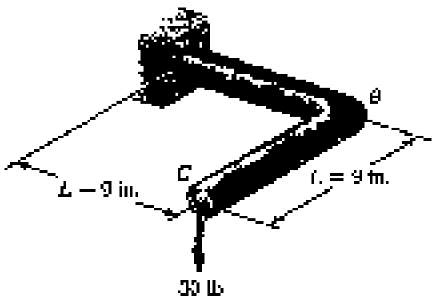
$\curvearrowleft \Sigma M_A = 0$

$$13 R_B - 60 + (6.5)(8.327) = 0$$

$$R_B = 0.452 \text{ kips} \uparrow$$

PROBLEM 9.95

9.95 A 5/8-inch-diameter rod ABC has been bent into the shape shown. Determine the deflection of end C after the 30-lb force is applied. Use $E = 29 \times 10^6$ psi and $G = 11.2 \times 10^6$ psi.



SOLUTION

Let 30 lb = P.

Consider torsion of rod AB.

$$\varphi_B = \frac{TL}{GJ} = \frac{(PL)L}{GJ} = \frac{PL^2}{GJ}$$

$$(y_C)_I = -L\varphi_B = -\frac{PL^3}{GJ}$$

Consider bending of AB (Case 1, App. D)

$$y_B = -\frac{PL^3}{3EI}$$

$$(y_C)_{II} = y_B = -\frac{PL^3}{3EI}$$

Consider bending of BC (Case 1, App. D)

$$(y_B)_{III} = -\frac{PL^3}{3EI}$$

Superposition

$$y_B = (y_B)_I + (y_B)_{II} + (y_B)_{III}$$

$$= -PL^3\left(\frac{1}{GJ} + \frac{1}{3EI} + \frac{1}{3EI}\right)$$

$$= -\frac{PL^3}{EI}\left(\frac{EI}{GJ} + \frac{2}{3}\right)$$

Data: $G = 11.2 \times 10^6$ psi, $J = \frac{\pi}{2}\left(\frac{d}{2}\right)^4 = \frac{\pi}{2}\left(\frac{5}{16}\right)^4 = 0.014980\ \text{in}^4$

$E = 29 \times 10^6$ psi, $I = \frac{1}{2}J = 0.007490\ \text{in}^4$

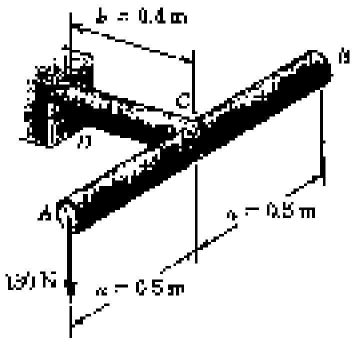
$EI = 217.21 \times 10^3\ \text{lb}\cdot\text{in}^2$ $GJ = 167.78 \times 10^3\ \text{lb}\cdot\text{in}^2$

$$y_B = -\frac{(30)(9)^3}{217.21 \times 10^3}\left(\frac{217.21 \times 10^3}{167.78 \times 10^3} + \frac{2}{3}\right) = -0.1975\ \text{in.}$$

$$y_B = 0.1975\ \text{in.}\downarrow$$

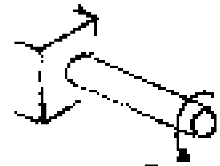
PROBLEM 9.96

9.96 Two 24-mm-diameter aluminum rods are welded together to form the T-shaped hanger shown. Knowing that $E = 70$ GPa and $G = 26$ GPa, determine the deflection at (*a*) end *A*, (*b*) end *B*.



SOLUTION

Consider torsion of rod CD (180 N = P)



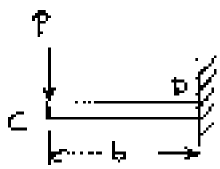
$T = Pa$

$$\varphi_c = \frac{TL}{GJ} = \frac{(Pa)b}{GJ}$$

$$(y_A)_I = -a\varphi_c = -\frac{Pa^2b}{GJ}$$

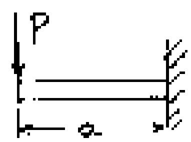
$$(y_B)_I = a\varphi_c = \frac{Pa^2b}{GJ}$$

Consider bending of rod CD



$$(y_A)_{II} = (y_B)_{II} = (y_C)_{II} = -\frac{Pb^3}{3EI}$$ (Case 1, App D.)

Consider bending of rod portion AC



$$(y_A)_{III} = -\frac{Pa^3}{3EI}$$

By superposition.

$$y_A = (y_A)_I + (y_A)_{II} + (y_A)_{III}$$
$$= P\left\{-\frac{a^2b}{GJ} - \frac{b^3}{3EI} - \frac{a^3}{3EI}\right\}$$

$$y_B = (y_B)_I + (y_B)_{II}$$
$$= P\left\{\frac{a^2b}{GJ} - \frac{b^3}{3EI}\right\}$$

Data: $G = 26\times10^9$ Pa, $J = \frac{\pi}{2}c^4 = \frac{\pi}{2}(12)^4 = 32.572\times10^3\ \text{mm}^4 = 32.572\times10^{-9}\ \text{m}^4$

$E = 70\times10^9$ Pa, $I = \frac{1}{2}J = 16.286\times10^{-9}\ \text{m}^4$

$GJ = 846.87$ N-m^2 $EI = 1140.02$ N-m^2

$a = 0.5$ m, $b = 0.4$ m

$$y_A = 180\left\{-\frac{(0.5)^2(0.4)}{846.87} - \frac{(0.4)^3}{(3)(1140.02)} - \frac{(0.5)^3}{(3)(1140.02)}\right\} = -31.2\times10^{-3}\ \text{m}$$

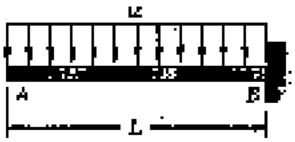
$= 31.2$ mm ↓

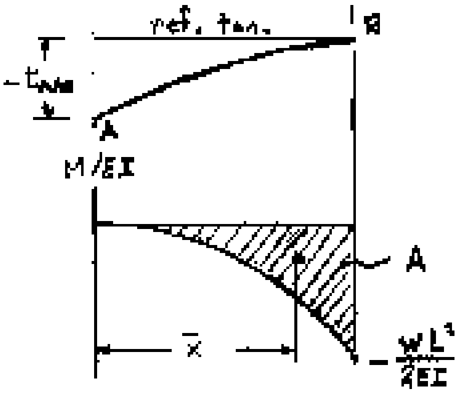
$$y_B = 180\left\{\frac{(0.5)^2(0.4)}{846.87} - \frac{(0.4)^3}{(3)(1140.02)}\right\} = 17.89\times10^{-3}\ \text{m}$$

17.89 mm ↑

PROBLEM 9.97

9.97 and 9.98 For the uniform cantilever beam and loading shown, determine (*a*) the slope at the free end, (*b*) the deflection at the free end.





SOLUTION

Place reference tangent at B. $\theta_B = 0$

Draw $\frac{M}{EI}$ curve as parabola.

$$A = -\frac{1}{3}\left(\frac{wL^2}{2EI}\right)L = -\frac{1}{6}\frac{wL^3}{EI}$$

$$\bar{x} = L - \frac{1}{4}L = \frac{3}{4}L$$

By first moment-area theorem

$$\theta_{B/A} = A = -\frac{1}{6}\frac{wL^3}{EI}$$

$$\theta_B = \theta_A + \theta_{B/A}$$

$$\theta_A = \theta_B - \theta_{B/A} = 0 + \frac{1}{6}\frac{wL^3}{EI} = \frac{1}{6}\frac{wL^3}{EI}$$

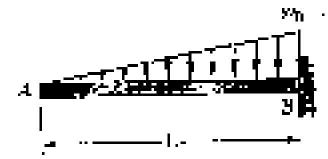
By second moment-area theorem

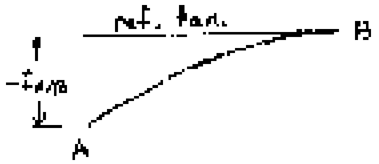
$$t_{A/B} = \bar{x}A = \left(\frac{3}{4}L\right)\left(-\frac{1}{6}\frac{wL^3}{EI}\right) = -\frac{1}{8}\frac{wL^4}{EI}$$

$$y_A = t_{A/B} = -\frac{1}{8}\frac{wL^4}{EI}$$

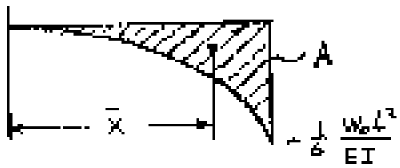
PROBLEM 9.98

9.97 and 9.98 For the uniform cantilever beam and loading shown, determine (a) the slope at the free end, (b) the deflection at the free end.





M/EI



SOLUTION

Place reference tangent at B. $\theta_B = 0$

$$\circlearrowleft \Sigma M_B = 0 \qquad \left(\tfrac{1}{2}w_0L\right)\frac{L}{3} + M_B = 0$$

$$M_B = -\frac{1}{6}wL^2$$

Draw $\frac{M}{EI}$ curve as cubic parabola.

$$A = -\frac{1}{4}\left(\frac{1}{6}\frac{wL^2}{EI}\right)L = -\frac{1}{24}\frac{wL^3}{EI}$$

$$\bar{x} = L - \frac{1}{5}L = \frac{4}{5}L$$

By first moment-area theorem

$$\theta_{B/A} = A = -\frac{1}{24}\frac{w_0L^2}{EI}$$

$$\theta_B = \theta_A + \theta_{B/A}$$

$$\theta_A = \theta_B - \theta_{B/A} = 0 + \frac{1}{24}\frac{w_0L^3}{EI} = \frac{1}{24}\frac{wL^3}{EI}$$

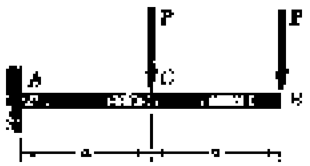
By second moment-area theorem

$$t_{A/B} = \bar{x}A = \left(\frac{4}{5}L\right)\left(-\frac{1}{24}\frac{wL^3}{EI}\right) = -\frac{1}{30}\frac{w_0L^4}{EI}$$

$$y_A = t_{A/B} = -\frac{1}{30}\frac{w_0L^4}{EI}$$

PROBLEM 9.99

9.99 and 9.100 For the uniform cantilever beam and loading shown, determine (*a*) the slope at the free end, (*b*) the deflection at the free end.



SOLUTION

Place reference tangent at A. $\theta_A = 0$.

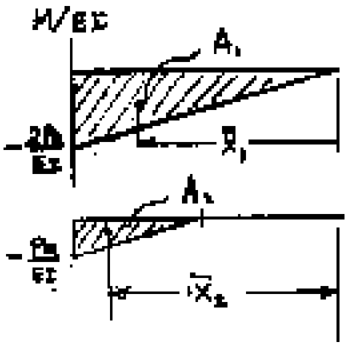
Draw $\frac{M}{EI}$ diagram by parts (two triangles)

$$A_1 = \frac{1}{2}\left(-\frac{2Pa}{EI}\right)(2a) = -\frac{2Pa^2}{EI}$$

$$\bar{x}_1 = \frac{1}{3}(2a) = \frac{4}{3}a$$

$$A_2 = \frac{1}{2}\left(-\frac{Pa}{EI}\right)a = -\frac{1}{2}\frac{Pa^2}{EI}$$

$$\bar{x}_2 = a + \frac{2}{3}a = \frac{5}{3}a$$



By first moment-area theorem

$$\theta_{B/A} = A_1 + A_2 = -\frac{2Pa^2}{EI} - \frac{1}{2}\frac{Pa^2}{EI} = -\frac{5}{2}\frac{Pa^2}{EI}$$

$$\theta_B = \theta_A + \theta_{B/A} = -\frac{5}{2}\frac{Pa^2}{EI}$$

By second moment area theorem

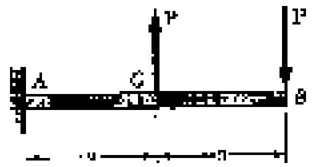
$$t_{B/A} = A_1\bar{x}_1 + A_2\bar{x}_2$$

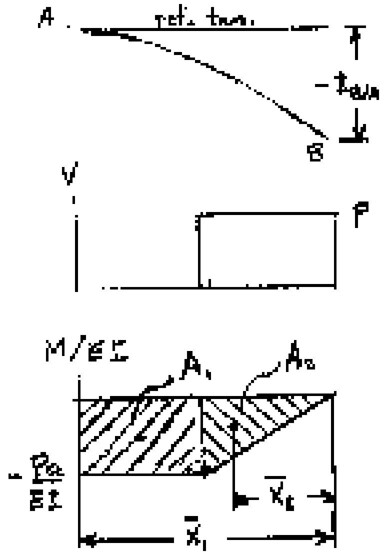
$$= \left(-\frac{2Pa^2}{EI}\right)\left(\frac{4}{3}a\right) + \left(-\frac{1}{2}\frac{Pa^2}{EI}\right)\left(\frac{5}{3}a\right) = -\frac{7}{2}\frac{Pa^3}{EI}$$

$$y_B = t_{B/A} = -\frac{7}{2}\frac{Pa^3}{EI}$$

PROBLEM 9.100

9.99 and 9.100 For the uniform cantilever beam and loading shown, determine (*a*) the slope at the free end, (*b*) the deflection at the free end.





SOLUTION

Place reference tangent at A. $\theta_A = 0$

Draw V (shear) and $\frac{M}{EI}$ diagrams.

$$A_1 = -\left(\frac{Pa}{EI}\right)(a) = -\frac{Pa^2}{EI}$$

$$A_2 = -\frac{1}{2}\left(\frac{Pa}{EI}\right)(a) = -\frac{1}{2}\frac{Pa^2}{EI}$$

$$\bar{x}_1 = a + \frac{1}{2}a = \frac{3}{2}a$$

$$\bar{x}_2 = \frac{2}{3}a$$

By first moment-area theorem

$$\theta_{B/A} = A_1 + A_2 = -\frac{Pa^2}{EI} - \frac{1}{2}\frac{Pa^2}{EI} = -\frac{3}{2}\frac{Pa^2}{EI}$$

$$\theta_B = \theta_A + \theta_{B/A} = -\frac{3}{2}\frac{Pa^2}{EI}$$

By second moment-area theorem

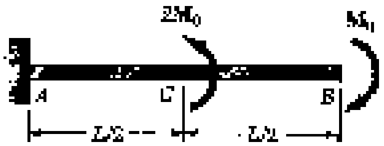
$$t_{B/A} = A_1\bar{x}_1 + A_2\bar{x}_2 = \left(-\frac{Pa^2}{EI}\right)\left(\frac{3}{2}a\right) + \left(-\frac{1}{2}\frac{Pa^2}{EI}\right)\left(\frac{2}{3}a\right)$$

$$= -\frac{11}{6}\frac{Pa^3}{EI}$$

$$y_B = t_{B/A} = -\frac{11}{6}\frac{Pa^3}{EI}$$

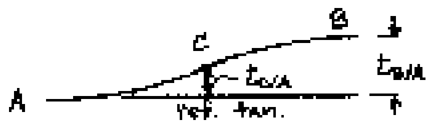
PROBLEM 9.101

9.101 and 102 For the uniform cantilever beam and loading shown, determine (*a*) the slope at point *B*, (*b*) the deflection at *C*.



SOLUTION

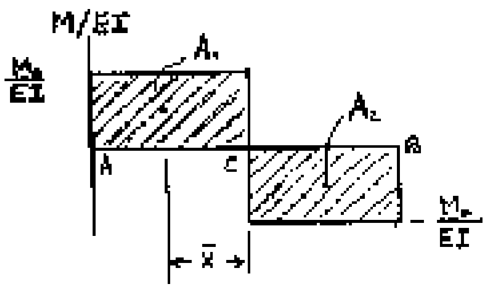
Place reference tangent at A. $\theta_A = 0$



Draw $\frac{M}{EI}$ diagram.

$$A_1 = \left(\frac{M_0}{EI}\right)\left(\frac{L}{2}\right) = \frac{1}{2}\frac{M_0 L}{EI}$$

$$A_2 = \left(-\frac{M_0}{EI}\right)\left(\frac{L}{2}\right) = -\frac{1}{2}\frac{M_0 L}{EI}$$



By first moment-area theorem

$$\theta_{B/A} = A_1 + A_2 = \frac{1}{2}\frac{M_0 L}{EI} - \frac{1}{2}\frac{M_0 L}{EI} = 0$$

$$\theta_B = \theta_A + \theta_{B/A} = 0$$

Deflection at C.

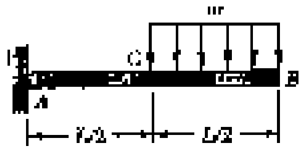
By second moment-area theorem

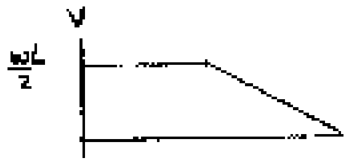
$$t_{C/A} = A_1 \bar{x} = \left(\frac{1}{2}\frac{M_0 L}{EI}\right)\left(\frac{L}{4}\right) = \frac{1}{8}\frac{M_0 L^2}{EI}$$

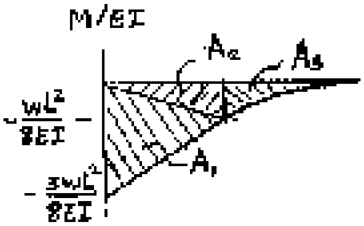
$$\underline{y_C} = t_{C/A} = \frac{1}{8}\frac{M_0 L^2}{EI}$$

PROBLEM 9.102

9.101 and 102 For the uniform cantilever beam and loading shown, determine (*a*) the slope at point *B*, (*b*) the deflection at *C*.







SOLUTION

Place reference tangent at A. $\theta_A = 0$

Draw V (shear) and $\frac{M}{EI}$ diagrams.

(a) Slope at B

$$A_1 = -\frac{1}{2}\left(\frac{3wL^2}{8EI}\right)\left(\frac{L}{2}\right) = -\frac{3}{32}\frac{wL^3}{EI}$$

$$A_2 = -\frac{1}{2}\left(\frac{wL^2}{8EI}\right)\left(\frac{L}{2}\right) = -\frac{1}{32}\frac{wL^3}{EI}$$

$$A_3 = -\frac{1}{3}\left(\frac{wL^2}{8EI}\right)\left(\frac{L}{2}\right) = -\frac{1}{48}\frac{wL^3}{EI}$$

$$\theta_{B/A} = A_1 + A_2 + A_3 = -\frac{7}{48}\frac{wL^3}{EI}$$

$$\theta_B = \theta_A + \theta_{B/A} = -\frac{7}{48}\frac{wL^3}{EI}$$ ◀

(b) Deflection at C

$$\bar{x}_{1C} = \frac{2}{3}\cdot\frac{L}{2} = \frac{1}{3}L$$

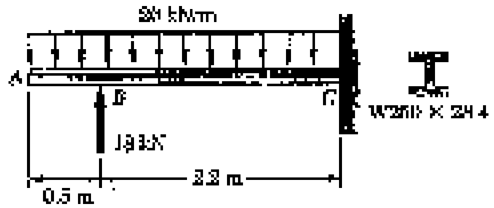
$$\bar{x}_{2C} = \frac{1}{3}\cdot\frac{L}{2} = \frac{1}{6}L$$

$$t_{C/A} = A_1\bar{x}_{1C} + A_2\bar{x}_{2C}$$
$$= \left(-\frac{3}{32}\frac{wL^3}{EI}\right)\left(\frac{1}{3}L\right) + \left(-\frac{1}{32}\frac{wL^3}{EI}\right)\left(\frac{1}{6}L\right) = -\frac{7}{192}\frac{wL^4}{EI}$$

$$y_C = t_{C/A} = -\frac{7}{192}\frac{wL^4}{EI}$$ ◀

PROBLEM 9.103

9.103 For the cantilever beam and loading shown, determine (a) the slope at point A, (b) the deflection at point A. Use $E = 200$ GPa.



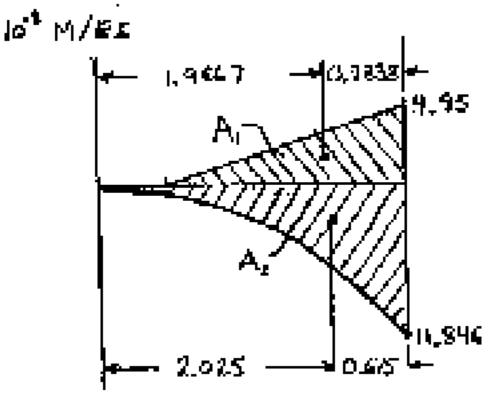
SOLUTION

Units: Forces in kN, lengths in m.

$E = 200 \times 10^{9}$ Pa

$I = 40.0 \times 10^{6}\ \text{mm}^4 = 40.0 \times 10^{-6}\ \text{m}^4$

$EI = (200 \times 10^{9})(40.0 \times 10^{-6}) = 8.00 \times 10^{6}\ \text{N}\cdot\text{m}^2$
$= 8000\ \text{kN}\cdot\text{m}^2$



Draw M/EI diagram by parts.

$\dfrac{M_1}{EI} = \dfrac{(18)(2.2)}{8000} = 4.95 \times 10^{-3}\ \text{m}^{-1}$

$A_1 = \frac{1}{2}(4.95 \times 10^{-3})(2.2) = 5.445 \times 10^{-3}$

$\bar{x}_1 = \frac{1}{3}(2.2) = 0.7333\ \text{m}$

$\dfrac{M_2}{EI} = -\dfrac{(26)(2.7)^2}{(2)(8000)} = -11.846 \times 10^{-3}\ \text{m}^{-1}$

$A_2 = \frac{1}{3}(-11.846 \times 10^{-3})(2.7) = -10.662 \times 10^{-3}$

$\bar{x}_2 = \frac{1}{4}(2.7) = 0.675\ \text{m}$

Draw reference tangent at C.

$\theta_C = \theta_A + \theta_{C/A} = \theta_A + A_1 + A_2 = 0$

$\theta_A = -A_1 - A_2 = -5.445 \times 10^{-3} + 10.662 \times 10^{-3}$
$= 5.22 \times 10^{-3}$ rad

$\theta_A = 5.22 \times 10^{-3}$ ◁

$y_A = y_C - \theta_C L + t_{A/C}$

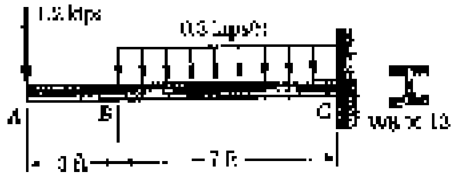
$= 0 - 0 + A_1 \bar{x}_1 + A_2 \bar{x}_2$

$= 0 - 0 + (5.445 \times 10^{-3})(1.9667) - (10.662 \times 10^{-3})(2.025)$

$= -10.88 \times 10^{-3}$ m $=$ 10.88 mm ↓

PROBLEM 9.104

9.104 For the cantilever beam and loading shown, determine (*a*) the slope at point *A*, (*b*) the deflection at point *A*. Use $E = 29 \times 10^6$ psi.



SOLUTION

Units: Forces in kips, lengths in ft.

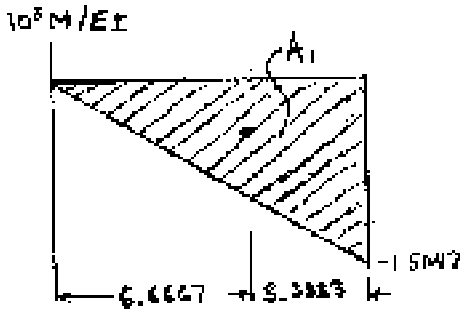
$E = 29 \times 10^6$ psi $= 29 \times 10^3$ ksi

$I = 39.6 \text{ in}^4$

$EI = (29 \times 10^3)(39.6) = 1.1484 \times 10^6 \text{ kip} \cdot \text{in}^2$

$= 7975 \text{ kip} \cdot \text{ft}^2$

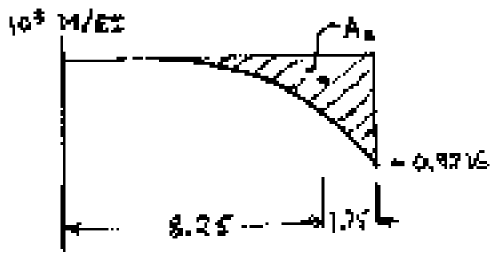
Draw $\frac{M}{EI}$ diagram by parts.



$$\frac{M_1}{EI} = -\frac{(1.2)(10)}{7975} = -1.5047 \times 10^{-3} \text{ ft}^{-1}$$

$$A_1 = -\tfrac{1}{2}(1.5047 \times 10^{-3})(10) = -7.5235 \times 10^{-3}$$

$$\bar{x}_1 = \tfrac{1}{3}(10) = 3.3333 \text{ ft}$$

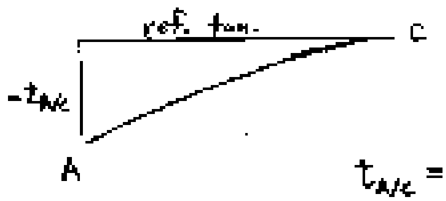


$$\frac{M_2}{EI} = -\frac{(0.3)(7)^2}{(2)(7975)} = -0.9216 \times 10^{-3} \text{ ft}^{-1}$$

$$A_2 = \tfrac{1}{3}(-0.9216 \times 10^{-3})(7) = -2.1505 \times 10^{-3}$$

$$\bar{x}_2 = \tfrac{1}{4}(7) = 1.75 \text{ ft}$$

Place reference tangent at C. $\theta_C = 0$



$$\theta_{C/A} = A_1 + A_2 = -9.67 \times 10^{-3}$$

$$\theta_A = \theta_C - \theta_{C/A} = 9.67 \times 10^{-3} \text{ rad}$$

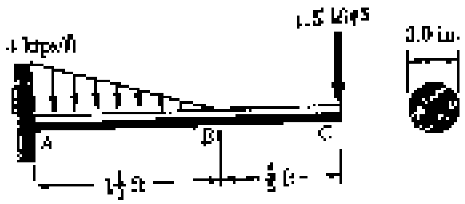
$$t_{A/C} = (6.6667)(-7.5235 \times 10^{-3}) + (8.25)(-2.1505 \times 10^{-3})$$

$$= -67.90 \times 10^{-3} \text{ ft}$$

$$y_A = t_{A/C} = -67.90 \times 10^{-3} \text{ ft} = -0.814 \text{ in.}$$

PROBLEM 9.105

9.105 For the cantilever beam and loading shown, determine (*a*) the slope at point C, (*b*) the deflection at point C. Use $E = 29 \times 10^6$ psi.



SOLUTION

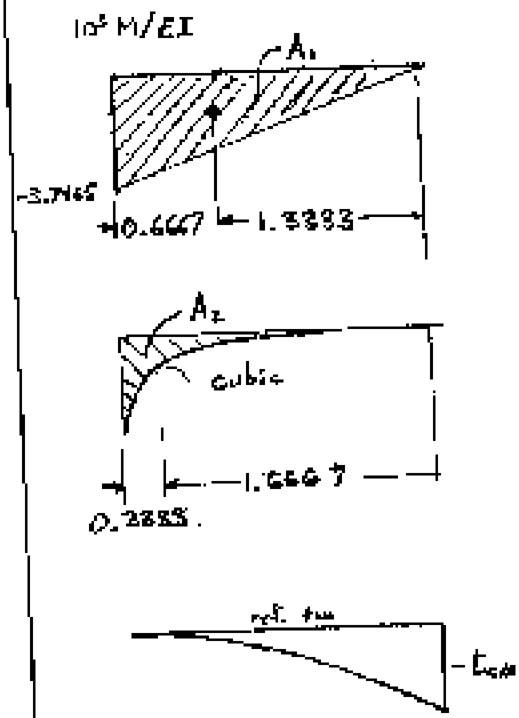
Units: Forces in kips, lengths in ft.

$E = 29 \times 10^6$ psi $= 29 \times 10^3$ ksi

$I = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi}{4}(1.5)^4 = 3.97608 \text{ in}^4$

$EI = (29 \times 10^3)(3.97608) = 115.306 \text{ kip} \cdot \text{in}^2$

$= 800.74 \text{ kip} \cdot \text{ft}^2$



Draw $\frac{M}{EI}$ diagram by parts

$\frac{M_1}{EI} = -\frac{(1.5)(2)}{800.74} = -3.7465 \times 10^{-3} \text{ ft}^{-1}$

$A_1 = \frac{1}{2}(-3.7465 \times 10^{-3})(2) = -3.7465 \times 10^{-3}$

$\bar{x}_1 = \frac{1}{3}(2) = 0.66667$ ft

$\frac{M_2}{EI} = \frac{\frac{1}{2}(4)\left(\frac{4}{3}\right)\left(\frac{1}{3} \cdot \frac{4}{3}\right)}{800.74} = -1.4802 \times 10^{-3} \text{ ft}^{-1}$

$A_2 = \frac{1}{4}(-1.4802 \times 10^{-3})\left(\frac{4}{3}\right) = -0.49337 \times 10^{-3}$

$\bar{x} = \frac{1}{4} \cdot \frac{4}{3} = 0.33333$ ft

Place reference tangent at A. $\theta_A = 0$

$\theta_{C/A} = A_1 + A_2 = -4.24 \times 10^{-3}$ rad

$\theta_C = \theta_A + \theta_{C/A} = -4.24 \times 10^{-3}$ rad

$t_{C/A} = (1.3333)(-3.7465 \times 10^{-3})$
$+ (1.6667)(-0.49337 \times 10^{-3})$
$= -5.82 \times 10^{-3}$ ft

$y_C = y_A + (2)(\theta_A) + t_{C/A}$
$= 0 + 0 - 5.82 \times 10^{-3} = -5.82 \times 10^{-3}$ ft
$= 0.0698$ in. ↓

PROBLEM 9.106

9.106 Two C6 × 8.2 channels are welded back to back and loaded as shown. Knowing that $E = 29 \times 10^6$ psi, determine (a) the slope at D, (b) the deflection at D.

SOLUTION

Units: Forces in kips, lengths in ft.

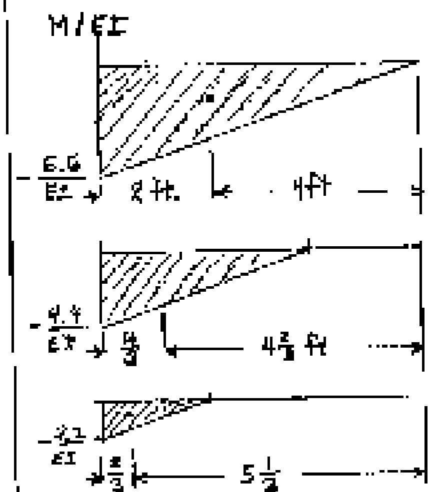
$E = 29 \times 10^6 \text{ psi} = 29 \times 10^3 \text{ ksi}$

$I = (2)(13.1) = 26.2 \text{ in}^4$

$EI = (29 \times 10^3)(26.2) = 759.8 \times 10^3 \text{ kip}\cdot\text{in}^2$

$= 5276 \text{ kip}\cdot\text{ft}^2$

Draw $\frac{M}{EI}$ diagram by parts.



$\frac{M_1}{EI} = -\frac{(1.1)(6)}{EI} = -\frac{6.6}{EI} \quad \text{ft}^{-1}$

$A_1 = \frac{1}{2}\left(-\frac{6.6}{EI}\right)(6) = -\frac{19.8}{EI}$

$\bar{x}_1 = \frac{1}{3}(6) = 2 \text{ ft}$

$\frac{M_2}{EI} = -\frac{(1.1)(4)}{EI} = -\frac{4.4}{EI} \quad \text{ft}^{-1}$

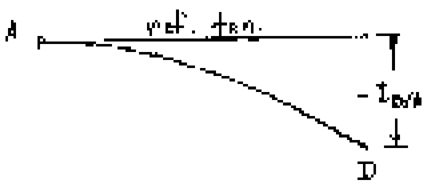
$A_2 = \frac{1}{2}\left(-\frac{4.4}{EI}\right)(4) = -\frac{8.8}{EI}$

$\bar{x}_2 = \frac{1}{3}(4) = \frac{4}{3} \text{ ft}$

$\frac{M_3}{EI} = -\frac{(1.1)(2)}{EI} = -\frac{2.2}{EI} \quad \text{ft}^{-1}$

$A_3 = \frac{1}{2}\left(-\frac{2.2}{EI}\right)(2) = -\frac{2.2}{EI}$

$\bar{x}_3 = \frac{1}{3}(2) = \frac{2}{3} \text{ ft}$



Place reference tangent at A. $\theta_A = 0$

$$\theta_{D/A} = A_1 + A_2 + A_3 = -\frac{30.8}{EI} = -\frac{30.8}{5276} = -5.84 \times 10^{-3} \text{ rad.}$$

$$\theta_D = \theta_A + \theta_{D/A} = -5.84 \times 10^{-3} \text{ rad.}$$

$$t_{D/A} = \left(-\frac{19.8}{EI}\right)(4) + \left(-\frac{8.8}{EI}\right)\left(4\tfrac{2}{3}\right) + \left(-\frac{2.2}{EI}\right)\left(5\tfrac{1}{3}\right) = -\frac{132.0}{EI} = -\frac{132.0}{5276} = 25.02 \times 10^{-3} \text{ ft}$$

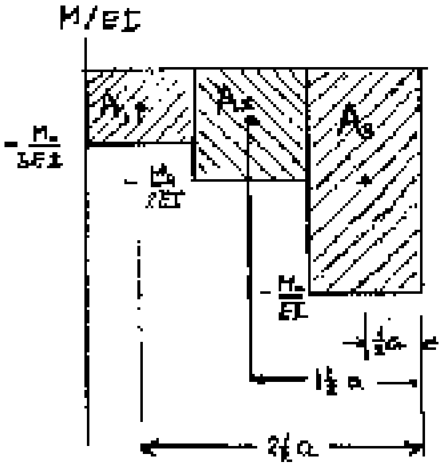
$$y_D = t_{D/A} = 25.02 \times 10^{-3} \text{ ft} = 0.300 \text{ in} \downarrow$$

PROBLEM 9.107

9.107 For the cantilever beam and loading shown, determine the deflection and slope at end D caused by the couple M_0.

SOLUTION

Draw $\frac{M}{EI}$ diagram.



$$A_1 = -\frac{M_0 a}{3EI}$$

$$A_2 = -\frac{M_0 a}{2EI}$$

$$A_3 = -\frac{M_0 a}{EI}$$

Place reference tangent at A. $\theta_A = 0$

$$\theta_{D/A} = A_1 + A_2 + A_3$$

$$= -\frac{11}{6}\frac{M_0 a}{EI}$$

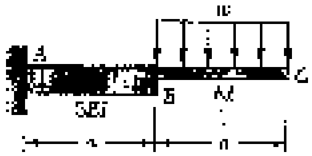
$$\theta_D = \theta_A + \theta_{D/A} = -\frac{11}{6}\frac{M_0 a}{EI}$$

$$t_{D/A} = -\left(\frac{M_0 a}{3EI}\right)\left(2\tfrac{1}{2}a\right) - \left(\frac{M_0 a}{2EI}\right)\left(1\tfrac{1}{2}a\right) - \left(\frac{M_0 a}{EI}\right)\left(\tfrac{1}{2}a\right) = -\frac{25}{12}\frac{M_0 a^2}{EI}$$

$$y_D = t_{D/A} = -\frac{25}{12}\frac{M_0 a^2}{EI}$$

PROBLEM 9.108

9.108 For the cantilever beam and loading shown, determine the deflection at (*a*) point *B*, (*b*) point *C*.



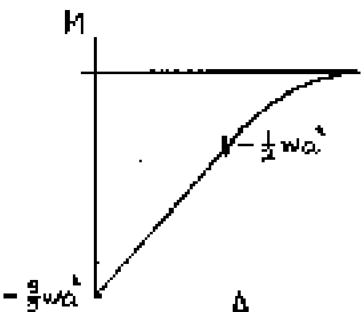
SOLUTION

Draw $\frac{M}{EI}$ diagram

$$A_1 = \tfrac{1}{2}\left(-\tfrac{1}{6}\tfrac{wa^2}{EI}\right)a = -\tfrac{1}{12}\tfrac{wa^3}{EI}$$

$$A_2 = \tfrac{1}{2}\left(-\tfrac{1}{2}\tfrac{wa^2}{EI}\right)a = -\tfrac{1}{4}\tfrac{wa^3}{EI}$$

$$A_3 = \tfrac{1}{3}\left(-\tfrac{1}{2}\tfrac{wa^2}{EI}\right)a = -\tfrac{1}{6}\tfrac{wa^3}{EI}$$



Place reference tangent at A

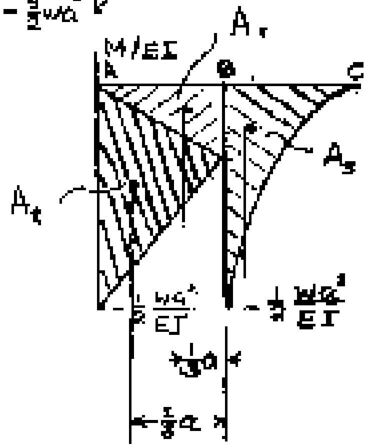
(a) Deflection at B.

$$t_{B/A} = A_1\left(\tfrac{1}{3}a\right) + A_2\left(\tfrac{2}{3}a\right)$$

$$= \left(-\tfrac{1}{12}\tfrac{wa^3}{EI}\right)\left(\tfrac{1}{3}a\right) + \left(-\tfrac{1}{4}\tfrac{wa^3}{EI}\right)\left(\tfrac{2}{3}a\right)$$

$$= -\tfrac{7}{36}\tfrac{wa^4}{EI}$$

$$y_B = t_{B/A} = -\tfrac{7}{36}\tfrac{wa^4}{EI}$$

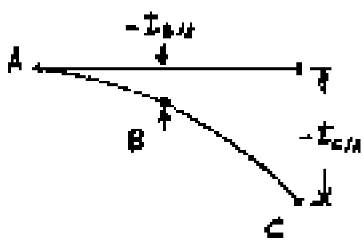


(b) Deflection at C.

$$t_{C/A} = A_1\left(a+\tfrac{1}{3}a\right) + A_2\left(a+\tfrac{2}{3}a\right) + A_3\left(a-\tfrac{1}{4}a\right)$$

$$= \left(-\tfrac{1}{12}\tfrac{wa^3}{EI}\right)\left(\tfrac{4}{3}a\right) + \left(-\tfrac{1}{4}\tfrac{wa^3}{EI}\right)\left(\tfrac{5}{3}a\right)$$

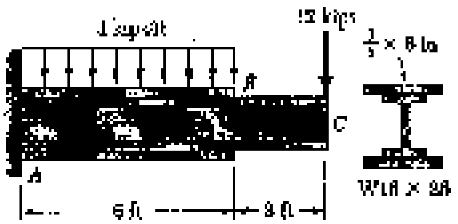
$$-\left(\tfrac{1}{6}\tfrac{wa^3}{EI}\right)\left(\tfrac{3}{4}a\right) = -\tfrac{47}{72}\tfrac{wa^4}{EI}$$



$$y_C = t_{C/A} = -\tfrac{47}{72}\tfrac{wa^4}{EI}$$

PROBLEM 9.109

9.109 Two cover plates are welded to the rolled-steel beam as shown. Using $E = 29 \times 10^6$ psi, determine (a) the slope at end C, (b) the deflection at end C.



SOLUTION

For W 16×26 rolled steel section

$d = 15.69$ in $\qquad I = 301 \text{ in}^4$

For the two cover plates

$$I = 2\left[\tfrac{1}{12}(6)(\tfrac{3}{8})^3 + (6)(\tfrac{3}{8})(\tfrac{15.69}{2} + \tfrac{3}{16})^2\right]$$

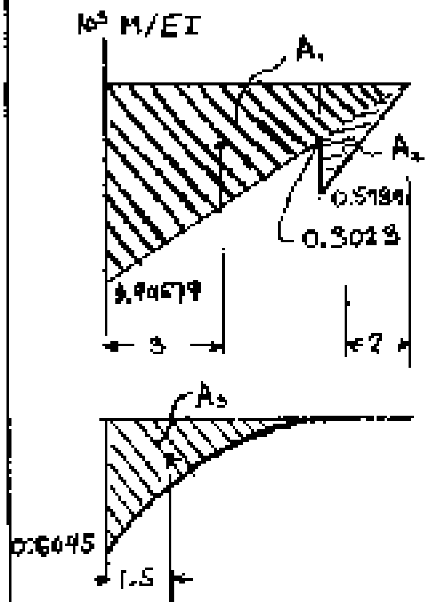
$$= 290.4 \text{ in}^4$$

A to B $\quad EI_1 = (29 \times 10^3)(301 + 290.4) = 17.151 \times 10^6 \text{ kip}\cdot\text{in}^2$

$= 119101 \text{ kip}\cdot\text{ft}^2$

B to C $\quad EI_2 = (29 \times 10^3)(301) = 8.729 \times 10^6 \text{ kip}\cdot\text{in}^2$

$= 60618 \text{ kip}\cdot\text{ft}^2$



Draw $\frac{M}{EI}$ diagram by parts.

$$\frac{M_1}{EI_1} = -\frac{(12)(9)}{119101} = -0.90679 \times 10^{-3} \text{ ft}^{-1}$$

$$A_1 = -\tfrac{1}{2}(0.90679 \times 10^{-3})(9) = -4.081 \times 10^{-3}$$

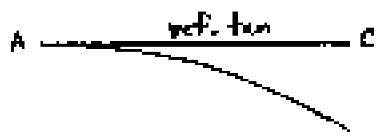
$$\frac{M_2}{EI_1} = -\frac{(12)(3)}{119101} = -0.3023 \times 10^{-3} \text{ ft}^{-1}$$

$$\frac{M_2}{EI_2} = -\frac{(12)(3)}{60618} = -0.5939 \times 10^{-3} \text{ ft}^{-1}$$

$$A_2 = -\tfrac{1}{2}(0.5939 - 0.3023)(10^{-3})(3) = -0.437 \times 10^{-3}$$

$$\frac{M_3}{EI_1} = -\frac{(4)(6)(6)}{(2)(119101)} = -0.6045 \times 10^{-3} \text{ ft}^{-1}$$

$$A_3 = -\tfrac{1}{3}(0.6045 \times 10^{-3})(6) = -1.209 \times 10^{-3}$$



Place reference tangent at A where $y_A = 0$, $\theta_A = 0$

(a) $\theta_C = \theta_A + \theta_{C/A} = 0 + A_1 + A_2 + A_3 = -5.73 \times 10^{-3}$ rad

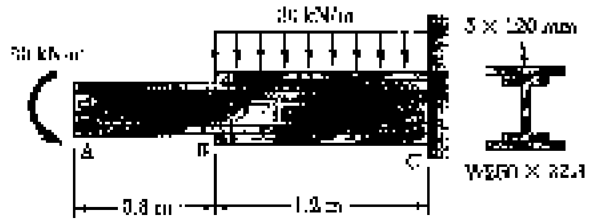
(b) $y_C = y_A + L\theta_A + t_{C/A}$

$$= 0 + 0 - (4.081 \times 10^{-3})(6) - (0.437 \times 10^{-3})(8) - (1.209 \times 10^{-3})(7.5)$$

$$= -34.48 \times 10^{-3} \text{ ft} = 0.413 \text{ in.} \downarrow$$

PROBLEM 9.110

9.110 Two cover plates are welded to the rolled-steel beam as shown. Using $E = 200$ GPa, determine (*a*) the slope at end *A*, (*b*) the deflection at end *A*.



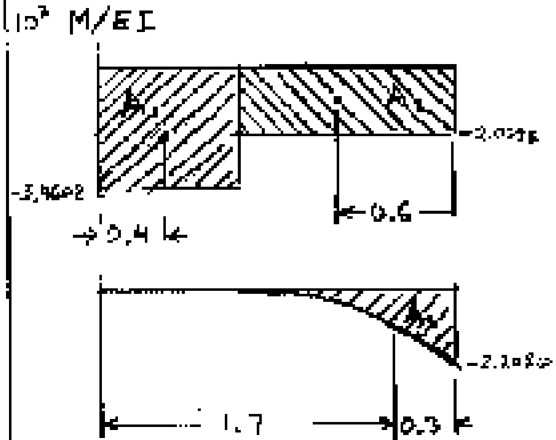
SOLUTION

Units: Forces in kN, lengths in m.

$E = 200 \times 10^9$ Pa

From A to B $\quad I = 28.9 \times 10^6 \text{ mm}^4 = 28.9 \times 10^{-6} \text{ m}^4$

$EI = (200 \times 10^9)(28.9 \times 10^{-6}) = 5.78 \times 10^6 \text{ N}\cdot\text{m}^2 = 5780 \text{ kN}\cdot\text{m}^2$



From B to C $\quad I = I_w + 2A_p d^2 + 2\bar{I}_p$

$A_p = 5 \times 120 = 600 \text{ mm}^2$

$d = \frac{254}{2} + \frac{5}{2} = 129.5 \text{ mm}$

$Ad^2 = 10.062 \times 10^6 \text{ mm}^4$

$\bar{I}_p = \frac{1}{12}(120)(5)^3 = 0.00125 \times 10^6 \text{ mm}^4$

$I = [28.9 + (2)(10.062) + (2)(0.00125)] \times 10^6 \text{ mm}^4 = 49.03 \times 10^6 \text{ mm}^4 = 49.03 \times 10^{-6} \text{ m}^4$

$EI = (200 \times 10^9)(49.03 \times 10^{-6}) = 9.806 \times 10^6 \text{ N}\cdot\text{m}^2 = 9805 \text{ kN}\cdot\text{m}^2$

Draw M/EI diagram by parts

A to B $\quad \frac{M_1}{EI} = -\frac{20}{5780} = -3.4602 \times 10^{-3} \text{ m}^{-1}$

B to C $\quad \frac{M_2}{EI} = -\frac{20}{9805} = -2.0398 \times 10^{-3} \text{ m}^{-1}$

$\frac{M_3}{EI} = -\frac{(30)(1.2)^2}{(2)(9805)} = -2.2030 \times 10^{-3} \text{ m}^{-1}$

$A_1 = (-3.4602 \times 10^{-3})(0.8) = -2.7682 \times 10^{-3}$

$A_2 = (-2.0398 \times 10^{-3})(1.2) = -2.4478 \times 10^{-3}$

$A_3 = \frac{1}{3}(-2.2030 \times 10^{-3})(1.2) = -0.8812 \times 10^{-3}$

Place reference tangent at C. $\quad \theta_C = 0$

(a) Slope at A $\quad \theta_A = \theta_C - \theta_{A/C} = 0 - (A_1 + A_2 + A_3) = 6.10 \times 10^{-3}$ rad

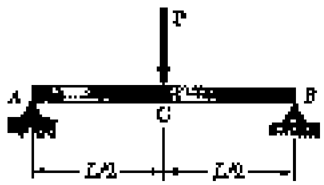
(b) Deflection at A

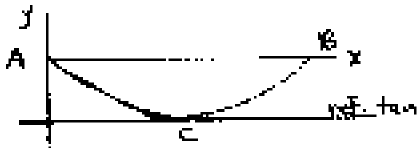
$y_A = t_{A/C} = (-2.7682 \times 10^{-3})(0.4) + (-2.4478 \times 10^{-3})(1.4) + (-0.8812 \times 10^{-3})(1.7)$

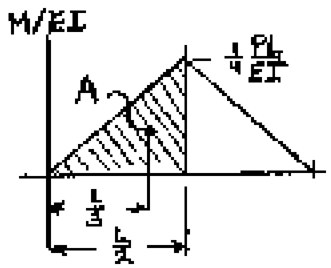
$= -6.03 \times 10^{-3}$ m $= 6.03$ mm ↓

PROBLEM 9.111

9.111 through 9.114 For the prismatic beam and loading shown, determine (*a*) the slope at end *A*, (*b*) the deflection at the center *C* of the beam.







SOLUTION

Symmetrical beam and loading.

Place reference tangent at C.

$\theta_C = 0, \quad y_C = -t_{A/C}$

Reactions $R_A = R_B = \frac{1}{2}P$

Bending moment at C $\quad M_C = \frac{1}{4}PL$

$A = \frac{1}{2}\left(\frac{1}{4}\frac{PL}{EI}\right)\left(\frac{L}{2}\right) = \frac{1}{16}\frac{PL^2}{EI}$

(a) Slope at A. $\quad \theta_A = \theta_C - \theta_{C/A}$

$\theta_A = 0 - \frac{1}{16}\frac{PL^2}{EI} = -\frac{1}{16}\frac{PL^2}{EI}$

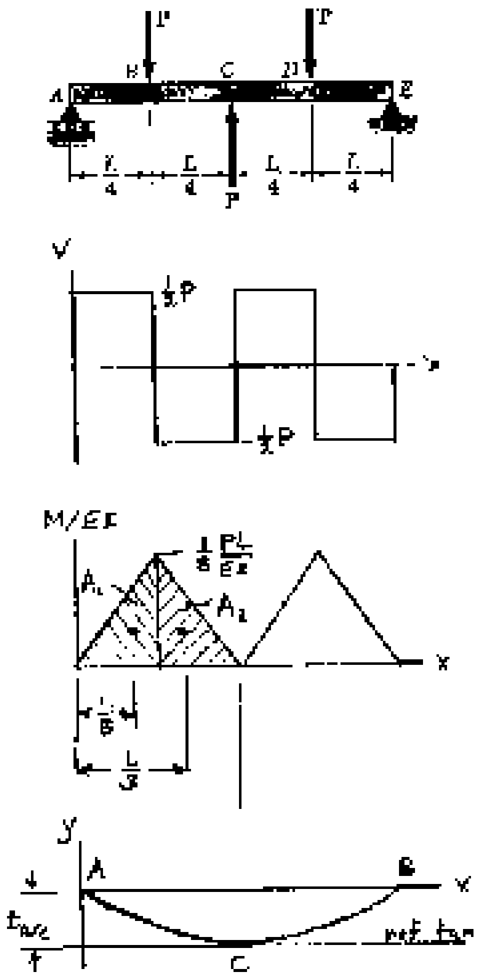
(b) Deflection at C

$y_C = -t_{A/C} = -A\left(\frac{L}{3}\right) = -\left(\frac{1}{16}\frac{PL^2}{EI}\right)\left(\frac{L}{3}\right)$

$y_C = \frac{1}{48}\frac{PL^3}{EI}$

PROBLEM 9.112

9.111 through 9.114 For the prismatic beam and loading shown, determine (*a*) the slope at end *A*, (*b*) the deflection at the center *C* of the beam.



SOLUTION

Symmetrical beam and loading.

Place reference tangent at C. $\theta_C = 0$

Reactions $R_A = R_E = \frac{1}{2}P$

Draw V (shear) and M/EI diagrams.

$$A_1 = A_2 = \frac{1}{2}\left(\frac{1}{8}\frac{PL}{EI}\right)\frac{L}{4} = \frac{1}{64}\frac{PL^2}{EI}$$

(a) Slope at A

$$\theta_A = \theta_C - \theta_{A/C} = 0 - A_1 - A_2$$

$$= -\frac{1}{32}\frac{PL^2}{EI}$$

(b) Deflection at C

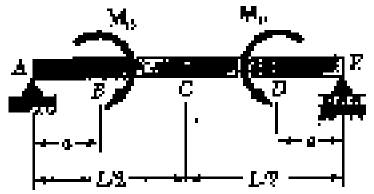
$$y_C = -t_{A/C} = -\left(A_1\frac{L}{6} + A_2\frac{L}{3}\right)$$

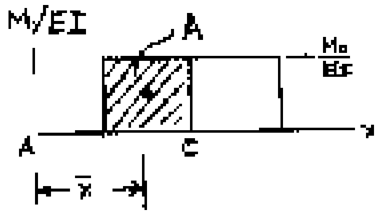
$$= -\left(\frac{1}{64}\frac{PL^2}{EI}\cdot\frac{L}{6} + \frac{1}{64}\frac{PL^2}{EI}\frac{L}{3}\right)$$

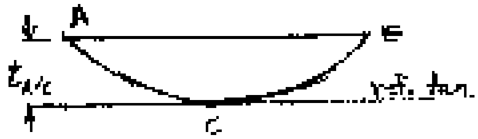
$$= -\frac{1}{128}\frac{PL^3}{EI}$$

PROBLEM 9.113

9.113 through 9.114 For the prismatic beam and loading shown, determine (*a*) the slope at end *A*, (*b*) the deflection at the center *C* of the beam.







SOLUTION

Symmetrical beam and loading.

Place reference tangent at C. $\theta_C = 0$.

Draw $\frac{M}{EI}$ diagram.

(a) Slope at A $\qquad \theta_A$

$$A = \frac{M_0}{EI}\left(\frac{L}{2} - a\right) = \frac{1}{2}\frac{M_0}{EI}(L - 2a)$$

$$\theta_A = \theta_C - \theta_{C/A} = 0 - A =$$

$$= -\frac{1}{2}\frac{M_0}{EI}(L - 2a)$$

(b) Deflection at C

$$\bar{x} = a + \frac{1}{2}\left(\frac{L}{2} - a\right) = \frac{1}{4}(L + 2a)$$

$$y_C = -t_{C/A} = A\bar{x}$$

$$= -\frac{1}{2}\frac{M_0}{EI}(L - 2a)\frac{1}{4}(L + 2a)$$

$$= -\frac{1}{8}\frac{M_0}{EI}(L^2 - 4a^2)$$

PROBLEM 9.114

9.111 through 9.114 For the prismatic beam and loading shown, determine (*a*) the slope at end *A*, (*b*) the deflection at the center *C* of the beam.

SOLUTION

Symmetric beam and loading.

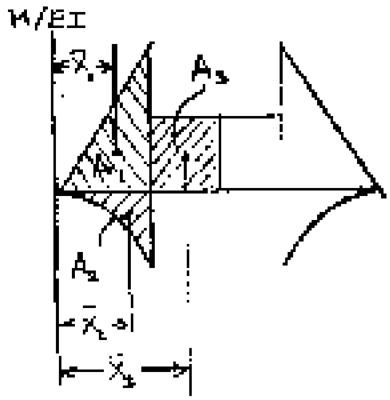
Place reference tangent at C. $\theta_C = 0$

Reactions $R_A = R_E = wa$

Bending moment

Over AB $M = wax - \frac{1}{2}wa^2$

Over BD $M = \frac{1}{2}wa^2$



Draw $\frac{M}{EI}$ diagram by parts

$$\frac{M_1}{EI} = \frac{wa^2}{EI} \qquad \frac{M_2}{EI} = -\frac{1}{2}\frac{wa^2}{EI}$$

$$\frac{M_3}{EI} = \frac{1}{2}\frac{wa^2}{EI}$$

$$A_1 = \frac{1}{2}\frac{M_1}{EI}a = \frac{1}{2}\frac{wa^3}{EI}$$

$$A_2 = \frac{1}{3}\frac{M_2}{EI}a = -\frac{1}{6}\frac{wa^3}{EI}$$

$$A_3 = \frac{M_3}{EI}\left(\frac{L}{2}-a\right) = \frac{1}{4}\frac{wa^2}{EI}(L-2a)$$

(a) Slope at A. $\theta_A = \theta_C - \theta_{C/A} = 0 - (A_1 + A_2 + A_3)$

$$= -\frac{1}{2}\frac{wa^3}{EI} + \frac{1}{6}\frac{wa^3}{EI} - \frac{1}{4}\frac{wa^2}{EI}(L-2a) = -\frac{wa^2}{EI}\left(\frac{1}{4}L - \frac{1}{6}a\right)$$

$$= -\frac{1}{12}\frac{wa^2}{EI}(3L - 2a)$$

(b) Deflection at C $y_C = -t_{C/A}$

$$\bar{x}_1 = \frac{2}{3}a, \qquad \bar{x}_2 = \frac{3}{4}a, \qquad \bar{x}_3 = a + \frac{1}{2}\left(\frac{L}{2}-a\right) = \frac{1}{4}(L+2a)$$

$$y_C = -t_{C/A} = -A_1\bar{x}_1 - A_2\bar{x}_2 - A_3\bar{x}_3$$

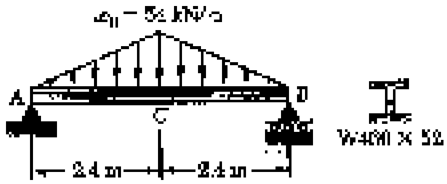
$$= -\left(\frac{1}{2}\frac{wa^3}{EI}\right)\left(\frac{2}{3}a\right) + \left(\frac{1}{6}\frac{wa^3}{EI}\right)\left(\frac{3}{4}a\right) - \frac{1}{4}\left(\frac{wa^2}{EI}\right)(L-2a)\frac{1}{4}(L+2a)$$

$$= -\frac{1}{3}\frac{wa^3}{EI} + \frac{1}{8}\frac{wa^3}{EI} - \frac{1}{16}\frac{wa^2}{EI}(L^2 - 4a^2)$$

$$= -\frac{wa^2}{EI}\left(\frac{1}{16}L^2 - \frac{1}{24}a^2\right) = -\frac{1}{48}\frac{wa^2}{EI}(3L^2 - 2a^2)$$

PROBLEM 9.115

9.115 For the beam and loading shown, determine (a) the slope at end A, (b) the deflection at the midpoint of the beam. Use $E = 200$ GPa.



SOLUTION

Symmetric beam and loading $R_A = R_B$

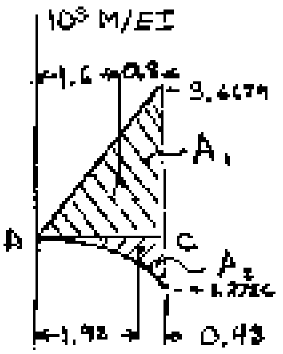
Place reference tangent at C. $\theta_C = 0$

Units: Force in kN, lengths in m.

$E = 200 \times 10^9$ Pa

$I = 212 \times 10^6 \text{ mm}^4 = 212 \times 10^{-6} \text{ m}^4$

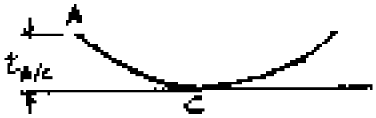
$EI = (200 \times 10^9)(212 \times 10^{-6}) = 42.4 \times 10^6 \text{ N}\cdot\text{m}^2$
$= 42400 \text{ kN}\cdot\text{m}^2$



$+\uparrow \Sigma F_y = 0 \qquad R_A + R_B - \frac{1}{2}(54)(4.8) = 0$

$R_A = 64.8$ kN

$k = \frac{54}{2.4} = 22.5 \text{ kN/m}^2$



For A to C $\qquad M = R_A x - \frac{1}{6}kx^3$

At C

$$\frac{M}{EI} = \frac{(64.8)(2.4)}{42400} - \frac{(22.5)(2.4)^3}{(6)(42400)}$$
$$= 3.6679 \times 10^{-3} - 1.2226 \times 10^{-3} \quad \text{m}^{-1}$$

Draw $\frac{M}{EI}$ diagram by parts.

$A_1 = \frac{1}{2}(3.6679 \times 10^{-3})(2.4) = 4.4015 \times 10^{-3}$

$A_2 = -\frac{1}{4}(1.2226 \times 10^{-3})(2.4) = -0.73356 \times 10^{-3}$

(a) Slope at A $\qquad \theta_A = \theta_C - \theta_{C/A} = 0 - (A_1 + A_2)$

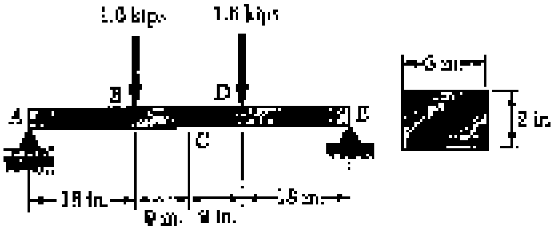
$= -4.4015 \times 10^{-3} + 0.73356 \times 10^{-3} = -3.67 \times 10^{-3}$ rad ◄

(b) Deflection at C $\qquad y_C = -t_{A/C} = -[(4.4015 \times 10^{-3})(1.6) - (0.73356 \times 10^{-3})(1.92)]$

$= -5.63 \times 10^{-3}$ m $=$ 5.63 mm ↓ ◄

PROBLEM 9.116

9.116 For the beam and loading shown, determine (a) the slope at end A, (b) the deflection at the midpoint C of the beam. Use E = 29 × 10⁶ psi.



SOLUTION

$I = \frac{1}{12}(6)(2)^3 = 2.0 \text{ in}^4$

$E = 29 \times 10^3 \text{ ksi}$

$EI = (29 \times 10^3)(2.0) = 58 \times 10^3 \text{ kip} \cdot \text{in}^2$

Symmetric beam and loading.

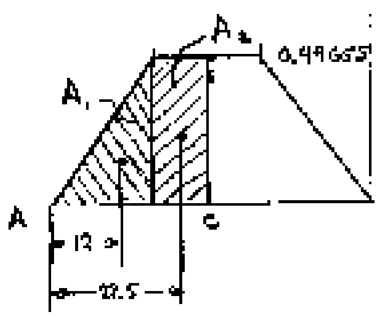
$R_A = R_E = 1.6 \text{ kips}$

Draw $\frac{M}{EI}$ diagram.

$\frac{M_{max}}{EI} = \frac{(1.6)(18)}{58 \times 10^3} = 0.49655 \times 10^{-3} \text{ in}^{-1}$

$A_1 = \frac{1}{2}(0.49655 \times 10^{-3})(18) = 4.469 \times 10^{-3}$

$A_2 = (0.49655 \times 10^{-3})(9) = 4.469 \times 10^{-3}$



Place reference tangent at C. $\theta_C = 0$

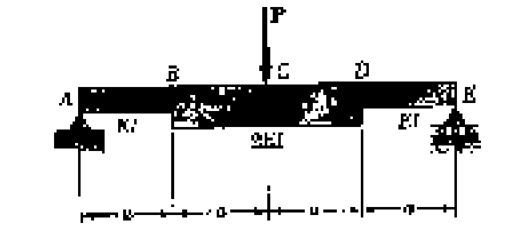
(a) Slope at A $\quad \theta_A = \theta_C - \theta_{C/A} = 0 - (A_1 + A_2) = -8.94 \times 10^{-3} \text{ rad}$

(b) Deflection at C $\quad |y_C| = t_{A/C} = (4.469 \times 10^{-3})(12) + (4.469 \times 10^{-3})(22.5)$

$= 0.1542 \text{ in} \downarrow$

PROBLEM 9.117

9.117 and 9.118 For the beam and loading shown, determine (*a*) the slope at end *A*, (*b*) the deflection at the midpoint *C* of the beam.



SOLUTION

Symmetric beam and loading. $R_A = R_E = \frac{1}{2}P$

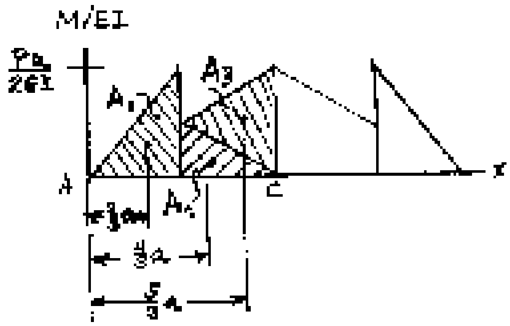
$M_{max} = (\frac{1}{2}P)(2a) = Pa$

Draw M and $\frac{M}{EI}$ diagrams.

$$A_1 = \frac{1}{2}\left(\frac{Pa}{2EI}\right)a = \frac{1}{4}\frac{Pa^2}{EI}$$

$$A_2 = \frac{1}{2}\left(\frac{Pa}{4EI}\right)a = \frac{1}{8}\frac{Pa^2}{EI}$$

$$A_3 = \frac{1}{2}\left(\frac{Pa}{2EI}\right)a = \frac{1}{4}\frac{Pa^2}{EI}$$



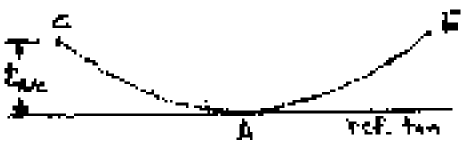
Place reference tangent at C. $\theta_C = 0$

(a) Slope at A.

$$\theta_A = \theta_C - \theta_{C/A} = 0 - (A_1 + A_2 + A_3)$$

$$= -\frac{5}{8}\frac{Pa^2}{EI}$$

(b) Deflection at C



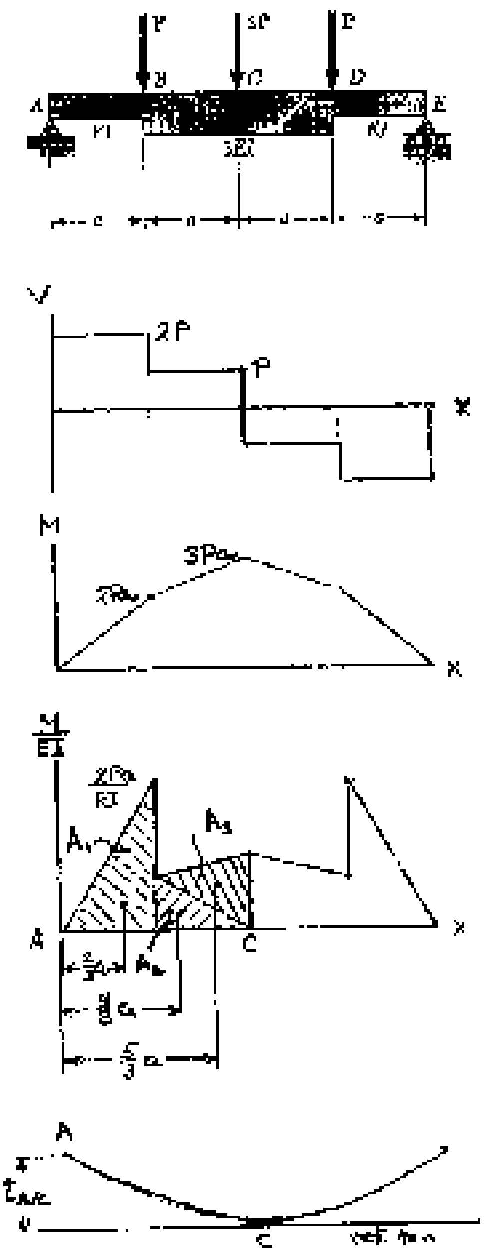
$$|y_C| = t_{A/C} = A_1\left(\tfrac{2}{3}a\right) + A_2\left(\tfrac{4}{3}a\right) + A_3\left(\tfrac{5}{3}a\right)$$

$$\frac{1}{6}\frac{Pa^3}{EI} + \frac{1}{6}\frac{Pa^3}{EI} + \frac{5}{12}\frac{Pa^3}{EI}$$

$$= \frac{3}{4}\frac{Pa^3}{EI} \downarrow$$

PROBLEM 9.118

9.117 and 9.118 For the beam and loading shown, determine (*a*) the slope at end *A*, (*b*) the deflection at the midpoint *C* of the beam.



SOLUTION

Symmetric beam and loading. $R_A = R_E = 2P$.

Draw V, M, and $\frac{M}{EI}$ diagrams.

$$A_1 = \frac{1}{2}\left(\frac{2Pa}{EI}\right)a = \frac{Pa^2}{EI}$$

$$A_2 = \frac{1}{2}\left(\frac{2}{3}\frac{Pa}{EI}\right)a = \frac{1}{3}\frac{Pa^2}{EI}$$

$$A_3 = \frac{1}{2}\left(\frac{Pa}{EI}\right)a = \frac{1}{2}\frac{Pa^2}{EI}$$

Place reference tangent at C. $\theta_C = 0$

(a) Slope at A

$$\theta_A = \theta_C - \theta_{C/A} = 0 - (A_1 + A_2 + A_3)$$

$$= -\frac{11}{6}\frac{Pa^2}{EI}$$

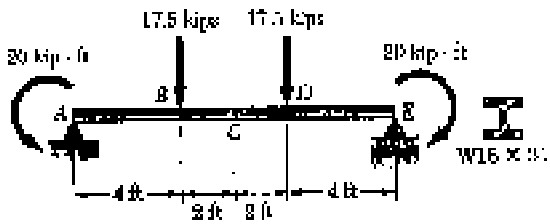
(b) Deflection at C.

$$|y_C| = t_{A/C} = A_1\left(\frac{2}{3}a\right) + A_2\left(\frac{4}{3}a\right) + A_3\left(\frac{5}{3}a\right)$$

$$= \frac{35}{18}\frac{Pa^3}{EI} \downarrow$$

PROBLEM 9.119

9.119 For the beam and loading shown, determine (a) the slope at end A, (b) the deflection at the midpoint C of the beam. Use $E = 29 \times 10^6$ psi.



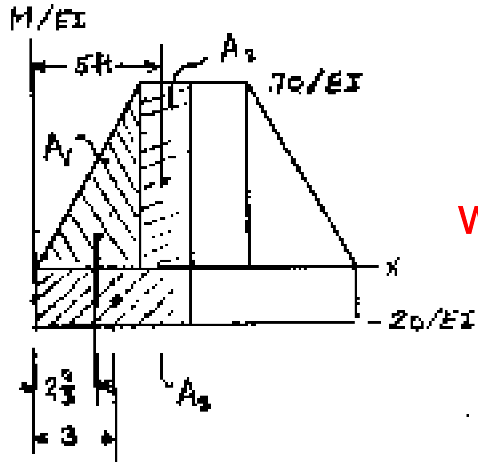
SOLUTION

$E = 29 \times 10^6$ psi $= 29 \times 10^3$ ksi

$I = 375$ in^4

$EI = (29 \times 10^3)(375) = 10.875 \times 10^6$ kip·in^2

$= 75521$ kip·ft^2



Symmetric beam and loading.

$R_A = R_E = 17.5$ kips

Bending moments:

$M_A = -20$ kip·ft

$M_B = -20 + (17.5)(4) = -20 + 70$ kip·ft

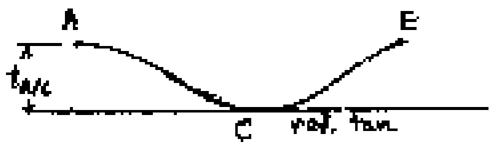
$M_C = -20 + 70$ kip·ft.

Draw M/EI diagram by parts.

$A_1 = \frac{1}{2}(70)(4) = 140/EI$

$A_2 = (70)(2) = 140/EI$

$A_3 = -(20)(6) = -120/EI$



Place reference tangent at C. $\theta_C = 0$

(a) Slope at A. $\theta_A = \theta_C - \theta_{C/A}$

$\theta_A = 0 - (A_1 + A_2 + A_3) = -160/EI$

$= -\frac{160}{75521} = -2.119 \times 10^{-3}$ rad.

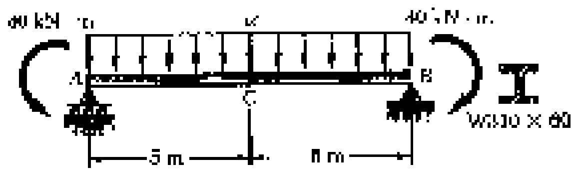
(b) Deflection at C $|y_C| = t_{A/C}$

$|y_C| = \frac{1}{EI}\left\{(140)(2\frac{2}{3}) + (140)(5) - (120)(3)\right\} = \frac{713\frac{1}{3}}{EI}$

$= \frac{713\frac{1}{3}}{75521} = 9.445 \times 10^{-3}$ ft $= 0.1133$ in. ↓

PROBLEM 9.120

9.120 For the beam and loading shown and knowing that $w = 8$ kN/m, determine (*a*) the slope at end *A*, (*b*) the deflection at the midpoint *C* of the beam. Use $E = 200$ GPa.



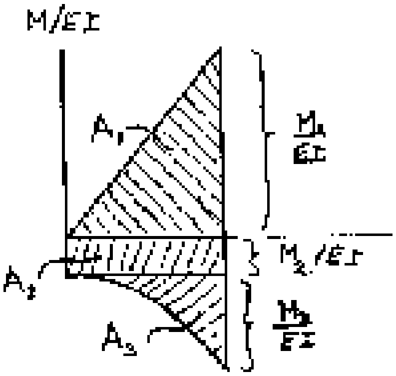
SOLUTION

$E = 200 \times 10^9$ Pa

$I = 129 \times 10^6\ \text{mm}^4 = 129 \times 10^{-6}\ \text{m}^4$

$EI = (200 \times 10^9)(129 \times 10^{-6}) = 25.8 \times 10^6\ \text{N}\cdot\text{m}^2$

$= 25800\ \text{kN}\cdot\text{m}^2$



Symmetrical beam and loading.

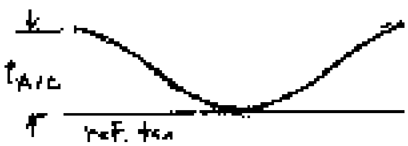
$R_A = R_B = \frac{1}{2}(8)(10) = 40$ kN

Bending moment

$M = 40x - 40 - \frac{1}{2}(8)x^2$

At $x = 5$

$M = 200 - 40 - 100$



Draw $\frac{M}{EI}$ diagram by parts.

$\frac{M_1}{EI} = \frac{200}{25800} = 7.7519 \times 10^{-3}\ \text{m}^{-1}$

$\frac{M_2}{EI} = \frac{-40}{25800} = -1.5504 \times 10^{-3}\ \text{m}^{-1}$

$\frac{M_3}{EI} = \frac{-100}{25800} = -3.8760 \times 10^{-3}\ \text{m}^{-1}$

$A_1 = \frac{1}{2}(7.7519 \times 10^{-3})(5) = 19.380 \times 10^{-3}$ $\qquad \bar{x}_1 = (\frac{2}{3})(5) = 3.3333$ m

$A_2 = -(1.5504)(5) = -7.7520 \times 10^{-3}$ $\qquad \bar{x}_2 = (\frac{1}{2})(5) = 2.5$ m

$A_3 = -\frac{1}{3}(3.8760)(5) = -6.4600 \times 10^{-3}$ $\qquad \bar{x}_3 = (\frac{3}{4})(5) = 3.75$ m

Place reference tangent at C. $\quad \theta_C = 0$

(a) Slope at A. $\quad \theta_A = \theta_C - \theta_{C/A} = 0 - (A_1 + A_2 + A_3)$

$\theta_A = -(19.380 \times 10^{-3} - 7.7520 \times 10^{-3} - 6.4600 \times 10^{-3}) = -5.17 \times 10^{-3}$ rad ◄

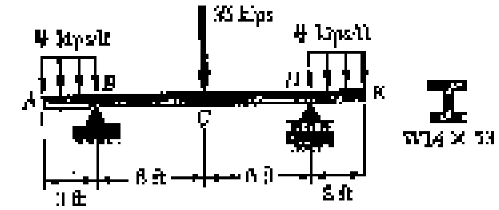
(b) Deflection at C $\quad |y_C| = t_{A/C}$

$= (19.380 \times 10^{-3})(3.3333) - (7.7520 \times 10^{-3})(2.5) - (6.4600 \times 10^{-3})(3.75)$

$= 21.0 \times 10^{-3}$ m $= 21.0$ mm ↓ ◄

PROBLEM 9.121

9.121 For the beam and loading shown, determine (*a*) the slope at end *A*, (*b*) the deflection at point *A*. Use $E = 29 \times 10^6$ psi.

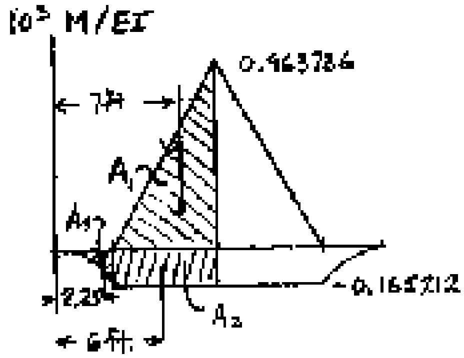


SOLUTION

$E = 29 \times 10^6$ psi $= 29 \times 10^3$ ksi

$I = 541$ in^4

$EI = (29\times10^3)(541) = 15.689 \times 10^6$ ksi
$= 108951$ kip·ft^2



Draw bending diagram by parts

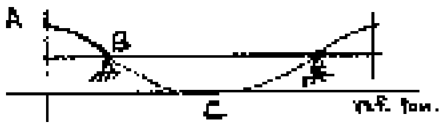
$$\frac{M_1}{EI} = \frac{(\frac{1}{2})(35)(6)}{108951} = 0.963736\times10^{-3}\ \text{ft}^{-1}$$

$$A_1 = (\tfrac{1}{2})(0.963736\times10^{-3})(6) = 2.8912\times10^{-3}$$

$$\frac{M_2}{EI} = -\frac{(4)(3)(1.5)}{108951} = -0.165212 \times 10^{-3}\ \text{ft}^{-1}$$

$$A_2 = -(0.165212\times10^{-3})(6) = -0.99127\times10^{-3}$$

$$A_3 = -\tfrac{1}{2}(0.165212\times10^{-3})(2) = -0.165212\times10^{-3}$$



Place reference at symmetry point C.

(a) $\theta_C = \theta_A + \theta_{C/A} = 0$

$$\theta_A = -\theta_{C/A} = -A_1 - A_2 - A_3$$
$$= -2.8912\times10^{-3} + 0.99127\times10^{-3} + 0.165212\times10^{-3} = -1.735\times10^{-3}\ \text{rad}$$

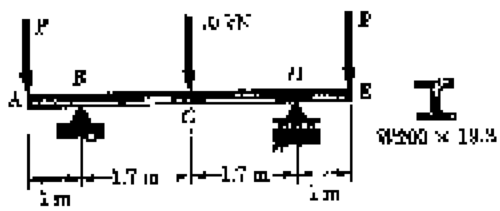
(b) $t_{A/C} = (7)(2.8912\times10^{-3}) + (6)(-0.99127\times10^{-3}) - (2.25)(-0.165212)$
$= 13.919 \times10^{-3}$ ft

$t_{B/C} = (4)(2.8912\times10^{-3}) + (3)(-0.99127\times10^{-3})$
$= 8.591\times10^{-3}$ ft

$y_A = t_{A/C} - t_{B/C} = 5.328\times10^{-3}$ ft $= 0.0639$ in ↑

PROBLEM 9.122

9.122 Knowing that $P = 8$ kN, determine (a) the slope at end A, (b) the deflection at midpoint C. Use $E = 200$ GPa.



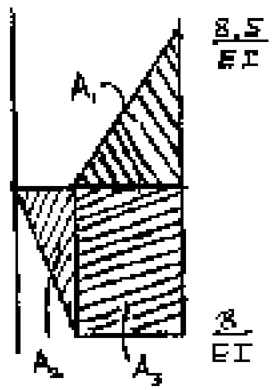
SOLUTION

$E = 200 \times 10^9$ Pa

$I = 16.6 \times 10^6 \text{ mm}^4 = 16.6 \times 10^{-6} \text{ m}^4$

$EI = (200 \times 10^9)(16.6 \times 10^{-6}) = 3.32 \times 10^6 \text{ N}\cdot\text{m}^2$

$= 3320 \text{ kN}\cdot\text{m}^2$



Symmetric beam and loading

$R_A = R_B = P + 5 = 8 + 5 = 13$ kN.

Bending moment

Over AB $\quad M = -Px = -8x$

Over BC $\quad M = -8x + 13(x - 1)$

$= 5(x-1) - 8$

Draw $\frac{M}{EI}$ diagram by parts

$A_1 = \frac{1}{2}\left(\frac{8.5}{EI}\right)(1.7) = \frac{7.225}{EI}$

$A_2 = -\frac{1}{2}\left(\frac{8}{EI}\right)(1) = -\frac{4}{EI}$

$A_3 = -\left(\frac{8}{EI}\right)(1.7) = -\frac{13.600}{EI}$

Place reference tangent at C $\qquad \theta_C = 0$

(a) Slope at A. $\quad \theta_A = \theta_C - \theta_{C/A} = 0 - (A_1 + A_2 + A_3)$

$\theta_A = -\left(\frac{7.225}{EI} - \frac{4}{EI} - \frac{13.600}{EI}\right) = \frac{10.375}{EI} = \frac{10.375}{3320} = 3.125 \times 10^{-3}$ rad ◄

(b) Deflection at C $\qquad y_C = -t_{B/C}$

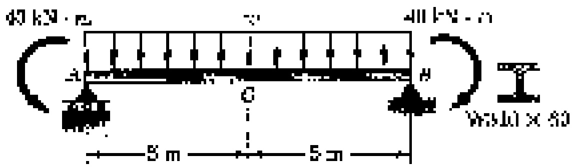
$= -(A_1\bar{x}_1 + A_3\bar{x}_3)$

$= -\left[\left(\frac{7.225}{EI}\right)\left(\frac{2}{3}(1.7)\right) - \left(\frac{13.600}{EI}\right)\left(\frac{1.7}{2}\right)\right] = \frac{3.3717}{EI} = \frac{3.3717}{3320}$

$= 1.016 \times 10^{-3}$ m $= 1.016$ mm ◄

PROBLEM 9.123

9.123 For the beam and loading of Prob. 9.120, determine the value of w for which the deflection is zero at the midpoint C of the beam. Use E = 200 GPa.



SOLUTION

Symmetric beam and loading.

$R_A = R_B = 5w \quad (w \text{ in kN/m})$

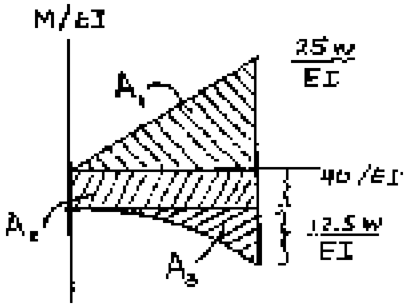
Bending moment in kN·m.

$M = 5wx - 40 - \frac{1}{2}wx^2$

At $x = 5$ m

$M = 25w - 40 - 12.5w$

Draw $\frac{M}{EI}$ diagram by parts.



$A_1 = \frac{1}{2}\left(\frac{25w}{EI}\right)(5) = \frac{62.5w}{EI}$

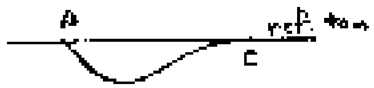
$A_2 = -\frac{(40)(5)}{EI} = -\frac{200}{EI}$

$A_3 = -\frac{1}{3}\left(\frac{12.5w}{EI}\right)(5) = -\frac{20.833w}{EI}$

$\bar{x}_1 = \frac{2}{3}(5) = 3.3333$ m

$\bar{x}_2 = \frac{1}{2}(5) = 2.5$ m

$\bar{x}_3 = \frac{3}{4}(5) = 3.75$ m



Place reference tangent at C.

Deflection at C is zero $\qquad t_{A/C} = y_A - y_C = 0$

$A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 = 0$

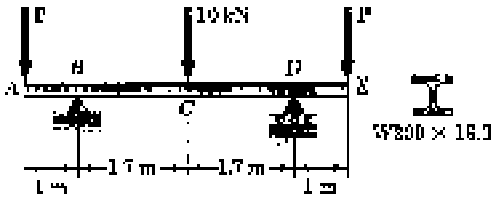
$\left(\frac{62.5w}{EI}\right)(3.3333) - \left(\frac{200}{EI}\right)(2.5) - \left(\frac{20.833w}{EI}\right)(3.75) = 0$

$\frac{130.21w}{EI} - \frac{500}{EI} = 0$

$w = \frac{500}{130.21} = 3.84$ kN/m

PROBLEM 9.124

9.124 For the beam and loading of Prob. 9.122, determine the magnitude of the forces **P** for which the deflection is zero at end A. Use $E = 200$ GPa.



SOLUTION

Symmetric beam and loading.

$R_A = R_E = P + 5 \quad (P \text{ in kN})$

Bending moment

Over AB $\quad M = -Px \quad$ kN·m

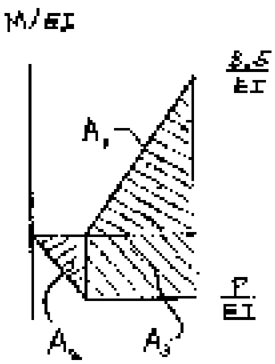
Over BC $\quad M = -Px + (P+5)(x-1)$

$= 5(x-1) - P(1)$

At $x = 2.7$ m

$M = 8.5 - P(1)$

Draw $\frac{M}{EI}$ diagram by parts

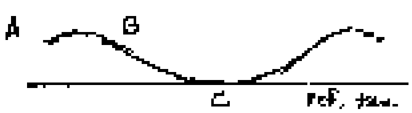


$A_1 = \frac{1}{2}\left(\frac{8.5}{EI}\right)(1.7) = \frac{7.225}{EI}$

$A_2 = -\frac{1}{2}\frac{P}{EI}(1) = -\frac{0.5P}{EI}$

$A_3 = -\left(\frac{P}{EI}\right)(1.7) = -\frac{1.7P}{EI}$

Place reference tangent at C



$y_A = y_B = 0$

$y_A - y_B = 0$

$t_{A/C} - t_{B/C} = 0$

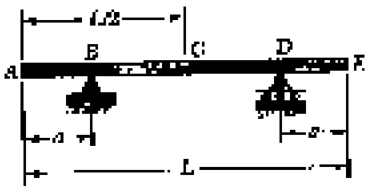
$A_1\left(1 + \frac{2}{3}\cdot 1.7\right) + A_3\left(1 + \frac{1}{2}\cdot 1.7\right) + A_2\left(\frac{2}{3}\right) - A_1\left(\frac{2}{3}\cdot 1.7\right) - A_3\left(\frac{1}{2}\cdot 1.7\right) = 0$

$A_1(1) + A_3(1) + A_2\left(\frac{2}{3}\right) = 0$

$\frac{7.225}{EI} - \frac{1.7P}{EI} - \frac{0.33333P}{EI} = 0 \qquad P = \frac{7.225}{2.0333} = 3.55 \text{ kN}$

PROBLEM 9.125

*9.125 A uniform rod AE is supported at two points B and D. Determine the distance a from the ends of the rod to the points of support if the downward deflections of points A, C, and E are to be equal.



SOLUTION

Let w = weight per unit length of rod.

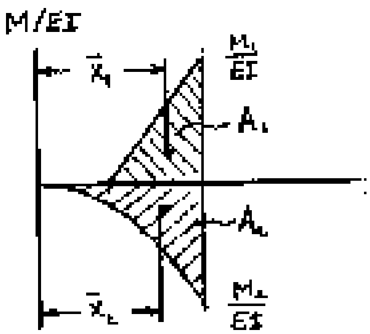
Symmetrical beam and loading.

$$R_B = R_D = \frac{1}{2}wL$$

Bending moment:

Over AB $\quad M = -\frac{1}{2}wx^2$

Over BCD $\quad M = -\frac{1}{2}wx^2 + \frac{1}{2}wL(x-a)$



Draw $\frac{M}{EI}$ diagram by parts

$$\frac{M_1}{EI} = \frac{1}{2}\frac{wL\left(\frac{L}{2}-a\right)}{EI} = \frac{1}{4}\frac{wL(L-2a)}{EI}$$

$$\frac{M_2}{EI} = -\frac{1}{2}\frac{w\left(\frac{L}{2}\right)^2}{EI} = -\frac{1}{8}\frac{wL^2}{EI}$$

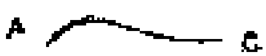
$$A_1 = \frac{1}{2}\frac{M_1}{EI}\left(\frac{L}{2}-a\right) = \frac{1}{16}\frac{wL(L-2a)^2}{EI}$$

$$A_2 = \frac{1}{3}\left(\frac{M_2}{EI}\right)\left(\frac{L}{2}\right) = -\frac{1}{48}\frac{wL^3}{EI}$$

$$\bar{x}_1 = a + \frac{2}{3}\left(\frac{L}{2}-a\right) = \frac{1}{3}(L+a)$$

$$\bar{x}_2 = \frac{L}{2} - \frac{1}{4}\left(\frac{L}{2}\right) = \frac{3}{8}L$$

Place reference tangent at C.



$$y_C - y_C = t_{A/C} = 0$$

$$A_1\bar{x}_1 + A_2\bar{x}_2 = 0$$

$$\frac{1}{16}\frac{wL(L-2a)^2}{EI}\frac{1}{3}(L+a) - \frac{1}{48}\frac{wL^3}{EI}\frac{3}{8}L = 0$$

Let $u = a/L$. Divide by $\frac{wL^4}{48EI}$

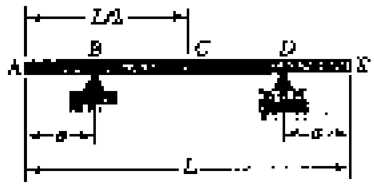
$$(1-2u)^2(1+u) - \frac{3}{8} = 0$$

$$4u^3 - 3u + \frac{5}{8} = 0$$

Solving for $u = 0.22315 \qquad \frac{a}{L} = 0.223 \qquad a = 0.223L$ ◄

PROBLEM 9.126

*9.126 A uniform rod AE is supported at two points B and D. Determine the distance a for which the slope at ends A and E is to be zero.



SOLUTION

Let w = weight per unit length of rod.

Symmetrical beam and loading.

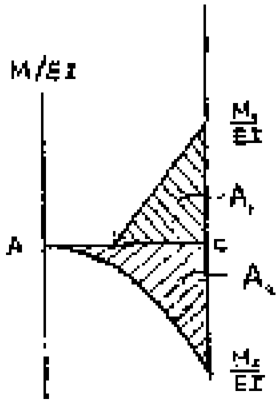
$$R_B = R_D = \tfrac{1}{2} wL$$

Bending moment

Over AB $\quad M = -\tfrac{1}{2} w x^2$

Over BCD $\quad M = -\tfrac{1}{2} w x^2 + \tfrac{1}{2} wL(x-a)$

Draw $\frac{M}{EI}$ diagram by parts.

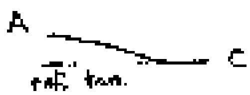


$$\frac{M_1}{EI} = \frac{1}{2}\,\frac{wL\left(\frac{L}{2}-a\right)}{EI} = \frac{1}{4}\,\frac{wL(L-2a)}{EI}$$

$$\frac{M_2}{EI} = \frac{1}{2}\,\frac{w\left(\frac{L}{2}\right)^2}{EI} = -\frac{1}{8}\,\frac{wL^2}{EI}$$

$$A_1 = \frac{1}{2}\,\frac{M_1}{EI}\left(\frac{L}{2}-a\right) = \frac{1}{16}\,\frac{wL(L-2a)^2}{EI}$$

$$A_2 = \frac{1}{3}\left(\frac{M_2}{EI}\right)\frac{L}{2} = -\frac{1}{48}\,\frac{wL^3}{EI}$$



Place reference tangent at C. $\quad \theta_C = 0$

$$\theta_A = \theta_C - \theta_{C/A} = 0 - (A_1 + A_2) = 0$$

$$-\frac{1}{16}\,\frac{wL(L-2a)^2}{EI} + \frac{1}{48}\,\frac{wL^3}{EI} = 0$$

Let $u = \frac{a}{L}$ and divide by $\frac{wL^3}{48EI}$

$$1 - 3(1-2u)^2 = 0$$

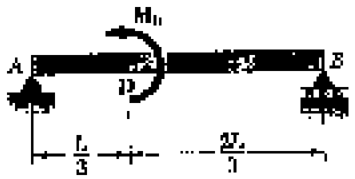
$$1 - 2u = \frac{\sqrt{3}}{3}$$

$$u = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{3}\right) = 0.21132$$

$\frac{a}{L} = 0.211 \qquad\qquad a = 0.211\,L$ ◀

PROBLEM 9.127

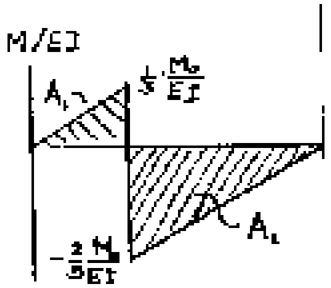
9.127 through 9.130 For the prismatic beam and loading shown, determine (*a*) the deflection at point *D*, (*b*) the slope at end *A*.



SOLUTION

Reactions: $R_A = \frac{M_0}{L}\uparrow$, $R_B = \frac{M_0}{L}\downarrow$

Draw $\frac{M}{EI}$ diagram.



$$A_1 = \frac{1}{2}\left(\frac{1}{3}\frac{M_0}{EI}\right)\frac{L}{3} = \frac{1}{18}\frac{M_0 L}{EI}$$

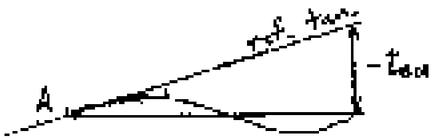
$$A_2 = -\frac{1}{2}\left(\frac{2}{3}\frac{M_0}{EI}\right)\frac{2L}{3} = -\frac{2}{9}\frac{M_0 L}{EI}$$

Place reference tangent at A

$$t_{B/A} = A_1\left(\frac{L}{9} + \frac{2L}{3}\right) + A_2\left(\frac{2}{3}\cdot\frac{2L}{3}\right)$$

$$= \frac{7}{162}\frac{M_0 L^2}{EI} - \frac{8}{81}\frac{M_0 L^2}{EI} = -\frac{1}{18}\frac{M_0 L^2}{EI}$$

$$t_{D/A} = A_1\frac{L}{9} = \frac{1}{162}\frac{M_0 L^2}{EI}$$



(a) Deflection at D

$$y_D = t_{D/A} - \frac{x_D}{L}t_{B/A}$$

$$= \frac{1}{162}\frac{M_0 L^2}{EI} - \frac{1}{3}\left(-\frac{1}{18}\frac{M_0 L^2}{EI}\right)$$

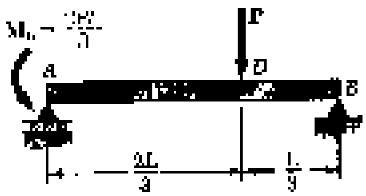
$$= \frac{2}{81}\frac{M_0 L^2}{EI}\uparrow$$

(b) Slope at A

$$\theta_A = -\frac{t_{B/A}}{L} = \frac{1}{18}\frac{M_0 L}{EI}$$

PROBLEM 9.128

9.127 through 9.130 For the prismatic beam and loading shown, determine (*a*) the deflection at point *D*, (*b*) the slope at end *A*.

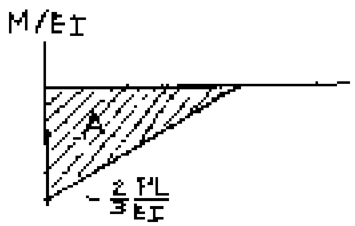


SOLUTION

$+\circlearrowleft \Sigma M_B = 0 \qquad \frac{2PL}{3} - R_A L + P\frac{L}{3} = 0 \qquad R_A = P$

$+\circlearrowleft \Sigma M_A = 0 \qquad \frac{2PL}{3} - P\frac{2L}{3} + R_B L = 0 \qquad R_B = 0$

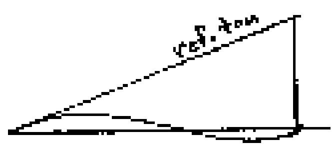
Draw $\frac{M}{EI}$ diagram. Reference tangent at A.



$$A = -\frac{1}{2}\left(\frac{2}{3}\frac{PL}{EI}\right)\left(\frac{2L}{3}\right) = -\frac{2}{9}\frac{PL^2}{EI}$$

$$t_{B/A} = \left(-\frac{2}{9}\frac{PL^2}{EI}\right)\left(\frac{1}{3}\cdot\frac{2L}{3} + \frac{L}{3}\right) = -\frac{14}{81}\frac{PL^3}{EI}$$

$$t_{D/A} = \left(-\frac{2}{9}\frac{PL^2}{EI}\right)\left(\frac{2}{3}\cdot\frac{2L}{3}\right) = -\frac{8}{81}\frac{PL^3}{EI}$$



(a) Deflection at D

$$y_D = t_{D/A} - \frac{x_D}{L} t_{B/A}$$

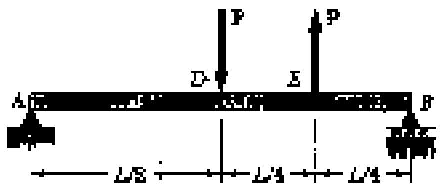
$$= -\frac{8}{81}\frac{PL^3}{EI} + \frac{2}{3}\cdot\frac{14}{81}\frac{PL^3}{EI} = \frac{4}{243}\frac{PL^3}{EI} \uparrow$$

(b) Slope at A

$$\theta_A = -\frac{t_{B/A}}{L} = \frac{14}{81}\frac{PL^2}{EI}$$

PROBLEM 9.129

9.127 through 9.130 For the prismatic beam and loading shown, determine (*a*) the deflection at point *D*, (*b*) the slope at end *A*.

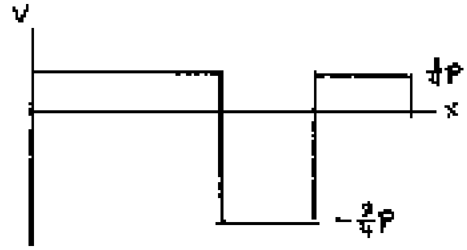


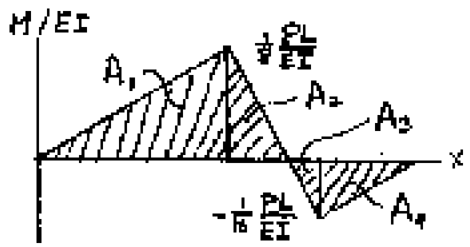
SOLUTION

$+\circlearrowleft \Sigma M_B = 0 \qquad -R_A L + \frac{PL}{2} - \frac{PL}{4} = 0 \qquad R_A = \frac{1}{4}P \uparrow$

$+\circlearrowleft \Sigma M_A = 0 \qquad -\frac{PL}{2} + P\frac{3L}{4} + R_B L = 0 \qquad R_B = \frac{1}{4}P \downarrow$

Draw V (shear) diagram and $\frac{M}{EI}$ diagram.





$$A_1 = \frac{1}{2}\left(\frac{1}{8}\frac{PL}{EI}\right)\left(\frac{L}{2}\right) = \frac{1}{32}\frac{PL^2}{EI}$$

$$A_2 = \frac{1}{2}\left(\frac{1}{8}\frac{PL}{EI}\right)\left(\frac{L}{6}\right) = \frac{1}{96}\frac{PL^2}{EI}$$

$$A_3 = \frac{1}{2}\left(-\frac{1}{16}\frac{PL}{EI}\right)\left(\frac{L}{12}\right) = -\frac{1}{384}\frac{PL^2}{EI}$$

$$A_4 = \frac{1}{2}\left(-\frac{1}{16}\frac{PL}{EI}\right)\left(\frac{L}{4}\right) = -\frac{1}{128}\frac{PL^2}{EI}$$

Place reference tangent at A.

$$t_{B/A} = \left(\frac{1}{32}\frac{PL^2}{EI}\right)\left(\frac{2L}{3}\right) + \left(\frac{1}{96}\frac{PL^2}{EI}\right)\left(\frac{L}{2} - \frac{1}{3}\cdot\frac{L}{6}\right) + \left(-\frac{1}{384}\frac{PL^2}{EI}\right)\left(\frac{L}{4} + \frac{1}{3}\cdot\frac{L}{12}\right) + \left(-\frac{1}{128}\frac{PL^2}{EI}\right)\left(\frac{2}{3}\cdot\frac{L}{4}\right)$$

$$= \frac{1}{48}\frac{PL^3}{EI} + \frac{1}{216}\frac{PL^3}{EI} - \frac{5}{6912}\frac{PL^3}{EI} - \frac{1}{768}\frac{PL^3}{EI} = \frac{3}{128}\frac{PL^3}{EI}$$

$$t_{D/A} = \left(\frac{1}{32}\frac{PL^2}{EI}\right)\left(\frac{1}{3}\cdot\frac{L}{2}\right) = \frac{1}{192}\frac{PL^3}{EI}$$

(a) Deflection at D.

$$y_D = t_{D/A} - \frac{x_D}{L}t_{B/A} = \frac{1}{192}\frac{PL^3}{EI} - \frac{1}{2}\left(\frac{3}{128}\frac{PL^3}{EI}\right) = -\frac{5}{768}\frac{PL^3}{EI} \blacktriangleleft$$

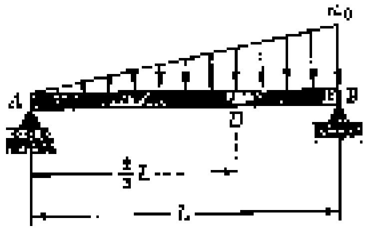
(b) Slope at A

$$\theta_A = -\frac{t_{B/A}}{L} = -\frac{3}{128}\frac{PL^2}{EI} \blacktriangleleft$$

PROBLEM 9.130

9.127 through 9.130 For the prismatic beam and loading shown, determine (a) the deflection at point *D*, (b) the slope at end *A*.

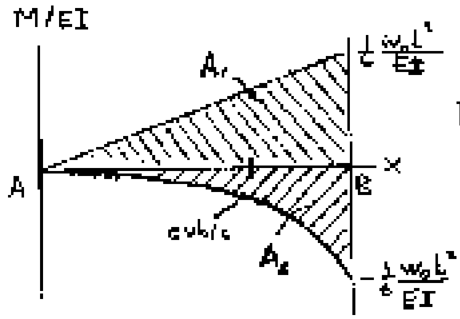
SOLUTION



$+\circlearrowleft \Sigma M_B = 0 \quad -R_A L + (\frac{1}{2} w_0 L)(\frac{1}{3} L) = 0 \qquad R_A = \frac{1}{6} w_0 L$

Bending moment $\quad M = R_A x - \frac{1}{6}\frac{w_0}{L} x^3$

$= \frac{1}{6}\frac{w_0}{L}(L^2 x - x^3)$

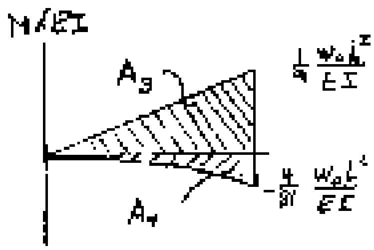


At $x = L \qquad M = \frac{1}{6} w_0 L^2 - \frac{1}{6} w_0 L^2$

Draw $\frac{M}{EI}$ diagram by parts.

$A_1 = \frac{1}{2}(\frac{1}{6}\frac{w_0L^2}{EI})L = \frac{1}{12}\frac{w_0L^3}{EI} \qquad \bar{x}_1 = \frac{1}{3}L$

$A_2 = \frac{1}{4}(-\frac{1}{6}\frac{w_0L^2}{EI})L = -\frac{1}{24}\frac{w_0L^3}{EI} \qquad \bar{x}_2 = \frac{1}{5}L$



Place reference tangent at A.

$t_{B/A} = A_1\bar{x}_1 + A_2\bar{x}_2$

$= \frac{1}{36}\frac{w_0L^4}{EI} - \frac{1}{120}\frac{w_0L^4}{EI} = \frac{7}{360}\frac{w_0L^4}{EI}$

$y_B = L\theta_A + t_{B/A}$

$\theta_A = -\frac{t_{B/A}}{L} = -\frac{7}{360}\frac{w_0L^3}{EI}$

$A_3 = \frac{1}{2}(\frac{1}{9}\frac{w_0L^2}{EI})(\frac{2}{3}L) = \frac{1}{27}\frac{w_0L^3}{EI}$

$A_4 = \frac{1}{4}(-\frac{4}{81}\frac{w_0L^2}{EI})(\frac{2}{3}L) = -\frac{2}{243}\frac{w_0L^3}{EI}$

$t_{D/A} = A_3\bar{x}_3 + A_4\bar{x}_4$

$= (\frac{1}{27}\frac{w_0L^3}{EI})(\frac{1}{3}\cdot\frac{2}{3}L) + (-\frac{2}{243}\frac{w_0L^3}{EI})(\frac{1}{5}\cdot\frac{2}{3}L) = \frac{2}{243}\frac{w_0L^4}{EI} - \frac{4}{3645}\frac{w_0L^4}{EI}$

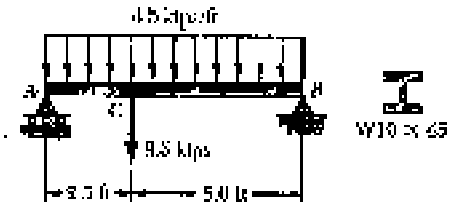
$= \frac{26}{3645}\frac{w_0L^4}{EI}$

(a) $\quad y_D = t_{D/A} + \frac{2}{3}L\theta_A = \frac{26}{3645}\frac{w_0L^4}{EI} + (\frac{2}{3}L)(-\frac{7}{360}\frac{w_0L^3}{EI}) = -\frac{17}{2916}\frac{w_0L^4}{EI}$ ◄

(b) $\quad \theta_A = -\frac{7}{360}\frac{w_0L^3}{EI}$ ◄

PROBLEM 9.131

9.131 For the beam and loading shown, determine (a) the slope at point A, (b) the deflection at point C. Use $E = 29 \times 10^6$ psi.



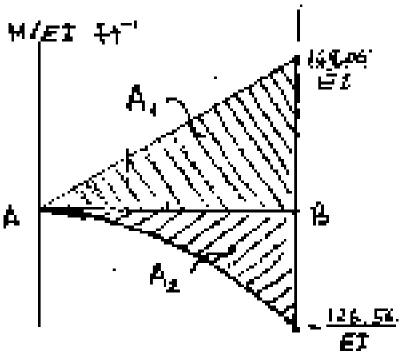
SOLUTION

$E = 29 \times 10^6$ psi $= 29 \times 10^3$ ksi

$I = 248$ in^4

$EI = (29 \times 10^3)(248) = 7.192 \times 10^6$ kip·in^2

$= 49\,944$ kip·ft^2



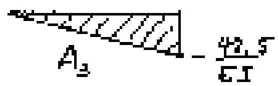
$\circlearrowleft \Sigma M_B = 0 \qquad -7.5 R_A + (4.5)(7.5)\left(\frac{7.5}{2}\right) + (8.5)(5.0) = 0$

$R_A = 22.542$ kips $\uparrow$

Bending moment

Over AC $\quad M = 22.542x - 2.25x^2 \quad$ kip·ft

Over CB $\quad M = 22.542x - 2.25x^2 - 8.5(x - 2.5)$

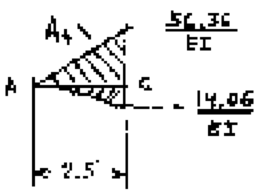


Draw $\frac{M}{EI}$ diagram by parts.

$A_1 = \frac{1}{2}(7.5)\left(\frac{169.06}{EI}\right) = \frac{633.98}{EI}$

$A_2 = -\frac{1}{3}(7.5)\left(\frac{126.56}{EI}\right) = -\frac{316.40}{EI}$

$A_3 = -\frac{1}{2}(5)\left(\frac{42.5}{EI}\right) = -\frac{106.25}{EI}$

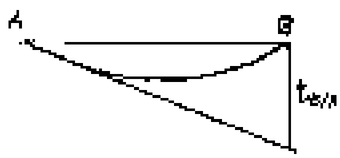


Place reference tangent at A.

$t_{B/A} = A_1\left(\frac{7.5}{3}\right) + (A_2)\left(\frac{7.5}{4}\right) + A_3\left(\frac{5}{3}\right) = \frac{814.62}{EI}$

$A_4 = \frac{1}{2}\left(\frac{56.36}{EI}\right)(2.5) = \frac{70.45}{EI}$

$A_5 = -\frac{1}{3}\left(\frac{14.06}{EI}\right)(2.5) = -\frac{11.72}{EI}$



$t_{C/A} = A_4\left(\frac{2.5}{3}\right) + A_5\left(\frac{2.5}{4}\right) = \frac{51.375}{EI}$

(a) Slope at A

$\theta_A = -\frac{t_{B/A}}{L} = -\frac{814.62}{7.5\,EI} = -\frac{108.62}{EI}$

$= -\frac{108.62}{49944} = -2.17 \times 10^{-3}$ rad

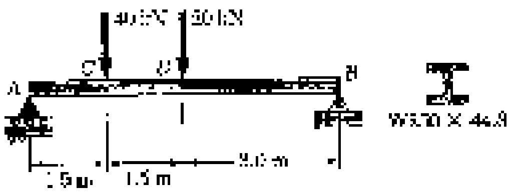
(b) Deflection at C $\qquad y_C = t_{C/A} - \frac{x_C}{L} t_{B/A}$

$y_C = \frac{51.375}{EI} - \left(\frac{2.5}{7.5}\right)\frac{814.62}{EI} = -\frac{220.16}{EI} = -\frac{220.16}{49944} = -4.41 \times 10^{-3}$ ft

$= 0.0529$ in. $\downarrow$

PROBLEM 9.132

9.132 and 9.133 For the beam and loading shown, determine (*a*) the slope at point *A*, (*b*) the deflection at point *D*. Use $E = 200$ GPa.



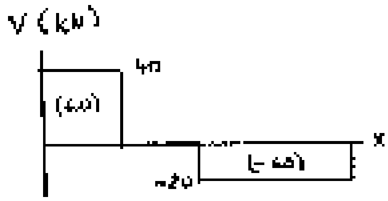
SOLUTION

$E = 200 \times 10^9$ Pa

$I = 71.1 \times 10^6 \text{ mm}^4 = 71.1 \times 10^{-6} \text{ m}^4$

$EI = (200 \times 10^9)(71.1 \times 10^{-6}) = 14.22 \times 10^6 \text{ N}\cdot\text{m}^2$

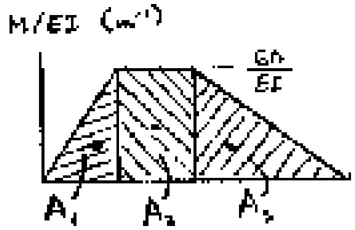
$= 14220 \text{ kN}\cdot\text{m}^2$



$\circlearrowleft \Sigma M_B = 0 \quad -6R_A + (4.5)(40) + (3)(20) = 0$

$R_A = 40$ kN.

Draw shear and $\frac{M}{EI}$ diagrams.

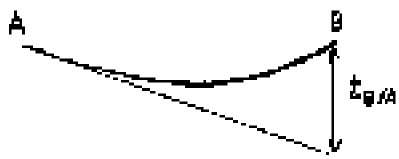


$A_1 = \frac{1}{2}\left(\frac{60}{EI}\right)(1.5) = \frac{45}{EI}$

$A_2 = \left(\frac{60}{EI}\right)(1.5) = \frac{90}{EI}$

$A_3 = \frac{1}{2}\left(\frac{60}{EI}\right)(3) = \frac{90}{EI}$

Place reference tangent at A.



$t_{B/A} = A_1(4.5 + 0.5) + A_2(3 + 0.75) + A_3(2.0) = \frac{742.5}{EI}$ m.

$t_{D/A} = A_1(1.5 + 0.5) + A_2(0.75) = \frac{157.5}{EI}$ m

(a) Slope at A $\quad \theta_A = -\frac{t_{B/A}}{L} = -\frac{742.5}{6EI} = -\frac{123.75}{EI} = -\frac{123.75}{14220}$

$= -8.70 \times 10^{-3}$ rad.

(b) Deflection at D.

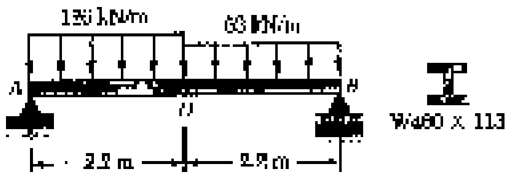
$y_D = t_{D/A} - \frac{x_D}{L} t_{B/A} = \frac{157.5}{EI} - \left(\frac{3}{6}\right)\left(\frac{742.5}{EI}\right) = -\frac{213.75}{EI}$

$= -\frac{213.75}{14220} = -15.03 \times 10^{-3}$ m

$= 15.03$ mm $\downarrow$

PROBLEM 9.133

9.132 and 9.133 For the beam and loading shown, determine (*a*) the slope at point *A*, (*b*) the deflection at point *D*. Use $E = 200$ GPa.



SOLUTION

$E = 200 \times 10^9$ Pa

$I = 556 \times 10^6 \text{ mm}^4 = 556 \times 10^{-6} \text{ m}^4$

$EI = (200 \times 10^9)(556 \times 10^{-6}) = 111.2 \times 10^6 \text{ N}\cdot\text{m}^2$

$= 111200 \text{ kN}\cdot\text{m}^2$

$+\circlearrowleft \Sigma M_B = 0$

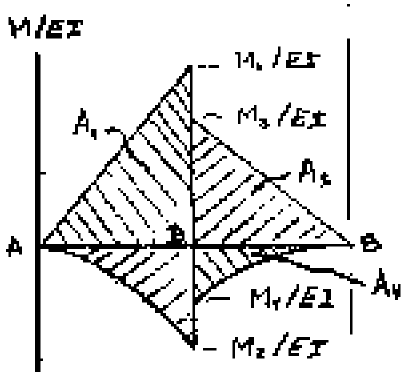
$-4.4 R_A + (126)(2.2)(3.3) + (63)(2.2)(1.1) = 0$

$R_A = 242.55 \text{ kN} \uparrow$

$+\circlearrowleft \Sigma M_A = 0$

$-(126)(2.2)(1.1) - (63)(2.2)(3.3) + 4.4 R_B = 0$

$R_B = 173.25 \text{ kN} \uparrow$



Draw $\frac{M}{EI}$ diagram by parts.

$\frac{M_1}{EI} = \frac{(242.55)(2.2)}{111200} = 4.7987 \times 10^{-3} \text{ m}^{-1}$

$\frac{M_2}{EI} = -\frac{1}{2}\frac{(126)(2.2)^2}{111200} = -2.7421 \times 10^{-3} \text{ m}^{-1}$

$\frac{M_3}{EI} = \frac{(173.25)(2.2)}{111200} = 3.4276 \times 10^{-3} \text{ m}^{-1}$

$\frac{M_4}{EI} = -\frac{1}{2}\frac{(63)(2.2)^2}{111200} = -1.3710 \times 10^{-3} \text{ m}^{-1}$

$A_1 = \frac{1}{2}\frac{M_1}{EI}(2.2) = 5.2785 \times 10^{-3}$

$A_2 = \frac{1}{3}\frac{M_2}{EI}(2.2) = -2.0109 \times 10^{-3}$

$A_3 = \frac{1}{2}\frac{M_3}{EI}(2.2) = 3.7704 \times 10^{-3}$

$A_4 = \frac{1}{3}\frac{M_4}{EI}(2.2) = -1.0054 \times 10^{-3}$

Place reference tangent at A

$t_{B/A} = A_1(2.9333) + A_2(2.75) + A_3(1.46667) + A_4(1.65) = 13.824 \times 10^{-3} \text{ m}$

$t_{D/A} = A_1(0.7333) + A_2(0.55) = 2.7647 \times 10^{-3} \text{ m}$

(a) Slope at A $\qquad \theta_A = -\frac{t_{B/A}}{L} = -\frac{13.824 \times 10^{-3}}{4.4} = -3.14 \times 10^{-3} \text{ rad}$ ◄

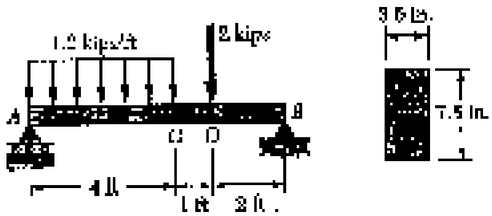
(b) Deflection at D $\qquad y_D = t_{D/A} - \frac{x_D}{L} t_{B/A}$

$y_D = 2.7647 \times 10^{-3} - \left(\frac{2.2}{4.4}\right)(13.824 \times 10^{-3}) = -4.15 \times 10^{-3} \text{ m}$

$= 4.15 \text{ mm} \downarrow$ ◄

PROBLEM 9.134

9.134 For the timber beam and loading shown, determine (a) the slope at point A, (b) the deflection at point D. Use $E = 1.5 \times 10^6$ psi.

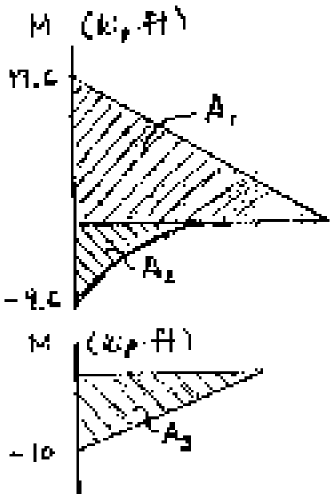


SOLUTION

$I = \frac{1}{12}(3.5)(7.5)^3 = 123.047\ \text{in}^4$

$E = 1.5 \times 10^6\ \text{psi} = 1.5 \times 10^3\ \text{ksi}$

$EI = 184.57 \times 10^3\ \text{kip}\cdot\text{in}^2 = 1281.7\ \text{kip}\cdot\text{ft}^2$



$+\circlearrowleft \Sigma M_A = 0 \qquad 7R_B - (2)(5) - (1.2)(4)(2) = 0$

$R_B = 2.8\ \text{kip}$

Draw bending moment diagram by parts.

$M_1 = (2.8)(7) = 19.6\ \text{kip}\cdot\text{ft}$

$M_2 = -(1.2)(4)(2) = -9.6\ \text{kip}\cdot\text{ft}$

$M_3 = -(2)(5) = -10\ \text{kip}\cdot\text{ft}$

$A_1 = \frac{1}{2}(7)(19.6) = 68.6\ \text{kip}\cdot\text{ft}^2$

$A_2 = \frac{1}{3}(4)(-9.6) = -12.8\ \text{kip}\cdot\text{ft}^2$

$A_3 = \frac{1}{2}(5)(-10) = -25.0\ \text{kip}\cdot\text{ft}^2$

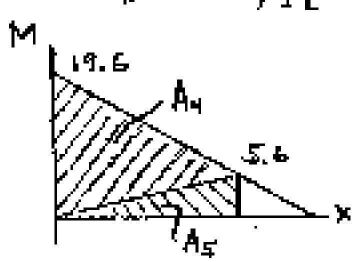
Draw reference tangent at A.

$\theta_A = -\dfrac{t_{B/A}}{L}$

$y_D = t_{D/A} - \dfrac{x_D}{L} t_{B/A}$

$EI\, t_{B/A} = A_1\left(7 - \frac{7}{3}\right) + A_2(7 - 1) + A_3\left(7 - \frac{5}{3}\right) = 110.0\ \text{kip}\cdot\text{ft}^3$

(a) $\theta_A = -\dfrac{EI\, t_{B/A}}{EIL} = -\dfrac{110.0}{(1281.7)(7)} = -12.26 \times 10^{-3}\ \text{rad}$



$A_4 = \frac{1}{2}(19.6)(5) = 49\ \text{kip}\cdot\text{ft}^2$

$A_5 = \frac{1}{2}(5.6)(5) = 14\ \text{kip}\cdot\text{ft}^2$

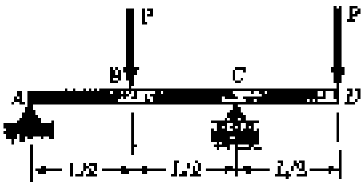
$EI\, t_{D/A} = A_4\left(5 - \frac{5}{3}\right) + A_5\left(\frac{5}{3}\right) + A_2(5 - 1) + A_3\left(5 - \frac{5}{3}\right) = 52.133\ \text{kip}\cdot\text{ft}^3$

$EI\, y_D = 52.133 - \frac{5}{7}(110.0) = -26.438\ \text{kip}\cdot\text{ft}^3$

$y_D = -\dfrac{26.438}{1281.7} = -20.63 \times 10^{-3}\ \text{ft} = 0.248\ \text{in.} \downarrow$

PROBLEM 9.135

9.135 and 9.136 For the beam and loading shown, determine (*a*) the slope at point *A*, (*b*) the deflection at point *D*.



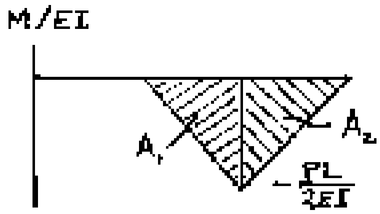
SOLUTION

$\circlearrowleft \Sigma M_C = 0 \qquad -R_A L + P\frac{L}{2} - P\frac{L}{2} = 0 \qquad R_A = 0.$

Draw $\frac{M}{EI}$ diagram

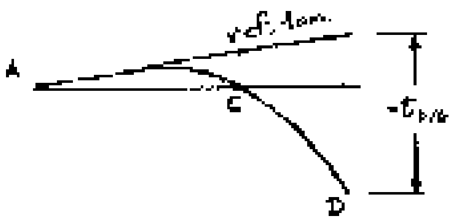
$$A_1 = -\frac{1}{2}\left(\frac{PL}{2EI}\right)\left(\frac{L}{2}\right) = -\frac{1}{8}\frac{PL^2}{EI}$$

$$A_2 = -\frac{1}{2}\left(\frac{PL}{2EI}\right)\left(\frac{L}{2}\right) = -\frac{1}{8}\frac{PL^2}{EI}$$



Place reference tangent at A

$$t_{C/A} = A_1\left(\frac{1}{3}\cdot\frac{L}{2}\right) = -\frac{1}{48}\frac{PL^3}{EI}$$



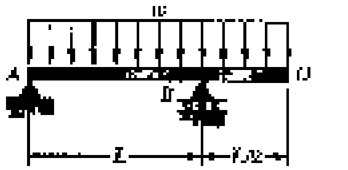
(a) $\theta_A = -\frac{t_{C/A}}{L} = \frac{1}{48}\frac{PL^2}{EI}$

$$t_{D/A} = A_1\left(\frac{L}{2} + \frac{L}{6}\right) + A_2\left(\frac{2}{3}\cdot\frac{L}{2}\right) = -\frac{1}{8}\frac{PL^3}{EI}$$

$$y_D = t_{D/A} - \frac{x_D}{L}t_{C/A} = -\frac{1}{8}\frac{PL^3}{EI} - \left(\frac{3}{2}\right)\left(-\frac{1}{48}\frac{PL^3}{EI}\right) = -\frac{3}{32}\frac{PL^3}{EI}$$

PROBLEM 9.136

9.135 and 9.136 For the beam and loading shown, determine (*a*) the slope at point *A*, (*b*) the deflection at point *D*.



SOLUTION

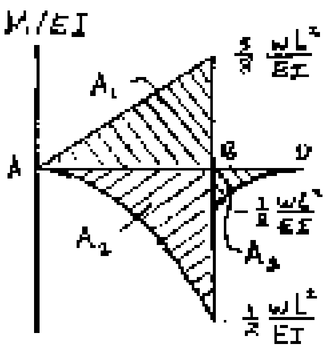
$\circlearrowleft \Sigma M_B = 0 \quad -R_A L + \left(\frac{3}{2} wL\right)\left(\frac{1}{4} L\right) = 0 \qquad R_A = \frac{3}{8} wL$

Draw $\frac{M}{EI}$ diagram by parts.

$$A_1 = \frac{1}{2}\left(\frac{3}{8}\frac{wL^2}{EI}\right)L = \frac{3}{16}\frac{wL^3}{EI}$$

$$A_2 = -\frac{1}{3}\left(\frac{1}{2}\frac{wL^2}{EI}\right)L = -\frac{1}{6}\frac{wL^3}{EI}$$

$$A_3 = -\frac{1}{3}\left(\frac{1}{8}\frac{wL^2}{EI}\right)\frac{L}{2} = -\frac{1}{48}\frac{wL^3}{EI}$$

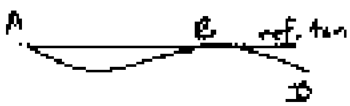


Place reference tangent at A.

$$t_{B/A} = A_1 \frac{L}{3} + A_2 \frac{L}{4}$$

$$= \frac{1}{16}\frac{wL^4}{EI} - \frac{1}{24}\frac{wL^4}{EI} = \frac{1}{48}\frac{wL^4}{EI}$$

(a) Slope at A



$$\theta_A = -\frac{t_{B/A}}{L} = -\frac{1}{48}\frac{wL^3}{EI}$$

$$t_{D/A} = A_1\left(\frac{L}{3} + \frac{L}{2}\right) + A_2\left(\frac{L}{4} + \frac{L}{2}\right) + A_3\left(\frac{3}{4}\cdot\frac{L}{2}\right)$$

$$= \frac{5}{32}\frac{wL^4}{EI} - \frac{1}{8}\frac{wL^4}{EI} - \frac{1}{128}\frac{wL^4}{EI} = \frac{3}{128}\frac{wL^4}{EI}$$

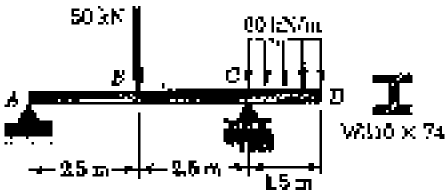
(b) Deflection at D.

$$y_D = t_{D/A} - \frac{x_D}{L} t_{B/A} = \frac{3}{128}\frac{wL^4}{EI} - \frac{3}{2}\cdot\frac{1}{48}\frac{wL^4}{EI} = -\frac{1}{128}\frac{wL^4}{EI}$$

$$y_D = \frac{1}{128}\frac{wL^4}{EI} \downarrow$$

PROBLEM 9.137

9.137 For the beam and loading shown, determine (a) the slope at point C; (b) the deflection at point D. Use $E = 200$ GPa.



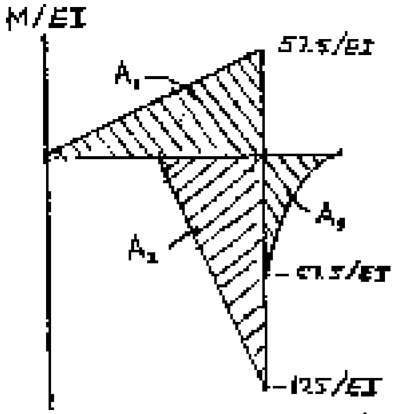
SOLUTION

$E = 200 \times 10^9$ Pa

$I = 165 \times 10^6\ mm^4 = 165 \times 10^{-6}\ m^4$

$EI = (200 \times 10^9)(165 \times 10^{-6}) = 33.0 \times 10^6\ N \cdot m^2$

$= 33000\ kN \cdot m^2$



$+\circlearrowleft \Sigma M_C = 0 \quad -5R_A + (50)(2.5) - (60)(1.5)(4.75) = 0$

$R_A = 11.5$ kN

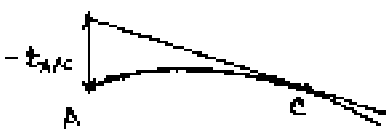
Draw $\frac{M}{EI}$ diagram by parts.

$A_1 = \frac{1}{2}\left(\frac{57.5}{EI}\right)(5) = \frac{143.75}{EI}$

$A_2 = -\frac{1}{2}\left(\frac{125}{EI}\right)(2.5) = -\frac{156.25}{EI}$

$A_3 = -\frac{1}{3}\left(\frac{67.5}{EI}\right)(1.5) = -\frac{33.75}{EI}$

Place reference tangent at C



$t_{A/C} = A_1\left(\frac{1}{3} \cdot 5\right) + A_2\left(2.5 + \frac{2}{3} \cdot 2.5\right)$

$= -\frac{171.875}{EI}$ m

(a) Slope at C

$\theta_C = \frac{t_{A/C}}{L} = -\frac{171.875}{5EI} = -\frac{34.375}{EI}$

$= -\frac{34.375}{33000} = -1.042 \times 10^{-3}$ rad

$t_{D/C} = A_3\left(\frac{3}{4} \cdot 1.5\right) = -\frac{37.96875}{EI}$ m.

(b) Deflection at D $\quad y_D = \theta_C\, x_{D/C} + t_{D/C}$

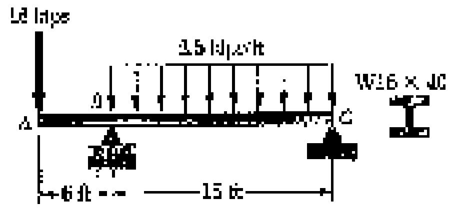
$= -\left(\frac{34.375}{EI}\right)(1.5) - \frac{37.96875}{EI} = -\frac{89.53}{EI}$

$= -\frac{89.53}{33000} = -2.71 \times 10^{-3}$ m

$= 2.71$ mm ↓

PROBLEM 9.138

9.138 For the beam and loading shown, determine (a) the slope at point B, (b) the deflection at point A. Use $E = 29 \times 10^6$ psi.

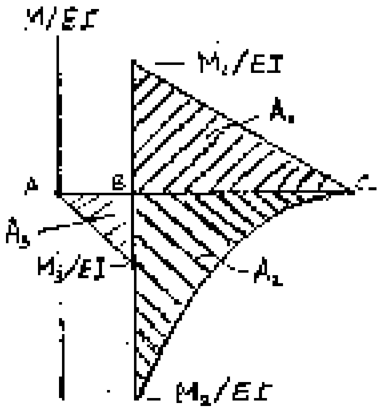


SOLUTION

$E = 29 \times 10^6$ psi $= 29 \times 10^3$ ksi

$I = 518$ in^4

$EI = (29\times10^3)(518) = 15.022 \times 10^6$ kip·in^2 $= 104319$ kip·ft^2



$+\circlearrowleft \Sigma M_B = 0 \qquad (16)(6) - (2.5)(15)(7.5) + 15R_C = 0$

$R_C = 12.35$ kN

Draw $\frac{M}{EI}$ diagram by parts

$M_1 = (12.35)(15) = 185.25$ kip·ft

$M_2 = -\frac{1}{2}(2.5)(15)^2 = -281.25$ kip·ft

$M_3 = -(16)(6) = -96$ kip·ft

$A_1 = \frac{1}{2}(185.25)(15)/EI = 1389.375/EI$

$A_2 = -\frac{1}{3}(281.25)(15)/EI = -1406.25/EI$

$A_3 = -\frac{1}{2}(96)(6) = -288/EI$

Place reference tangent at B.

$t_{C/B} = A_1\left(\frac{2}{3}\cdot 15\right) + A_2\left(\frac{3}{4}\cdot 15\right)$

$= -1926.5625/EI$

$= -18.468 \times 10^{-3}$ ft.

(a) Slope at B $\qquad \theta_B = -\frac{t_{C/B}}{L} = \frac{18.468\times10^{-3}}{15} = 1.231 \times 10^{-3}$ rad ◄

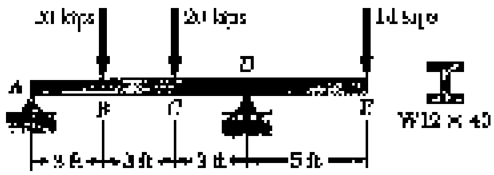
(b) $\quad t_{A/C} = A_3\left(\frac{2}{3}\cdot 6\right) = -1152/EI$

$y_A = t_{A/C} + \frac{x_{AB}}{L} t_{B/C} = -\frac{1152}{EI} - \frac{6}{15}\left(\frac{1926.5625}{EI}\right)$

$= -\frac{1922.6}{EI} = -18.43 \times 10^{-3}$ ft $= 0.221$ in.↓ ◄

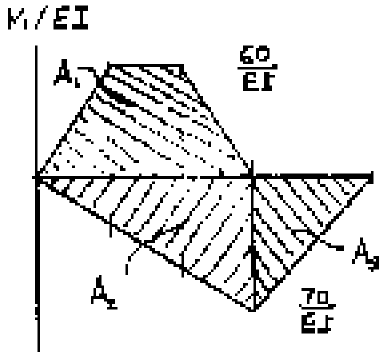
PROBLEM 9.139

9.139 For the beam and loading shown, determine (a) the slope at point D, (b) the deflection at point E. Use $E = 29 \times 10^6$ psi.



SOLUTION

Draw bending moment diagram as the sum of two diagrams: one for the pair of 20 kip loads and one for the 14 kip load.



$$A_1 = \left[2 \cdot \tfrac{1}{2}(3)(60) + (3)(60)\right]/EI = 360/EI$$

$$A_2 = \tfrac{1}{2}(9)(70)/EI = -315/EI$$

$$A_3 = \tfrac{1}{2}(5)(70) = -175/EI$$

$$E = 29 \times 10^6 \text{ psi} = 29 \times 10^3 \text{ ksi}$$

$$I = 310 \text{ in}^4$$

$$EI = (29 \times 10^3)(310) = 8.99 \times 10^6 \text{ kip}\cdot\text{in}^2$$
$$= 62430 \text{ kip}\cdot\text{ft}^2$$

Place reference tangent at D

$$t_{A/D} = A_1(4.5) + A_2(6) = -270/EI \text{ ft.}$$

(a) Slope at D

$$\theta_D = \frac{t_{A/D}}{L} = -\frac{270}{9EI} = -\frac{30}{EI}$$
$$= -0.48054 \times 10^{-3} \text{ rad}$$

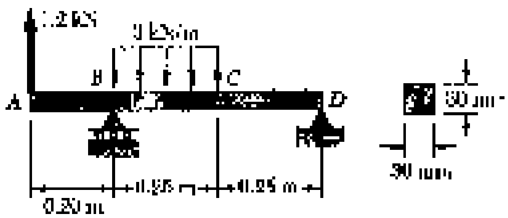
$$t_{E/D} = A_3\left(\tfrac{2}{3} \cdot 5\right) = -583.333/EI = -9.3438 \times 10^{-3} \text{ ft}$$

(b) Deflection at E

$$y_E = L_{DE}\,\theta_D + t_{E/D}$$
$$= -(5)(0.48054 \times 10^{-3}) - 9.3438 \times 10^{-3} = -11.75 \times 10^{-3} \text{ ft}$$
$$= 0.1410 \text{ in.} \downarrow$$

PROBLEM 9.140

9.140 Knowing the beam *AD* is made of a solid steel bar, determine the (*a*) slope at point *B*, (*b*) the deflection at point *A*. Use $E = 200$ GPa.



SOLUTION

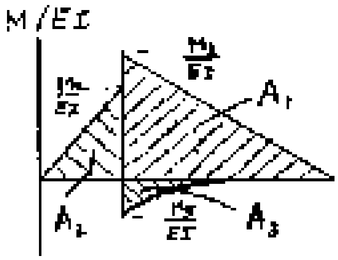
$E = 200 \times 10^9$ Pa

$I = \frac{1}{12}(30)(30)^3 = 67.5 \times 10^3\ \text{mm}^4$

$= 67.5 \times 10^{-9}\ \text{m}^4$

$EI = (200\times10^9)(67.5\times10^{-9}) = 13\,500\ \text{N}\cdot\text{m}^2$

$= 13.5\ \text{kN}\cdot\text{m}^2$



$\circlearrowleft \Sigma M_B = 0 \qquad -(0.2)(1.2) - (3)(0.25)(0.125) + .5R_D = 0$

$R_D = 0.6675$ kN

Draw $\frac{M}{EI}$ diagram by parts.

$M_1 = (0.6675)(0.5) = 0.33375$ kN·m

$M_2 = (1.2)(0.2) = 0.240$ kN·m

$M_3 = -\frac{1}{2}(3)(0.25)^2 = -0.09375$ kN·m

$A_1 = \frac{1}{2}(0.33375)(0.5)/EI = 0.0834375/EI$

$A_2 = \frac{1}{2}(0.240)(0.2)/EI = 0.024/EI$

$A_3 = \frac{1}{3}(-0.09375)(0.25)/EI = -0.0078125/EI$

Place reference tangent at B.

$t_{D/B} = A_1\left(\frac{2}{3}\cdot 0.5\right) + A_3\left(\frac{3}{4}(0.25) + 0.25\right) = 0.024395/EI$

(a) Slope at B $\qquad \theta_B = -\frac{t_{D/B}}{L} = -\frac{0.024395}{0.5\ EI} = -\frac{0.048789}{EI}$

$= -3.6140 \times 10^{-3}$ rad.

$t_{A/B} = A_2\left(\frac{2}{3}(0.20)\right) = 0.0032/EI = 0.23704 \times 10^{-3}$ m

(b) Deflection at A

$y_A = t_{A/B} - L_{AB}\,\theta_B$

$= 0.23704\times10^{-3} - (0.2)(-3.6140\times10^{-3}) = 0.960 \times 10^{-3}$ m

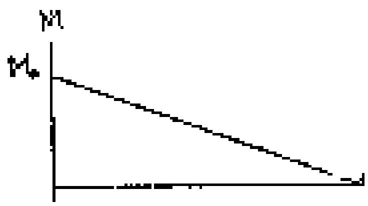
$= 0.960$ mm ↑

PROBLEM 9.141

9.141 and 9.142 For the beam and loading shown, determine (*a*) the slope at end *A*, (*b*) the slope at end *B*, (*c*) the deflection at the midpoint *C*.

SOLUTION

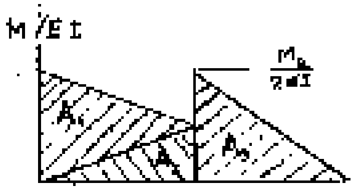
Draw bending moment and $\frac{M}{EI}$ diagrams.



$$A_1 = \frac{1}{2}\left(\frac{M_0}{2EI}\right)\left(\frac{L}{2}\right) = \frac{1}{8}\frac{M_0L}{EI}$$

$$A_2 = \frac{1}{2}\left(\frac{M_0}{4EI}\right)\left(\frac{L}{2}\right) = \frac{1}{16}\frac{M_0L}{EI}$$

$$A_3 = \frac{1}{2}\left(\frac{M_0}{2EI}\right)\left(\frac{L}{2}\right) = \frac{1}{8}\frac{M_0L}{EI}$$



Place reference tangent at A.

$$t_{B/A} = A_1\left(\frac{L}{2}+\frac{2}{3}\frac{L}{2}\right) + A_2\left(\frac{L}{2}+\frac{1}{3}\frac{L}{2}\right) + A_3\left(\frac{2}{3}\frac{L}{2}\right)$$

$$= \left(\frac{1}{8}\frac{M_0L}{EI}\right)\left(\frac{5}{6}L\right) + \left(\frac{1}{16}\frac{M_0L}{EI}\right)\left(\frac{2}{3}L\right) + \left(\frac{1}{8}\frac{M_0L}{EI}\right)\left(\frac{1}{3}L\right)$$

$$= \frac{3}{16}\frac{M_0L^2}{EI}$$

(a) Slope at A

$$\theta_A = -\frac{t_{B/A}}{L} = -\frac{3}{16}\frac{M_0L}{EI}$$

(b) Slope at B

$$\theta_B = \theta_A + \theta_{B/A} = \theta_A + A_1 + A_2 + A_3$$

$$= -\frac{3}{16}\frac{M_0L}{EI} + \frac{1}{8}\frac{M_0L}{EI} + \frac{1}{16}\frac{M_0L}{EI} + \frac{1}{8}\frac{M_0L}{EI}$$

$$= \frac{1}{8}\frac{M_0L}{EI}$$

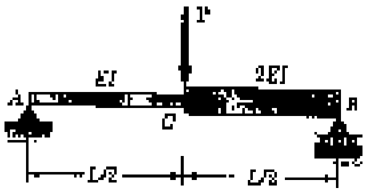
$$t_{C/A} = A_1\left(\frac{2}{3}\frac{L}{2}\right) + A_2\left(\frac{1}{3}\frac{L}{2}\right) = \left(\frac{1}{8}\frac{M_0L}{EI}\right)\left(\frac{1}{3}L\right) + \left(\frac{1}{16}\frac{M_0L}{EI}\right)\left(\frac{1}{6}L\right) = \frac{5}{96}\frac{M_0L^2}{EI}$$

(c) Deflection at C

$$y_C = t_{C/A} + \frac{L}{2}\theta_A = \frac{5}{96}\frac{M_0L^2}{EI} - \frac{3}{32}\frac{M_0L^2}{EI} = -\frac{1}{24}\frac{M_0L^2}{EI} \qquad \frac{1}{24}\frac{M_0L^2}{EI}\downarrow$$

PROBLEM 9.142

9.141 and 9.142 For the beam and loading shown, determine (*a*) the slope at end *A*, (*b*) the slope at end *B*, (*c*) the deflection at the midpoint *C*.

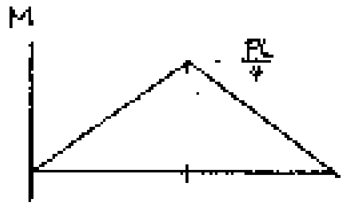


SOLUTION

Draw bending moment and $\frac{M}{EI}$ diagrams.

$$A_1 = \frac{1}{2}\left(\frac{1}{4}\frac{PL}{EI}\right)\left(\frac{L}{2}\right) = \frac{1}{16}\frac{PL^2}{EI}$$

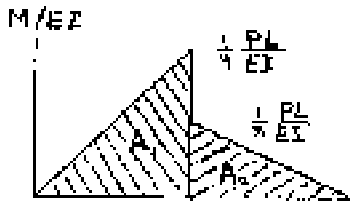
$$A_2 = \frac{1}{2}\left(\frac{1}{8}\frac{PL}{EI}\right)\left(\frac{L}{2}\right) = \frac{1}{32}\frac{PL^2}{EI}$$



Place reference tangent at A

$$t_{B/A} = A_1\left(\frac{L}{2} + \frac{1}{3}\frac{L}{2}\right) + A_2\left(\frac{2}{3}\frac{L}{2}\right)$$

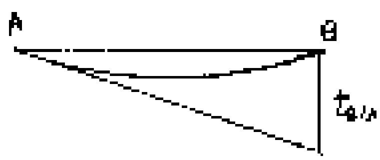
$$= \left(\frac{1}{16}\frac{PL^2}{EI}\right)\left(\frac{2}{3}L\right) + \left(\frac{1}{32}\frac{PL^2}{EI}\right)\left(\frac{1}{3}L\right) = \frac{5}{96}\frac{PL^3}{EI}$$



(a) Slope at A

$$\theta_A = -\frac{t_{B/A}}{L} = -\frac{5}{96}\frac{PL^2}{EI}$$ ◄

(b) Slope at B



$$\theta_B = \theta_A + \theta_{B/A} = \theta_A + A_1 + A_2$$

$$= -\frac{5}{96}\frac{PL^2}{EI} + \frac{1}{16}\frac{PL^2}{EI} + \frac{1}{32}\frac{PL^2}{EI}$$

$$= \frac{1}{24}\frac{PL^2}{EI}$$ ◄

$$t_{C/A} = A_1\left(\frac{1}{3}\frac{L}{2}\right) = \left(\frac{1}{16}\frac{PL^2}{EI}\right)\left(\frac{1}{6}L\right) = \frac{1}{96}\frac{PL^3}{EI}$$

(c) Deflection at C

$$y_A = t_{C/A} - \frac{x_C}{L}t_{B/A} = \frac{1}{96}\frac{PL^3}{EI} - \frac{1}{2}\left(\frac{5}{96}\frac{PL^3}{EI}\right) = -\frac{1}{64}\frac{PL^3}{EI}$$

$$= \frac{1}{64}\frac{PL^3}{EI}\downarrow$$ ◄

PROBLEM 9.143

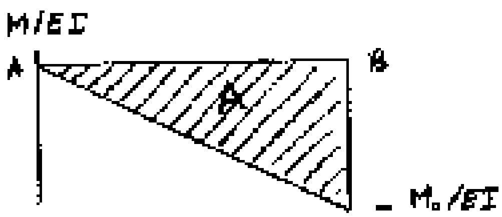
9.143 For the beam and loading shown, determine the magnitude and location of the maximum deflection.

SOLUTION

Draw $\frac{M}{EI}$ diagram

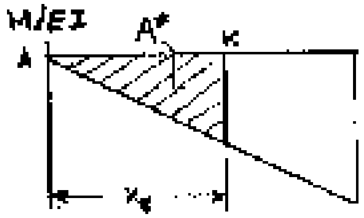
Place reference tangent at A.

$$A = \frac{1}{2}\left(-\frac{M_0}{EI}\right)L = -\frac{1}{2}\frac{M_0L}{EI}$$



$$t_{B/A} = A\left(\frac{L}{3}\right) = -\frac{1}{6}\frac{M_0L^2}{EI}$$

$$\theta_A = -\frac{t_{B/A}}{L} = \frac{1}{6}\frac{M_0L}{EI}$$

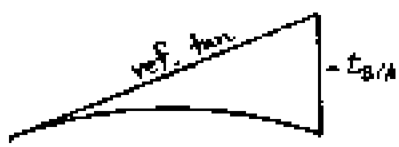


$$A^* = \frac{1}{2}\left(\frac{M_0}{EI}\frac{x_K}{L}\right)x_K = -\frac{1}{2}\frac{M_0x_K^2}{EIL}$$

$$\theta_K = \theta_A + \theta_{K/A} = \theta_A + A^* = 0$$

$$\frac{1}{6}\frac{M_0L}{EI} - \frac{1}{2}\frac{M_0x_K^2}{EIL} = 0$$

$$x_K = \frac{\sqrt{3}}{3}L$$



$$t_{K/A} = A^*\left(\frac{1}{3}x_K\right)$$

$$= -\frac{1}{2}\frac{M_0x_K^2}{EIL}\left(\frac{1}{3}x_K\right)$$

$$= -\frac{1}{6}\frac{M_0x_K^3}{EIL}$$

Maximum deflection

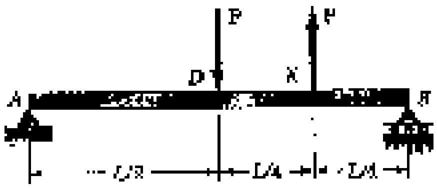
$$y_K = t_{K/A} - \frac{x_K}{L}t_{B/A} = -\frac{1}{6}\frac{M_0x_K^3}{EIL} - \frac{x_K}{L}\left(\frac{M_0L^2}{6EI}\right)$$

$$= \frac{M_0x_K}{6EIL}\left(L^2 - x_K^2\right) = \frac{\sqrt{3}}{18}\frac{M_0}{EI}\left(L^2 - \frac{1}{3}L^2\right) = \frac{\sqrt{3}}{27}\frac{M_0L^2}{EI} \uparrow$$

PROBLEM 9.144

9.144 through 9.147 For the beam and loading shown, determine the magnitude and location of the largest downward deflection.

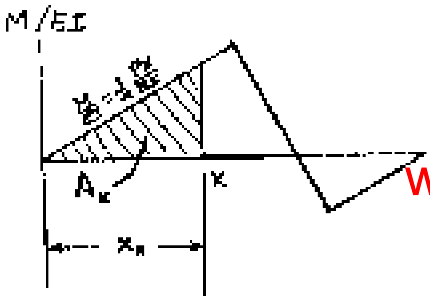
9.144 Beam and loading of Prob. 9.129



SOLUTION

Referring to the solution of Prob. 9.129

$$R_A = \frac{1}{4}P, \quad t_{B/A} = \frac{3}{128}\frac{PL^3}{EI}, \quad \theta_A = -\frac{3}{128}\frac{PL^2}{EI}$$



$$\theta_K = \theta_A + \theta_{K/A}$$

$$= -\frac{3}{128}\frac{PL^2}{EI} + A_K$$

$$= -\frac{3}{128}\frac{PL^2}{EI} + \frac{1}{2}\left(\frac{1}{4}\frac{Px_K}{EI}\right)x_K$$

$$= \frac{P}{EI}\left(-\frac{3}{128}L^2 + \frac{1}{8}x_K^2\right) = 0$$

$$x_K = \sqrt{\frac{3}{16}}\,L = \frac{1}{4}\sqrt{3}\,L = 0.433\,L$$

$$t_{K/A} = A_K\left(\frac{1}{3}x_K\right) = \frac{1}{2}\left(\frac{1}{4}\frac{Px_K^2}{EI}\right)\frac{x_K}{3}$$

$$= \frac{1}{24}\frac{Px_K^3}{EI} = \frac{\sqrt{3}}{512}\frac{PL^3}{EI}$$

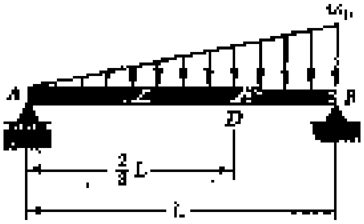
$$y_K = t_{K/A} - \frac{x_K}{L}t_{B/A} = \frac{\sqrt{3}}{512}\frac{PL^3}{EI} - \left(\frac{1}{4}\sqrt{3}\right)\frac{3}{128}\frac{PL^3}{EI} = -\frac{\sqrt{3}}{256}\frac{PL^3}{EI}$$

$$= 0.00677\frac{PL^3}{EI} \downarrow$$

PROBLEM 9.145

9.144 through 9.147 For the beam and loading shown, determine the magnitude and location of the largest downward deflection.

9.145 Beam and loading of Prob. 9.130



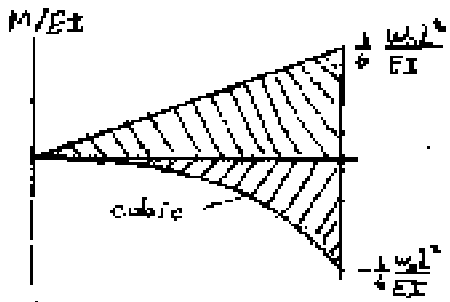
SOLUTION

$+\circlearrowleft \Sigma M_B = 0 \quad -R_A L + \left(\frac{1}{2}w_0 L\right)\left(\frac{1}{3}L\right) = 0 \qquad R_A = \frac{1}{6}w_0 L$

Bending moment $\quad M = R_A x - \frac{1}{6}\frac{w_0}{L}x^3$

$$= \frac{1}{6}\frac{w_0}{L}\left(L^2 x - x^3\right)$$

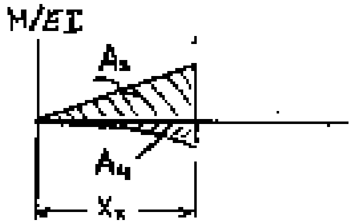
At $x = L \qquad M = \frac{1}{6}w_0 L^2 - \frac{1}{6}w_0 L^2$



Draw $\frac{M}{EI}$ diagram by parts

$A_1 = \frac{1}{2}\left(\frac{1}{6}\frac{w_0 L^2}{EI}\right)L = \frac{1}{12}\frac{w_0 L^3}{EI} \qquad \bar{x}_1 = \frac{1}{3}L$

$A_2 = \frac{1}{4}\left(-\frac{1}{6}\frac{w_0 L^2}{EI}\right)L = -\frac{1}{24}\frac{w_0 L^3}{EI} \qquad \bar{x}_2 = \frac{1}{5}L$



Place reference tangent at A.

$t_{B/A} = A_1\bar{x}_1 + A_2\bar{x}_2$

$= \frac{1}{36}\frac{w_0 L^4}{EI} - \frac{1}{120}\frac{w_0 L^4}{EI} = \frac{7}{360}\frac{w_0 L^4}{EI}$

Slope at A $\qquad \theta_A = -\frac{t_{B/A}}{L} = -\frac{7}{360}\frac{w_0 L^3}{EI}$

$A_3 = A_1\left(\frac{x_K}{L}\right)^2 = \frac{1}{12}\frac{w_0 L^3}{EI}u^2$

$A_4 = A_2\left(\frac{x_K}{L}\right)^4 = -\frac{1}{24}\frac{w_0 L^3}{EI}u^4$

$\theta_{K/A} = A_3 + A_4 = \frac{w_0 L^3}{EI}\left(\frac{1}{12}u^2 - \frac{1}{24}u^4\right) = -\theta_A = \frac{7}{360}\frac{w_0 L^3}{EI}$

$u^4 - 2u^2 + \frac{7}{15} = 0 \qquad$ Solving for $u \qquad u = 0.51933$

$x_K = 0.51933\,L$ ◄

$A_3 = \frac{1}{12}\frac{w_0 L^3}{EI}(0.51933)^2 = 0.022475\frac{w_0 L^3}{EI}, \qquad \bar{x}_3 = \frac{1}{3}(0.51933)L$

$A_4 = -\frac{1}{24}\frac{w_0 L^3}{EI}(0.51933)^4 = -0.0030308\frac{w_0 L^3}{EI}, \qquad \bar{x}_4 = \frac{1}{5}(0.51933)L$

$t_{K/A} = A_3\bar{x}_3 + A_4\bar{x}_4 = 0.0035759\frac{w_0 L^4}{EI}$

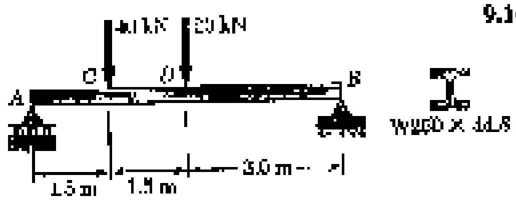
$y_K = t_{K/A} - \frac{x_K}{L}t_{B/A} = 0.0035759\frac{w_0 L^4}{EI} - (0.51933)\left(\frac{7}{360}\frac{w_0 L^4}{EI}\right)$

$= -0.00652\frac{w_0 L^4}{EI} \qquad = 0.00652\frac{w_0 L^4}{EI}\downarrow$ ◄

PROBLEM 9.146

9.144 through 9.147 For the beam and loading shown, determine the magnitude and location of the largest downward deflection.

9.146 Beam and loading of Prob. 9.132



SOLUTION

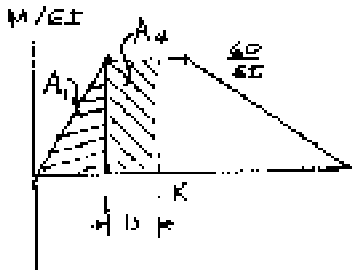
Referring to the solution to Prob. 9.129

$EI = 14220\ \text{kN}\cdot\text{m}^2$

$R_A = 40\ \text{kN}, \quad A_1 = \frac{45}{EI}$

$t_{B/A} = \frac{742.5}{EI}\ \text{m}$

$\theta_A = -\frac{128.75}{EI}$



Let K be the location of the maximum deflection. Assume that K lies between C and D.

$$\theta_K = \theta_A + \theta_{K/A}$$

$$= -\frac{128.75}{EI} + A_1 + A_4$$

$$= -\frac{128.75}{EI} + \frac{45}{EI} + \frac{60\,u}{EI} = 0$$

$$u = \frac{128.75-45}{60} = 1.3125\ \text{m}.$$

$$x_K = 1.5 + u = 2.8125\ \text{m}$$

$$t_{K/A} = A_1(u + 0.5) + A_4\left(\tfrac{1}{2}u\right)$$

$$= \frac{45}{EI}(1.8125) + \frac{(60)(1.3125)(\frac{1}{2})(1.3125)}{EI} = \frac{133.242}{EI}$$

$$y_K = t_{K/A} - \frac{x_K}{L}\,t_{B/A}$$

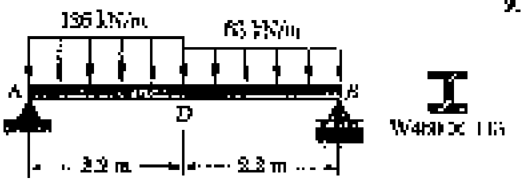
$$= \frac{133.242}{EI} - \frac{2.8125}{6}\left(\frac{742.5}{EI}\right) = -\frac{214.80}{EI} = -\frac{214.80}{14220}$$

$$= -15.11\times10^{-3}\ \text{m} = 15.11\ \text{mm}\downarrow$$

PROBLEM 9.147

9.144 through 9.147 For the beam and loading shown, determine the magnitude and location of the largest downward deflection.

9.147 Beam and loading of Prob. 9.133



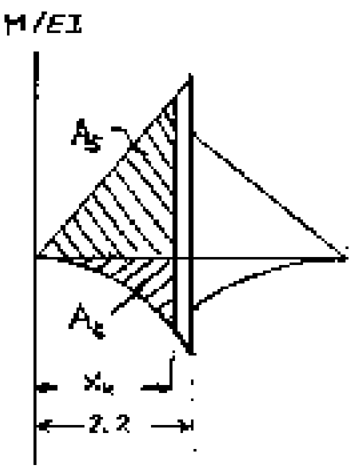
SOLUTION

From the solution to Prob. 9.133

$EI = 111200 \text{ kN}\cdot\text{m}^2, \qquad R_A = 242.55 \text{ kN}$

$t_{B/A} = 13.824 \times 10^{-3} \text{ m}$

$\theta_A = -\frac{t_{B/A}}{L} = -\frac{13.824 \times 10^{-3}}{4.4} = -3.1418 \times 10^{-3} \text{ rad.}$



Over portion AD of the beam

$M = 242.55x - 63x^2 \text{ kN}\cdot\text{m}$

$\frac{M}{EI} = (2.1812x - 0.56655x^2) \times 10^{-3} \text{ m}^{-1}$

$\theta_{K/A} = \int_0^{x_K} \frac{M}{EI} dx$

$= (1.0906 x_K^2 - 0.188849 x_K^3) \times 10^{-3} \text{ rad}$

$\theta_K = \theta_A + \theta_{K/A} = -3.1418 \times 10^{-3} + (1.0906 x_K^2 - 0.188849 x_K^3) \times 10^{-3} = 0$

Solving for x_K $\qquad x_K = 2.13907 \text{ m}$ $\qquad x_K = 2.14 \text{ m}$ ◀

$A_S = 1.0906 \times 10^{-3} \, x_K^2 = 4.99017 \times 10^{-3}, \qquad \bar{x}_S = \frac{1}{3} x_K = 0.71302 \text{ m}$

$A_c = -0.188849 \times 10^{-3} \, x_K^3 = -1.84837 \times 10^{-3}, \qquad \bar{x}_c = \frac{1}{4} x_K = 0.53477 \text{ m}$

$t_{K/A} = A_S \bar{x}_S + A_c \bar{x}_c = 2.5696 \times 10^{-3} \text{ m}$

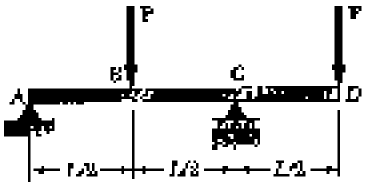
$y_K = t_{K/A} - \frac{x_K}{L} t_{B/A}$

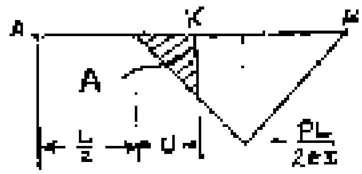
$= 2.5696 \times 10^{-3} - \frac{2.13907}{4.4}(13.824 \times 10^{-3}) = -4.15 \times 10^{-3} \text{ m}$

$= 4.15 \text{ mm} \downarrow$ ◀

PROBLEM 9.148

9.148 For the beam and loading of Prob. 9.135, determine the magnitude and location of the largest upward deflection in span *AC*.





SOLUTION

From solution of Problem 9.135

$$R_A = 0 , \qquad \theta_A = \frac{1}{48}\frac{PL^2}{EI}$$

Draw M/EI diagram. Let K be location of maximum deflection.

$$\theta_K = \theta_A + \theta_{K/A} = \frac{1}{48}\frac{PL^2}{EI} + A = 0$$

where $\quad A = \frac{1}{2}\left(-\frac{PL}{2EI}\cdot\frac{2u}{L}\right)u = -\frac{1}{2}\frac{PLu^2}{EI}$

$$\frac{1}{2}\frac{PLu^2}{EI} = \frac{1}{48}\frac{PL^2}{EI}$$

$$u^2 = \frac{1}{24}L^2 \qquad u = 0.20412\,L$$

$$x_K = \frac{L}{2} + u = 0.704\,L$$

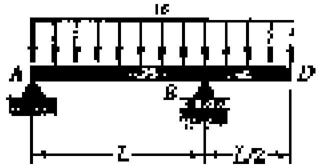
$$A = -\frac{1}{48}\frac{PL^2}{EI}$$

$$t_{A/K} = A\left(\frac{L}{2} + \frac{2}{3}u\right) = \left(-\frac{1}{48}\frac{PL^2}{EI}\right)(0.63608\,L) = -0.01325\frac{PL^3}{EI}$$

$$y_{max} = -t_{A/K} = 0.01325\frac{PL^3}{EI}$$

PROBLEM 9.149

9.149 For the beam and loading of Prob. 9.136, determine the magnitude and location of the largest downward deflection in span AB.



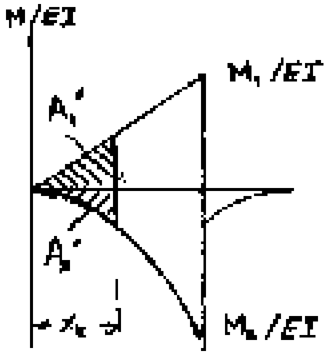
SOLUTION

From solution of Prob. 9.146

$$\frac{M_1}{EI} = \frac{3}{8}\frac{wL^2}{EI}, \quad \frac{M_2}{EI} = -\frac{1}{2}\frac{wL^2}{EI}, \quad \theta_A = -\frac{1}{48}\frac{wL^3}{EI}$$

Let K be the location of maximum deflection.

$$u = \frac{x_K}{L}$$



From $\frac{M}{EI}$ diagram

$$A_1' = \frac{1}{2}\left(\frac{M_1}{EI}u\right)(Lu) = \frac{3}{16}\frac{wL^3}{EI}u^2$$

$$A_2' = \frac{1}{3}\left(\frac{M_2}{EI}u^2\right)(Lu) = -\frac{1}{6}\frac{wL^3}{EI}u^3$$

$$\theta_K = \theta_A + \theta_{K/A} = \theta_A + A_1' + A_2'$$

$$= -\frac{1}{48}\frac{wL^3}{EI} + \frac{3}{16}\frac{wL^3}{EI}u^2 - \frac{1}{6}\frac{wL^3}{EI}u^3$$

$$= -\left(\frac{1}{6}u^3 - \frac{3}{16}u^2 + \frac{1}{48}\right)\frac{wL^3}{EI} = 0$$

Multiplying by 48 $\qquad 8u^3 - 9u^2 + 1 = 0$

Solving for u $\qquad u = 0.421535 \qquad x_K = 0.4215\,L$ ◄

$$t_{A/K} = A_1'\left(\tfrac{2}{3}x_K\right) + A_2'\left(\tfrac{3}{4}x_K\right)$$

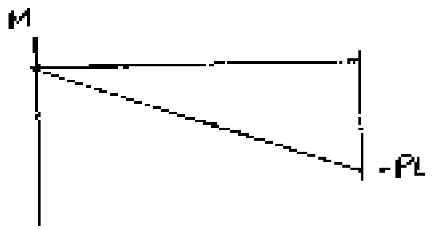
$$= \left(\frac{3}{16}\frac{wL^3}{EI}u^2\right)\left(\frac{2}{3}Lu\right) + \left(-\frac{1}{6}\frac{wL^3}{EI}u^3\right)\left(\frac{3}{4}Lu\right)$$

$$= \left(\frac{1}{8}u^3 - \frac{1}{8}u^4\right)\frac{wL^4}{EI} = 0.00542\frac{wL^4}{EI}$$

$$y_{max} = -t_{A/K} = -0.00542\frac{wL^4}{EI} \qquad 0.00542\frac{wL^4}{EI}\downarrow$$ ◄

PROBLEM 9.150

*9.150 The cantilever AB is a beam of constant strength. It has a rectangular cross section of uniform width b and variable depth h. Express the deflection at end A in terms of P, L and the flexural rigidity EI_0 at B. (*Hint:* Since the beam is of constant strength, Mc/I has a constant value along AB.)



SOLUTION

Bending moment $\quad M = -Px$

$$M_0 = -PL$$

$$M = M_0 \frac{x}{L}$$

For a constant strength beam

$$\frac{Mc}{I} = \frac{(M_0 x/L)(h/2)}{\frac{1}{12} b h^3} = \frac{6 M_0 x/L}{h^2}$$

$$= \frac{6 M_0}{h_0^2}$$

$$\left(\frac{h}{h_0}\right) = \left(\frac{x}{L}\right)^{\frac{1}{2}}$$

Moment of inertia $\quad I = \frac{1}{12} b h^3, \quad I_0 = \frac{1}{12} b h_0^3$

$$\frac{I}{I_0} = \left(\frac{h}{h_0}\right)^3 = \left(\frac{x}{L}\right)^{3/2}$$

Curvature $\quad \frac{M}{EI} = \frac{M_0 (x/L)}{EI_0 (x/L)^{3/2}} = \frac{M_0}{EI_0}\left(\frac{L}{x}\right)^{1/2}$

$$= -\frac{PL}{EI_0}\left(\frac{L}{x}\right)^{1/2}$$

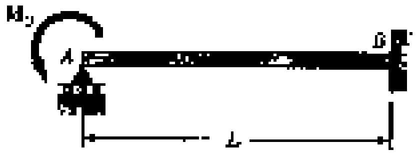
Deflection at A $\quad y_A = t_{A/B} = \int_0^L x \frac{M}{EI} dx$

$$= -\frac{PL^{3/2}}{EI_0} \int_0^L x^{\frac{1}{2}} dx = -\frac{PL^{3/2}}{EI_0} \left.\frac{x^{3/2}}{3/2}\right|_0^L$$

$$= -\frac{2}{3}\frac{PL^3}{EI_0} = \frac{2}{3}\frac{PL^3}{EI_0} \downarrow$$

PROBLEM 9.151

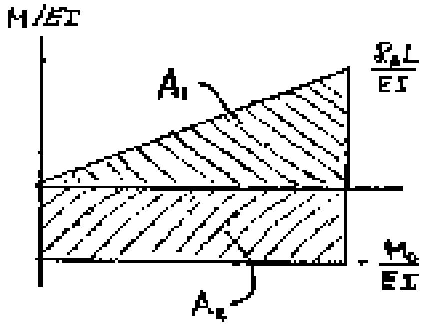
9.151 through 9.154 For the beam and loading shown, determine the reaction at the roller support.



SOLUTION

Remove support A and treat R_A as redundant.

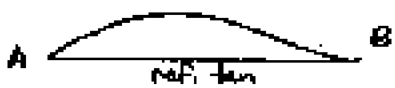
Draw M/EI diagram for loads M_0 and R_A.



$$A_1 = \frac{1}{2}\left(\frac{R_A L}{EI}\right)L = \frac{1}{2}\frac{R_A L^2}{EI}$$

$$A_2 = -\left(\frac{M_0}{EI}\right)L = -\frac{M_0 L}{EI}$$

Place reference tangent at B



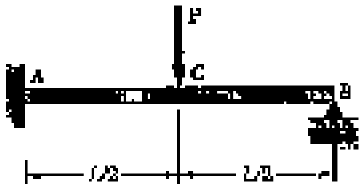
$$t_{B/A} = A_1\left(\tfrac{2}{3}L\right) + A_2\left(\tfrac{1}{2}L\right)$$

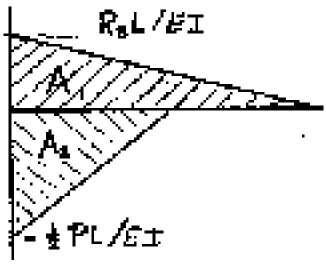
$$= \frac{1}{3}\frac{R_A L^3}{EI} - \frac{1}{2}\frac{M_0 L^2}{EI} = 0$$

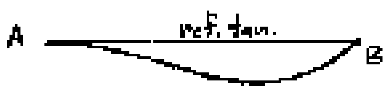
$$R_A = \frac{3}{2}\frac{M_0}{L} \uparrow$$

PROBLEM 9.152

9.151 through 9.154 For the beam and loading shown, determine the reaction at the roller support.







SOLUTION

Remove support B and treat R_B as redundant.

Draw M/EI diagram for loads P and R_B.

$$A_1 = \frac{1}{2}\left(\frac{R_B L}{EI}\right)(L) = \frac{1}{2}\frac{R_B L^2}{EI}$$

$$A_2 = \frac{1}{2}\left(-\frac{1}{2}\frac{PL}{EI}\right)\left(\frac{L}{2}\right) = -\frac{1}{8}\frac{PL^2}{EI}$$

Place reference tangent at A.

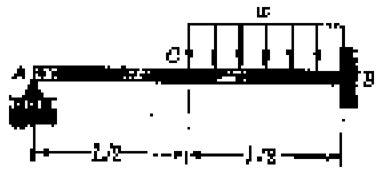
$$t_{B/A} = A_1\left(\tfrac{2}{3}L\right) + A_2\left(\tfrac{L}{2} + \tfrac{2}{3}\tfrac{L}{2}\right)$$

$$= \frac{1}{3}\frac{R_B L^3}{EI} - \frac{5}{48}\frac{PL^3}{EI} = 0$$

$$R_B = \frac{5}{16}P \uparrow$$

PROBLEM 9.153

9.151 through 9.154 For the beam and loading shown, determine the reaction at the roller support.



SOLUTION

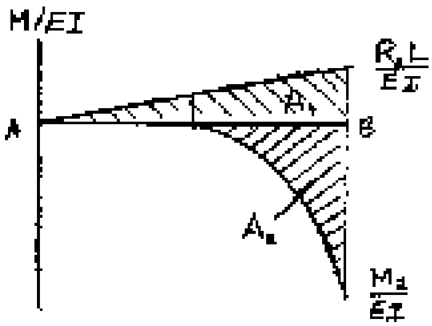
Remove support A and treat R_A as redundant.

Draw M/EI diagram for loads R_A and w.

$$M_2 = -\frac{1}{2} w \left(\frac{L}{2}\right)^2 = -\frac{1}{8} wL^2$$

$$A_1 = \frac{1}{2}\left(\frac{R_A L}{EI}\right) L = \frac{1}{2}\frac{R_A L^2}{EI}$$

$$A_2 = \frac{1}{3}\left(-\frac{1}{8}\frac{wL^2}{EI}\right)\left(\frac{L}{2}\right) = -\frac{1}{48}\frac{wL^3}{EI}$$



Place reference tangent at B

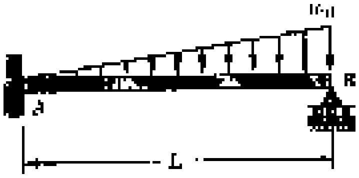
$$t_{A/B} = A_1\left(\frac{2}{3}L\right) + A_2\left(\frac{L}{2} + \frac{3}{4}\frac{L}{2}\right)$$

$$= \frac{1}{3}\frac{R_A L^3}{EI} - \frac{7}{384}\frac{wL^4}{EI} = 0$$

$$R_A = \frac{7}{128} wL \uparrow$$

PROBLEM 9.154

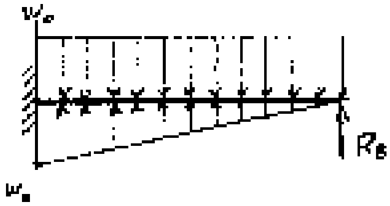
9.151 through 1.154 For the beam and loading shown, determine the reaction at the roller support.



SOLUTION

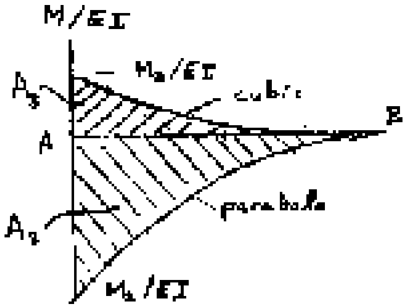
Remove support B and treat R_B as redundant.

Replace loading by equivalent shown at left.



Draw M/EI diagram for load w_0 and R_B.

Use parts as shown.



$$A_1 = \frac{1}{2}\left(\frac{R_B L}{EI}\right)(L) = \frac{1}{2}\frac{R_B L^2}{EI}$$

$$M_2 = -\frac{1}{2} w_0 L^2$$

$$A_2 = \frac{1}{3}\left(-\frac{1}{2}\frac{w_0 L^2}{EI}\right)L = -\frac{1}{6}\frac{w_0 L^3}{EI}$$

$$M_3 = \frac{1}{6}\frac{w_0}{L} L^3 = \frac{1}{6} w_0 L^2$$

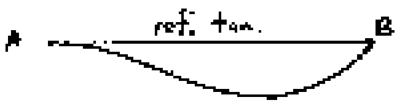
$$A_3 = \frac{1}{4}\left(\frac{1}{6}\frac{w_0 L^2}{EI}\right)L = \frac{1}{24}\frac{w_0 L^3}{EI}$$

Place reference tangent at A

$$t_{B/A} = A_1\left(\frac{2}{3}L\right) + A_2\left(\frac{3}{4}L\right) + A_3\left(\frac{4}{5}L\right)$$

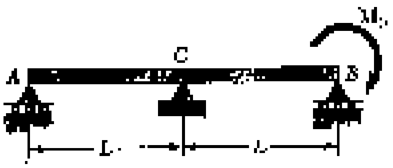
$$= \frac{1}{3}\frac{R_B L^3}{EI} - \frac{1}{8}\frac{w_0 L^4}{EI} + \frac{1}{30}\frac{w_0 L^4}{EI} = 0$$

$$R_A = \frac{11}{40} w_0 L = 0.275\, w_0 L \uparrow$$



PROBLEM 9.155

9.155 and 9.156 For the beam and loading shown, determine the reaction at each support.

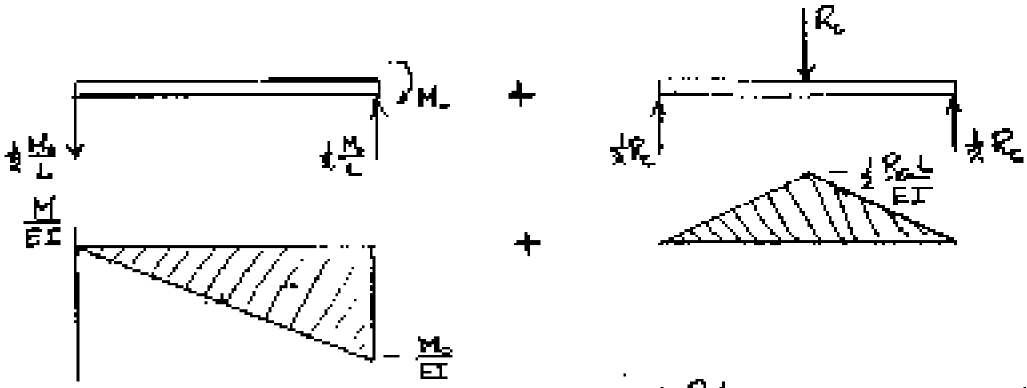


SOLUTION

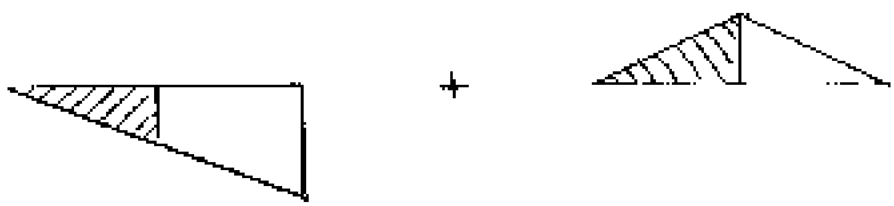
Remove support C and treat R_C as redundant.

Consider the loads M_0 and R_C separately.

Place reference tangent at A.



$$t_{B/A} = -\frac{1}{2}\left(\frac{M_0}{EI}\right)(2L)\left(\frac{2L}{3}\right) + \frac{1}{2}\left(\frac{1}{2}\frac{R_C L}{EI}\right)(2L)(L) = -\frac{2}{3}\frac{M_0 L^2}{EI} + \frac{1}{2}\frac{R_C L^3}{EI}$$



$$t_{C/A} = -\frac{1}{2}\left(\frac{1}{2}\frac{M_0}{EI}\right)(L)\left(\frac{L}{3}\right) + \frac{1}{2}\left(\frac{1}{2}\frac{R_C L}{EI}\right)(L)\left(\frac{L}{3}\right) = -\frac{1}{12}\frac{M_0 L^2}{EI} + \frac{1}{12}\frac{R_C L^3}{EI}$$

$$y_C = t_{C/A} - \frac{x_{C/A}}{x_{B/A}}t_{B/A} = t_{C/A} - \frac{1}{2}t_{B/A} = 0$$

$$\left(-\frac{1}{12} + \frac{1}{3}\right)\frac{M_0 L^2}{EI} + \left(\frac{1}{12} - \frac{1}{4}\right)\frac{R_C L^3}{EI} = 0$$

$$R_C = \frac{3}{2}\frac{M_0}{L} \downarrow$$

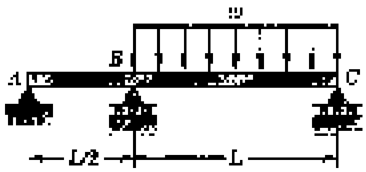
From Statics

$$R_A = \frac{1}{2}R_C - \frac{1}{2}\frac{M_0}{L} = \frac{1}{4}\frac{M_0}{L} \uparrow$$

$$R_B = \frac{1}{2}R_C + \frac{1}{2}\frac{M_0}{L} = \frac{5}{4}\frac{M_0}{L} \uparrow$$

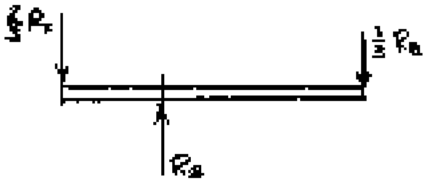
PROBLEM 9.156

9.155 and 9.156 For the beam and loading shown, determine the reaction at each support.



SOLUTION

Remove support B and consider R_B as redundant.
Consider loads R_B and w separately.
Place reference tangent at A.



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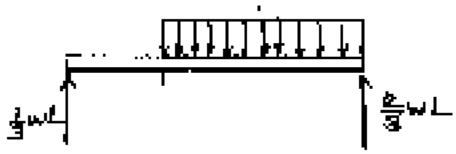

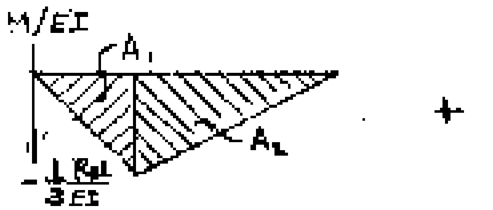

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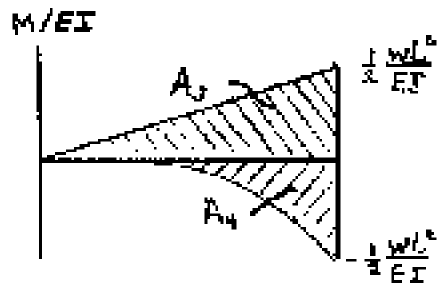


$$A_1 = \frac{1}{2}\cdot\left(-\frac{1}{3}\frac{R_B L}{EI}\right)\left(\frac{L}{2}\right) = -\frac{1}{12}\frac{R_B L^2}{EI}$$

$$A_2 = \frac{1}{2}\left(-\frac{1}{3}\frac{R_B L}{EI}\right)L = -\frac{1}{6}\frac{R_B L^2}{EI}$$

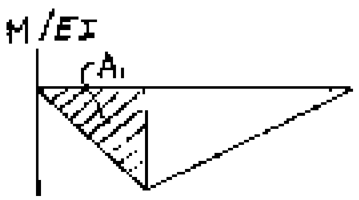
$$A_3 = \frac{1}{2}\left(\frac{1}{2}\frac{wL^2}{EI}\right)\frac{3L}{2} = \frac{3}{8}\frac{wL^3}{EI}$$

$$A_4 = \frac{1}{3}\left(-\frac{1}{2}\frac{wL^2}{EI}\right)L = -\frac{1}{6}\frac{wL^3}{EI}$$

$$t_{C/A} = A_1\left(L + \frac{1}{3}\frac{L}{2}\right) + A_2\left(\frac{2}{3}L\right) + A_3\left(\frac{1}{3}\cdot\frac{3L}{2}\right) + A_4\left(\frac{1}{4}L\right)$$

$$= -\frac{7}{72}\frac{R_B L^3}{EI} - \frac{1}{9}\frac{R_B L^3}{EI} + \frac{3}{16}\frac{wL^4}{EI} - \frac{1}{24}\frac{wL^4}{EI}$$

$$= -\frac{5}{24}\frac{R_B L^3}{EI} + \frac{7}{48}\frac{wL^4}{EI}$$



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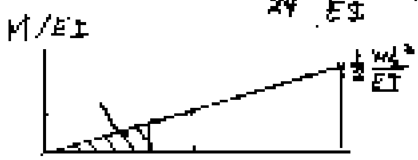


$$A_5 = \frac{1}{2}\left(\frac{1}{6}\frac{wL^2}{EI}\right)\frac{L}{2} = \frac{1}{24}\frac{wL^3}{EI}$$

$$t_{B/A} = A_1\left(\frac{1}{3}\frac{L}{2}\right) + A_5\left(\frac{1}{3}\frac{L}{2}\right) = -\frac{1}{72}\frac{R_B L^3}{EI} + \frac{1}{144}\frac{wL^4}{EI}$$

$$y_B = t_{B/A} - \frac{L/2}{3L/2}t_{C/A} = \left(-\frac{1}{72} + \frac{5}{72}\right)\frac{R_B L^3}{EI} + \left(\frac{1}{144} - \frac{7}{144}\right)\frac{wL^4}{EI}$$

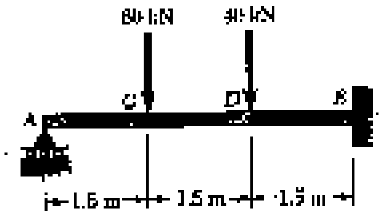
$$= \frac{1}{18}\frac{R_B L^3}{EI} - \frac{1}{24}\frac{wL^4}{EI} = 0 \qquad R_B = \frac{3}{4}wL \uparrow$$

$$R_A = \frac{1}{3}wL - \frac{2}{3}R_B = \frac{1}{3}wL - \frac{1}{2}wL = -\frac{1}{6}wL = \frac{1}{6}wL \downarrow$$

$$R_C = \frac{2}{3}wL - \frac{1}{3}R_B = \frac{2}{3}wL - \frac{1}{4}wL = \frac{5}{12}wL \uparrow$$

PROBLEM 9.157

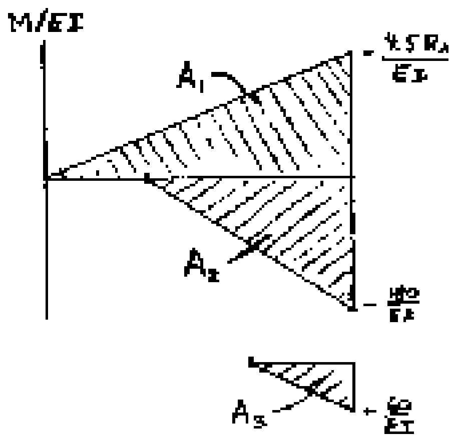
9.157 and 9.158 Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.



SOLUTION

Remove support at A and treat R_A as redundant.

Draw bending moment diagram by parts.



$$A_1 = \frac{1}{2}\left(\frac{4.5 R_A}{EI}\right)(4.5) = \frac{10.125 R_A}{EI}$$

$$A_2 = -\frac{1}{2}\left(\frac{180}{EI}\right)(3.0) = -\frac{270}{EI}$$

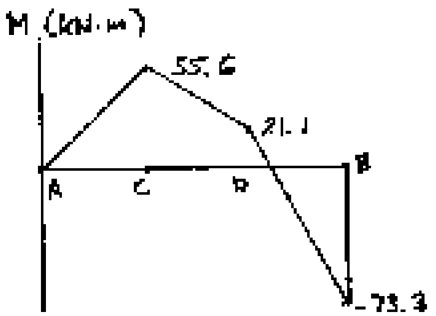
$$A_3 = -\frac{1}{2}\left(\frac{60}{EI}\right)(1.5) = -\frac{45}{EI}$$

Place reference tangent at B. $\theta_B = 0$

$$t_{A/B} = A_1(3.0) + A_2(1.5 + 2.0) + A_3(3.0 + 1.0)$$

$$= \left\{30.375 R_A - 945 - 180\right\}\frac{1}{EI} = 0$$

$$R_A = 37.037 \text{ kN} \uparrow$$



$$M_A = 0$$

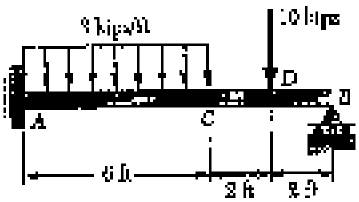
$$M_C = (1.5)(37.037) = 55.6 \text{ kN·m}$$

$$M_D = (3.0)(37.037) - (60)(1.5) = 21.1 \text{ kN·m}$$

$$M_B = (4.5)(37.037) - (60)(3) - (40)(1.5)$$
$$= -73.3 \text{ kN·m}$$

PROBLEM 9.158

9.157 and 9.158 Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.



SOLUTION

Remove support at B and treat R_B as redundant.

Draw bending moment diagram by parts

$$A_1 = \tfrac{1}{2}\left(\frac{10R_B}{EI}\right)(10) = \frac{50R_B}{EI}$$

$$A_2 = -\tfrac{1}{2}\left(\frac{80}{EI}\right)(8) = -\frac{320}{EI}$$

$$M_3 = -\tfrac{1}{2}(3)(6)^2 = -54 \text{ kN}\cdot\text{m}$$

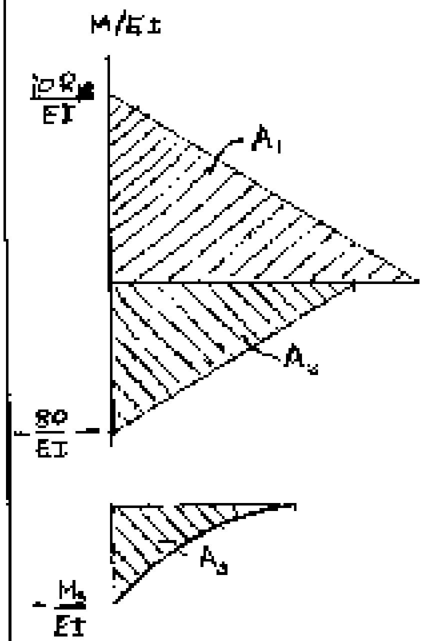
$$A_3 = \tfrac{1}{3}\left(-\frac{54}{EI}\right)(6) = -\frac{108}{EI}$$

Place reference tangent at A. $\theta_A = 0$

$$t_{B/A} = A_1\left(\tfrac{2}{3}\cdot 10\right) + A_2\left(\tfrac{2}{3}\cdot 8 + 2\right) + A_3\left(10 - \tfrac{1}{4}\cdot 6\right)$$

$$= \frac{333.33R_A}{EI} - \frac{2346.7}{EI} - \frac{918}{EI} = 0$$

$$R_A = 9.79 \text{ kN} \uparrow$$

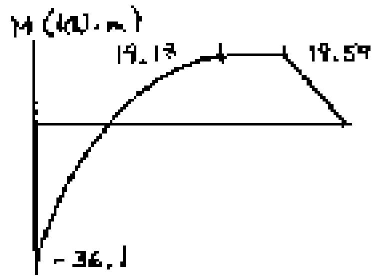


$$M_B = 0$$

$$M_D = (9.79)(2) = 19.59 \text{ kN}\cdot\text{m}$$

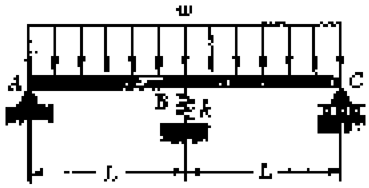
$$M_C = (9.79)(4) - (10)(2) = 19.18 \text{ kN}\cdot\text{m}$$

$$M_A = (9.79)(10) - (10)(8) - 54 = -36.1 \text{ kN}\cdot\text{m}$$

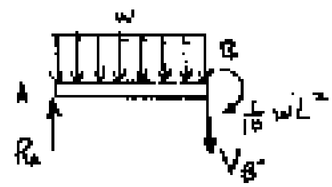


PROBLEM 9.159

9.159 For the beam and loading shown, determine the spring constant k for which the bending moment at B is $M_B = -wL^2/10$.



SOLUTION



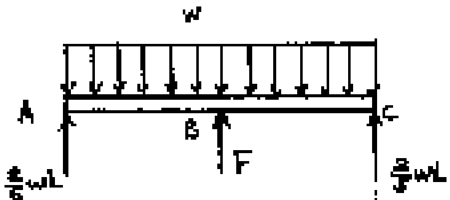
Using free body AB

$+\circlearrowleft M_B = 0$

$-R_A L + (wL)(\frac{L}{2}) - \frac{1}{10}wL^2 = 0$

$R_A = \frac{2}{5}wL \uparrow$

Symmetric beam and loading. $R_C = R_A$



Using free body ABC $\quad +\uparrow \Sigma F_y = 0$

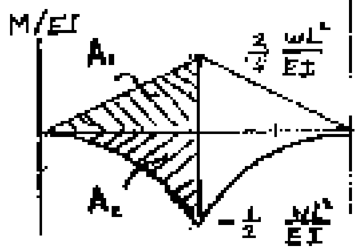
$\frac{2}{5}wL + F + \frac{2}{5}wL - 2wL = 0$

$F = \frac{6}{5}wL$

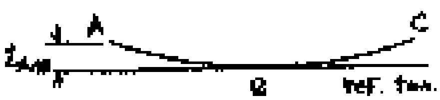
Draw $\frac{M}{EI}$ diagram by parts

$A_1 = \frac{1}{2}\left(\frac{2}{5}\frac{wL^2}{EI}\right)L = \frac{1}{5}\frac{wL^3}{EI}$

$A_2 = -\frac{1}{3}\left(\frac{1}{2}\frac{wL^2}{EI}\right)L = -\frac{1}{6}\frac{wL^3}{EI}$



Place reference tangent at B. $\quad \theta_B = 0$



$y_B = -t_{A/B}$

$= -\left(A_1 \cdot \frac{2}{3}L + A_2 \cdot \frac{3}{4}L\right)$

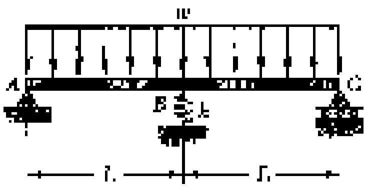
$= -\frac{1}{120}\frac{wL^4}{EI}$

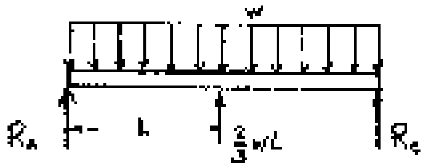
$F = -k y_B$

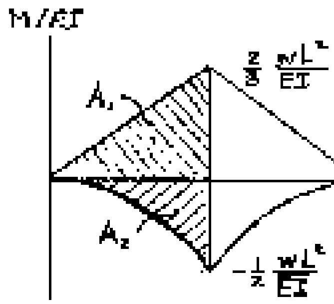
$k = -\frac{F}{y_B} = \frac{\frac{6}{5}wL}{\frac{1}{120}\frac{wL^4}{EI}} = 144\frac{EI}{L^3}$

PROBLEM 9.160

9.160 For the beam and loading shown, determine the spring constant k for which the force in the spring is equal to one-third of the total load on the beam.







SOLUTION

Symmetric beam and loading. $R_C = R_A$

Spring force $F = \frac{1}{3}(2wL) = \frac{2}{3}wL$

$+\uparrow \Sigma F_y = 0 \qquad R_A + F - 2wL + R_C = 0$

$$R_A = R_C = \frac{2}{3}wL$$

Draw $\frac{M}{EI}$ diagram by parts.

$$A_1 = \frac{1}{2}\left(\frac{2}{3}\frac{wL^2}{EI}\right)L = \frac{1}{3}\frac{wL^3}{EI}$$

$$A_2 = -\frac{1}{3}\left(\frac{1}{2}\frac{wL^2}{EI}\right)L = -\frac{1}{6}\frac{wL^3}{EI}$$

Place reference tangent at B. $\theta_B = 0$

$$y_B = -t_{A/B}$$

$$= -\left(A_1 \cdot \frac{2}{3}L + A_2 \cdot \frac{3}{4}L\right)$$

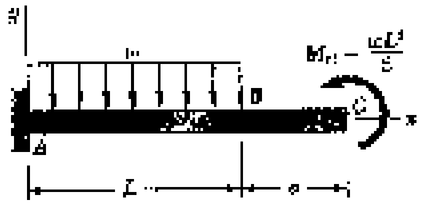
$$= -\frac{7}{72}\frac{wL^4}{EI}$$

$$F = -k\,y_B$$

$$k = -\frac{F}{y_B} = \frac{\frac{2}{3}wL}{\frac{7}{72}\frac{wL^4}{EI}} = \frac{48}{7}\frac{EI}{L^3}$$

PROBLEM 9.161

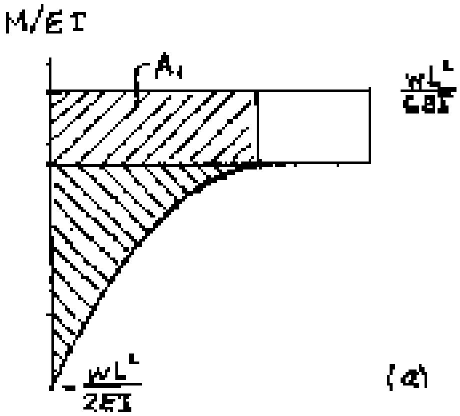
9.161 For the cantilever beam and loading shown, determine (a) the deflection at point B, (b) the slope at point B.



SOLUTION

Use moment area method.

Draw $\frac{M}{EI}$ diagram by parts.



Place reference tangent at A. $\theta_A = 0$

$$\theta_B = \theta_A + \theta_{B/A} = \theta_{B/A}$$

$$y_B = y_A + L\theta_A + t_{B/A} = t_{B/A}$$

$$A_1 = \left(\frac{wL^2}{6EI}\right)(L) = \frac{1}{6}\frac{wL^3}{EI}$$

$$A_2 = \frac{1}{3}\left(-\frac{wL^2}{2EI}\right)(L) = -\frac{1}{6}\frac{wL^3}{EI}$$

(a) Deflection at B

$$y_B = t_{B/A} = A_1\left(\frac{L}{2}\right) + A_2\left(\frac{3}{4}L\right)$$

$$= \frac{1}{12}\frac{wL^4}{EI} - \frac{1}{8}\frac{wL^4}{EI} = -\frac{1}{24}\frac{wL^4}{EI}$$

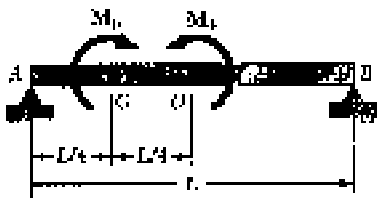
(b) Slope at B

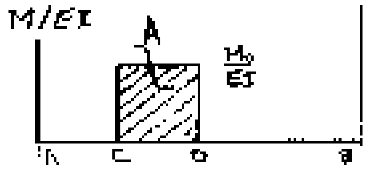
$$\theta_B = \theta_{B/A} = A_1 + A_2$$

$$= \frac{1}{6}\frac{wL^3}{EI} - \frac{1}{6}\frac{wL^3}{EI} = 0$$

PROBLEM 9.162

9.162 For the beam and loading shown, determine (*a*) the slope at point *A*, (*b*) the deflection at point *D*.





SOLUTION

From Statics $R_A = R_B = 0$.

Draw $\frac{M}{EI}$ diagram

$$A = \left(\frac{M_0}{EI}\right)\left(\frac{L}{4}\right) = \frac{1}{4}\frac{M_0 L}{EI}$$

Place reference tangent at A.

$$t_{B/A} = A\left(\frac{L}{2} + \frac{L}{8}\right) = \frac{5}{32}\frac{M_0 L^2}{EI}$$

$$t_{D/A} = A\left(\frac{L}{8}\right) = \frac{1}{32}\frac{M_0 L^2}{EI}$$

(a) Slope at A

$$\theta_A = -\frac{t_{B/A}}{L} = -\frac{5}{32}\frac{M_0 L}{EI}$$

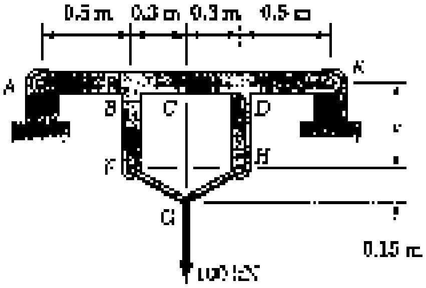
(b) Deflection at D

$$y_D = t_{D/A} - \frac{x_D}{L} t_{B/A} = t_{D/A} - \frac{1}{2} t_{B/A}$$

$$= \frac{1}{32}\frac{M_0 L^2}{EI} - \frac{5}{64}\frac{M_0 L^2}{EI} = -\frac{3}{64}\frac{M_0 L^2}{EI}$$

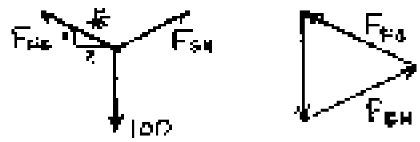
PROBLEM 9.163

9.163 The rigid bars BF and DH are welded to the rolled-steel beam AE as shown. Knowing that $c = 0.4$ m, determine for the loading shown (a) the deflection at point B, (b) the deflection at the midpoint C of the beam. Use $E = 200$ GPa.



SOLUTION

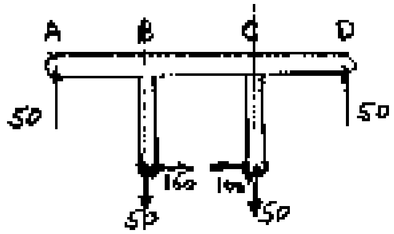
Using joint G as a free body



By symmetry $F_{GH} = F_{FG}$

$$2F_{GHy} - 100 = 0 \qquad F_{GHy} = 50 \text{ kN}$$

$$F_{GHx} = 2F_{GHx} = 100 \text{ kN.}$$



Forces in kN. Lengths in m

$$V = 50 - 50\langle x-0.5\rangle^0 - 50\langle x-1.1\rangle^0 \qquad \text{kN}$$

$$M = 50x - 50\langle x-0.5\rangle^1 - 50\langle x-1.1\rangle^1$$
$$+ 40\langle x-0.5\rangle^0 - 40\langle x-1.1\rangle \qquad \text{kN}\cdot\text{m}$$

$$EI\frac{dy}{dx} = 25x^2 - 25\langle x-0.5\rangle^2 - 25\langle x-1.1\rangle^2 + 40\langle x-0.5\rangle^1 + 40\langle x-1.1\rangle^1 + C_1 \qquad \text{kN}\cdot\text{m}^2$$

$$EIy = \frac{25}{3}x^3 - \frac{25}{3}\langle x-0.5\rangle^3 - \frac{25}{3}\langle x-1.1\rangle^3 - 20\langle x-0.5\rangle^2 + 20\langle x-1.1\rangle^2 + C_1x + C_2 \qquad \text{kN}\cdot\text{m}^3$$

$[x = 0, y = 0]$ $\qquad C_2 = 0$

$[x = 1.6, y = 0]$

$$\left(\frac{25}{3}\right)(1.6)^3 - \left(\frac{25}{3}\right)(1.1)^3 - \left(\frac{25}{3}\right)(0.5)^3 - (20)(1.1)^2 + (20)(0.5)^2 + C_1(1.6) + 0 = 0$$

$$C_1 = -1.75 \quad \text{kN}\cdot\text{m}^2$$

For EIy_B, $\quad x = 0.5$ m

$$EIy_B = \left(\frac{25}{3}\right)(0.5)^3 - 0 - 0 + 0 - 0 - (1.75)(0.5) = \quad 0.1667 \text{ kN}\cdot\text{m}^3$$

For EIy_C, $\quad x = 0.8$ m

$$EIy_C = \left(\frac{25}{3}\right)(0.8)^3 - \left(\frac{25}{3}\right)(0.3)^3 - 0 - (20)(0.3)^2 - 0 - (1.75)(0.8) + 0$$
$$= -0.8417 \quad \text{kN}\cdot\text{m}^3$$

For W 100 × 19.3 rolled steel section $\quad I = 4.77 \times 10^6 \text{ mm}^4 = 4.77 \times 10^{-6} \text{ m}^4$

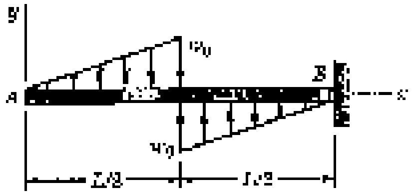
$$EI = (200 \times 10^9)(4.77 \times 10^{-6}) = 954 \times 10^3 \text{ N}\cdot\text{m}^2 = 954 \text{ kN}\cdot\text{m}^2$$

(a) $\quad y_B = \dfrac{0.1667}{954} = 0.175 \times 10^{-3} \text{ m} = 0.175 \text{ mm} \uparrow$

(b) $\quad y_C = \dfrac{0.8417}{954} = 0.882 \times 10^{-3} \text{ m} = 0.882 \text{ mm} \uparrow$

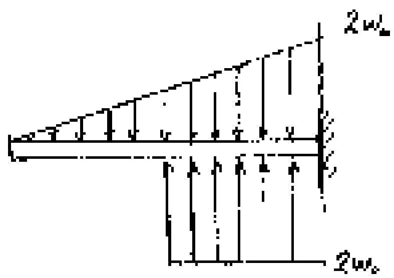
PROBLEM 9.164

9.164 For the beam and loading shown, determine the deflection of point A.



SOLUTION

Express loading in terms of singularity functions.



$$w = \frac{2w_0}{L}x - 2w_0\langle x - \tfrac{L}{2}\rangle^0$$

$$\frac{dV}{dx} = -w = -\frac{2w_0}{L}x + 2w_0\langle x - \tfrac{L}{2}\rangle^0$$

$$V = -\frac{w_0}{L}x^2 + 2w_0\langle x - \tfrac{L}{2}\rangle^1 + C_1$$

$[x = 0,\ V = 0]$

$$0 + 0 + C_1 = 0 \qquad C_1 = 0$$

$$\frac{dM}{dx} = V = -\frac{w_0}{L}x^2 + 2w_0\langle x - \tfrac{L}{2}\rangle^1$$

$$M = -\frac{1}{3}\frac{w_0}{L}x^3 + w_0\langle x - \tfrac{L}{2}\rangle^2 + C_2$$

$[x = 0,\ M = 0]$

$$0 + 0 + C_2 = 0 \qquad C_2 = 0$$

$$EI\frac{d^2y}{dx^2} = M = -\frac{1}{3}\frac{w_0}{L}x^3 + w_0\langle x - \tfrac{L}{2}\rangle^2$$

$$EI\frac{dy}{dx} = -\frac{1}{12}\frac{w_0}{L}x^4 + \frac{1}{3}w_0\langle x - \tfrac{L}{2}\rangle^3 + C_3$$

$[x = L,\ EI\frac{dy}{dx} = 0]$ $\quad -\frac{1}{12}w_0L^3 + \frac{1}{3}w_0\left(\frac{L}{2}\right)^3 + C_3 = 0 \qquad C_3 = \frac{1}{24}w_0L^3$

$$EIy = -\frac{1}{60}\frac{w_0}{L}x^5 + \frac{1}{12}w_0\langle x - \tfrac{L}{2}\rangle^4 + C_3x + C_4$$

$[x = L,\ EIy = 0]$ $\quad -\frac{1}{60}w_0L^4 + \frac{1}{12}w_0\left(\frac{L}{2}\right)^4 + \frac{1}{24}w_0L^3\cdot L + C_4 = 0$

$$C_4 = -\frac{29}{960}w_0L^4$$

At point A, $\quad x = 0$

$$EIy_A = 0 + 0 + 0 + C_4 = -\frac{29}{960}w_0L^4$$

$$y_A = -\frac{29}{960}\frac{w_0L^4}{EI}$$

PROBLEM 9.165

9.165 For the beam and loading shown, determine (*a*) the reaction at *C*, (*b*) the deflection at point *B*. Use $E = 29 \times 10^6$ psi.

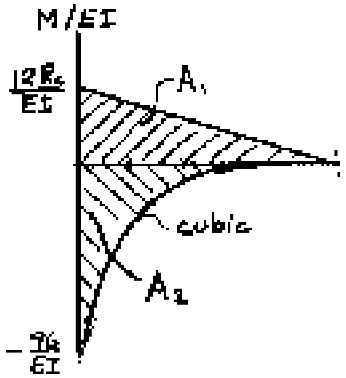
SOLUTION

$E = 29 \times 10^6$ psi $= 29 \times 10^3$ ksi

$I = 310$ in^4

$EI = (29 \times 10^3)(310) = 8.99 \times 10^6$ kip·in^2

$= 62430$ kip·ft^2



Statically indeterminate beam. Remove support at C and treat R_c as redundant.

Draw $\frac{M}{EI}$ diagram by parts

For the uniformly varying load

$$k = \frac{9 \text{ kips/ft}}{8 \text{ ft}} = \frac{9}{8} \text{ kips/ft}^2$$

$$M_2 = -\frac{1}{6}ka^3 = -\frac{1}{6}\cdot\frac{9}{8}(8)^3 = -96 \text{ kip·ft}$$

$$A_1 = \frac{1}{2}\left(\frac{12R_c}{EI}\right)(12) = \frac{72R_c}{EI}$$

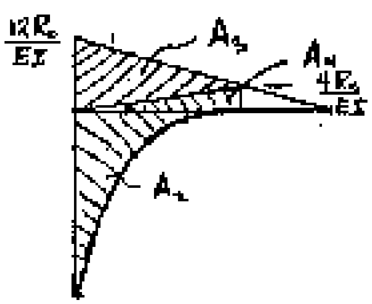
$$A_2 = \frac{1}{4}\left(-\frac{96}{EI}\right)(8) = -\frac{192}{EI}$$

Place reference tangent at A. $\theta_A = 0$

$y_A = 0$

$$y_C = y_A + \theta_A L + t_{C/A} = 0 + 0 + \left[A_1\left(\tfrac{2}{3}\cdot 12\right) + A_2\left(12 - \tfrac{1}{5}\cdot 8\right)\right]$$

$$= \frac{576R_c}{EI} - \frac{1996.8}{EI} = 0 \qquad R_c = 3.4667 \text{ kips} \uparrow$$



$$A_3 = \frac{1}{2}\left(\frac{12R_c}{EI}\right)(8) = \frac{48R_c}{EI}$$

$$A_4 = \frac{1}{2}\left(\frac{4R_c}{EI}\right)(8) = \frac{16R_c}{EI}$$

$$y_B = t_{B/A} = A_3\left(\tfrac{2}{3}\cdot 8\right) + A_4\left(\tfrac{1}{3}\cdot 8\right) + A_2\left(\tfrac{4}{5}\cdot 8\right)$$

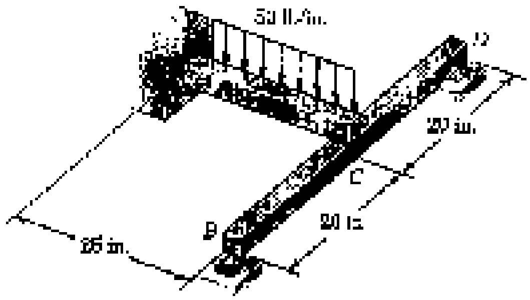
$$= 298\tfrac{2}{3}\frac{R_c}{EI} - \frac{6144}{5EI} = -\frac{193.42}{EI}$$

$$= -\frac{192.42}{62430} = -3.098 \times 10^{-3} \text{ ft.}$$

$$= 0.0372 \text{ in.} \downarrow$$

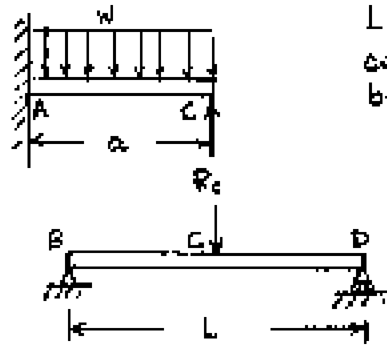
PROBLEM 9.166

9.166 For the loading shown, knowing that beams AC and BD have the same flexural rigidity, determine the reaction at B.



SOLUTION

Consider the two beams shown below.



Let R_c be the contact force between beams AC and BCD.

Applying Cases 1 and 2 of Appendix D to cantilever beam AC

$$y_c = \frac{R_c a^3}{3EI} - \frac{wa^4}{8EI}$$

Applying Case 4 of Appendix D to simply supported beam BCD.

$$y_c = -\frac{R_c L^3}{48EI}$$

Equating expressions for y_c

$$\frac{R_c a^3}{3EI} - \frac{wa^4}{8EI} = -\frac{R_c L^3}{48EI}$$

$$(16a^3 + L^3)R_c = 6wa^4$$

$$R_c = \frac{6wa}{16 + L^3/a^3}$$

Data: $w = 50$ lb/in, $a = 25$ in, $L = 20 + 20 = 40$ in.

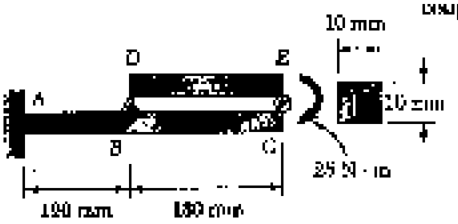
$$R_c = \frac{(6)(50)(25)}{16 + (40/25)^3} = 373.21 \text{ lb}$$

Using beam BCD as a free body

$\circlearrowleft \Sigma M_D = 0 \qquad -R_B L + R_c \frac{L}{2} = 0 \qquad R_B = \frac{1}{2}R_c = 186.6 \text{ lb} \uparrow$ ◄

PROBLEM 9.167

9.167 Beam *DE* rests on the cantilever beam *AC* as shown. Knowing that a square rod of side 10 mm is used for each beam, determine the deflection at end *C* if the 25-N·m couple is applied (*a*) to end *E* of beam *DE*, (*b*) to end *C* of beam *AC*. Use *E* = 200 GPa.

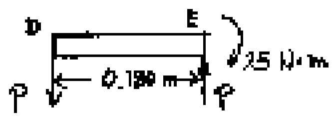


SOLUTION

$E = 200 \times 10^9$ Pa

$I = \frac{1}{12}(10)(10)^3 = 833.33 \text{ mm}^4 = 833.33 \times 10^{-12} \text{ m}^4$

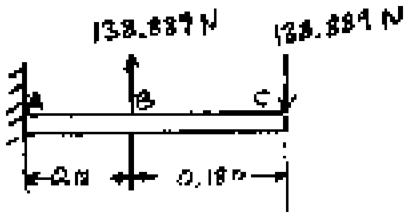
$EI = 166.667 \text{ N} \cdot \text{m}^2$



(a) Couple applied to beam DE

Free body DE $\circlearrowleft \Sigma M = 0$

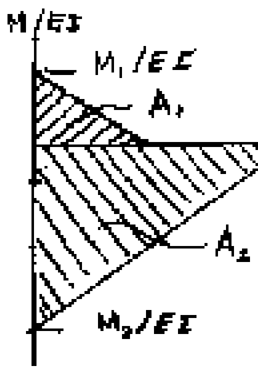
$.180\, P - 25 = 0 \qquad P = 138.889$ N



For beam ABC draw the $\frac{M}{EI}$ diagram by parts.

$$\frac{M_1}{EI} = \frac{(138.889)(0.12)}{166.667} = 100 \times 10^{-3} \text{ m}^{-1}$$

$$\frac{M_2}{EI} = -\frac{(138.889)(0.30)}{166.667} = -250 \times 10^{-3} \text{ m}^{-1}$$



$A_1 = \frac{1}{2}(100 \times 10^{-3})(0.12) = 6 \times 10^{-3}$

$A_2 = \frac{1}{2}(-250 \times 10^{-3})(0.30) = 37.5 \times 10^{-3}$

$y_A = 0 \qquad \theta_A = 0$

Place reference tangent at A

$y_C = y_A + L\theta_A + t_{C/A}$

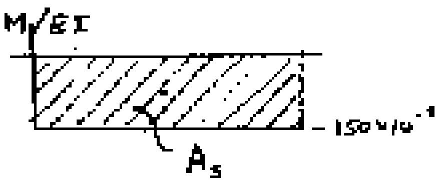
$= 0 + 0 + A_1(0.180 + 0.080) + A_2(0.200)$

$= -5.94 \times 10^{-3}$ m $= 5.94$ mm $\downarrow$ ◄

(b) Couple applied to beam AC

Draw $\frac{M}{EI}$ diagram

$$\frac{M}{EI} = \frac{25}{166.667} = 150 \times 10^{-3} \text{ m}^{-1}$$



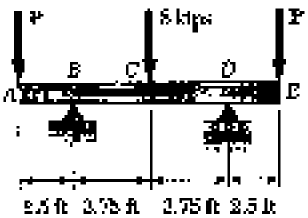
$A_3 = (-150 \times 10^{-3})(0.30) = -45 \times 10^{-3}$

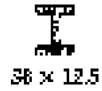
$y_C = t_{C/A} = A_3(0.15) = -6.75 \times 10^{-3}$ m

$= 6.75$ mm $\downarrow$ ◄

PROBLEM 9.168

9.168 For the beam and loading shown, determine the value of P for which the deflection is zero at end A of the beam. Use $E = 29 \times 10^6$ psi.





SOLUTION

Symmetric beam and loading. $\theta_C = 0$

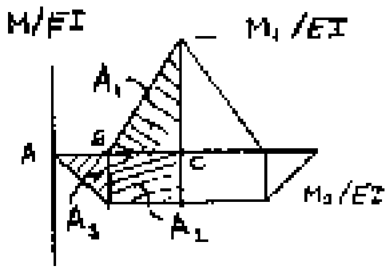
Place reference tangent at C.

Draw $\frac{M}{EI}$ diagram by parts.

Assume EI in kip $\cdot$ ft^2

$M_1 = (4)(3.75) = 15$ kip$\cdot$ft

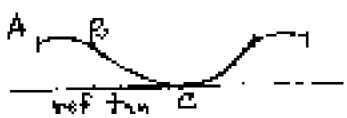
$M_2 = -(P)(2.5) = -2.5P$ kip$\cdot$ft.



$$A_1 = \frac{1}{2}\left(\frac{15}{EI}\right)(3.75) = \frac{28.125}{EI}$$

$$A_2 = -\frac{2.5P}{EI}(3.75) = -\frac{9.375P}{EI}$$

$$A_3 = \frac{1}{2}\left(-\frac{2.5P}{EI}\right)(2.5) = -\frac{3.125P}{EI}$$



$$y_A = y_C + t_{A/C}$$

$$y_B = y_C + t_{B/C}$$

$$y_A - y_B = t_{A/C} - t_{B/C} = 0$$

$$A_1(2.5 + 2.5) + A_2(2.5 + 1.875) + A_3\left(\tfrac{2}{3} \cdot 2.5\right)$$
$$- A_1(2.5) \qquad - A_2(1.875) \qquad = 0$$

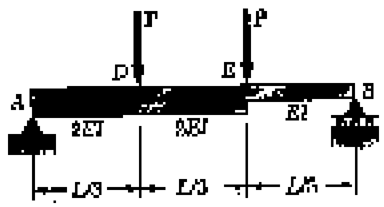
$$A_1(2.5) + A_2(2.5) + A_3(1.66667) = 0$$

$$\frac{70.3125}{EI} - \frac{23.4375P}{EI} - \frac{5.208333P}{EI} = 0$$

$$P = 2.45 \text{ kips}$$

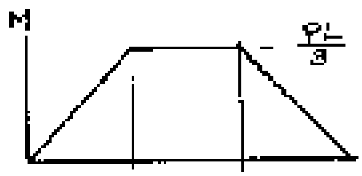
PROBLEM 9.169

9.169 For the beam and loading shown, determine the deflection (*a*) at point *D*, (*b*) at point *E*.



SOLUTION

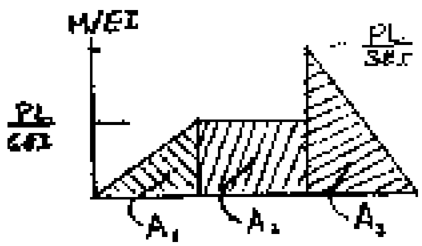
Draw bending moment and $\frac{M}{EI}$ diagrams.



$$A_1 = \frac{1}{2}\left(\frac{PL}{6EI}\right)\left(\frac{L}{3}\right) = \frac{1}{36}\frac{PL^2}{EI}$$

$$A_2 = \left(\frac{PL}{6EI}\right)\left(\frac{L}{3}\right) = \frac{1}{18}\frac{PL^2}{EI}$$

$$A_3 = \frac{1}{2}\left(\frac{PL}{3EI}\right)\left(\frac{L}{3}\right) = \frac{1}{18}\frac{PL^2}{EI}$$



Place reference tangent at A

$$t_{B/A} = A_1\left(\frac{2}{3}L + \frac{1}{3}\cdot\frac{1}{3}L\right) + A_2\left(\frac{L}{2}\right) + A_3\left(\frac{2}{3}\cdot\frac{L}{3}\right)$$

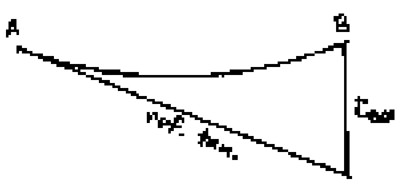
$$= \frac{7}{324}\frac{PL^3}{EI} + \frac{1}{36}\frac{PL^3}{EI} + \frac{1}{81}\frac{PL^3}{EI} = \frac{5}{81}\frac{PL^3}{EI}$$

$$t_{D/A} = A_1\left(\frac{1}{3}\cdot\frac{L}{3}\right)$$

$$= \frac{1}{324}\frac{PL^3}{EI}$$

$$t_{E/A} = A_1\left(\frac{1}{3}\cdot\frac{L}{3} + \frac{L}{3}\right) + A_2\left(\frac{1}{2}\cdot\frac{L}{3}\right)$$

$$= \frac{1}{81}\frac{PL^3}{EI} + \frac{1}{108}\frac{PL^3}{EI} = \frac{7}{324}\frac{PL^3}{EI}$$



(a) Deflection at D

$$y_D = t_{D/A} - \frac{x_D}{L}t_{B/A} = \frac{1}{324}\frac{PL^3}{EI} - \frac{1}{3}\cdot\frac{5}{81}\frac{PL^3}{EI}$$

$$= -\frac{17}{972}\frac{PL^3}{EI} = -0.01749\frac{PL^3}{EI}$$

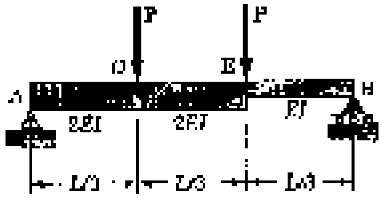
(b) Deflection at E

$$y_E = t_{E/A} - \frac{x_E}{L}t_{B/A} = \frac{7}{324}\frac{PL^3}{EI} - \frac{2}{3}\cdot\frac{5}{81}\frac{PL^3}{EI}$$

$$= -\frac{19}{972}\frac{PL^3}{EI} = -0.01955\frac{PL^3}{EI}$$

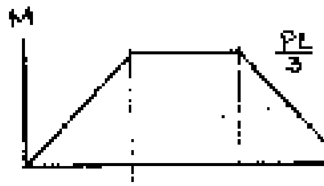
PROBLEM 9.170

9.170 For the beam and loading shown, determine the magnitude and location of the largest downward deflection.



SOLUTION

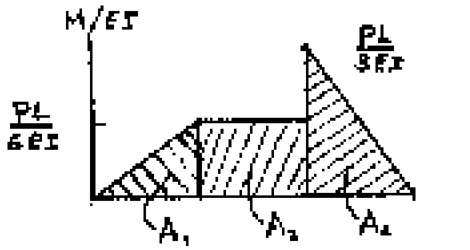
Draw bending moment and $\frac{M}{EI}$ diagrams.



$$A_1 = \frac{1}{2}\left(\frac{PL}{6EI}\right)\left(\frac{L}{3}\right) = \frac{1}{36}\frac{PL^2}{EI}$$

$$A_2 = \left(\frac{PL}{6EI}\right)\left(\frac{L}{3}\right) = \frac{1}{18}\frac{PL^2}{EI}$$

$$A_3 = \frac{1}{2}\left(\frac{PL}{3EI}\right)\left(\frac{L}{3}\right) = \frac{1}{18}\frac{PL^2}{EI}$$



Place reference tangent at A

$$t_{B/A} = A_1\left(\frac{2}{3}L + \frac{1}{3}\cdot\frac{1}{3}L\right) + A_2\left(\frac{L}{2}\right) + A_3\left(\frac{2}{3}\cdot\frac{L}{3}\right)$$

$$= \frac{7}{324}\frac{PL^3}{EI} + \frac{1}{36}\frac{PL^3}{EI} + \frac{1}{81}\frac{PL^3}{EI} = \frac{5}{81}\frac{PL^3}{EI}$$

Slope at A

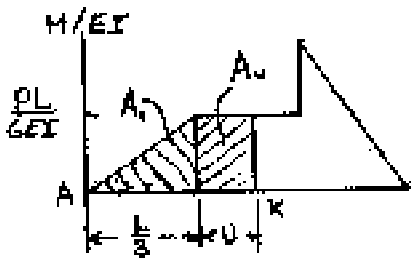
$$\theta_A = -\frac{t_{B/A}}{L} = -\frac{5}{81}\frac{PL^2}{EI}$$

Deflection is maximum at point K.

$$\theta_K = \theta_A + \theta_{K/A} = \theta_A + A_1 + A_4 = 0$$

assuming that point K lies between D and E.

$$A_4 = \frac{1}{6}\left(\frac{PL}{EI}\right)u = \frac{1}{6}\frac{PLu}{EI}$$



$$-\frac{5}{81}\frac{PL^2}{EI} + \frac{1}{36}\frac{PL^2}{EI} + \frac{1}{6}\frac{PLu}{EI} = 0$$

$$u = 6\left(\frac{5}{81} - \frac{1}{36}\right)L = \frac{11}{54}L$$

$$x_K = \frac{L}{3} + u = \left(\frac{1}{3} + \frac{11}{54}\right)L = \frac{29}{54}L$$

$$= 0.537\,L$$

$$A_4 = \frac{11}{324}\frac{PL^2}{EI}$$

$$t_{K/A} = A_1\left(\frac{1}{3}\cdot\frac{L}{3} + u\right) + A_4\left(\frac{u}{2}\right) = \frac{17}{1944}\frac{PL^3}{EI} + \frac{121}{34992}\frac{PL^3}{EI} = \frac{427}{34992}\frac{PL^3}{EI}$$

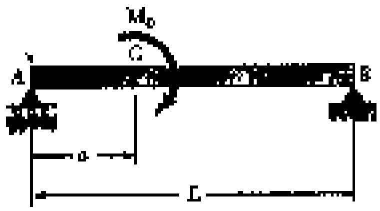
Maximum deflection

$$y_K = t_{K/A} - \frac{x_K}{L}t_{B/A} = \frac{427}{34992}\frac{PL^3}{EI} - \frac{29}{54}\cdot\frac{5}{81}\frac{PL^3}{EI} = -\frac{733}{34992}\frac{PL^3}{EI}$$

$$= -0.02095\frac{PL^3}{EI}$$

PROBLEM 9.171

9.171 For the beam and loading shown, determine (*a*) the value of *a* for which the slope at end *A* is zero, (*b*) the corresponding deflection at point *C*.



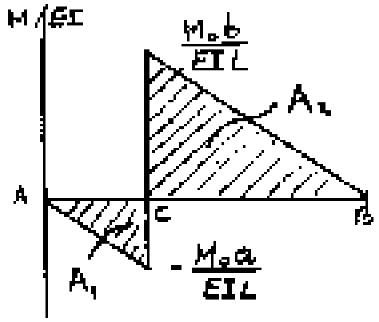
SOLUTION

Let $b = L - a$

Place reference tangent at A.

$$y_B = y_A + L\theta_A + t_{B/A} = 0 + 0 + t_{B/A} = 0$$

Draw $\frac{M}{EI}$ diagram.



$$A_1 = -\frac{1}{2}\frac{M_0 a}{EIL}a = -\frac{1}{2}\frac{M_0 a^2}{EIL}$$

$$A_2 = \frac{1}{2}\frac{M_0 b}{EIL}b = \frac{1}{2}\frac{M_0 b^2}{EIL}$$

$$t_{B/A} = A_1\left(\frac{a}{3} + b\right) + A_2\left(\frac{2}{3}b\right)$$

$$= -\frac{1}{6}\frac{M_0 a^3}{EIL} - \frac{1}{2}\frac{M_0 a^2 b}{EIL} + \frac{1}{3}\frac{M_0 b^3}{EIL} = 0$$

Let $u = \frac{a}{b}$

$$u^3 + 3u^2 - 2 = 0$$

Solving for u: $\quad u = 0.73205$

$$\frac{a}{b} = \frac{a}{L-a} = 0.73205$$

$$a = 0.73205(L-a)$$

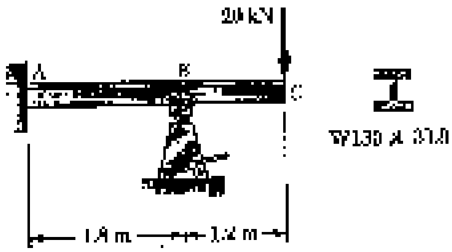
$$a = \frac{0.73205}{1.73205}L = 0.42265\,L$$

$$A_1 = -\frac{1}{2}\frac{M_0 a^2}{EIL} = -0.089316\frac{M_0 L}{EI}$$

$$t_{C/A} = A_1\left(\frac{1}{3}a\right) = -0.01258\frac{M_0 L^2}{EI} \qquad 0.01258\frac{M_0 L^2}{EI}\downarrow$$

PROBLEM 9.172

9.172 A hydraulic jack may be used to raise point B of the cantilever beam ABC. Knowing that after the 20-kN load is applied, point C is to have the same elevation as point A, determine (*a*) how much B should be raised, (*b*) the reaction at B after point B has been raised and the 20-kN load has been applied. Use $E = 200$ GPa.

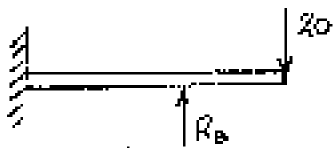


SOLUTION

For W 130 × 23.8 $\quad I_x = 8.80 \times 10^6\ \text{mm}^4 = 8.80 \times 10^{-6}\ \text{m}^4$

$E = 200 \times 10^9\ \text{Pa}$

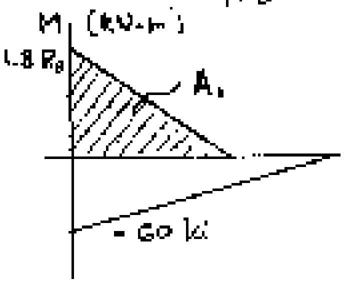
$EI = (200 \times 10^9)(8.80 \times 10^{-6}) = 1.760 \times 10^6\ \text{N}\cdot\text{m}^2 = 1760\ \text{kN}\cdot\text{m}^2$



Let R_B be the jack force in kN.

$A_1 = \frac{1}{2}(1.8 R_B)(1.8) = 1.62\ R_B \quad \text{kN}\cdot\text{m}^2$

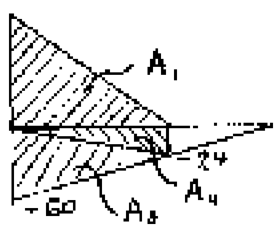
$A_2 = \frac{1}{2}(-60)(3) = -90\ \text{kN}\cdot\text{m}^2$



$EI\, t_{C/A} = (1.2 + 1.2)A_1 + (\frac{2}{3} \cdot 3) A_2$

$= 3.888\ R_B - 180 = 0 \quad \text{kN}\cdot\text{m}^3$

$R_B = \frac{180}{3.888} = 46.296\ \text{kN}$



$A_1 = 75 \quad \text{kN}\cdot\text{m}^2$

$A_3 = \frac{1}{2}(-60)(1.8) = -54 \quad \text{kN}\cdot\text{m}^2$

$A_4 = \frac{1}{2}(-24)(1.8) = -21.6\ \text{kN}\cdot\text{m}^2$

$EI\, t_{B/A} = 1.2 A_1 + 1.2 A_3 + 0.6 A_4$

$= 12.24 \quad \text{kN}\cdot\text{m}^3$

(a) $\quad y_B = t_{B/A} = \dfrac{EI\, t_{B/A}}{EI} = \dfrac{12.24}{1760} = 6.95 \times 10^{-3}\ \text{m}$

$= 6.95 \quad \text{mm}$ ◄

(b) $\quad R_B = 46.3 \quad \text{kN}$ ◄

PROBLEM 9.C1

9.C1 Several concentrated loads can be applied to the cantilever beam AB. Write a computer program to calculate the slope and deflection of beam AB from $x = 0$ to $x = L$, using given increments Δx. Apply this program with increments $\Delta x = 50$ mm to the beam and loading of Probs. 9.79 and 9.80.

SOLUTION

FOR EACH LOAD, ENTER

P_i, c_i

COMPUTE REACTION AT A

FOR $i = 1$ TO NUMBER LOADS

$R_A = R_A + P_i$

$M_A = M_A - P_i c_i$

COMPUTE SLOPE AND DEFLECTION

USE METHOD OF INTEGRATION

STARTING WITH $x = 0$ AND UPDATING

THROUGH INCREMENTS, SUPERPOSE:

(1) DUE TO REACTION AT A:

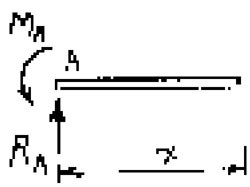
$\theta = (1/EI)(R_A x^2/2.0 + M_A x)$

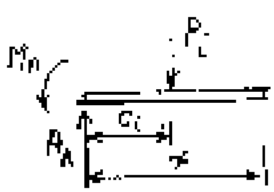
$y = (1/EI)(R_A x^3/6.0 + M_A x^2/2.0)$

(2) DUE TO EACH LOAD WITH $c_i < x$:

$\theta = -(1/EI)(P_i/2.0)(x - c_i)^2$

$y = -(1/EI)(P_i/6.0)(x - c_i)^3$





AT $x = 0$, $y = \frac{dy}{dx} = 0$

$\therefore$ THE CONSTANTS OF INTEGRATION EQUAL ZERO

CONTINUED

PROGRAM OUTPUT

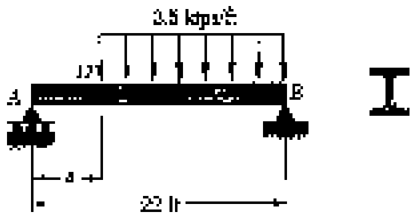
Problem 9.79 and 9.80

At A: Force = 6.0 kN Couple = -6.5 kN·m

x m	Slope radians	Deflection m
.00	.000000	.000000
.05	-.000576	-.000015
.10	-.001126	-.000057
.15	-.001645	.000127
.20	-.002136	-.000221
.25	-.002594	-.000340
.30	-.003024	-.000490
.35	-.003424	-.000642
.40	-.003794	-.000822
.45	-.004135	-.001021
.50	-.004447	-.001239
.55	-.004728	-.001465
.60	-.004980	-.001720
.65	-.005203	-.001962
.70	-.005395	-.002227
.75	-.005558	-.002501
.80	-.005699	-.002783
.85	-.005825	-.003071
.90	-.005936	-.003365
.95	-.006032	-.003664
1.00	-.006114	-.003968
1.05	-.006181	-.004275
1.10	-.006233	-.004586
1.15	-.006270	-.004895
1.20	-.006292	-.005213
1.25	-.006299	-.005527

PROBLEM 9.C2

9.C2 The 22-ft beam AB consists of a W21 × 62 rolled-steel shape and supports a 3.5 kips/ft distributed load as shown. Write a computer program and use it to calculate for values of a from 0 to 22 ft, using 1-ft increments, (*a*) the slope and deflection at D, (*b*) the location and magnitude of the maximum deflection. Use $E = 29 \times 10^6$ psi.



SOLUTION

ENTER LOAD w, LENGTH L, a

COMPUTE REACTION AT A

$$R_A = w\,(L-a)^2/(2.0\,L)$$

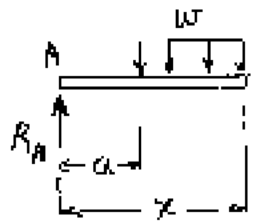
COMPUTE SLOPE AND DEFLECTION AT D

USING SINGULARITY FUNCTIONS:

$$C_1 = -\frac{w}{24L}(L-a)^4 - \frac{1}{6}R_A L^2$$

$$\theta = (1/EI)(R_A a^2/2.0 + C_1)$$

$$y = (1/EI)(R_A a^3/6.0 + C_1 a)$$



$$EI\frac{d^2y}{dx^2} = R_A x - \frac{w}{2}\langle x-a\rangle^2$$

$$EI\frac{dy}{dx} = \frac{1}{2}R_A x^2 - \frac{w}{6}\langle x-a\rangle^3 + C_1$$

$$EI\,y = \frac{1}{6}R_A x^3 - \frac{w}{24}\langle x-a\rangle^4 + C_1 x + C_2$$

FROM BOUNDARY CONDITIONS:

$$C_2 = 0$$

$$C_1 = -\frac{w}{24L}(L-a)^4 - \frac{1}{6}R_A L^2$$

MAXIMUM y AT $\theta = 0$:

$$0 = \frac{1}{2}R_A x^2 - \frac{w}{6}\langle x-a\rangle^3 + C_1$$

IF $x_{max} \le a$

$$\frac{1}{2}R_A x^2 + C_1 = 0$$

$$x_{max} = \sqrt{\frac{-2.0\,C_1}{R_A}}$$

$$y_{max} = \frac{1}{6}R_A x_{max}^3 + C_1 x_{max}$$

COMPUTE LOCATION AND MAGNITUDE OF MAXIMUM DEFLECTION

ASSUME $x < a$:

$$x_{max} = (-2.0\,C_1/R_A)^{\frac{1}{2}}$$

IF $x_{max} < a$, THEN

$$y_{max} = (1/EI)\left(\frac{1}{6}R_A x_{max}^3 + C_1 x_{max}\right)$$

IF $x_{max} > a$, THEN

BEGIN WITH $x = a$

$$\theta = (1/EI)\left(\frac{1}{2}R_A x^2 - \frac{1}{6}(x-a)^3 + C_1\right)$$

INCREASE x BY SMALL AMOUNT UNTIL θ IS APPROXIMATELY 0

$$y_{max} = (1/EI)\left(\frac{1}{6}R_A x^3 - \frac{w}{24}(x-a)^4 + C_1 x\right)$$

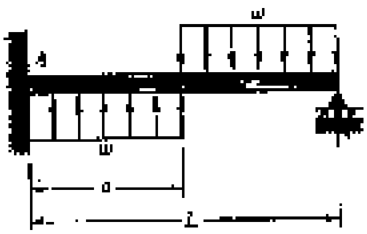
CONTINUED

PROBLEM 9.C2 CONTINUED

PROGRAM OUTPUT

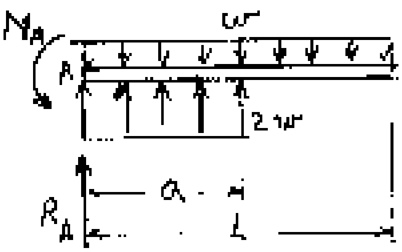
x ft	theta D radians	yD in.	xn ft	yn in.
0.	.00580	.000000	11.000	-.478290
1.	-.00569	-.066756	11.008	-.475922
2.	-.00539	-.133047	11.030	-.468066
3.	-.00494	-.189440	11.068	-.457231
4.	-.00439	-.235551	11.121	-.441245
5.	-.00378	-.269927	11.189	-.421192
6.	-.00314	-.291944	11.272	-.397443
7.	-.00250	-.301695	11.370	-.370441
8.	-.00188	-.299889	11.481	-.340895
9.	-.00131	-.287738	11.606	-.308795
10.	-.00080	-.266655	11.742	-.275364
11.	-.00036	-.239145	11.885	-.241090
12.	-.00001	-.206699	12.029	-.206700
13.	.00025	-.171604	12.159	-.172954
14.	.00043	-.136340	12.275	-.140603
15.	.00052	-.102374	12.376	-.110339
16.	.00054	.071846	12.463	-.082792
17.	.00049	-.045069	12.537	-.058515
18.	.00039	-.026001	12.596	-.037987
19.	.00027	-.012036	12.643	-.021604
20.	.00014	-.003696	12.675	-.009677
21.	.00004	-.000510	12.695	-.002401
22.	.00000	.000000	12.702	.000000

PROBLEM 9.C3



9.C3 The cantilever beam AB carries the distributed loads shown. Write a computer program to calculate the slope and deflection of beam AB from $x = 0$ to $x = L$ using given increments Δx. Apply this program with increments $\Delta x = 100$ mm, assuming that $L = 2.4$ m, $w = 36$ kN/m, and (*a*) $a = 0.6$ m, (*b*) $a = 1.2$ m, (*c*) $a = 1.8$ m. Use $E = 200$ GPa.

SOLUTION



ENTER w, a, L

COMPUTE REACTION AT A

$$R_A = wL - 2.0\,w\,a$$

$$M_A = \frac{1}{2} w L^2 - \frac{1}{2} w a^2$$

COMPUTE SLOPE AND DEFLECTION

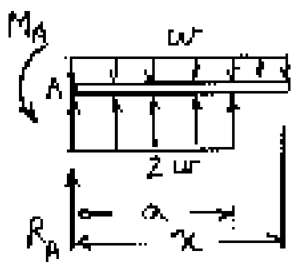
USE EQUATION OF ELASTIC CURVE

STARTING WITH $x = 0$ AND UPDATING THROUGH INCREMENTS, SUPERPOSE:

(1) DUE TO REACTIONS AT A

$$\theta = (1/EI)\left(\frac{1}{2} R_A x^2 + M_A x\right)$$

$$y = (1/EI)\left(\frac{1}{6} R_A x^3 + \frac{1}{2} M_A x^2\right)$$



AT $x = 0$, $y = \frac{dy}{dx} = 0$

∴ THE CONSTANTS OF INTEGRATION ARE ZERO

(2) DUE TO LOAD w

$$\theta = -(1/EI)\left(\frac{1}{6} w x^3\right)$$

$$y = -(1/EI)\left(\frac{1}{24} w x^4\right)$$

(3) DUE TO LOAD $2w$

IF $x \le a$

$$\theta = (1/EI)\left(\frac{1}{3} w x^3\right)$$

$$y = (1/EI)\left(\frac{1}{12} w x^4\right)$$

IF $x > a$

$$\theta = (1/EI)\left(\frac{1}{3} w x^3 - \frac{1}{3} w (x-a)^3\right)$$

$$y = (1/EI)\left(\frac{1}{12} w x^4 - \frac{1}{12} w (x-a)^4\right)$$

CONTINUED

PROBLEM 9.C3 CONTINUED

PROGRAM OUTPUT

Problem 9.C3 (a) a = 0.6 m

At A: Force = 63.2 kN Couple = -90.7 kN-m

x m	slope radians	deflection m
.00	.000000	.000000
.10	.000905	-.000046
.20	-.001762	-.000179
.30	-.002567	.000396
.40	-.003319	-.000691
.50	-.004209	-.001058
.60	-.004636	-.001491
.70	-.005202	-.001982
.80	-.005733	-.002529
.90	-.006145	-.003122
1.00	-.006533	-.003756
1.10	-.006869	-.004427
1.20	-.007156	-.005128
1.30	.007399	-.005856
1.40	-.007602	-.006607
1.50	-.007769	-.007378
1.60	-.007902	-.008163
1.70	-.008006	-.008955
1.80	-.008053	-.009760
1.90	-.008139	-.010571
2.00	-.008177	-.011387
2.10	-.008193	-.012206
2.20	-.008211	-.013027
2.30	-.008215	-.013848
2.40	-.008216	-.014669

Problem 9.C3 (b) a = 1.2 m

At A: Force = 0.0 kN Couple = -51.2 kN-m

x m	slope radians	deflection m
.00	.000000	.000000
.10	-.000529	-.000026
.20	-.001055	-.000106
.30	-.001574	-.000237
.40	-.002081	-.000420
.50	-.002574	-.000653
.60	-.003048	-.000934
.70	-.003500	.001262
.80	-.003926	-.001633
.90	-.004323	-.002046
1.00	-.004687	-.002497
1.10	-.005014	-.002982
1.20	-.005301	-.003498
1.30	-.005544	-.004041
1.40	-.005747	-.004606
1.50	-.005913	-.005189
1.60	-.006047	-.005787
1.70	-.006150	-.006398
1.80	-.006228	-.007017
1.90	-.006284	-.007642
2.00	-.006321	-.008273
2.10	-.006344	-.008906
2.20	-.006356	-.009541
2.30	-.006360	-.010177
2.40	-.006361	-.010813

CONTINUED

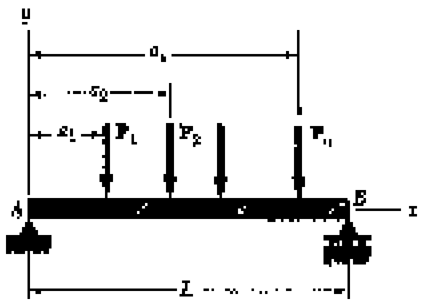
PROBLEM 9.C3 PROGRAM OUTPUTS CONTINUED

Problem 9.C3 (c) a = 1.8 m

At A: Force = -43.2 kN Couple = 13.0 kN·m

x m	slope radians	deflection m
.00	.000000	.000000
.10	.000117	.000006
.20	.000192	.000021
.30	.000215	.000041
.40	.000216	.000063
.50	.000137	.000083
.60	.000093	.000099
.70	.000056	.000109
.80	-.000019	.000113
.90	.000149	.000101
1.00	-.000270	.000085
1.10	-.000390	.000046
1.20	-.000530	.000000
1.30	-.000662	-.000060
1.40	-.000740	-.000132
1.50	-.000911	-.000217
1.60	-.001021	-.000314
1.70	-.001116	-.000421
1.80	-.001193	-.000537
1.90	-.001248	-.000659
2.00	-.001296	-.000786
2.10	-.001309	-.000916
2.20	-.001320	-.001047
2.30	-.001325	-.001179
2.40	-.001325	-.001312

PROBLEM 9.C4



9.C4 The simply supported beam AB is of constant flexural rigidity EI and carries several concentrated loads as shown. Using the *Method of Integration*, write a computer program to calculate the slope and deflection at points along the beam from $x = 0$ to $x = L$ using given increments Δx. Apply this program to the beam and loading of (a) Prob. 9.14 with $\Delta x = 0.25$ m, (b) Prob. 9.15 with $\Delta x = 0.05$ m, (c) Prob. 9.132 with $\Delta x = 0.25$ m.

SOLUTION

FOR EACH LOAD, ENTER P_i, a_i

COMPUTE REACTION AT A

FOR $i = 1$ TO NUMBER LOADS:

$M_A = M_A + P_i a_i$

LOAD = LOAD + P_i

THEN:

$R_B = M_A / L$

$R_A = \text{LOAD} - R_B$

COMPUTE SLOPE AND DEFLECTION

STARTING WITH $x = 0$ AND UPDATING THROUGH INCREMENTS, SUPERPOSE:

(1) DUE TO REACTION AT A

$\theta = (1/EI)(\frac{1}{2} R_A x^2)$

$y = (1/EI)(\frac{1}{6} R_A x^3)$

(2) DUE TO LOADS - CONSTANT PART

$\text{CONST}_1 = -\frac{1}{6} R_A L^2$

FOR i TO NUMBER LOADS

$\text{CONST}_2 = \frac{1}{6L} P_i (L - a_i)^3 + \text{CONST}_2$

THEN, TOTAL CONTRIBUTION FOR CONSTANT

$\text{CONST} = (1/EI)(\text{CONST}_1 + \text{CONST}_2)$

(3) DUE TO LOADS — REMAINING PART

IF $x \le a_i$

$\theta = (1/EI)(\frac{1}{2\cdot 0} R_A x^2)$

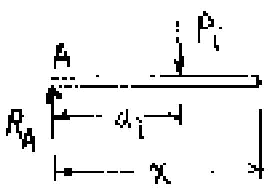
$y = (1/EI)(\frac{1}{6\cdot 0} R_A x^3)$

IF $x > a_i$

$\theta = (1/EI)(\frac{1}{2\cdot 0} R_A x^2 - \frac{1}{2\cdot 0} P_i (x - a_i)^2)$

$y = (1/EI)(\frac{1}{6\cdot 0} R_A x^3 - \frac{1}{6\cdot 0} P_i (x - a_i)^3)$

FOR LOAD P_i:



FOR $x < a_i$

$$EI \frac{d^2 y}{dx^2} = R_A x$$

$$EI \frac{dy}{dx} = \frac{1}{2} R_A x^2 + C_1$$

$$EI\, y = \frac{1}{6} R_A x^3 + C_1 x + C_2$$

FOR $x > a_i$

$$EI \frac{d^2 y}{dx^2} = R_A x - P_i (x - a_i)$$

$$EI \frac{dy}{dx} = \frac{1}{2} R_A x^2 - \frac{1}{2} P_i (x - a_i)^2 + C_3$$

$$EI\, y = \frac{1}{6} R_A x^3 - \frac{1}{6} (x - a_i)^3 + C_3 x + C_4$$

FROM BOUNDARY CONDITIONS

$C_2 = C_4 = 0$

$C_1 = C_3 = \frac{P_i}{6L} (L - a_i)^3 - \frac{1}{6} R_A L^2$

NOTE: R_A FOR LOAD P_i

CONTINUED

PROBLEM 9.C4 CONTINUED

PROGRAM OUTPUT

Problem 9.14

x m	theta rad*10**3	y mm
.000	-6.058	.000
.250	-5.031	-1.496
.500	-5.156	-2.878
.750	-4.014	-4.033
1.000	-2.423	-4.847
1.250	-.719	-5.235
1.500	.757	-5.226
1.750	2.007	-4.875
2.000	3.029	-4.241
2.250	3.824	-3.379
2.500	4.392	-2.348
2.750	4.730	-1.222
3.000	4.847	.000

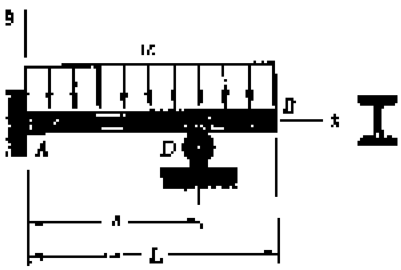
Problem 9.132

x m	theta rad*10**3	y mm
.000	-8.703	.000
.250	-8.615	-2.168
.500	-8.351	-4.293
.750	-7.911	-6.329
1.000	-7.296	-8.234
1.250	-6.505	-9.962
1.500	-5.538	-11.472
1.750	-4.483	-12.724
2.000	3.428	-13.713
2.250	-2.373	-14.439
2.500	-1.319	14.900
2.750	-.264	-15.098
3.000	.791	-15.032
3.250	1.802	-14.706
3.500	2.723	-14.138
3.750	3.560	-13.350
4.000	4.307	-12.365
4.250	4.967	-11.204
4.500	5.508	-9.883
4.750	6.021	-8.442
5.000	6.417	-6.886
5.250	6.725	-5.241
5.500	6.944	-3.531
5.750	7.076	-1.776
6.000	7.120	.000

Problem 9.15

x m	theta rad*10**3	y mm
.000	-2.490	.000
.050	-2.485	.124
.100	-2.471	-.248
.150	-2.448	.371
.200	-2.416	-.493
.250	-2.375	-.613
.300	2.326	-.732
.350	-2.265	-.845
.400	-2.197	-.957
.450	-2.119	-1.065
.500	-2.032	-1.168
.550	-1.936	-1.268
.600	-1.831	-1.362
.650	-1.716	-1.451
.700	-1.593	-1.533
.750	-1.460	-1.610
.800	-1.318	-1.679
.850	-1.172	-1.741
.900	-1.025	-1.796
.950	-.879	-1.844
1.000	.732	-1.884
1.050	-.586	-1.917
1.100	-.439	1.943
1.150	-.293	-1.961
1.200	-.146	-1.972
1.250	.000	-1.976
1.300	.146	-1.972
1.350	.293	-1.961
1.400	.439	-1.943
1.450	.586	-1.917
1.500	.732	-1.884
1.550	.879	-1.844
1.600	1.025	-1.796
1.650	1.172	-1.741
1.700	1.318	-1.679
1.750	1.460	-1.610
1.800	1.593	-1.533
1.850	1.716	-1.451
1.900	1.831	1.362
1.950	1.936	-1.268
2.000	2.032	-1.168
2.050	2.119	-1.065
2.100	2.197	-.957
2.150	2.265	-.845
2.200	2.325	-.730
2.250	2.375	-.613
2.300	2.416	-.433
2.350	2.448	-.371
2.400	2.471	-.249
2.450	2.485	-.124
2.500	2.490	.000

PROBLEM 9.C5



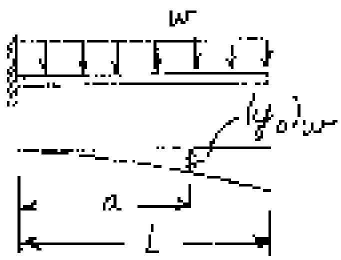
9.C5 The supports of beam *AB* consist of a fixed support at end *A* and a roller located at point *D*. Write a computer program to calculate the slope and deflection at the free end of the beam for values of *a* from 0 to *L* using given increments Δa. Apply this program to calculate the slope and deflection at point *B* for each of the following cases:

	L	Δa	w	E	Shape
(a)	12 ft	0.5 ft	1.6 kips/ft	29×10^6 psi	W16 × 57
(b)	3 m	0.2 m	18 kN/m	200 GPa	W460 × 113

SOLUTION

BEAM IS INDETERMINATE

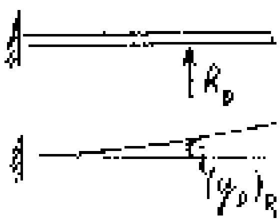
USE APPENDIX D AND SUPERPOSITION



DETERMINE REACTION AT D

DUE TO DISTRIBUTED LOAD

$$(y_D)_w = -\frac{w}{24EI}\left(a^4 - 4La^3 + 6L^2a^2\right)$$



DUE TO REDUNDANT LOAD:

$$(y_D)_R = \frac{R_D L^3}{3EI}$$

REDUNDANT REACTION:

SINCE $(y_D)_w + (y_D)_R = 0$:

$$R_D = \frac{3EI}{L^3}(y_D)_w$$

COMPUTE SLOPE AND DEFLECTION AT B

SUPERPOSE:

DUE TO DISTRIBUTED LOAD:

$$\theta_B = -\frac{wL^3}{6EI}$$

$$y_B = -\frac{wL^4}{8EI}$$

DUE TO R_D:

$$\theta_B = \frac{Pa^2}{2EI}$$

$$y_B = \frac{Pa^3}{3EI} + (L-a)\frac{Pa^2}{2EI}$$

$$\theta_B = \theta_D$$

$$y_B = y_D + (L-a)\theta_D$$

CONTINUED

PROBLEM 9.C5 CONTINUED

PROGRAM OUTPUT

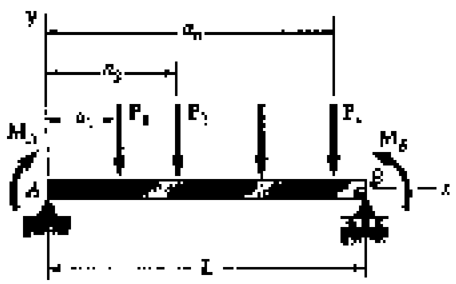
Problem 9.C5 (a)

a ft	theta B rad*10^-3	y at B in.
.0	-3.019	-.3260
.5	-2.743	-.2869
1.0	-2.483	-.2511
1.5	-2.238	-.2183
2.0	-2.007	-.1885
2.5	-1.790	-.1614
3.0	-1.586	-.1369
3.5	-1.395	-.1149
4.0	-1.216	-.0953
4.5	-1.049	-.0778
5.0	-.893	-.0624
5.5	-.746	-.0490
6.0	-.613	-.0374
6.5	-.486	-.0274
7.0	-.375	-.0191
7.5	-.266	-.0122
8.0	-.168	-.0067
8.5	-.077	-.0025
9.0	.006	.0006
9.5	.082	.0027
10.0	.152	.0037
10.5	.216	.0039
11.0	.274	.0035
11.5	.328	.0020
12.0	.377	.0000

Problem 9.C5 (b)

a m	theta B rad*10^-3	y at B mm.
.0	-.728	-1.6389
.2	-.624	-1.3324
.4	-.529	-1.0665
.6	-.442	-.8374
.8	-.364	-.6426
1.0	-.293	-.4789
1.2	-.230	-.3435
1.4	-.174	-.2338
1.6	-.124	-.1472
1.8	-.079	-.0813
2.0	-.040	-.0337
2.2	-.006	-.0024
2.4	.023	.0149
2.6	.049	.0198
2.8	.072	.0143

PROBLEM 9.C6

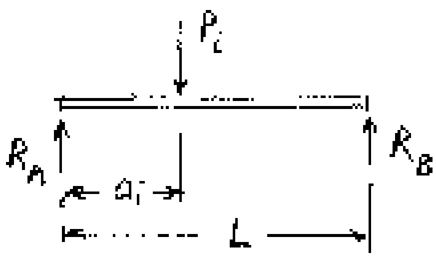


9.C6 For the beam and loading shown, use the *Moment-Area Method* to write a computer program to calculate the slope and deflection at points along the beam from $x = 0$ to $x = L$ using given increments Δx. Apply this program to calculate the slope and deflection at each concentrated load for the beam of (*a*) Prob. 9.76 with $\Delta x = 0.5$ m, (*b*) Prob. 9.116 with $\Delta x = 3$ in., (*c*) Prob. 9.119 with $\Delta x = 0.5$ ft.

SOLUTION

ENTER M_A AND M_B

FOR EACH LOAD ENTER P_i AND a_i



DETERMINE REACTION AT A

DUE TO MOMENTS AT ENDS:

$$(R_A)_1 = -(M_A - M_B)/L$$

DUE TO LOADS P_i:

FOR $i = 1$ TO NUMBER OF LOADS

$$R_B = R_B + P_i\, a_i / L$$

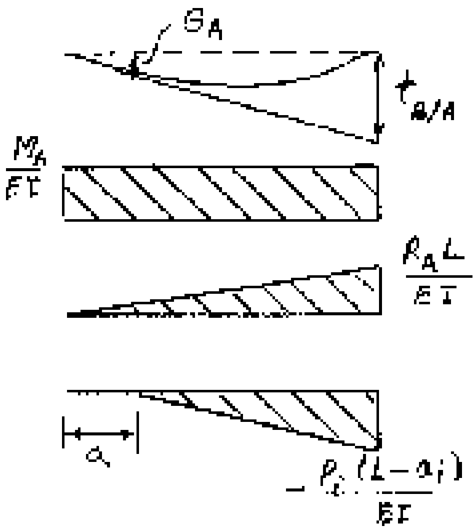
$$LOAD = LOAD + P_i$$

$$(R_A)_2 = LOAD - R_B$$

$$R_A = (R_A)_1 + (R_A)_2$$

DETERMINE SLOPE AT A

USE SECOND MOMENT-AREA THEOREM TO GET TANGENTIAL DEVIATION AT B



DUE TO M_A:

$$t_{B/A} = M_A L^2/(2.0\, EI)$$

DUE TO R_A:

$$t_{B/A} = R_A L^3/(6.0\, EI)$$

DUE TO LOADS P_i:

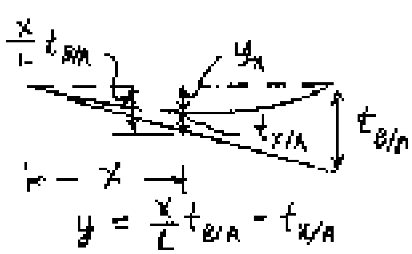
FOR $i = 1$ TO NUMBER OF LOADS

$$t_{B/A} = -P_i\,(L - a_i)^3/(6.0\, EI)$$

SUM $t_{B/A}$:

$$\theta_A = -t_{B/A}/L$$

DETERMINE SLOPE AND DEFLECTIONS



FOR $x = 0$ TO L ; SUPERPOSE:

DUE TO M_A AND R_A:

$$\theta_x = \theta_A + (M_A\, x + R_A\, x^2/2.0)/EI$$

CONTINUED

PROBLEM 9.C6 CONTINUED

$$y_x = \frac{x}{L} t_{B/A} - M_A x^2/(2.0EI) - R_A x^3/(6.0EI)$$

DUE TO LOADS P_i :

DO FOR ALL LOADS WITH $a_i < x$

$$\theta_x = P_i (x - a_i)^2/(2.0\,EI)$$

$$y_x = P_i (x - a_i)^3/(6.0\,EI)$$

PROGRAM OUTPUT

Problem 9.76

x m	theta rad*1000	y at x mm
.000	-.600962	.000000
.500	-1.602564	.574252
1.000	-2.043269	1.509081
1.500	-1.923077	2.524039
2.000	-1.201923	3.338675
2.500	.000000	3.672543
3.000	1.201923	3.338676
3.500	1.923077	2.524039
4.000	2.043269	1.509082
4.500	1.602564	.574253
5.000	.600962	.000000

Problem 9.116

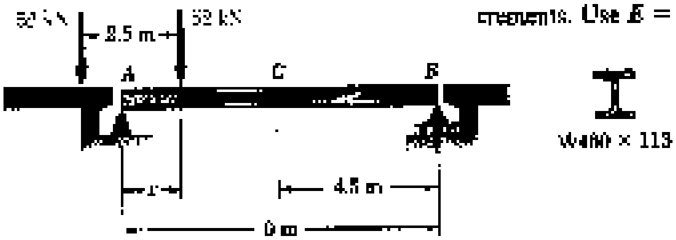
x ft	theta rad*1000	y at x in.
.000	-8.937931	.000000
.250	-8.813793	.026690
.500	-8.441380	.052634
.750	-7.820690	.077096
1.000	-6.951724	.099310
1.250	-5.834483	.118552
1.500	-4.468966	.134069
1.750	-2.979310	.145241
2.000	-1.489655	.151945
2.250	.000000	.154178
2.500	1.489655	.151945
2.750	2.979310	.145241
3.000	4.468966	.134069
3.250	5.834483	.118552
3.500	6.951724	.099310
3.750	7.820690	.077096
4.000	8.441380	.052634
4.250	8.813793	.026690
4.500	8.937931	.000000

Problem 9.119

x ft	theta rad*1000	y at x in.
.000	-2.118621	.000000
.500	-2.222069	.013051
1.000	-2.267586	.026549
1.500	-2.255172	.040146
2.000	-2.184828	.053495
2.500	-2.056552	.066248
3.000	-1.870345	.078058
3.500	-1.626207	.088577
4.000	-1.324138	.097457
4.500	-.993103	.104408
5.000	-.662069	.109374
5.500	-.331034	.112353
6.000	.000000	.113346
6.500	.331034	.112353
7.000	.662069	.109374
7.500	.993103	.104408
8.000	1.324138	.097457
8.500	1.626207	.088577
9.000	1.870345	.078058
9.500	2.056552	.066248
10.000	2.184620	.053495
10.500	2.255172	.040146
11.000	2.267586	.026549
11.500	2.222069	.013051
12.000	2.118621	.000000

PROBLEM 9.C7

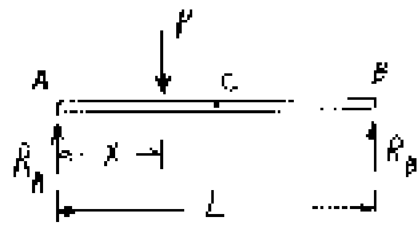
9.C7 Two 52-kN loads are maintained 2.5 m apart as they are moved slowly across beam *AB*. Write a computer program to calculate the deflection at the midpoint *C* of the beam for values of *x* from 0 to 9 m, using 0.5-m increments. Use $E = 200$ GPa.



SOLUTION

ENTER LOAD P, BEAM LENGTH L AND SPACE BETWEEN LOADS D

WILL SOLVE WITH MOMENT-AREA METHOD



DETERMINE DEFLECTION AT C

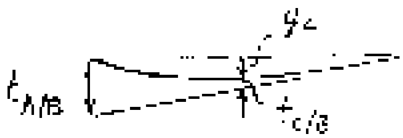
FOR $x = 0$ TO L

IF $0 \le x \le D$:

HAVE ONE LOAD TO LEFT OF C

$R_B = Px/L$

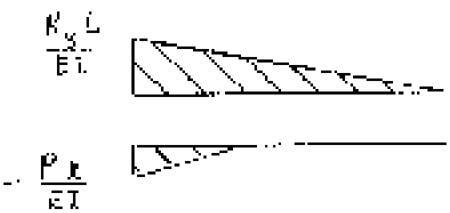
$t_{A/B} = (R_B L^3 - P x^3)/(6.0EI)$



$t_{C/B} = R_B L^3/(48.0EI)$

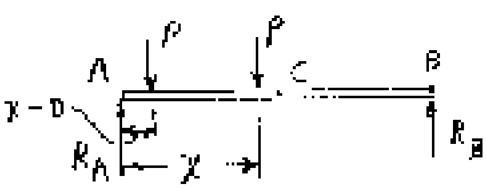
$y_c = \frac{1}{2} t_{A/B} - t_{C/B}$

$y_c = \frac{1}{2} t_{A/B} - t_{C/B}$



IF $D < x \le L/2$

HAVE TWO LOADS TO LEFT OF C



$R_B = Px/L + P(x - D)/L$

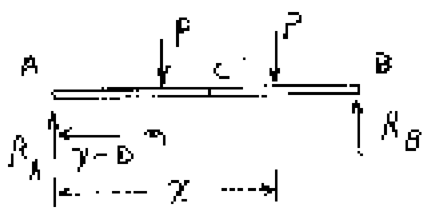
$t_{A/B} = (R_B L^3 - Px^3 - P(x - D)^3)/(6.0EI)$

$t_{C/B} = R_B L^3/(48.0EI)$

$y_c = \frac{1}{2} t_{A/B} - t_{C/B}$

IF $L/2 < x \le (L/2 + D)$

HAVE ONE LOAD TO LEFT OF C AND ONE TO RIGHT OF C OR AT C



CONTINUED

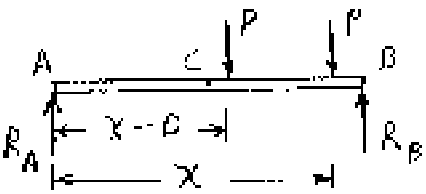
PROBLEM 9.C7 CONTINUED

$$R_B = Px/L + P(x-D)/L$$

$$t_{A/B} = (R_B L^3 - Px^3 - P(x-D)^3)/(6.0\,EI)$$

$$t_{C/B} = (R_B L^3/48.0 - P(x-\tfrac{L}{2})^3/6.0)/EI$$

$$y_C = \tfrac{1}{2}\, t_{A/B} - t_{C/B}$$



IF $(L/2 + D) < x < L$

HAVE BOTH LOADS TO RIGHT OF $\mathcal{C}$

$$R_B = Px/L + P(x-D)/L$$

$$t_{A/B} = (R_B L^3 - Px^3 - P(x-D)^3)/(6.0\,EI)$$

$$t_{C/B} = (R_B L^3/48.0 - P(x-\tfrac{L}{2})^3/6.0 - P(x-D-\tfrac{L}{2})^3/6.0)/EI$$

$$y_C = \tfrac{1}{2}\, t_{A/B} - t_{C/B}$$

<u>PROGRAM OUTPUT</u>

x m	RB kN	ThetaB rad	YC mm
.0	.000	.00000	.00000
.5	2.889	.00315	1.17861
1.0	5.779	.00624	2.32339
1.5	8.667	.00921	3.41551
2.0	11.556	.01200	4.42296
2.5	14.444	.01456	5.30950
3.0	20.222	.01999	7.22872
3.5	26.000	.02499	8.94335
4.0	31.778	.02947	10.39423
4.5	37.556	.03331	11.52503
5.0	43.333	.03639	12.28492
5.5	49.111	.03859	12.66487
6.0	54.889	.03980	12.66407
6.5	60.667	.03989	12.28432
7.0	65.444	.03875	11.52503
7.5	72.222	.03629	10.39409
8.0	78.000	.03235	8.94335
8.5	83.778	.02684	7.22872
9.0	89.556	.01963	5.30950

PROBLEM 9.C8

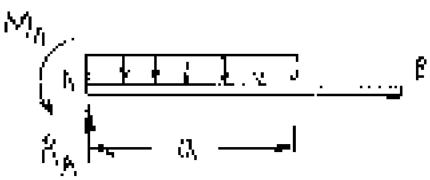
9.C8 A uniformly distributed load w and several concentrated loads P_i may be applied to the cantilever beam AB. Write a computer program to determine the reaction at the roller support and apply this program to the beam and loading of (a) Prob. 9.57a, (b) Prob. 9.58a.

SOLUTION

THE BEAM IS INDETERMINATE

USE EQUATION OF ELASTIC CURVE

ENTER w AND FOR EACH LOAD P_i AND c_i



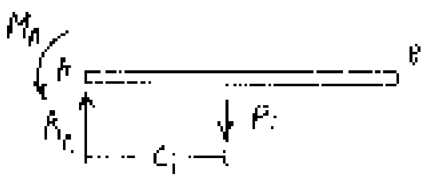
COMPUTE DISPLACEMENT AT B DUE TO LOADS

REACTION AT A:

DUE TO w

$$R_A = w\,a$$

$$M_A = \frac{1}{2}\,w\,a^2$$

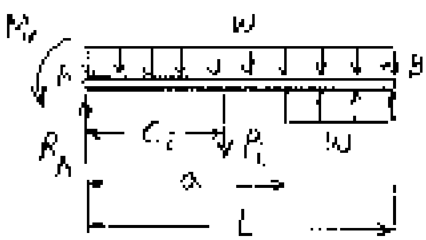


FOR $i = 1$ TO NUMBER LOADS P_i

$$R_A = R_A - P_i$$

$$M_A = M_A - P_i\,c_i$$

FOR DISPLACEMENT AT B, SUPERPOSE:



DUE TO REACTION AT A

$$EI\,y_B = \frac{1}{6}\,R_A\,L^3 + \frac{1}{2}\,M_A\,L^2$$

DUE TO DISTRIBUTED LOADS

$$EI\,y_B = \frac{1}{24}\left(-wL^4 + w\,(L-a)^4\right)$$

DUE TO P_i

FOR $i = 1$ TO NUMBER LOADS

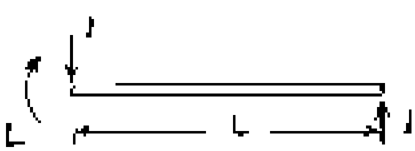
$$EI\,y_B - \frac{1}{6}\,P_i\,(L - c_i)^3$$

AT $x = 0$, $y = \frac{dy}{dx} = 0$

$\therefore$ THE CONSTANTS OF INTEGRATION ARE ZERO

COMPUTE DISPLACEMENT AT B DUE TO UNIT R_B

$$EI\,(y_B)_{UNIT} = \frac{1}{3}\,L^3$$



$$EI\,\frac{d^2y}{dx^2} = -x + L$$

$$EI\,\frac{dy}{dx} = -\frac{1}{2}x^2 + Lx + C_1$$

$$EI\,y = -\frac{1}{6}x^3 + \frac{1}{2}Lx^2 + C_1x + C_2$$

BOUNDARY CONDITIONS GIVE $C_1 = C_2 = 0$

COMPUTE REACTION AT B

FROM $EI\,y_B + R_B\,EI\,(y_B)_{UNIT} = 0$

$$R_B = -y_B/(y_B)_{UNIT}$$

CONTINUED

PROBLEM 9.C8 CONTINUED

PROGRAM OUTPUT

```
Problem 9.57 (a)

     Reaction at Roller Support = 41.2500 kN

Problem 9.58 (a)

     Reaction at Roller Support = 11.5356 kN
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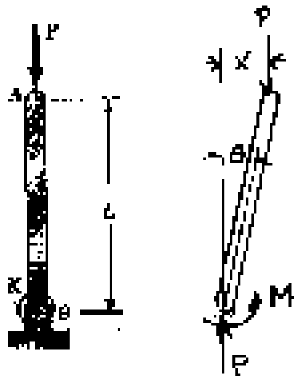
CHAPTER 10

PROBLEM 10.1

10.1 Knowing that the torsional spring at B is of constant K and that the bar AB is rigid, determine the critical load P_{cr}.



SOLUTION

Let θ be the angle change of bar AB.

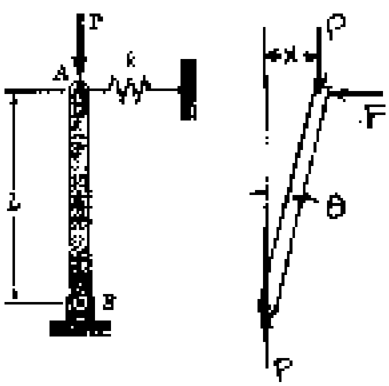
$$M = K\theta \quad , \quad x = L\sin\theta \approx L\theta$$

$$\circlearrowright M_B = 0 \qquad M - Px = 0 \qquad K\theta - PL\theta = 0$$

$$(K - PL)\theta = 0 \qquad P_{cr} = K/L$$

PROBLEM 10.2

10.2 Knowing that the spring at A is of constant k and that the bar AB is rigid, determine the critical load P_{cr}.



SOLUTION

Let θ be the angle change of bar AB.

$$F = kx = kL\sin\theta$$

$$\circlearrowright \Sigma M_B = 0 \qquad FL\cos\theta - Px = 0$$

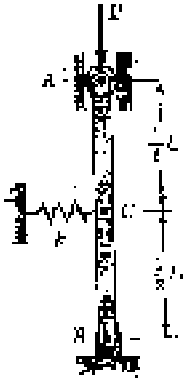
$$kL^2\sin\theta\cos\theta - PL\sin\theta = 0$$

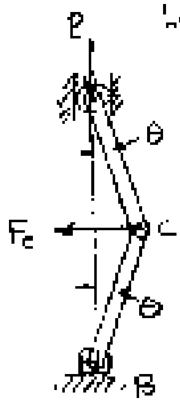
Using $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ $\qquad kL^2\theta - PL\theta = 0$

$$(kL^2 - PL)\theta = 0 \qquad P_{cr} = kL$$

PROBLEM 10.3

10.3 Two rigid bars AC and BC are connected as shown to a spring of constant k. Knowing that the spring can act in either tension or compression, determine the critical load P_{cr} for the system.





SOLUTION

Let x be the lateral deflection of point C

$$x = \tfrac{1}{2}L\sin\theta \qquad F_s = kx = \tfrac{1}{2}kL\sin\theta$$

Joint C: $+\uparrow \Sigma F_y = 0 \quad F_{AC}\cos\theta - F_{CB}\cos\theta = 0$

$$F_{AC} = F_{CB}$$

$$\overset{+}{\rightarrow}\Sigma F_x = 0 \qquad F_{AB}\sin\theta + F_{CB}\sin\theta - F_s = 0$$

$$-2F_{AB}\sin\theta - \tfrac{1}{2}kL\sin\theta = 0$$

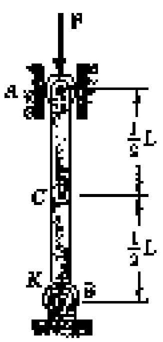
$$-(F_{AB} + \tfrac{1}{4}kL)\sin\theta = 0 \qquad F_{AB} = -\tfrac{1}{4}kL$$

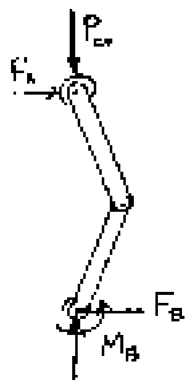
Joint A: $\Sigma F_y = 0 \quad -P - F_{AB}\cos\theta = 0 \qquad P = -F_{AB}\cos\theta = \tfrac{1}{4}kL\cos\theta$

With $\theta \to 0 \qquad P_{cr} = \tfrac{1}{4}kL$

PROBLEM 10.4

10.4 Two rigid bars AC and BC are connected by a pin at C as shown. Knowing that the torsional spring at B is of constant K, determine the critical load P_{cr} for the system.





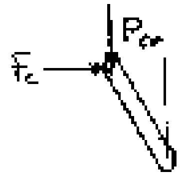
SOLUTION

Let θ be the angle change of each bar.

$$M_B = K\theta$$

$$\circlearrowleft M_B = 0 \qquad K\theta - F_A L = 0 \qquad F_A = \frac{K\theta}{L}$$

Bar AC $\quad \circlearrowleft \Sigma M_c = 0$

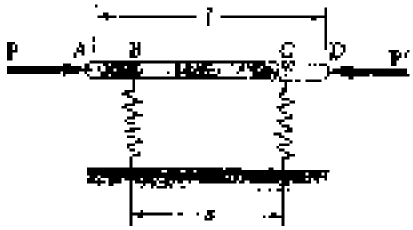


$$P_{cr}\tfrac{1}{2}L\theta - \tfrac{1}{2}L F_A = 0$$

$$P_{cr} = \frac{F_A}{\theta} = \frac{K}{L}$$

PROBLEM 10.5

10.5 The rigid bar *AD* is attached to two springs of constant *k* and is in equilibrium in the position shown. Knowing that the equal and opposite loads **P** and **P'** remain *horizontal*, determine the magnitude P_{cr} of the critical load for the system.



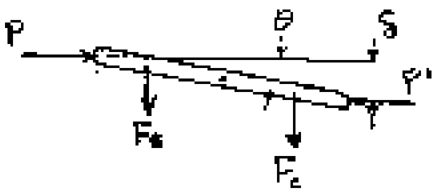
SOLUTION

Let y_B and y_C be the deflections of points B and C, positive upward.

Then $F_B = -k y_B$ $\quad F_C = -k y_C$

$+\uparrow \Sigma F_y = 0 \quad F_B + F_C = 0 \quad F_C = -F_B$

$y_C = -y_B \quad F_B$ and F_C form a couple ↻



Let θ be the angle change: $y_B = -y_C = \frac{1}{2} a \sin\theta$, $\quad \delta = l \sin\theta$

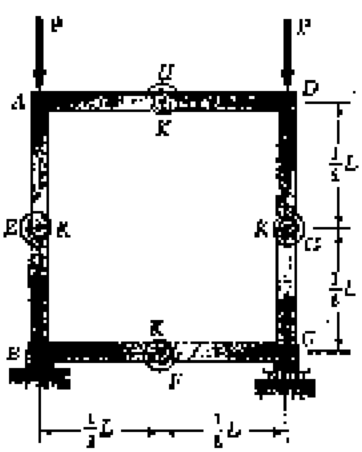
P and P' form a couple ↺ of amount $P\delta$

↻ $\Sigma M = 0$, $k\left(\frac{1}{2} a \sin\theta\right) a \cos\theta - P l \sin\theta = 0 \qquad P = \frac{ka^2}{2l}\cos\theta$

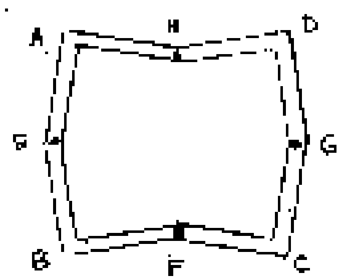
Let $\theta \rightarrow 0 \qquad P_{cr} = \frac{ka^2}{2l}$ ◀

PROBLEM 10.6

10.6 A frame consists of four L-shaped members connected by four torsional springs, each of constant *K*. Knowing that equal loads **P** are applied at points *A* and *D* as shown, determine the critical value P_{cr} of the loads applied to the frame.

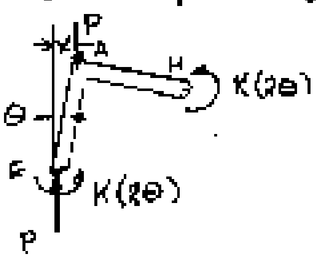


SOLUTION



Let θ be the rotation of each L-shaped member.

Angle change across each torsional spring is 2θ



$x = \frac{1}{2} L \sin\theta \approx \frac{1}{2} L\theta$

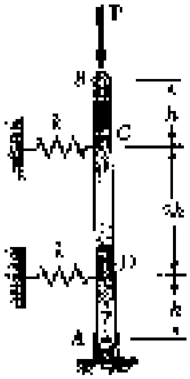
$\Sigma M_F = 0$

$K(2\theta) + K(2\theta) - Px = 0$

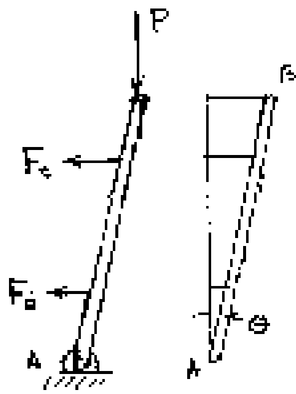
$P_{cr} = \frac{4K\theta}{x} = \frac{8K}{L}$ ◀

PROBLEM 10.7

10.7 The rigid rod AB is attached to a hinge at A and to two springs, each of constant $k = 2.0$ kip/in., that can act in either tension or compression. Knowing that $h = 2.0$ ft, determine the critical load.



SOLUTION



Let θ be the small rotation angle

$$x_D \approx h\theta, \quad x_C \approx 3h\theta, \quad x_B \approx 4h\theta$$

$$F_C = kx_C \approx 3kh\theta$$

$$F_D = kx_D \approx kh\theta$$

$$\circlearrowleft \Sigma M_A = 0 \qquad hF_D + 3hF_C - Px_B = 0$$

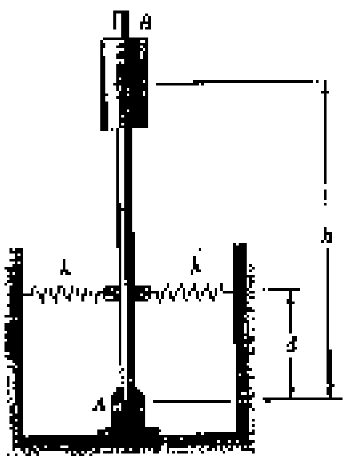
$$kh^2\theta + 9kh^2\theta - 4hP = 0, \quad P = \frac{5}{2}kh$$

Data: $k = 2.0$ kip/in, $h = 2$ ft $= 24$ in

$$P = \frac{5}{2}(2.0)(24) = 120 \text{ kips.}$$

PROBLEM 10.8

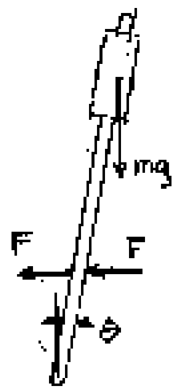
10.8 If $m = 125$ kg, $h = 700$ mm, and the constant of each spring is $k = 2.8$ kN/m, determine the range of values of the distance d for which the equilibrium of the rigid rod AB is stable in the position shown. Each spring can act in either tension or compression.



SOLUTION

$$h = 700 \text{ mm} = 700 \times 10^{-3} \text{ m}$$

Let θ be the small rotation of AB



$$x = d\theta \qquad F = kx = kd\theta$$

$$\circlearrowleft \Sigma M_A = 0 \quad \ldots \quad 2Fd - mgh\theta = 0$$

$$2kd^2\theta - mgh = 0$$

$$d_{cr}^2 = \frac{mgh}{2k}$$

$$d_{cr} = \sqrt{\frac{mgh}{k}} = \sqrt{\frac{(125)(9.81)(700\times10^{-3})}{(2)(2.8\times10^{3})}}$$

$$= 0.392 \text{ m} = 392 \text{ mm}$$

$d > 392$ mm for stability

PROBLEM 10.9

10.9 Determine the critical load of a round wooden dowel that is 48-in. long and has a diameter of (*a*) 0.375 in., (*b*) 0.5 in. Use $E = 1.6 \times 10^6$ psi.

SOLUTION

(a) $c = \frac{1}{2}d = 0.1875$ in $\qquad I = \frac{\pi}{4}c^4 = 970.7 \times 10^{-6}$ in^4

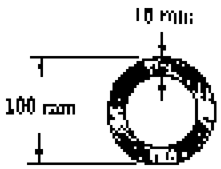
$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(1.6 \times 10^6)(970.7 \times 10^{-6})}{(48)^2} = 6.65 \text{ lb.}$$

(b) $c = \frac{1}{2}d = 0.25$ in. $\qquad I = \frac{\pi}{4}c^4 = 3.068 \times 10^{-3}$ in^4

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(1.6 \times 10^6)(3.068 \times 10^{-3})}{(48)^2} = 21.0 \text{ lb.}$$

PROBLEM 10.10

10.10 Determine the critical load of a steel tube that is 5.0 m long and has a 100-mm outer diameter and a 16 mm wall thickness. Use $E = 200$ GPa.



SOLUTION

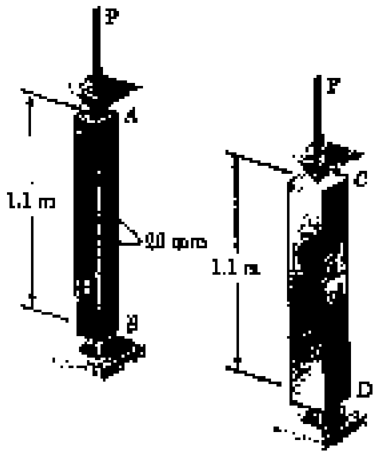
$c_o = \frac{1}{2}d_o = 50$ mm $\qquad c_i = c_o - t = 50 - 16 = 34$ mm

$I = \frac{\pi}{4}(c_o^4 - c_i^4) = 3.859 \times 10^6$ mm^4 $= 3.859 \times 10^{-6}$ m^4

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(200 \times 10^9)(3.859 \times 10^{-6})}{(5.0)^2} = 305 \times 10^3 \text{ N} = 305 \text{ kN}$$

PROBLEM 10.11

10.11 Determine (*a*) the critical load for the brass strut, (*b*) the dimension *d* for which the aluminum strut will have the same critical load, (*c*) the weight of the aluminum strut as a percent of the weight of the brass strut.



SOLUTION

(a) Brass strut $I = \frac{1}{12}(20)(20)^3 = 13.333 \times 10^3 \text{ mm}^4$
$= 13.333 \times 10^{-9} \text{ m}^4$

$$P_{cr} = \frac{\pi^2 E_b I_b}{L^2} = \frac{\pi^2 (120 \times 10^9)(13.333 \times 10^{-9})}{(1.1)^2}$$

$$= 13.06 \times 10^3 \text{ N} = 13.06 \text{ kN}$$

(b) Aluminum strut

$$P_{cr} = \frac{\pi^2 E_a I_a}{L^2} = \frac{\pi^2 E_a (d^4/12)}{L^2}$$

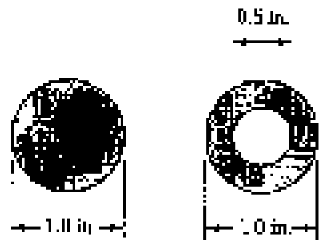
$$d^4 = \frac{12 P_{cr} L^2}{\pi^2 E_a} = \frac{(12)(13.06 \times 10^3)(1.1)^2}{\pi^2 (70 \times 10^9)} = 274.3 \times 10^{-9} \text{ m}^4$$

$$d = 22.9 \times 10^{-3} \text{ m} = 22.9 \text{ mm}$$

(c) $$\frac{m_a}{m_b} = \frac{\gamma_a L d^2}{\gamma_b L d_b^2} = \left(\frac{\gamma_a}{\gamma_b}\right)\left(\frac{d}{d_b}\right)^2 = \left(\frac{2710}{8740}\right)\left(\frac{22.9}{20}\right)^2 = 0.406 = 40.6\%$$

PROBLEM 10.12

10.12 A compression member of 20 in. effective length consists of a solid 1.0-in.-diameter aluminum rod. In order to reduce the weight of the member by 25%, the solid rod is replaced by a hollow rod of the cross section shown. Determine (*a*) the percent reduction in the critical load, (*b*) the value of the critical load for the hollow rod. Use $E = 10.6 \times 10^6$ psi.



SOLUTION

Solid $\quad A_S = \frac{\pi}{4} d_o^2 \qquad I_S = \frac{\pi}{4}\left(\frac{d_o}{2}\right)^4 = \frac{\pi}{64} d_o^4$

Hollow: $\quad A_H = \frac{\pi}{4}(d_o^2 - d_i^2) = \frac{3}{4} A_S = \frac{3}{4}\cdot\frac{\pi}{4} d_o^2$

$$d_i^2 = \frac{1}{4} d_o^2 \qquad d_i = \frac{1}{2} d_o = 0.5 \text{ in.}$$

Solid rod: $\quad I_S = \frac{\pi}{64}(1.0)^4 = 0.049087 \text{ in}^4$

$$P_{cr} = \frac{\pi^2 E I_S}{L^2} = \frac{\pi^2 (10.6\times10^6)(0.049087)}{(20)^2} = 12.839 \times 10^3 \text{ lb.}$$

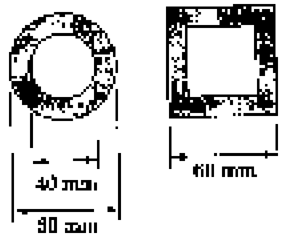
Hollow rod: $\quad I_H = \frac{\pi}{64}(d_o^4 - d_i^4) = \frac{\pi}{64}\left[(1)^4 - \left(\frac{1}{2}\right)^4\right] = 0.046019 \text{ in}^4$

(b) $\quad P_{cr} = \frac{\pi^2 E I_H}{L^2} = \frac{\pi^2 (10.6\times10^6)(0.046019)}{(20)^2} = 12.036 \times 10^3 \text{ lb.} = 12.04 \text{ kips}$ ◄

(a) $\quad \frac{P_S - P_H}{P_S} = \frac{12.839\times10^3 - 12.036\times10^3}{12.839\times10^3} = 0.0625 = 6.25\%$ ◄

PROBLEM 10.13

10.13 Two brass rods used as compression members, each of 3-m effective length, have the cross sections shown. (*a*) Determine the wall thickness of the hollow square rod for which the rods have the same cross-sectional area. (*b*) Using $E = 105$ GPa, determine the critical load of each rod.



SOLUTION

(a) Same area $\quad \frac{\pi}{4}(d_o^2 - d_i^2) = b_o^2 - b_i^2$

$$b_i^2 = b_o^2 - \frac{\pi}{4}(d_o^2 - d_i^2)$$
$$= 60^2 - \frac{\pi}{4}(60^2 - 40^2) = 2.0292 \text{ mm}^2$$

$b_i = 45.047$ mm $\qquad t = \frac{1}{2}(b_o - b_i) = 7.48$ mm

(b) Circular: $\quad I = \frac{\pi}{64}(d_o^4 - d_i^4) = 510.51 \times 10^3 \text{ mm}^4 = 510.51 \times 10^{-9} \text{ m}^4$

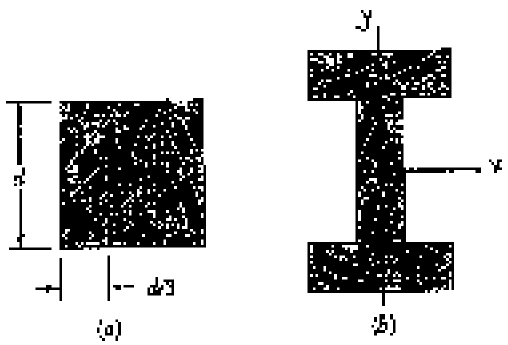
$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(105\times10^9)(510.51\times10^{-9})}{(3.0)^2} = 58.8\times10^3 \text{ N} = 58.8 \text{ kN}$$

Square: $\quad I = \frac{1}{12}(b_o^4 - b_i^4) = 736.85\times10^3 \text{ mm}^4 = 736.85\times10^{-9} \text{ m}^4$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(105\times10^9)(736.85\times10^{-9})}{(3.0)^2} = 84.8\times10^3 \text{ N} = 84.8 \text{ kN}$$

PROBLEM 10.14

10.14 A column of effective length L can be made by gluing together identical planks in each of the arrangements shown. Determine the ratio of the critical load using the arrangement *a* to the critical load using the arrangement *b*.



SOLUTION

Arrangement (a)

$$I_a = \frac{1}{12}d^4$$

$$P_{cr,a} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 E d^4}{12 L_e^2}$$

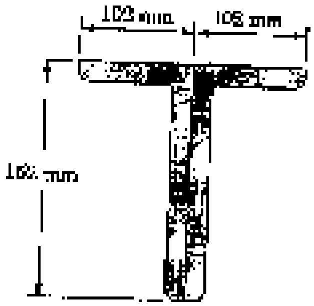
Arrangement (b) $\quad I_{min} = I_y = \frac{1}{12}\left(\frac{d}{3}\right)(d)^3 + \frac{1}{12}(d)\left(\frac{d}{3}\right)^3 + \frac{1}{12}\left(\frac{d}{3}\right)(d)^3 = \frac{19}{324}d^4$

$$P_{cr,b} = \frac{\pi^2 EI}{L_e^2} = \frac{19\pi^2 E d^4}{324 L_e^2}$$

$$\frac{P_{cr,a}}{P_{cr,b}} = \frac{1}{12}\cdot\frac{324}{19} = \frac{27}{19} = 1.421$$

PROBLEM 10.15

10.15 A compression member of 7-m effective length is made by welding together two L152 × 102 × 12.7 angles as shown. Using $E = 200$ GPa, determine the allowable centric load for the member if a factor of safety of 2.2 is required.



SOLUTION

Angle L 152 × 102 × 12.7 $\quad A = 3060\ \text{mm}^2$

$I_x = 7.20 \times 10^6\ \text{mm}^4 \quad I_y = 2.64 \times 10^6\ \text{mm}^4$

$y = 50.3\ \text{mm} \quad x = 25.3\ \text{mm}$

Two angles: $I_x = (2)(7.20 \times 10^6) = 14.00 \times 10^6\ \text{mm}^4$

$I_y = 2\left[2.64 \times 10^6 + (3060)(25.3)^2\right] = 9.197 \times 10^6\ \text{mm}^4$

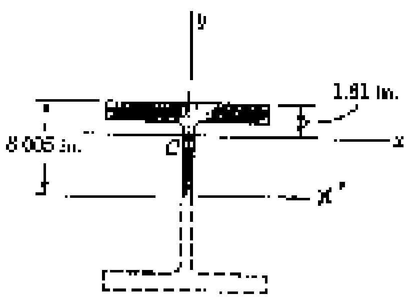
$I_{min} = I_y = 9.197 \times 10^6\ \text{mm}^4 = 9.197 \times 10^{-6}\ \text{m}^4$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2(200 \times 10^9)(9.197 \times 10^{-6})}{(7.0)^2} = 370.5 \times 10^3\ \text{N} = 370.5\ \text{kN}$$

$$P_{all} = \frac{P_{cr}}{F.S.} = \frac{370.5}{2.2} = 168.4\ \text{kN}$$ ◄

PROBLEM 10.16

10.16 A column of 26-ft effective length is made from half a W16 × 40 rolled-steel shape. Knowing that the centroid of the cross section is located as shown, determine the factor of safety if the allowable centric load is 20 kips. Use $E = 29 \times 10^6$ psi.



SOLUTION

Full W 16 × 40 $\quad A = 11.8\ \text{in}^2$

$I_x = 518\ \text{in}^4, \quad I_y = 28.9\ \text{in}^4$

Half W 16 × 40

$A = (\tfrac{1}{2})(11.8) = 5.90\ \text{in}^2$

$I_x = \tfrac{1}{2}(518) - (5.90)(8.005 - 1.81)^2 = 32.57\ \text{in}^4$

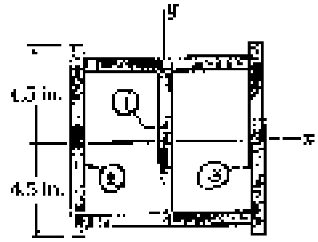
$I_y = \tfrac{1}{2}(28.9) = 14.45\ \text{in}^4 = I_{min}$

$$P_{cr} = \frac{\pi^2 E I_{min}}{L_e^2} = \frac{\pi^2(29 \times 10^6)(14.45)}{(26 \times 12)^2} = 42.5 \times 10^3\ \text{lb} = 42.5\ \text{kips}$$

$$P_{all} = \frac{P_{cr}}{F.S.} \qquad F.S. = \frac{P_{cr}}{P_{all}} = \frac{42.5}{20} = 2.125$$ ◄

PROBLEM 10.17

10.17 A column of 22-ft effective length is to be made by welding two 9 × 0.5 in. plates to a W8 × 35 as shown. Determine the allowable centric load if a factor of safety of 2.3 is required. Use $E = 29 \times 10^6$ psi.



SOLUTION

① W 8 × 35 $\quad I_x = 127\ \text{in}^4 \quad I_y = 42.6\ \text{in}^4$

$b_f = 8.02$ in

② and ③ For each plate $\quad A = (0.5)(9.0) = 4.5\ \text{in}^2$

$$I_x = \frac{1}{12}(0.5)(9)^3 = 30.375\ \text{in}^4$$

$$I_y = \frac{1}{12}(9)(0.5)^3 + (4.5)\left(\frac{8.02}{2} + \frac{0.5}{2}\right)^2\Big] = 81.758\ \text{in}^4$$

Total: $\quad I_x = 127 + (2)(30.375) = 187.75\ \text{in}^4 = I_{min}$

$I_y = 42.6 + (2)(81.758) = 206.12$ in

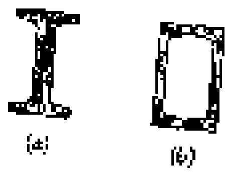
$L = 22$ ft $= 264$ in.

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2(29\times10^6)(187.75)}{264^2} = 771.0\times10^3\ \text{lb} = 771\ \text{kips}$$

$$P_{all} = \frac{P_{cr}}{F.S.} = \frac{771}{2.3} = 335\ \text{kips}$$

PROBLEM 10.18

10.18 A column of 3-m effective length is to be made by welding together two C130 × 13 rolled-steel channels. Using $E = 200$ GPa, determine for each arrangement shown the allowable centric load if a factor of safety of 2.4 is required.



SOLUTION

For channel C 130 × 13 $\quad A = 1710 \text{ mm}^2 \quad b_f = 48 \text{ mm}$

$I_x = 3.70 \times 10^6 \text{ mm}^4 \quad I_y = 0.264 \times 10^6 \text{ mm}^4 \quad \bar{x} = 12.2 \text{ mm}$

Arrangement (a) $\quad I_x = (2)(3.70 \times 10^6) = 7.40 \times 10^6 \text{ mm}^4$

$$I_y = 2\left[0.264 \times 10^6 + (1710)(12.2)^2\right] = 1.0370 \times 10^6 \text{ mm}^4$$

$$I_{min} = I_y = 1.0370 \times 10^6 \text{ mm}^4 = 1.0370 \times 10^{-6} \text{ m}^4$$

$$P_{cr} = \frac{\pi^2 E I_{min}}{L_e^2} = \frac{\pi^2 (200 \times 10^9)(1.0370 \times 10^{-6})}{(3.0)^2} = 227 \times 10^3 \text{ N} = 227 \text{ kN}$$

$$P_{all} = \frac{P_{cr}}{F.S.} = \frac{227}{2.4} = 94.8 \text{ kN}$$

Arrangement (b) $\quad I_x = (2)(3.70 \times 10^6) \text{ mm}^4$

$$I_y = 2\left[0.264 \times 10^6 + (1710)(48 - 12.2)^2\right] = 4.911 \times 10^6 \text{ mm}^4$$

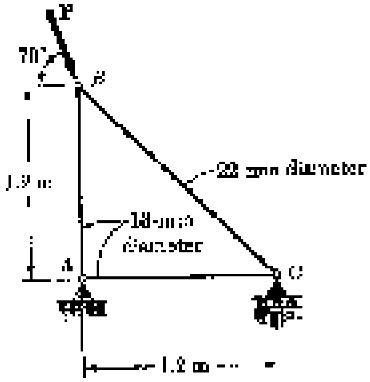
$$I_{min} = I_y = 4.911 \times 10^6 \text{ mm}^4 = 4.911 \times 10^{-6} \text{ m}^4$$

$$P_{cr} = \frac{\pi^2 E I_{min}}{L_e^2} = \frac{\pi^2 (200 \times 10^9)(4.911 \times 10^{-6})}{(3.0)^2} = 1077 \times 10^3 \text{ N} = 1077 \text{ kN}$$

$$P_{all} = \frac{P_{cr}}{F.S.} = \frac{1077}{2.4} = 449 \text{ kN}$$

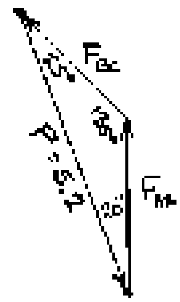
PROBLEM 10.19

10.19 Knowing that $P = 5.2$ kN, determine the factor of safety for the structure shown. Use $E = 200$ GPa and consider only buckling in the plane of the structure.



SOLUTION

Joint B: From force triangle



$$\frac{F_{AB}}{\sin 25^\circ} = \frac{F_{BC}}{\sin 20^\circ} = \frac{5.2}{\sin 135^\circ}$$

$F_{AB} = 3.1079$ kN (comp)

$F_{BC} = 2.5152$ kN (comp)

Member AB: $I_{AB} = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi}{4}\left(\frac{18}{2}\right)^4 = 5.153 \times 10^3\ \text{mm}^4 = 5.153 \times 10^{-9}\ \text{m}^4$

$$F_{AB,cr} = \frac{\pi^2 E I_{AB}}{L_{AB}^2} = \frac{\pi^2 (200 \times 10^9)(5.153 \times 10^{-9})}{(1.2)^2} = 7.0636 \times 10^3\ \text{N} = 7.0636\ \text{kN}$$

$$\text{F.S.} = \frac{F_{AB,cr}}{F_{AB}} = \frac{7.0636}{3.1079} = 2.27$$

Member BC: $I_{BC} = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi}{4}\left(\frac{22}{2}\right)^4 = 11.499 \times 10^3\ \text{mm}^4 = 11.499 \times 10^{-9}\ \text{m}^4$

$L_{BC}^2 = 1.2^2 + 1.2^2 = 2.88\ \text{m}^2$

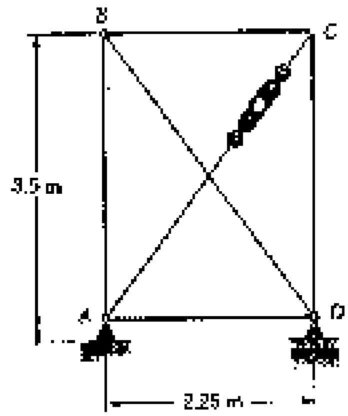
$$F_{BC,cr} = \frac{\pi^2 E I_{BC}}{L_{BC}^2} = \frac{\pi^2 (200 \times 10^9)(11.499 \times 10^{-9})}{2.88} = 7.8813 \times 10^3\ \text{N} = 7.8813\ \text{kN}$$

$$\text{F.S.} = \frac{F_{BC,cr}}{F_{BC}} = \frac{7.8813}{2.5152} = 3.13$$

Smallest F.S. governs. F.S. = 2.27

PROBLEM 10.20

10.20 Members AB and CD are 30-mm-diameter steel rods, and members BC and AD are 22-mm-diameter steel rods. When the turnbuckle is tightened, the diagonal member AC is put in tension. Knowing that a factor of safety with respect to buckling of 2.75 is required, determine the largest allowable tension in AC. Use E = 200 GPa and consider only buckling in the plane of the structure.

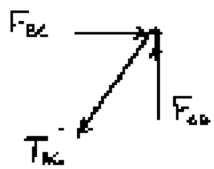


SOLUTION

$$L_{AC} = \sqrt{(3.5)^2 + (2.25)^2} = 4.1608 \text{ m}$$

Joint C

$$\xrightarrow{+} \Sigma F_x = 0 \qquad F_{BC} - \frac{2.25}{4.1608} T_{AC} = 0$$



$$T_{AC} = 1.84926\, F_{BC}$$

$$+\uparrow \Sigma F_y = 0 \qquad F_{CD} - \frac{3.5}{4.1608} T_{AC} = 0$$

$$T_{AC} = 1.1888\, F_{CD}$$

Members BC and AD: $I_{BC} = \frac{\pi}{4}\left(\frac{d_{BC}}{2}\right)^4 = \frac{\pi}{4}\left(\frac{22}{2}\right)^4 = 11.499 \times 10^3 \text{ mm}^4 = 11.499 \times 10^{-9} \text{ m}^4$

$$L_{BC} = 2.25\text{ m} \qquad F_{BC,cr} = \frac{\pi^2 E I_{BC}}{L_{BC}^2} = \frac{\pi^2(200 \times 10^9)(11.499 \times 10^{-9})}{(2.25)^2} = 4.4836 \times 10^3 \text{ N}$$

$$F_{BC,al} = \frac{F_{BC,cr}}{F.S.} = 1.6304 \times 10^3 \text{ N} \qquad T_{AC,al} = 3.02 \times 10^3 \text{ N}$$

Members AB and CD: $I_{CD} = \frac{\pi}{4}\left(\frac{d_{CD}}{2}\right)^4 = \frac{\pi}{4}\left(\frac{30}{2}\right)^4 = 39.761 \times 10^3 \text{ mm}^4 = 39.761 \times 10^{-9} \text{ m}^4$

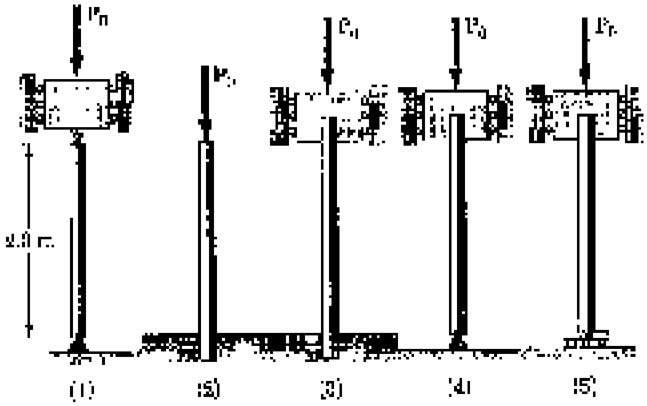
$$L_{CD} = 3.5\text{ m} \qquad F_{CD,cr} = \frac{\pi^2 E I_{CD}}{L_{CD}^2} = \frac{\pi^2(200 \times 10^9)(39.761 \times 10^{-9})}{(3.5)^2} = 6.4069 \times 10^3 \text{ N}$$

$$F_{CD,al} = \frac{F_{CD,cr}}{F.S.} = 2.3298 \times 10^3 \text{ N} \qquad T_{AC,al} = 2.77 \times 10^3 \text{ N}$$

Smaller value for $T_{AC,al}$ governs $\qquad T_{AC,al} = 2.77 \times 10^3 \text{ N} = 2.77 \text{ kN}$ ◄

PROBLEM 10.21

10.21 Each of the five struts consists of an aluminum tube that has a 32-mm outer diameter and a 4-mm wall thickness. Using $E = 70$ GPa and a factor of safety of 2.3, determine the allowable load P_0 for each support condition shown.



SOLUTION

$c_o = \frac{1}{2} d_o = \frac{1}{2}(32) = 16$ mm

$c_i = c_o - t = 16 - 4 = 12$ mm

$$I = \frac{\pi}{4}(c_o^4 - c_i^4) = 35.1858 \times 10^3 \text{ mm}^4 = 35.1858 \times 10^{-9} \text{ m}^4$$

$$\pi^2 EI = \pi^2 (70 \times 10^9)(35.1858 \times 10^{-9}) = 24309 \text{ N}\cdot\text{m}^2$$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{24309}{L_e^2}$$

$$P_{all} = \frac{P_{cr}}{F.S.} = \frac{10569}{L_e^2}$$

(1) $L_e = (1)(2.0) = 2.0$ m, $P_{all} = 2642$ N $= 2.64$ kN ◀

(2) $L_e = (2)(2.0) = 4.0$ m, $P_{all} = 661$ N $= 0.661$ kN ◀

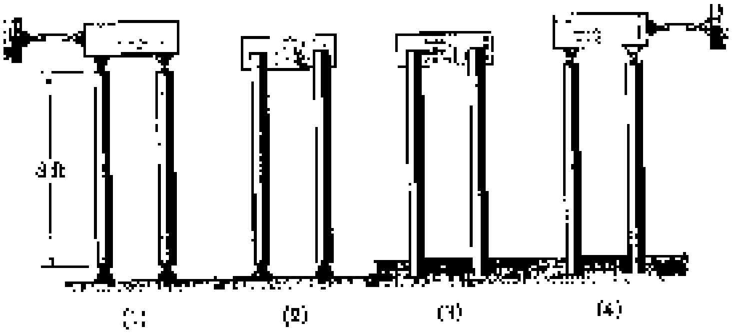
(3) $L_e = (\frac{1}{2})(2.0) = 1.0$ m, $P_{all} = 10569$ N $= 10.57$ kN ◀

(4) $L_e = (0.7)(2.0) = 1.4$ m, $P_{all} = 5392$ N $= 5.39$ kN ◀

(5) $L_e = (1.0)(2.0) = 2.0$ m, $P_{all} = 2642$ N $= 2.64$ kN ◀

PROBLEM 10.22

10.22 Two columns are used to support a block weighing 3.25 kips in each of the four ways shown. (*a*) Knowing that the column of Fig. (1) is made of steel with a 1.25-in.-diameter, determine the factor of safety with respect to buckling for the loading shown. (*b*) Determine the diameter of each of the other columns for which the factor of safety is the same as the factor of safety obtained in part *a*. Use $E = 29 \times 10^6$ psi.



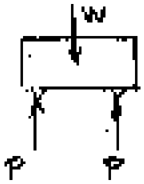
SOLUTION

(a) $I = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi}{4}\left(\frac{1.25}{2}\right)^4$
$= 0.119842 \text{ in}^4$

$L = 8 \text{ ft} = 96 \text{ in}$

$P_{cr} = \frac{\pi^2 EI}{L^2}$

$$P_{cr} = \frac{\pi^2 (29\times10^6)(0.119842)}{(96)^2} = 3722 \text{ lb} = 3.722 \text{ kip, for one column.}$$



$$P = \tfrac{1}{2}W = \frac{3.25}{2} = 1.625 \text{ kip.}$$

$$F.S. = \frac{P_{cr}}{P} = \frac{3.722}{1.625} = 2.29$$

$$P_{cr(1)} = \frac{\pi^2 EI}{L^2} \qquad P_{cr(\alpha)} = \frac{\pi^2 E I_\alpha}{(L_{e,\alpha})^2}$$

$$\frac{P_{cr(\alpha)}}{P_{cr(1)}} = 1 \qquad \frac{I_\alpha}{I_{(1)}}\cdot\frac{L^2}{L_e^2} = 1 \qquad \left(\frac{d_\alpha}{d_{(1)}}\right)^4\left(\frac{L}{L_e}\right)^2 = 1$$

$$d_\alpha = d_{(1)}\sqrt{\frac{L_{e(\alpha)}}{L}}$$

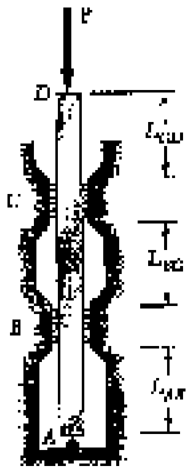
(2) $L_{e(2)}/L = 2.0$ $\qquad d_{(2)} = 1.25\sqrt{2.0} = 1.768$ in.

(3) $L_{e(3)}/L = 1.0$ $\qquad d_{(3)} = 1.25$ in.

(4) $L_{e(4)}/L = 0.7$ $\qquad d_{(4)} = 1.25\sqrt{0.7} = 1.046$ in.

PROBLEM 10.23

10.23 A 25-mm-square aluminum strut is maintained in the position shown by a pin support at A and by sets of rollers at B and C that prevent rotation of the strut in the plane of the figure. Knowing that $L_{AB} = 1.0$ m, $L_{BC} = 1.25$ m, and $L_{CD} = 0.5$ m, determine the allowable load **P** using a factor of safety with respect to buckling of 2.8. Consider only buckling in the plane of the figure and use $E = 75$ GPa.



SOLUTION

$$I = \frac{1}{12} b h^3 = \frac{1}{12}(25)(25)^3 = 32.552 \times 10^3 \text{ mm}^4 = 32.552 \times 10^{-9} \text{ m}^4$$

$$P_{cr} = \frac{\pi^2 E I}{L_e^2} \qquad P_{all} = \frac{P_{cr}}{F.S.} = \frac{\pi^2 E I}{(F.S.) L_e^2} = \frac{\pi^2 E I}{(F.S.)(L_{e,max})^2}$$

Portion AB: $L_e = 0.7 L_{AB} = (0.7)(1.0) = 0.7$ m

Portion BC: $L_e = 0.5 L_{BC} = (0.5)(1.25) = 0.625$ m

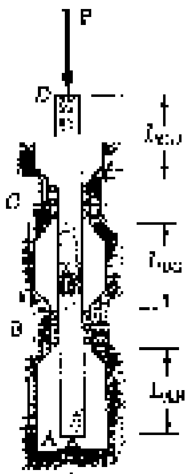
Portion CD: $L_e = 2 L_{CD} = (2.0)(0.5) = 1.0$ m

$L_{e,max} = 1.0$ m

$$P_{all} = \frac{\pi^2 (75 \times 10^9)(35.552 \times 10^{-9})}{(2.8)(1.0)^2} = 8.61 \times 10^3 \text{ N} = 8.61 \text{ kN}$$

PROBLEM 10.24

10.24 A 32-mm-square aluminum strut is maintained in the position shown by a pin support at A and by sets of rollers at B and C that prevent rotation of the strut in the plane of the figure. Knowing that $L_{AB} = 1.4$ m, determine (*a*) the largest values of L_{BC} and L_{CD} that may be used if the allowable load **P** is to be as large as possible, (*b*) the magnitude of the corresponding allowable load if the factor of safety is 2.8. Consider only buckling in the plane of the figure and use $E = 72$ GPa.



SOLUTION

$$I = \frac{1}{12} b h^3 = \frac{1}{12}(32)(32)^3 = 87.381 \times 10^3 \text{ mm}^4 = 87.381 \times 10^{-9} \text{ m}^4$$

Equivalent lengths: AB $L_e = 0.7 L_{AB} = 0.98$ m

BC $L_e = 0.5 L_{BC}$

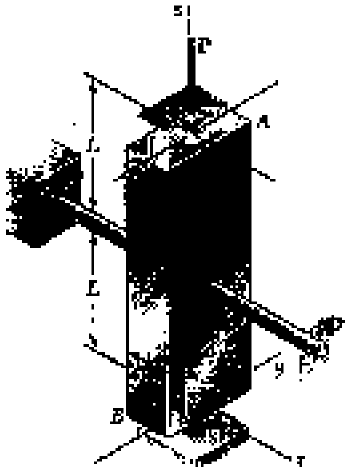
CD $L_e = 2 L_{CD}$

Equating $L_{BC} = \frac{0.7}{0.5} L_{AB} = (1.4)(1.4) = 1.96$ m

$L_{CD} = \frac{0.7}{2} L_{AB} = (0.35)(1.4) = 0.49$ m

(b) $$P_{all} = \frac{P_{cr}}{F.S.} = \frac{\pi^2 E I}{(F.S.) L_e^2} = \frac{\pi^2 (72 \times 10^9)(87.381 \times 10^{-9})}{(2.8)(0.98)^2} = 23.1 \times 10^3 \text{ N} = 23.1 \text{ kN}$$

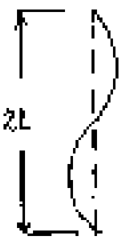
PROBLEM 10.25



10.25 Column *ABC* has a uniform rectangular cross section and is braced in the *xz* plane at its midpoint *C*. (*a*) Determine the ratio *b*/*d* for which the factor of safety is the same with respect to buckling in the *xz* and *yz* planes. (*b*) Using the ratio found in part *a*, design the cross section of the column so that the factor of safety will be 2.7 when $P = 1.2$ kips, $L = 24$ in., and $E = 10.6 \times 10^6$ psi.

SOLUTION

Buckling in xz-plane: $L_e = L = 24$ in.



2L

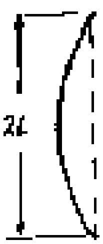
$$I = \frac{1}{12} d b^3$$

$$P = \frac{P_{cr}}{F.S} = \frac{\pi^2 E I}{F.S\, L_e^2} = \frac{\pi^2 E d b^3}{12 (F.S) L_e^2}$$

$$d b^3 = \frac{12 P (F.S.) L_e^2}{\pi^2 E} = \frac{(12)(1.2\times10^3)(2.7)(24)^2}{\pi^2 (10.6\times10^6)}$$

$$= 0.21406 \text{ in}^4$$

Buckling in yz-plane: $L_e = 2L = (2)(24) = 48$ in $\quad I = \frac{1}{12} b d^3$



2L

$$P = \frac{P_{cr}}{F.S} = \frac{\pi^2 E I}{F.S\, L_e^2} = \frac{\pi^2 E b d^3}{12 (F.S) L_e^2}$$

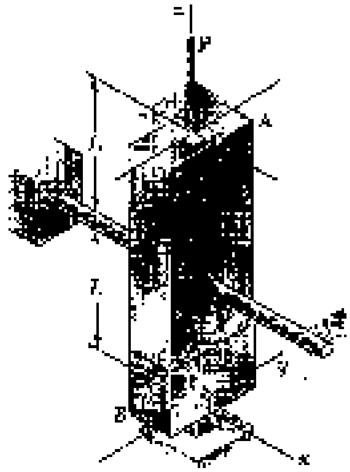
$$b d^3 = \frac{12 P (F.S.) L_e^2}{\pi^2 E} = \frac{(12)(1.2\times10^3)(2.7)(48)^2}{\pi^2 (10.6\times10^6)} = 0.85625 \text{ in}^4$$

(a) $$\frac{d b^3}{b d^3} = \frac{b^2}{d^2} = \frac{0.21406}{0.85625} = \frac{1}{4} \qquad \frac{b}{d} = \frac{1}{2}$$

$$d b^3 = d\left(\tfrac{1}{8} d^3\right) = \tfrac{1}{8} d^4 = 0.21406 \text{ in}^4, \quad d = 1.144 \text{ in.}$$

$$b = \tfrac{1}{2} d = 0.572 \text{ in.}$$

PROBLEM 10.26



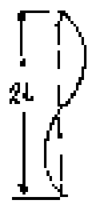
10.26 The aluminum column *ABC* has a uniform rectangular cross section with $b = \frac{1}{2}$ in. and $d = \frac{7}{8}$ in. The column is braced in the *xz* plane at its midpoint *C* and carries a centric load **P** of magnitude 1.1 kips. Knowing that a factor of safety of 2.5 is required, determine the largest allowable length *L*. Use $E = 10.6 \times 10^6$ psi.

SOLUTION

$$P_{cr} = (F.S.)P = (2.5)(1.1\times10^3) = 2.75\times10^3 \text{ lb}$$

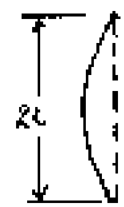
$$P_{cr} = \frac{\pi^2 EI}{L_e^2} \qquad L_e = \pi\sqrt{\frac{EI}{P_{cr}}}$$

Buckling in xz-plane: $I = \frac{1}{12}db^3 = \frac{1}{12}\left(\frac{7}{8}\right)\left(\frac{1}{2}\right)^3 = 9.1146\times10^{-3}\text{ in}^4$



$$L = L_e = \pi\sqrt{\frac{EI}{P_{cr}}} = \pi\sqrt{\frac{(10.6\times10^6)(9.1146\times10^{-3})}{2.75\times10^3}} = 18.62 \text{ in}$$

Buckling in yz-plane

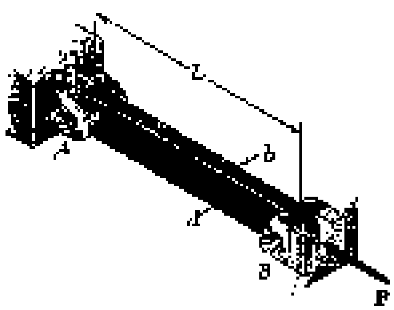


$$I = \frac{1}{12}bd^3 = \frac{1}{12}\left(\frac{1}{2}\right)\left(\frac{7}{8}\right)^3 = 27.913\times10^{-3}\text{ in}^4 \qquad L_e = 2L$$

$$L = \frac{1}{2}L_e = \frac{\pi}{2}\sqrt{\frac{EI}{P_{cr}}} = \frac{\pi}{2}\sqrt{\frac{(10.6\times10^6)(27.913\times10^{-3})}{2.75\times10^3}} = 16.29 \text{ in.}$$

Smaller length governs $L = 16.29$ in.

PROBLEM 10.27



10.27 The uniform brass bar *AB* has a rectangular cross section and is supported by pins and brackets as shown. Each end of the bar can rotate freely about a horizontal axis through the pin, but rotation about a vertical axis is prevented by the brackets. (*a*) Determine the ratio *b*/*d* for which the factor of safety is the same about the horizontal and vertical axes. (*b*) Determine the factor of safety if $P = 1.8$ kips, $L = 7$ ft, $d = 1.5$ in., and $E = 15 \times 10^6$ psi.

SOLUTION

Buckling in horizontal plane: $L_e = \frac{1}{2}L$, $I = \frac{1}{12}db^3$

$$P_{cr1} = \frac{\pi^2 EI}{L_e^2} = \frac{4\pi^2 Edb^3}{12L^2} \quad (1)$$

Buckling in vertical plane: $L_e = L$, $I = \frac{1}{12}bd^3$

$$P_{cr2} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 Ebd^3}{12L^2} \quad (2)$$

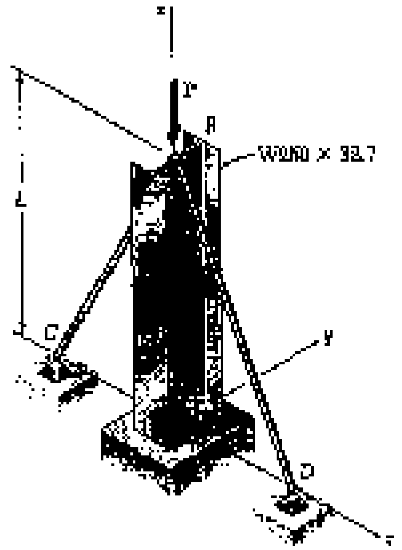
(a) Equating $P_{cr1} = P_{cr2}$: $\frac{4\pi^2 Edb^3}{12L^2} = \frac{\pi^2 Ebd^3}{12L^2}$ $\quad 4b^2 = d^2 \quad b = \frac{1}{2}d$

(b) $b = \frac{1}{2}d = 0.75$ in. $\quad L = 7$ ft $= 84$ in

Using (2) $P_{cr} = \frac{\pi^2(15\times10^6)(0.75)(1.5)^3}{(12)(84)^2} = 4.426\times10^3$ lb $= 4.426$ kips.

$$F.S. = \frac{P_{cr}}{P} = \frac{4.426}{1.8} = 2.46$$

PROBLEM 10.28



10.28 Column AB carries a centric load $\mathbf{P}$ of magnitude 72 kN. Cables BC and BD are taut and prevent motion of point B in the xz plane. Using Euler's formula and a factor of safety of 2.3, and neglecting the tension in the cables, determine the maximum allowable length L. Use $E = 200$ GPa.

SOLUTION

$W\ 250 \times 32.7 \qquad I_x = 48.9 \times 10^6\ mm^4 = 48.9 \times 10^{-6}\ m^4$

$\qquad\qquad\qquad\quad I_y = 4.73 \times 10^6\ mm^4 = 4.73 \times 10^{-6}\ m^4$

$P = 72 \times 10^3\ N \qquad P_{cr,min} = (F.S.)(P) = 165.6 \times 10^3\ N$

Buckling in xz-plane: $L_e = 0.7 L$

$$P_{cr} = \frac{\pi^2 E I_y}{(0.7 L)^2}$$

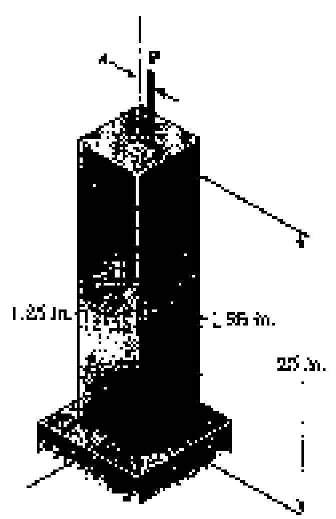
$$L = \frac{\pi}{0.7}\sqrt{\frac{E I_y}{P_{cr}}} = \frac{\pi}{0.7}\sqrt{\frac{(200 \times 10^9)(4.73 \times 10^{-6})}{165.6 \times 10^3}}$$

$$= 10.74\ m$$

Buckling in yz-plane: $L_e = 2L \qquad P_{cr} = \dfrac{\pi^2 E I_x}{(2L)^2}$

$$L = \frac{\pi}{2}\sqrt{\frac{E I_x}{P_{cr}}} = \frac{\pi}{2}\sqrt{\frac{(200 \times 10^9)(48.9 \times 10^{-6})}{165.6 \times 10^3}} = 12.08\ m$$

PROBLEM 10.29



10.29 An axial load $\mathbf{P}$ is applied to the 1.25-in.-square aluminum bar ABC as shown. When $P = 3.8$ kips, the horizontal deflection at end C is 0.16 in. Using $E = 10.1 \times 10^6$ psi, determine (*a*) the eccentricity e of the load, (*b*) the maximum stress in the rod.

SOLUTION

$I = \frac{1}{12}(1.25)^4 = 0.20345\ in^4 \qquad A = 1.25^2 = 1.5625\ in^2$

$L_e = 2L = 50\ in \qquad L_e = 2L = 50\ in.$

$$P_{cr} = \frac{\pi^2 E I}{L_e^2} = \frac{\pi^2 (10.1 \times 10^6)(0.20345)}{(50)^2} = 8.1122 \times 10^3\ lb$$

$$\frac{P}{P_{cr}} = \frac{3.8 \times 10^3}{8.1122 \times 10^3} = 0.46842$$

(a) $$y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] = e\left[\sec\left(\frac{\pi}{2}\sqrt{0.46842}\right) - 1\right]$$

$$= e\left[\sec(1.07508) - 1\right] = 1.1023\ e$$

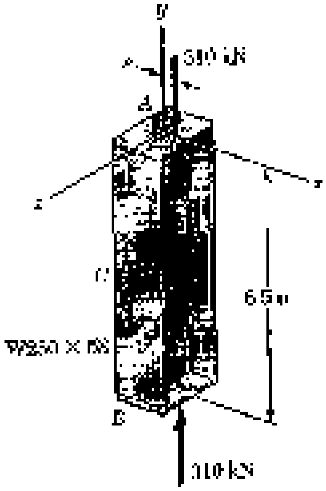
$$e = \frac{y_{max}}{1.1023} = \frac{0.16}{1.1023} = 0.1451\ in.$$

(b) $M_{max} = P(e + y_{max}) = (3.8 \times 10^3)(0.1451 + 0.16) = 1.15957\ lb \cdot in$

$$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I} = \frac{3.8 \times 10^3}{1.5625} + \frac{(1.15957)(0.625)}{0.20345} = 5.99 \times 10^3\ psi = 5.99\ ksi$$

PROBLEM 10.30

10.30 The line of action of the 310-kN axial load is parallel to the geometric axis of the column AB and intersects the x axis at $x = e$. Using $E = 200$ GPa, determine (a) the eccentricity e when the deflection of the midpoint C of the column is 9 mm, (b) the corresponding maximum stress in the column.



SOLUTION

For W250 × 58: $A = 7420\ \text{mm}^2 = 7420\times10^{-6}\ \text{m}^2$

$I_y = 18.8\times10^6\ \text{mm}^4 = 18.8\times10^{-6}\ \text{m}^4$

$S_y = 185\times10^3\ \text{mm}^3 = 185\times10^{-6}\ \text{m}^3$

$L = 6.5\ \text{m} \qquad L_e = 6.5\ \text{m}$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2(200\times10^9)(18.8\times10^{-6})}{(6.5)^2} = 878.3\times10^3\ \text{N}$$

$$\frac{P}{P_{cr}} = \frac{310\times10^3}{878.3\times10^3} = 0.35294$$

$$y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] = 0.67990\,e$$

(a) $$e = \frac{y_{max}}{0.67990} = \frac{9\times10^{-3}}{0.67990} = 13.24\times10^{-3}\ \text{m} = 13.24\ \text{mm}$$

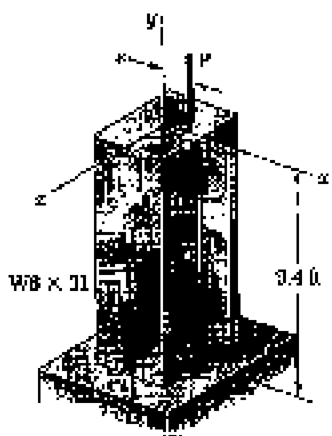
(b) $$M_{max} = P(e + y_{max}) = (310\times10^3)(9+13.24)(10^{-3}) = 6893.5\ \text{N}\cdot\text{m}$$

$$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{M}{S_y} = \frac{310\times10^3}{7420\times10^{-6}} + \frac{6893.5}{185\times10^{-6}}$$

$$= 41.78\times10^6 + 37.26\times10^6 = 79.04\times10^6\ \text{Pa} = 79.0\ \text{MPa}$$

PROBLEM 10.31

10.31 The axial load P is applied at a point located on the x axis at a distance e from the geometric axis of the rolled-steel column BC. When $P = 82$ kips, the horizontal deflection of the top of the column is 0.20 in. Using $E = 29\times10^6$ psi, determine (a) the eccentricity e of the load, (b) the maximum stress in the column.



SOLUTION

W8 × 31: $A = 9.13\ \text{in}^2,\quad I_y = 37.1\ \text{in}^4,\quad S_y = 9.27\ \text{in}^3$

$L = 9.4\ \text{ft} = 112.8\ \text{in} \qquad L_e = 2L = 225.6\ \text{in}$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2(29\times10^6)(37.1)}{(225.6)^2} = 208.63\times10^3$$

$$\frac{P}{P_{cr}} = \frac{82\times10^3}{208.63\times10^3} = 0.39304$$

(a) $$y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] = 0.80816\,e$$

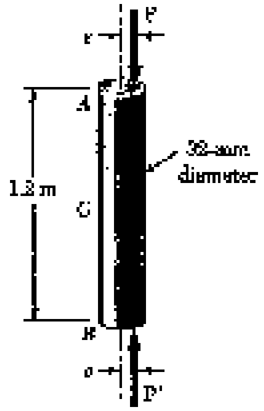
$$e = \frac{y_{max}}{0.80816} = \frac{0.20}{0.80816} = 0.247\ \text{in}$$

(b) $$M_{max} = P(e + y_{max}) = (82\times10^3)(0.247+0.20) = 36.693\times10^3\ \text{lb}\cdot\text{in}$$

$$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{M}{S_y} = \frac{82\times10^3}{9.13} + \frac{36.693\times10^3}{9.27} = 12.94\times10^3\ \text{psi}$$
$$= 12.94\ \text{ksi}$$

PROBLEM 10.32

10.32 An axial load **P** is applied to the 32-mm-diameter steel rod AB as shown. For $P = 37$ kN and $e = 1.2$ mm, determine (*a*) the deflection at the midpoint C of the rod, (*b*) the maximum stress in the rod. Use $E = 200$ GPa.



SOLUTION

$$I = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi}{4}\left(\frac{32}{2}\right)^4 = 51.47\times10^3\ \text{mm}^4 = 51.47\times10^{-9}\ \text{m}^4$$

$$L_e = L = 1.2\ \text{m}$$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2(200\times10^9)(51.47\times10^{-9})}{(1.2)^2} = 70.556\times10^3\ \text{N}$$

$$\frac{P}{P_{cr}} = \frac{37\times10^3}{70.556\times10^3} = 0.52440$$

(a) $$y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] = 1.3817e = (1.3817)(1.2)$$
$$= 1.658\ \text{mm}$$

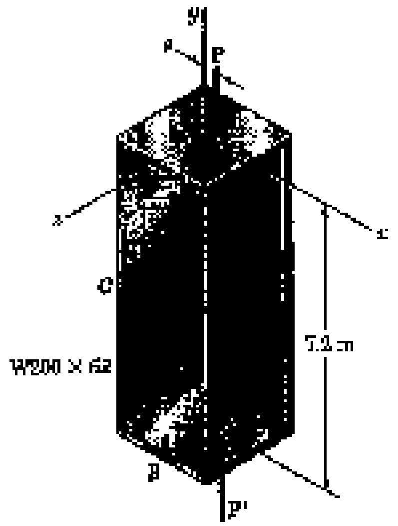
(b) $$M_{max} = P(e + y_{max}) = (37\times10^3)(1.2 + 1.658)(10^{-3}) = 105.75\ \text{N}\cdot\text{m}$$

$$A = \frac{\pi}{4}d^2 = \frac{\pi}{4}(32)^2 = 804.25\ \text{mm}^2 = 804.25\times10^{-6}\ \text{m}^2, \quad c = 16\times10^{-3}\ \text{m}$$

$$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I} = \frac{37\times10^3}{804.25\times10^{-6}} + \frac{(105.75)(16\times10^{-3})}{51.47\times10^{-9}} = 78.9\times10^6\ \text{Pa}$$
$$= 78.9\ \text{MPa}$$

PROBLEM 10.33

10.33 The line of action of the axial load **P** of magnitude 270 kN is parallel to the geometric axis of the column AB and intersects the x axis at $e = 14$ mm. Using $E = 200$ GPa, determine (*a*) the deflection of the midpoint C of the column, (*b*) the maximum stress in the column.



SOLUTION

W 200 × 52 $\quad A = 6660\ \text{mm}^2 = 6660\times10^{-6}\ \text{m}^2$
$I_y = 17.8\times10^6\ \text{mm}^4 = 17.8\times10^{-6}\ \text{m}^4$
$S_y = 175\times10^3\ \text{mm}^3 = 175\times10^{-6}\ \text{m}^3$

$L = 7.2$ m $\qquad L_e = 7.2$ m

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2(200\times10^9)(17.8\times10^{-6})}{(7.2)^2}$$
$$= 677.77\times10^3\ \text{N}$$

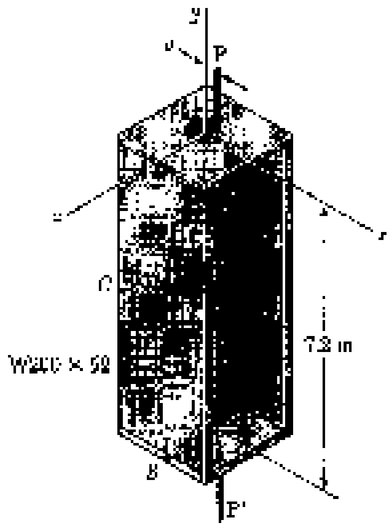
$$\frac{P}{P_{cr}} = \frac{270\times10^3}{677.77\times10^3} = 0.39836$$

(a) $$y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] = 0.82648e = (0.82648)(14) = 11.57\ \text{mm}$$

(b) $$M_{max} = P(e + y_{max}) = (270\times10^3)(14 + 11.57)(10^{-3}) = 6904\ \text{N}\cdot\text{m}$$

$$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{M}{S_y} = \frac{270\times10^3}{6660\times10^{-6}} + \frac{6904}{175\times10^{-6}} = 80.0\times10^6\ \text{Pa} = 80.0\ \text{MPa}$$

PROBLEM 10.34



10.33 The line of action of the axial load **P** of magnitude 270 kN is parallel to the geometric axis of the column *AB* and intersects the *x* axis at *e* = 14 mm. Using *E* = 200 GPa, determine (*a*) the deflection of the midpoint *C* of the column, (*b*) the maximum stress in the column.

10.34 Solve Prob. 10.33 if the load **P** is applied parallel to the geometric axis of the column *AB* so that it intersects the *x* axis at *e* = 21 mm.

SOLUTION

W 200 × 52 $\quad A = 6660\ \text{mm}^2 = 6660\times10^{-6}\ \text{m}^2$

$I_y = 17.8\times10^6\ \text{mm}^4 = 17.8\times10^{-6}\ \text{m}^4$

$S_y = 175\times10^3\ \text{mm}^3 = 175\times10^{-6}\ \text{m}^3$

$L = 7.2\ \text{m} \qquad L_e = 7.2\ \text{m}$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2(200\times10^9)(17.8\times10^{-6})}{(7.2)^2}$$

$$= 677.77\times10^3\ \text{N}$$

$$\frac{P}{P_{cr}} = \frac{270\times10^3}{677.77\times10^3} = 0.39836$$

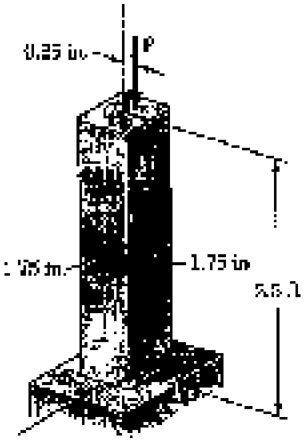
(a) $y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] = 0.82648\,e = (0.82648)(21) = 17.36\ \text{mm}$ ◄

(b) $M_{max} = P(e + y_{max}) = (270\times10^3)(21 + 17.36)(10^{-3}) = 10356\ \text{N}\cdot\text{m}$

$$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{M}{S_y} = \frac{270\times10^3}{6660\times10^{-6}} + \frac{10356}{175\times10^{-6}} = 99.7\times10^6\ \text{Pa} = 99.7\ \text{MPa}$$ ◄

PROBLEM 10.35

10.35 An axial load **P** is applied at a point D that is 0.25 in. from the geometric axis of the square aluminum bar BC. Determine (a) the load **P** for which the horizontal deflection of end C is 0.50 in., (b) the corresponding maximum stress in the column. Use $E = 10.1 \times 10^6$ ksi.



SOLUTION

$I = \frac{1}{12} b h^3 = \frac{1}{12}(1.75)(1.75)^3 = 0.78157 \text{ in}^4$

$A = (1.75)^2 = 3.0625 \text{ in}^2 \qquad c = \frac{1}{2}(1.75) = 0.875 \text{ in.}$

$L = 2.5 \text{ ft} = 30 \text{ in} \qquad L_e = 2L = 60 \text{ in.}$

$P_{cr} = \frac{\pi^2 E I}{L_e^2} = \frac{\pi^2 (10.1 \times 10^3)(0.78157)}{(60)^2} = 21.641 \text{ kips.}$

$y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right]$

$\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{y_{max} + e}{e}, \qquad \cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{e}{y_{max} + e}$

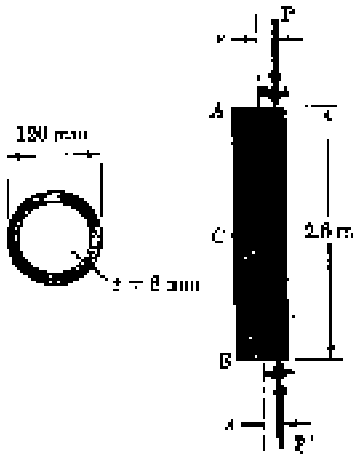
(a) $\frac{P}{P_{cr}} = \left[\frac{2}{\pi} \arccos \frac{e}{e + y_{max}}\right]^2 = \left[\frac{2}{\pi} \arccos \frac{0.25}{0.25 + 0.50}\right]^2$

$= 0.61411 \qquad P = 0.61411\, P_{cr} = 13.29 \text{ kips}$ ◄

(b) $M_{max} = P(e + y_{max}) = (13.29)(0.25 + 0.50) = 9.9675 \text{ kip}\cdot\text{in.}$

$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I} = \frac{13.29}{3.0625} + \frac{(9.9675)(0.875)}{0.78157} = 15.50 \text{ ksi}$ ◄

PROBLEM 10.36



10.36 A brass pipe having the cross section shown has an axial load **P** applied 5 mm from its geometric axis. Using $E = 120$ GPa, determine (*a*) the load **P** for which the horizontal deflection at the midpoint C is 5 mm, (*b*) the corresponding maximum stress in the column.

SOLUTION

$c_o = \frac{1}{2} d_o = 60 \text{ mm} \qquad c_i = c_o - t = 54 \text{ mm}$

$I = \frac{\pi}{4}(c_o^4 - c_i^4) = 3.5005 \times 10^6 \text{ mm}^4 = 3.5005 \times 10^{-6} \text{ m}^4$

$L = 2.8 \text{ m} \qquad L_e = 2.8 \text{ m}$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 (120 \times 10^9)(3.5005 \times 10^{-6})}{(2.8)^2}$$

$$= 528.8 \times 10^3 \text{ N} = 528.8 \text{ kN}$$

(a) $y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \qquad \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{y_{max} + e}{e}$

$\cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{e}{y_{max} + e} \qquad \frac{P}{P_{cr}} = \left[\frac{2}{\pi} \arccos \frac{e}{y_{max} + e}\right]^2$

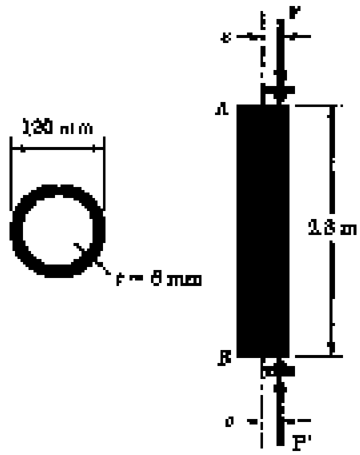
$\frac{P}{P_{cr}} = \left[\frac{2}{\pi} \arccos \frac{5}{5+5}\right]^2 = 0.44444 \qquad P = 0.44444\, P_{cr} = 235 \text{ kN}$ ◄

(b) $M_{max} = P(e + y_{max}) = (235 \times 10^3)(5+5)(10^{-3}) = 2350 \text{ N}\cdot\text{m}$

$A = \pi(c_o^2 - c_i^2) = \pi(60^2 - 54^2) = 2.1488 \times 10^3 \text{ mm}^2 = 2.1488 \times 10^{-3} \text{ m}^2$

$$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I} = \frac{235 \times 10^3}{2.1488 \times 10^{-3}} + \frac{(2350)(60 \times 10^{-3})}{3.5005 \times 10^{-6}} = 149.6 \times 10^6 \text{ Pa} = 149.6 \text{ MPa}$$ ◄

PROBLEM 10.37



10.36 A brass pipe having the cross section shown has an axial load **P** applied 5 mm from its geometric axis. Using $E = 120$ GPa, determine (*a*) the load **P** for which the horizontal deflection at the midpoint *C* is 5 mm, (*b*) the corresponding maximum stress in the column.

10.37 Solve Prob. 10.36, assuming that the axial load **P** is applied 10 mm from the geometric axis of the column.

SOLUTION

$c_o = \frac{1}{2} d_o = 60 \text{ mm} \qquad c_i = c_o - t = 54 \text{ mm}$

$I = \frac{\pi}{4}(c_o^4 - c_i^4) = 3.5005 \times 10^6 \text{ mm}^4 = 3.5005 \times 10^{-6} \text{ m}^4$

$L = 2.8 \text{ m} \qquad L_e = 2.8 \text{ m}$

$P_{cr} = \dfrac{\pi^2 EI}{L_e^2} = \dfrac{\pi^2 (120 \times 10^9)(3.5005 \times 10^{-6})}{(2.8)^2}$

$= 528.8 \times 10^3 \text{ N} = 528.8 \text{ kN}$

(a) $y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \qquad \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \dfrac{y_{max}+e}{e}$

$\cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \dfrac{e}{y_{max}+e} \qquad \dfrac{P}{P_{cr}} = \left[\dfrac{2}{\pi} \arccos \dfrac{e}{y_{max}+e}\right]^2$

$\dfrac{P}{P_{cr}} = \left[\dfrac{2}{\pi} \arccos \dfrac{10}{5+10}\right]^2 = 0.28670 \qquad P = 0.28670\, P_{cr} = 151.6 \text{ kN}$

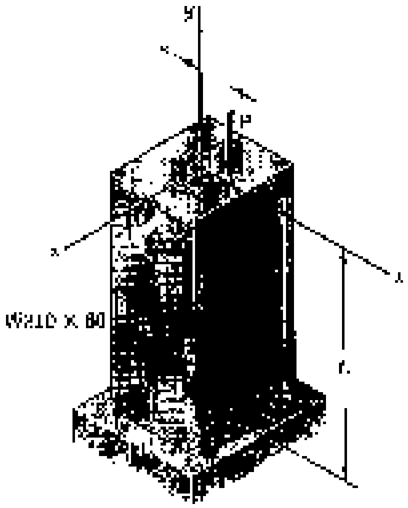
(b) $M_{max} = P(e + y_{max}) = (151.6 \times 10^3)(10+5)(10^{-3}) = 2274 \text{ N}\cdot\text{m}$

$A = \pi(c_o^2 - c_i^2) = \pi(60^2 - 54^2) = 2.1488 \times 10^3 \text{ mm}^2 = 2.1488 \times 10^{-3} \text{ m}^2$

$\sigma_{max} = \dfrac{P}{A} + \dfrac{Mc}{I} = \dfrac{151.6 \times 10^3}{2.1488 \times 10^{-3}} + \dfrac{(2274)(60 \times 10^{-3})}{3.5005 \times 10^{-6}} = 109.5 \times 10^6 \text{ Pa} = 109.5 \text{ MPa}$

PROBLEM 10.38

10.38 An axial load **P** is applied at a point located on the x axis at a distance $e = 12$ mm from the geometric axis of the W310 × 60 rolled-steel column BC. Assuming that $L = 3.5$ m and using $E = 200$ GPa, determine (a) the load **P** for which the horizontal deflection at end C is 15 mm, (b) the corresponding maximum stress in the column.



SOLUTION

W310×60 $\quad A = 7590 \text{ mm}^2 = 7590 \times 10^{-6} \text{ m}^2$

$I_y = 18.3 \times 10^6 \text{ mm}^4 = 18.3 \times 10^{-6} \text{ m}^4$

$S_y = 180 \times 10^3 \text{ mm}^3 = 180 \times 10^{-6} \text{ m}^3$

$L = 3.5 \text{ m} \qquad L_e = 2L = 7.0 \text{ m}$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2(200\times10^9)(18.3\times10^{-6})}{(7.0)^2}$$

$$= 737.2 \times 10^3 \text{ N} = 737.2 \text{ kN}$$

$$y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \qquad \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{y_{max}+e}{e} \qquad \cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{e}{y_{max}+e}$$

$$\frac{P}{P_{cr}} = \left[\frac{2}{\pi}\arccos\frac{e}{y_{max}+e}\right]^2 = \left[\frac{2}{\pi}\arccos\frac{12}{15+12}\right]^2 = 0.49957$$

(a) $\qquad P = 0.49957\, P_{cr} = 368.28 \text{ kN}$

$$M_{max} = P(e + y_{max}) = (368.28\times10^3)(12+15)(10^{-3}) = 9944 \text{ N}\cdot\text{m}$$

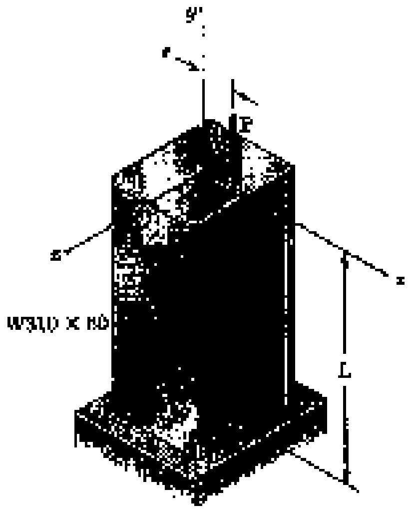
(b) $$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{M}{S_y} = \frac{368.28\times10^3}{7590\times10^{-6}} + \frac{9944}{180\times10^{-6}} = 103.8\times10^6 \text{ Pa}$$

$$= 103.8 \text{ MPa}$$

PROBLEM 10.39

10.38 An axial load **P** is applied at a point located on the x axis at a distance $e = 12$ mm from the geometric axis of the W310 × 60 rolled-steel column BC. Assuming that $L = 3.5$ m and using $E = 200$ GPa, determine (a) the load **P** for which the horizontal deflection at end C is 15 mm, (b) the corresponding maximum stress in the column.

10.39 Solve Prob. 10.38, assuming that L is 4.5 m.



SOLUTION

W 310 × 60 $\quad A = 7590 \text{ mm}^2 = 7590 \times 10^{-6} \text{ m}^2$

$I_y = 18.3 \times 10^6 \text{ mm}^4 = 18.3 \times 10^{-6} \text{ m}^4$

$S_y = 180 \times 10^3 \text{ mm}^3 = 180 \times 10^{-6} \text{ m}^3$

$L = 4.5 \text{ m} \qquad L_e = 2L = 9.0 \text{ m}$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 (200 \times 10^9)(18.3 \times 10^{-6})}{(9.0)^2}$$

$$= 445.96 \times 10^3 \text{ N} = 445.96 \text{ kN}$$

$$y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \qquad \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{y_{max} + e}{e} \qquad \cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{e}{y_{max} + e}$$

$$\frac{P}{P_{cr}} = \left[\frac{2}{\pi} \arccos \frac{e}{y_{max} + e}\right]^2 = \left[\frac{2}{\pi} \arccos \frac{12}{15 + 12}\right]^2 = 0.49957$$

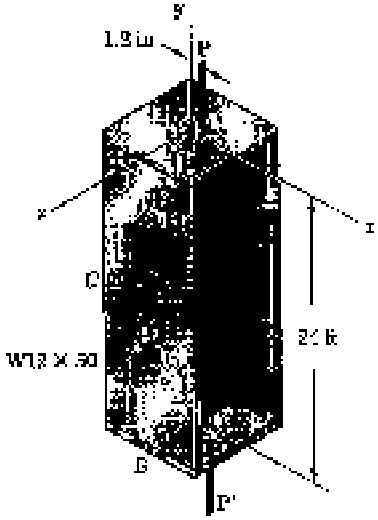
(a) $\qquad P = 0.49957\, P_{cr} = 222.79 \text{ kN}$

$$M_{max} = P(e + y_{max}) = (222.79 \times 10^3)(12 + 15)(10^{-3}) = 6015 \text{ N}\cdot\text{m}$$

(b) $$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{M}{S_y} = \frac{222.79 \times 10^3}{7590 \times 10^{-6}} + \frac{6015}{180 \times 10^{-6}} = 62.8 \times 10^6 \text{ Pa}$$

$$= 62.8 \text{ MPa}$$

PROBLEM 10.40



10.40 The line of action of an axial load **P** is parallel to the geometric axis of the column AB and intersects the x axis at $x = 1.2$ in. Using $E = 29 \times 10^6$ psi, determine (*a*) the load **P** for which the horizontal deflection of the midpoint C of the column is 0.8 in., (*b*) the corresponding maximum stress in the column.

SOLUTION

W 12 × 50 $\quad A = 14.7 \text{ in}^2, \quad I_y = 56.3 \text{ in}^4, \quad S_y = 13.9 \text{ in}^3$

$L = 24 \text{ ft} = 288 \text{ in} \qquad L_e = 288 \text{ in}$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 (29 \times 10^6)(56.3)}{(288)^2} = 194.28 \times 10^3 \text{ lb.}$$

$$= 194.28 \text{ kips}$$

$$y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \qquad \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{y_{max} + e}{e}$$

$$\cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{e}{y_{max} + e} \qquad \frac{P}{P_{cr}} = \left[\frac{2}{\pi} \arccos \frac{e}{y_{max} + e}\right]^2$$

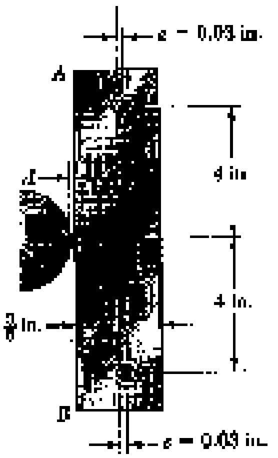
(a) $$\frac{P}{P_{cr}} = \left[\frac{2}{\pi} \arccos \frac{1.2}{0.8 + 1.2}\right]^2 = 0.34849 \qquad P = 0.34849\, P_{cr} = 67.7 \text{ kips}$$ ◄

$$M_{max} = P(e + y_{max}) = (67.7)(1.2 + 0.8) = 135.4 \text{ kip}\cdot\text{in}$$

(b) $$\sigma_{max} = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{M}{S_y} = \frac{67.7}{14.7} + \frac{135.4}{13.9} = 14.3 \text{ ksi}$$ ◄

PROBLEM 10.41

10.41 The steel bar AB has a $\frac{3}{8} \times \frac{3}{8}$-in. square cross section and is held by pins that are a fixed distance apart and are located at a distance $e = 0.03$ in. from the geometric axis of the bar. Knowing that at temperature T_0 the pins are in contact with the bar and that the force in the bar is zero, determine the increase in temperature for which the bar will just make contact with point C if $d = 0.01$ in. Use $E = 29 \times 10^6$ psi and the coefficient of thermal expansion $\alpha = 6.5 \times 10^{-6}/°\text{F}$.



SOLUTION

$$A = \left(\tfrac{3}{8}\right)\left(\tfrac{3}{8}\right) = 0.140625 \text{ in}^2$$

$$I = \tfrac{1}{12}\left(\tfrac{3}{8}\right)^4 = 1.64795 \times 10^{-3} \text{ in}^4$$

$$EI = (29 \times 10^6)(1.64795 \times 10^{-3}) = 47791 \text{ lb}\cdot\text{in}^2$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (47791)}{(8)^2} = 7370 \text{ lb.}$$

Calculate P using the secant formula

$$y_{max} = d = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \qquad \sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} = 1 + \frac{d}{e}$$

$$\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} = \cos^{-1}\left(1 + \frac{d}{e}\right)^{-1} = \cos^{-1}\left(1 + \frac{0.01}{0.03}\right)^{-1} = \cos^{-1}(0.75) = 0.72273$$

$$\frac{P}{P_{cr}} = \left[\frac{2}{\pi}(0.72273)\right]^2 = 0.21170 \qquad P = 0.21170\, P_{cr} = 1560.2 \text{ lb.}$$

Thermal analysis.

(1) Simple approximation by ignoring eccentricity.

$$\text{Total elongation} = \alpha L(\Delta T) - \frac{PL}{EA} = 0$$

$$\Delta T = \frac{PL}{EA}\frac{1}{\alpha L} = \frac{P}{EA\alpha} = \frac{1560.2}{(29 \times 10^6)(0.140625)(6.5 \times 10^{-6})} = 58.9\,°\text{F}$$

(2) Analysis with inclusion of eccentricity.

$$\text{Total elongation of centroidal axis} = \alpha L(\Delta T) - \frac{PL}{EA} = 2e\left.\frac{dy}{dx}\right|_{x=0}$$

To calculate $\frac{dy}{dx}$, differentiate eq. (10.26)

$$\frac{dy}{dx} = e\left(p \tan\frac{pL}{2}\cos px - p\sin px\right)$$

$$\text{At } x = 0 \qquad \left.\frac{dy}{dx}\right|_{x=0} = ep\tan\frac{pL}{2} = e\sqrt{\frac{P}{EI}}\tan\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}$$

The elongation of the centroidal axis is $2e^2\sqrt{\frac{P}{EI}}\tan\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)$

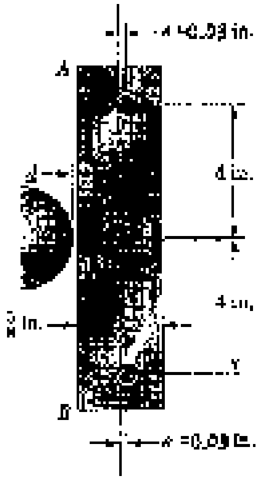
$$= (2)(0.03)^2\sqrt{\frac{1560.2}{47791}}\tan(0.72273) = 286.8 \times 10^{-6} \text{ in.}$$

$$\alpha L(\Delta T) = \frac{PL}{EA} + 2e\left.\frac{dy}{dx}\right|_{x=0}$$

$$\Delta T = \frac{P}{EA\alpha} + \frac{286.8 \times 10^{-6}}{\alpha L} = 58.9 + \frac{286.8 \times 10^{-6}}{(6.5 \times 10^{-6})(8)} = 58.9 + 5.5°$$

$$= 64.4°\text{ F}$$

PROBLEM 10.42



10.41 The steel bar AB has a $\frac{3}{8} \times \frac{3}{8}$-in. square cross section and is held by pins that are a fixed distance apart and are located at a distance $e = 0.03$ in. from the geometric axis of the bar. Knowing that at temperature T_0 the pins are in contact with the bar and that the force in the bar is zero, determine the increase in temperature for which the bar will just make contact with point C if $d = 0.01$ in. Use $E = 29 \times 10^6$ psi and the coefficient of thermal expansion $\alpha = 6.5 \times 10^{-6}/°F$.

10.42 For the bar of Prob. 10.41, determine the required distance d for which the bar will just make contact with point C when the temperature increases by 120 °F.

SOLUTION

$$A = \left(\tfrac{3}{8}\right)\left(\tfrac{3}{8}\right) = 0.140625 \text{ in}^2$$

$$I = \tfrac{1}{12}\left(\tfrac{3}{8}\right)^4 = 1.64795 \times 10^{-3} \text{ in}^4$$

$$EI = (29 \times 10^6)(1.64795 \times 10^{-3}) = 47791 \text{ lb}\cdot\text{in}^2$$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (47791)}{(8)^2} = 7370 \text{ lb.}$$

Calculate P from thermal analysis. To obtain an approximate value, neglect the effect of eccentricity in the thermal analysis.

$$\text{Total elongation} = \alpha L(\Delta T) - \frac{PL}{EA} = 0$$

$$P = EA\alpha(\Delta T) = (29 \times 10^6)(0.140625)(6.5 \times 10^{-6})(120) = 3181 \text{ lb.}$$

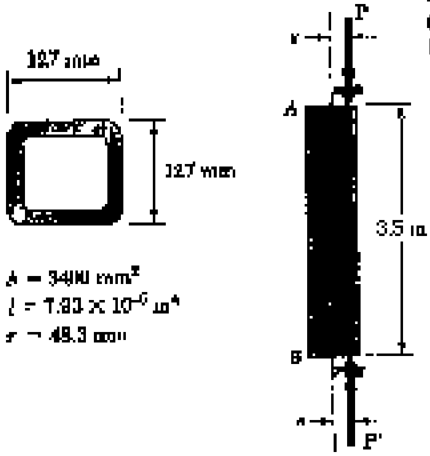
Calculate the deflection using the secant formula

$$d = y_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] = (0.03)\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{3181}{7370}}\right) - 1\right]$$

$$= (0.03)\left[\sec(1.03197) - 1\right] = (0.03)(0.94883) = 0.0285 \text{ in.}$$

For an improved thermal analysis including eccentricity, see solution of Prob. 10.41.

PROBLEM 10.43



10.43 A 3.5-m-long steel tube having the cross section and properties shown is used as a column. For the grade of steel used $\sigma_Y = 250$ MPa and $E = 200$ GPa. Knowing that a factor of safety of 2.6 with respect to permanent deformation is required, determine the allowable load **P** when the eccentricity e is (*a*) 15 mm, (*b*) 7.5 mm. (*Hint:* Since the factor of safety must be applied to the load **P**, not to the stress, use Fig. 10.24 to determine P_Y.)

SOLUTION

$A = 3400 \times 10^{-6}\ \text{m}^2 \qquad r = 48.3 \times 10^{-3}\ \text{m}$

$L_e = 3.5\ \text{m} \qquad \frac{L_e}{r} = \frac{3.5}{48.3 \times 10^{-3}} = 72.46$

$c = \frac{127}{2} = 63.5\ \text{mm}$

(a) $e = 15\ \text{mm} \qquad \frac{ec}{r^2} = \frac{(15)(63.5)}{(48.3)^2} = 0.40829$

Using Fig. 10.24 with $L_e/r = 72.46$ and $ec/r^2 = 0.40829$

$P/A = 144.75\ \text{MPa} = 144.75 \times 10^6\ \text{Pa}$

$P = (144.75 \times 10^6)(3400 \times 10^{-6}) = 492 \times 10^3\ \text{N}$

Using factor of safety $\quad P_{all} = \frac{492 \times 10^3}{2.6} = 189 \times 10^3\ \text{N} = 189\ \text{kN}$ ◀

(b) $e = 7.5\ \text{mm} \qquad \frac{ec}{r^2} = \frac{(7.5)(63.5)}{(48.3)^2} = 0.20415$

Using Fig. 10.24 with $L_e/r = 72.46$ and $ec/r^2 = 0.20415$

$P/A = 175.2\ \text{MPa} = 175.2 \times 10^6\ \text{Pa}$

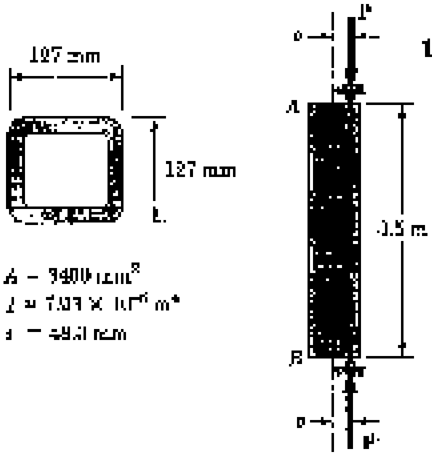
$P = (175.2 \times 10^6)(3400 \times 10^{-6}) = 596 \times 10^3\ \text{N}$

Using factor of safety $\quad P_{all} = \frac{596 \times 10^3}{2.6} = 229 \times 10^3\ \text{N} = 229\ \text{kN}$ ◀

PROBLEM 10.44

10.43 A 3.5-m-long steel tube having the cross section and properties shown is used as a column. For the grade of steel used $\sigma_Y = 250$ MPa and $E = 200$ GPa. Knowing that a factor of safety of 2.6 with respect to permanent deformation is required, determine the allowable load P when the eccentricity e is (*a*) 15 mm, (*b*) 7.5 mm. (*Hint:* Since the factor of safety must be applied to the load P, not to the stress, use Fig. 10.24 to determine P_Y).

10.44 Solve Prob. 10.43, assuming that the length of the steel tube is increased to 5 m.



SOLUTION

$A = 3400 \times 10^{-6}\ \text{m}^2 \qquad r = 48.3 \times 10^{-3}\ \text{m}$

$L_e = 5\ \text{m} \qquad \dfrac{L_e}{r} = \dfrac{5}{48.3 \times 10^{-3}} = 103.52$

$c = \dfrac{127}{2} = 63.5\ \text{mm}$

(a) $e = 15\ \text{mm} \qquad \dfrac{ec}{r^2} = \dfrac{(15)(63.5)}{(48.3)^2} = 0.40829$

Using Fig. 10.24 with $\dfrac{L_e}{r} = 103.52$

and $\dfrac{ec}{r^2} = 0.40829$ gives $\dfrac{P}{A} = 112.75\ \text{MPa} = 112.75 \times 10^{-6}\ \text{Pa}$

$P = (112.75 \times 10^{6})(3400 \times 10^{-6}) = 383 \times 10^{3}\ \text{N}$

Using factor of safety $\quad P_{all} = \dfrac{383 \times 10^3}{2.6} = 147 \times 10^3\ \text{N} = 147\ \text{kN}$ ◄

(b) $e = 7.5\ \text{mm} \qquad \dfrac{ec}{r^2} = \dfrac{(7.5)(63.5)}{(48.3)^2} = 0.20415$

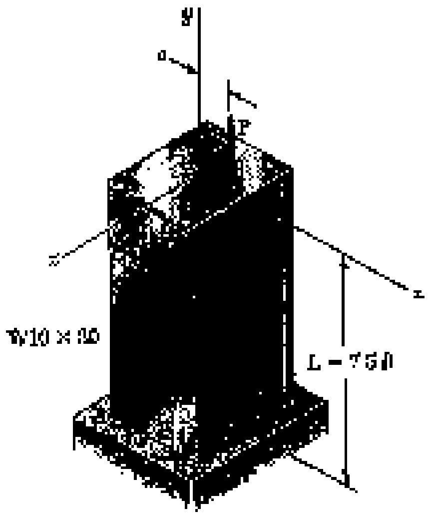
Using Fig. 10.24 gives $\dfrac{P}{A} = 133.2\ \text{MPa} = 133.2 \times 10^{6}\ \text{Pa}$

$P = (133.2 \times 10^{6})(3400 \times 10^{-6}) = 453 \times 10^{3}\ \text{N}$

Using factor of safety $\quad P_{all} = \dfrac{453 \times 10^3}{2.6} = 174 \times 10^3\ \text{N} = 174\ \text{kN}$ ◄

PROBLEM 10.45

10.45 An axial load **P** is applied to the W10 × 30 rolled-steel column *BC* that is free at its top *C* and fixed at its base *B*. Knowing that the eccentricity of the load is $e = 0.5$ in. and that for the grade of steel used $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi, determine (*a*) the magnitude of *P* of the allowable load when a factor of safety of 2.4 with respect to permanent deformation is required, (*b*) the ratio of the load found in part *a* to the magnitude of the allowable centric load for the column. (See hint of Prob. 10.43.)



SOLUTION

W10 × 30 $\quad A = 8.84 \text{ in}^2 \quad r_y = 1.37$ in.

$c = \frac{b_f}{2} = \frac{5.810}{2} = 2.905$ in $\quad I_y = 16.7 \text{ in}^4$

$L = 7.5 \text{ ft} = 90$ in $\quad L_e = 2L = 180$ in

$\frac{L_e}{r} = \frac{180}{1.37} = 131.39 \quad e = 0.5$ in

$\frac{ec}{r^2} = \frac{(0.5)(2.905)}{(1.37)^2} = 0.7739$

Using Fig 10.24 $\quad \frac{P}{A} = 10.47$ ksi

$P = (10.47)(8.84) = 92.6$ kips

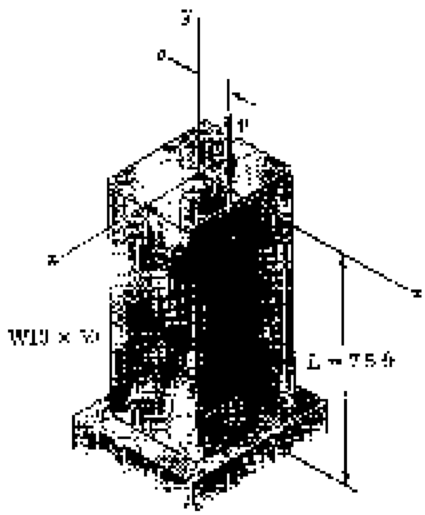
(a) Using factor of safety $\quad P_{all} = \frac{92.6}{2.4} = 38.6$ kips

$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 (29000)(16.7)}{(180)^2} = 147.5$ kips

Using factor of safety $\quad P_{all} = \frac{147.5}{2.4} = 61.5$ kips

(b) $\quad$ ratio $= \frac{38.6}{61.5} = 0.628$

PROBLEM 10.46



10.45 An axial load **P** is applied to the W10 × 30 rolled-steel column BC that is free at its top C and fixed at its base B. Knowing that the eccentricity of the load is $e = 0.5$ in. and that for the grade of steel used $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi, determine (*a*) the magnitude of P of the allowable load when a factor of safety of 2.4 with respect to permanent deformation is required, (*b*) the ratio of the load found in part *a* to the magnitude of the allowable centric load for the column. (See hint of Prob. 10.43.)

10.46 Solve Prob. 10.45, assuming that the length of the column is reduced to 5.0 ft.

SOLUTION

W10 × 30 $\quad A = 8.84 \text{ in}^2 \quad I_y = 16.7 \text{ in}^4$

$r_y = 1.37$ in $\quad c = \frac{b_f}{2} = \frac{5.810}{2} = 2.905$ in

$L = 5.0$ ft $= 60$ in $\quad L_e = 2L = 120$ in.

$$\frac{L_e}{r} = \frac{120}{1.37} = 87.6$$

$$\frac{ec}{r^2} = \frac{(0.5)(2.905)}{(1.37)^2} = 0.7739$$

Using Fig 10.24 $\quad \frac{P}{A} = 14.90$ ksi $\quad P = (14.90)(8.84) = 131.7$ kips

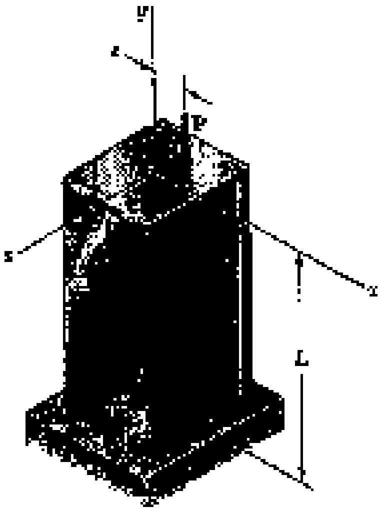
(a) Using factor of safety $\quad P_{all} = \frac{131.7}{2.4} = 54.9$ kips ◄

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2(29000)(16.7)}{(120)^2} = 332 \text{ kips}$$

Using factor of safety $\quad P_{all} = \frac{332}{2.4} = 138.3$ kips

(b) $\quad \text{ratio} = \frac{54.9}{138.3} = 0.397$ ◄

PROBLEM 10.47



10.47 A 55-kip axial load **P** is applied to a W8 × 24 rolled-steel column *BC* that is free at its top *C* and fixed at its base *B*. Knowing that the eccentricity of the load is *e* = 0.25 in., determine the largest permissible length *L* if the allowable stress in the column is 14 ksi. Use *E* = 29 × 10⁶ psi.

SOLUTION

Data: $P = 55$ kips, $e = 0.25$ in

$E = 29 \times 10^6$ psi $= 29000$ ksi

W 8 × 24 : $A = 7.08$ in^2 $b_f = 6.495$ in

$c = \frac{b_f}{2} = 3.25$ in, $I_y = 18.3$ in^4, $r_y = 1.61$ in.

$\sigma_{max} = 14$ ksi

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

$$\frac{A\sigma_{max}}{P} - 1 = \frac{ec}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)$$

$$\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{r^2}{ec}\left(\frac{A\sigma_{max}}{P} - 1\right) = \frac{(1.61)^2}{(0.25)(3.25)}\left[\frac{(7.08)(14)}{55} - 1\right] = 2.5592$$

$$\cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = 0.39075 \qquad \frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} = 1.16955$$

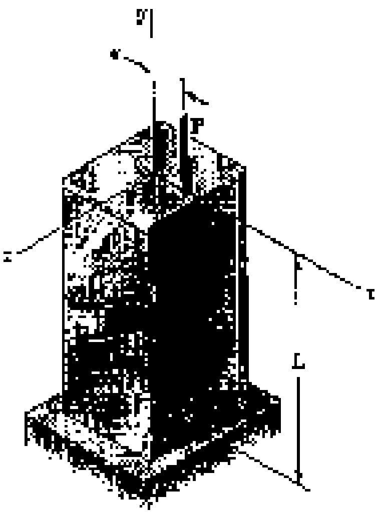
$$\frac{P}{P_{cr}} = \left[\frac{2}{\pi}(1.16935)\right]^2 = 0.55418$$

$$P_{cr} = \frac{P}{0.55418} = \frac{\pi^2 EI}{L_e^2}$$

$$L_e^2 = \frac{0.55418\,\pi^2 EI}{P} = \frac{0.55418\,\pi^2(29000)(18.3)}{55} = 52.78 \times 10^3 \text{ in}^2$$

$L_e = 229.7$ in $= 2L$ $\qquad L = 114.8$ in $= 9.57$ ft. ◄

PROBLEM 10.48



10.48 A 26-kip axial load **P** is applied to a W6 × 12 rolled-steel column BC that is free at its top C and fixed at its base B. Knowing that the eccentricity of the load is $e = 0.25$ in., determine the largest permissible length L if the allowable stress in the column is 14 ksi. Use $E = 29 \times 10^6$ psi.

SOLUTION

Data: $P = 26$ kips, $e = 0.25$ in

$E = 29 \times 10^6$ psi $= 29000$ ksi

W 6 × 12: $A = 3.55$ in^2 $b_f = 4.000$ in

$c = \frac{b_f}{2} = 2.000$ in, $I_y = 2.99$ in^4, $r_y = 0.918$ in.

$\sigma_{max} = 14$ ksi

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

$$\frac{A\sigma_{max}}{P} - 1 = \frac{ec}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)$$

$$\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{r^2}{ec}\left(\frac{A\sigma_{max}}{P} - 1\right) = \frac{(0.918)^2}{(0.25)(2.000)}\left[\frac{(3.55)(14)}{26} - 1\right] = 1.53635$$

$$\cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = 0.65089 \qquad \frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} = 0.86204$$

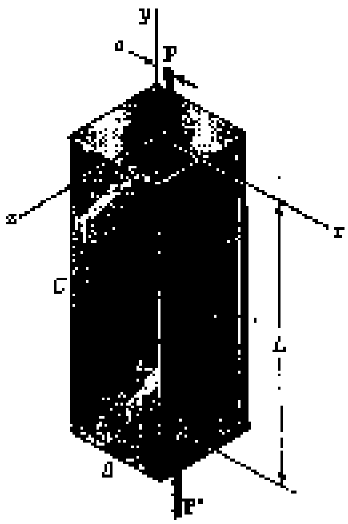
$$\frac{P}{P_{cr}} = \left[\frac{2}{\pi}(0.86204)\right]^2 = 0.30117$$

$$P_{cr} = \frac{P}{0.30117} = \frac{\pi^2 EI}{L_e^2}$$

$$L_e^2 = \frac{0.30117\,\pi^2\,EI}{P} = \frac{0.30117\,\pi^2(29000)(2.99)}{26} = 9.913 \times 10^3 \text{ in}^2$$

$L_e = 99.56$ in $= 2L$ $\qquad L = 49.78$ in. $= 4.15$ ft ◀

PROBLEM 10.49



10.49 Axial loads of magnitude $P = 84$ kN are applied parallel to the geometric axis of a W200 × 22.5 rolled-steel column AB and intersect the x axis at a distance e from its geometric axis. Knowing that allowable stress $\sigma_{all} = 75$ MPa and $E = 200$ GPa, determine the largest permissible length L when (*a*) $e = 5$ mm, (*b*) $e = 12$ mm.

SOLUTION

Data: $P = 84 \times 10^3$ N $\quad E = 200 \times 10^9$ Pa

W 200 × 22.5 $\quad A = 2860\ \text{mm}^2 = 2860 \times 10^{-6}\ \text{m}^2$

$b_f = 102$ mm $\quad c = \frac{b_f}{2} = 51$ mm $\quad r_y = 22.3$ mm

$I_y = 1.42 \times 10^6\ \text{mm}^4 = 1.42 \times 10^{-6}\ \text{m}^4$

$\sigma_{all} = \sigma_{max} = 75$ MPa $= 75 \times 10^6$ Pa

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

$$\frac{A\sigma_{max}}{P} - 1 = \frac{ec}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)$$

$$\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{r^2}{ec}\left(\frac{A\sigma_{max}}{P} - 1\right)$$

(a) $e = 5$ mm $\quad \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{(22.3)^2}{(5)(51)}\left[\frac{(2860 \times 10^{-6})(75 \times 10^6)}{84 \times 10^3} - 1\right] = 3.0297$

$\cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = 0.33006 \qquad \frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} = 1.2344$

$\frac{P}{P_{cr}} = \left[\frac{2}{\pi}(1.2344)\right]^2 = 0.61757$

$P_{cr} = \frac{P}{0.61757} = \frac{\pi^2 EI}{L_e^2}$

$L_e^2 = \frac{0.61757\,\pi^2 EI}{P} = \frac{0.61757\,\pi^2(200 \times 10^9)(1.42 \times 10^{-6})}{84 \times 10^3} = 20.61\ \text{m}^2$

$L_e = 4.54$ m $\qquad L = L_e = 4.54$ m ◄

(b) $e = 12$ mm $\quad \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{(22.3)^2}{(12)(51)}\left[\frac{(2860 \times 10^{-6})(75 \times 10^6)}{84 \times 10^3} - 1\right] = 1.26238$

$\cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = 0.79216 \qquad \frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} = 0.6564635$

$\frac{P}{P_{cr}} = \left[\frac{2}{\pi}(0.65646)\right]^2 = 0.17466$

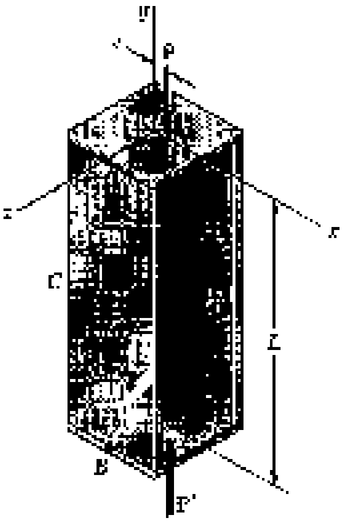
$P_{cr} = \frac{P}{0.17466} = \frac{\pi^2 EI}{L_e^2}$

$L_e^2 = \frac{0.17466\,\pi^2 EI}{P} = \frac{0.17466\,\pi^2(200 \times 10^9)(1.42 \times 10^{-6})}{84 \times 10^3} = 5.828\ \text{m}^2$

$L_e = 2.41$ m $\qquad L = L_e = 2.41$ m ◄

PROBLEM 10.50

10.50 Axial loads of magnitude $P = 580$ kN are applied parallel to the geometric axis of a W250 × 80 rolled-steel column AB and intersect the x axis at a distance e from its geometric axis. Knowing that allowable stress $\sigma_{all} = 75$ MPa and $E = 200$ GPa, determine the largest permissible length L when (a) $e = 5$ mm, (b) $e = 10$ mm.



SOLUTION

Data: $P = 580 \times 10^3$ N $\qquad E = 200 \times 10^9$ Pa

W 250 × 80 $\qquad A = 10200$ mm^2 $= 10200 \times 10^{-6}$ m^2

$b_f = 255$ mm $\qquad c = \frac{b_f}{2} = 127.5$ mm $\qquad r_y = 65.0$ mm

$I_y = 43.1 \times 10^6$ mm^4 $= 43.1 \times 10^{-6}$ m^4

$\sigma_{all} = \sigma_{max} = 75$ MPa $= 75 \times 10^6$ Pa

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r_y^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

$$\frac{A\sigma_{max}}{P} - 1 = \frac{ec}{r_y^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)$$

$$\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{r_y^2}{ec}\left(\frac{A\sigma_{max}}{P} - 1\right)$$

(a) $e = 5$ mm $\qquad \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{(65.0)^2}{(5)(127.5)}\left[\frac{(10200 \times 10^{-6})(75 \times 10^6)}{580 \times 10^3} - 1\right] = 2.1139$

$\cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = 0.47305 \qquad \frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} = 1.07804$

$\frac{P}{P_{cr}} = \left[\frac{2}{\pi}(1.07804)\right]^2 = 0.47101$

$P_{cr} = \frac{P}{0.47101} = \frac{\pi^2 E I_y}{L_e^2}$

$L_e^2 = \frac{0.47101\,\pi^2 E I_y}{P} = \frac{0.47101\,\pi^2(200 \times 10^9)(43.1 \times 10^{-6})}{580 \times 10^3} = 69.09$ m^2

$L_e = 8.31$ m. $\qquad L = L_e = 8.31$ m ◄

(b) $e = 10$ mm $\qquad \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = \frac{(65)^2}{(10)(127.5)}\left[\frac{(10200 \times 10^{-6})(75 \times 10^6)}{580 \times 10^3} - 1\right] = 1.05696$

$\cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) = 0.94611 \qquad \frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} = 0.32980$

$\frac{P}{P_{cr}} = \left[\frac{2}{\pi}(0.32980)\right]^2 = 0.044083$

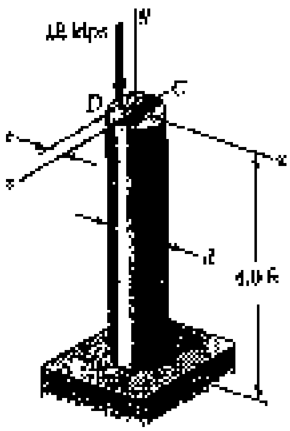
$P_{cr} = \frac{P}{0.044083} = \frac{\pi^2 E I}{L_e^2}$

$L_e^2 = \frac{0.044083\,\pi^2 E I}{P} = \frac{0.044083\,\pi^2(200 \times 10^9)(43.1 \times 10^{-6})}{580 \times 10^3} = 6.466$ m^2

$L_e = 2.54$ m $\qquad L = L_e = 2.54$ m ◄

PROBLEM 10.51

10.51 A 12-kip axial load is applied with an eccentricity $e = 0.375$ in. to the circular steel rod BC that is free at its top C and fixed at its base B. Knowing that the stock of rods available for use have diameters in increments of $\frac{1}{8}$ in. from 1.5 in. to 3.0 in., determine the lightest rod that may be used if $\sigma_{all} = 15$ ksi. Use $E = 29 \times 10^6$ psi.



SOLUTION

$E = 29 \times 10^6$ psi $= 29000$ ksi $\qquad d =$ diameter (in.)

$A = \frac{\pi}{4} d^2 \qquad I = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi d^4}{64} \qquad c = \frac{1}{2} d \qquad e = 0.375$ in

$L = 4.0$ ft $= 48$ in $\qquad L_e = 2L = 96$ in

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 (29000) \pi d^4}{(64)(96)^2} = 1.52449\, d^4 \text{ kips}$$

$$r^2 = \frac{I}{A} = \frac{\pi d^4}{64} \cdot \frac{4}{\pi d^2} = \frac{d^2}{16} \qquad P = 12 \text{ kips}$$

$$\frac{ec}{r^2} = \frac{(0.375)(\frac{1}{2} d)}{\frac{1}{16} d^2} = \frac{3}{d}$$

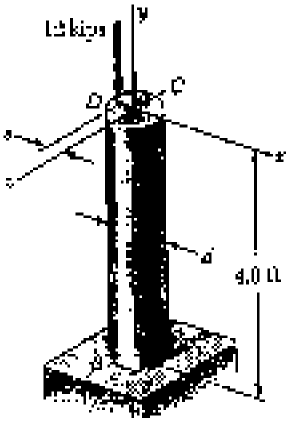
$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2} \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

d (in)	A (in^2)	P_{cr} (kips)	$\frac{ec}{r^2}$	σ_{max} (ksi)
2.25	3.976	39.07	1.3333	9.26
2.0	3.1416	24.39	1.5	16.49
2.125	3.546	31.09	1.4118	11.90

Use $d = 2.125$ in.

$\sigma_{max} = 11.90$ ksi < 15 ksi

PROBLEM 10.52



10.51 A 12-kip axial load is applied with an eccentricity $e = 0.375$ in. to the circular steel rod BC that is free at its top C and fixed at its base B. Knowing that the stock of rods available for use have diameters in increments of $\frac{1}{8}$ in. from 1.5 in. to 3.0 in., determine the lightest rod that may be used if $\sigma_{all} = 15$ ksi. Use $E = 29 \times 10^6$ psi.

10.52 Solve Prob. 10.51, assuming that the 12-kip axial load will be applied to the rod with an eccentricity $e = \frac{1}{2}d$.

SOLUTION

$E = 29 \times 10^6$ psi $= 29000$ ksi $\qquad d =$ diameter (in)

$A = \frac{\pi}{4}d^2 \qquad I = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi}{64}d^4 \qquad c = \frac{1}{2}d \qquad e = \frac{1}{2}d$

$L = 4$ ft $= 48$ in $\qquad L_e = 2L = 96$ in

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2(29000)(\pi d^4)}{(64)(96)^2} = 1.52449\, d^4$$

$$r^2 = \frac{I}{A} = \frac{\pi d^4}{64} \cdot \frac{4}{\pi d^2} = \frac{1}{16}d^2 \qquad P = 12 \text{ kips}$$

$$\frac{ec}{r^2} = \frac{\left(\frac{1}{2}d\right)\left(\frac{1}{2}d\right)}{\frac{1}{16}d^2} = 4.0$$

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right] = \frac{P}{A}\left[1 + 4.0\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

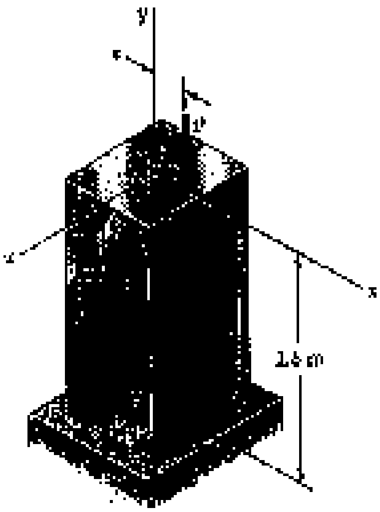
d (in)	A (in²)	P_{cr} (kips)	σ_{max} (ksi)
2.25	3.976	39.07	21.75
3.0	7.068	123.48	9.39
2.5	4.909	59.55	15.28
2.625	5.412	72.38	13.27

Use $d = 2.625$ in. ◄

$\sigma_{max} = 13.27$ ksi < 15 ksi

PROBLEM 10.53

10.53 An axial load of magnitude $P = 220$ kN is applied at a point located on the x axis at a distance $e = 6$ mm from the geometric axis of the wide-flange column BC. Knowing that $E = 200$ GPa, choose the lightest W200 shape that may be used if $\sigma_{all} = 120$ MPa.



SOLUTION

$P = 220 \times 10^3$ N $\qquad L = 1.8$ m $\qquad L_e = 2L = 3.6$ m

$$P_{cr} = \frac{\pi^2 E I_y}{L_e^2} = \frac{\pi^2 (200 \times 10^9) I_y}{3.6^2} = 152.3 \times 10^9\, I_y \quad N$$

$$e = 6\text{ mm} \qquad c = \frac{b_f}{2} \qquad \frac{ec}{r^2} = \frac{e\, b_f}{2 r_y^2} =$$

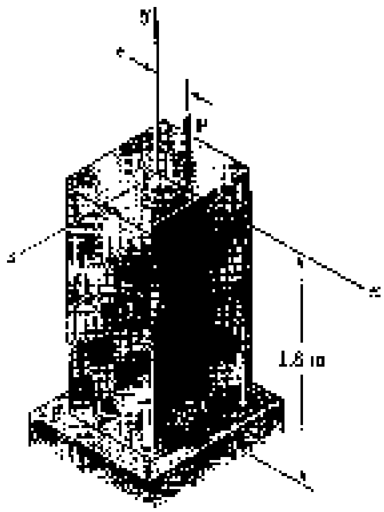
$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

Shape	$A\,(10^{-6}\,m^2)$	b_f (mm)	$I_y\,(10^{-6}\,m^4)$	r_y (mm)	P_{cr} (kN)	$\frac{ec}{r^2}$	σ_{max} (MPa)
W200 × 41.7	5310	166	9.01	41.2	1372	0.2934	56.5
W200 × 26.6	3390	133	3.30	31.2	502.6	0.4099	117.4
W200 × 22.5	2860	102	1.42	22.3	*216.3		

$* < P$

Use W200 × 26.6 ➡ $\sigma_{max} = 117.4$ MPa

PROBLEM 10.54



10.53 An axial load of magnitude $P = 220$ kN is applied at a point located on the x axis at a distance $e = 6$ mm from the geometric axis of the wide-flange column BC. Knowing that $E = 200$ GPa, chose the lightest W200 shape that may be used if $\sigma_{all} = 120$ MPa.

10.54 Solve Prob. 10.53, assuming that the magnitude of the axial load is $P = 345$ kN.

SOLUTION

$P = 345 \times 10^3$ N $\qquad L = 1.8$ m $\qquad L_e = 2L = 3.6$ m

$$P_{cr} = \frac{\pi^2 E I_y}{L_e^2} = \frac{\pi^2 (200 \times 10^9) I_y}{(3.6)^2} = 152.3 \times 10^9 I_y \quad \text{N}$$

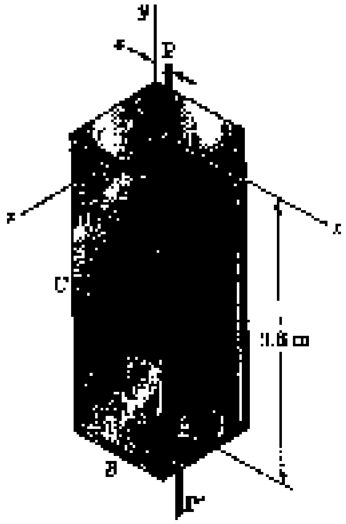
$e = 6$ mm $\qquad c = \frac{b_f}{2} \qquad \frac{ec}{r^2} = \frac{e\, b_f}{2 r_y^2}$

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2} \sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right]$$

Shape	A (10^{-6} m²)	b_f (mm)	I_y (10^{-6} m⁴)	r_y (mm)	P_{cr} (kN)	$\frac{ec}{r^2}$	σ_{max} (MPa)
W 200 × 41.7	5310	166	9.01	41.2	1372	0.2934	92.0
W 200 × 26.6	3390	133	3.30	31.2	502.6	0.4099	258
W 200 × 35.9	4580	165	7.64	40.8	1164	0.2974	109.5 ←
W 200 × 31.3	4000	134	4.10	32.0	624.4	0.3926	172.6

Use W 200 × 35.9 ◄ $\qquad \sigma_{max} = 109.5$ MPa

PROBLEM 10.55



10.55 Axial loads of magnitude $P = 175$ kN are applied to a point located on the x axis at a distance $e = 12$ mm from the geometric axis of the W250 × 44.8 rolled-steel column AB. Knowing that $\sigma_Y = 250$ MPa and $E = 200$ GPa, determine the factor of safety with respect to yield. (*Hint:* Since the factor of safety must be applied to the load P, not to the stresses, use Fig. 10.24 to determine P_Y.)

SOLUTION

For W 250 × 44.8 $\quad A = 5720 \text{ mm}^2, \quad r_y = 35.1$ mm

$L_e = 3800$ mm $\qquad L_e/r = 108.26$

$c = \frac{b_f}{2} = \frac{148}{2} = 74$ mm $\qquad e = 12$ mm

$\frac{ec}{r^2} = \frac{(12)(74)}{(35.1)^2} = 0.72077$

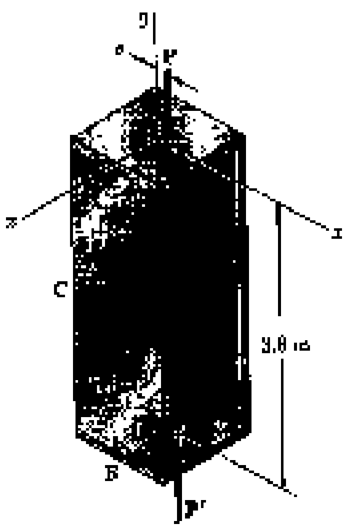
Using Fig 10.24 with $L_e/r = 108.26$ and $\frac{ec}{r^2} = 0.72077$

$P_Y/A = 90.37 \text{ MPa} = 90.37 \times 10^6 \text{ N/m}^2$

$P_Y = A\, P_Y/A = (5720 \times 10^{-6})(90.37 \times 10^6) = 517 \times 10^3 \text{ N} = 517 \text{ kN}$

$F.S. = \frac{P_Y}{P} = \frac{517}{175} = 2.95$

PROBLEM 10.56



10.55 Axial loads of magnitude $P = 175$ kN are applied to a point located on the x axis at a distance $e = 12$ mm from the geometric axis of the W250 × 44.8 rolled-steel column AB. Knowing that $\sigma_Y = 250$ MPa and $E = 200$ GPa, determine the factor of safety with respect to yield. (*Hint:* Since the factor of safety must be applied to the load P, not to the stresses, use Fig. 10.24 to determine P_Y.)

10.56 Solve Prob. 10.55, assuming that $e = 16$ mm and $P = 155$ kN.

SOLUTION

For W 250 × 44.8 $\quad A = 5720 \text{ mm}^2; \quad r_y = 35.1$ mm

$L_e = 3800$ mm $\qquad L_e/r = 108.26$

$c = \frac{b_f}{2} = \frac{148}{2} = 74$ mm $\qquad e = 16$ mm

$\frac{ec}{r^2} = \frac{(16)(74)}{(35.1)^2} = 0.96103$

Using Fig 10.24 with $L_e/r = 108.26$ and $\frac{ec}{r^2} = 0.96103$

$P_Y/A = 81.17 \text{ MPa} = 81.17 \text{ N/m}^2$

$P_Y = A(P_Y/A) = (5720 \times 10^{-6})(81.17 \times 10^6) = 464 \times 10^3 \text{ N} = 464 \text{ kN}$

$F.S. = \frac{P_Y}{P} = \frac{464}{155} = 3.00$

PROBLEM 10.57

10.57 Using allowable stress design, determine the allowable centric load for a column of 6.5-m effective length that is made from the following rolled-steel shape: (*a*) W250 × 49.1, (*b*) W250 × 80. Use $\sigma_Y = 250$ MPa and $E = 200$ GPa.

SOLUTION

Steel: $C_c = \sqrt{\dfrac{2\pi^2 E}{\sigma_Y}} = \sqrt{\dfrac{2\pi^2(200\times10^9)}{250\times10^6}} = 125.664$

(a) W250 × 49.1 $\quad A = 6250\times10^{-6}\ \text{m}^2 \quad r_{min} = 49.2\times10^{-3}$ m

$$\frac{L_e}{r} = \frac{6.5}{49.2\times10^{-3}} = 132.11 > C_c$$

$$\sigma_{all} = \frac{\pi^2 E}{1.92(L/r)^2} = \frac{\pi^2(200\times10^9)}{(1.92)(132.11)^2} = 58.9\times10^6\ \text{Pa}$$

$$P_{all} = A\sigma_{all} = (6250\times10^{-6})(58.9\times10^6) = 368\times10^3\ \text{N} = 368\ \text{kN}$$

(b) W250 × 80 $\quad A = 10200\times10^{-6}\ \text{m}^2 \quad r_{min} = 65.0\times10^{-3}$ m

$$\frac{L_e}{r} = \frac{6.5}{65.0\times10^{-3}} = 100 < C_c \qquad \frac{L/r}{C_c} = 0.79577$$

$$F.S. = \frac{5}{3} + \frac{3}{8}(0.79577) - \frac{1}{8}(0.79577)^3 = 1.90209$$

$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L/r}{C_c}\right)^2\right] = \frac{250\times10^6}{1.90209}\left[1 - \frac{1}{2}(0.79577)^2\right] = 89.82\times10^6\ \text{Pa}$$

$$P_{all} = A\sigma_{all} = (10200\times10^{-6})(89.82\times10^6) = 916\times10^3\ \text{N} = 916\ \text{kN}$$

PROBLEM 10.58

10.58 A W8 × 31 rolled-steel shape is used to form a column of 21-ft effective length. Using allowable stress design, determine the allowable centric load if the yield strength of the grade of steel used is (*a*) $\sigma_Y = 36$ ksi, (*b*) $\sigma_Y = 50$ ksi. Use $E = 29\times10^6$ psi.

SOLUTION

Steel: $E = 29000$ ksi $\quad$ W 8×31 $\quad A = 9.13\ \text{in}^2 \quad r_{min} = 2.02$ in

$L_e = 21$ ft $= 252$ in $\qquad L_e/r = 124.75$

(a) $\sigma_Y = 36$ ksi $\qquad C_c = \sqrt{\dfrac{2\pi^2 E}{\sigma_Y}} = \sqrt{\dfrac{2\pi^2(29000)}{36}} = 126.10$

$L_e/r < C_c \qquad \dfrac{L/r}{C_c} = 0.98932$

$$F.S. = \frac{5}{3} + \frac{3}{8}(0.98982) - \frac{1}{8}(0.98932)^3 = 1.91662$$

$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L/r}{C_c}\right)^2\right] = \frac{36}{1.91662}\left[1 - \frac{1}{2}(0.98892)^2\right] = 9.59\ \text{ksi}$$

$$P_{all} = \sigma_{all}A = (9.59)(9.13) = 87.6\ \text{kips}$$

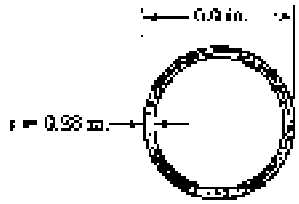
(b) $\sigma_Y = 50$ ksi $\qquad C_c = \sqrt{\dfrac{2\pi^2(29000)}{50}} = 107.00$

$L_e/r > C_c \qquad \sigma_{all} = \dfrac{\pi^2 E}{\frac{23}{12}(L/r)^2} = 9.59$ ksi

$$P_{all} = \sigma_{all}A = (9.59)(9.13) = 87.6\ \text{kips}$$

PROBLEM 10.59

10.59 A steel pipe having the cross section shown is used as a column. Using allowable stress design, determine the allowable centric load if the effective length of the column is (*a*) 18 ft, (*b*) 26 ft. Use $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi.



SOLUTION

$c_o = \frac{d_o}{2} = 3.0$ in $\qquad c_i = c_o - t = 2.72$ in

$A = \pi(c_o^2 - c_i^2) = 5.0316$ in^2

$I = \frac{\pi}{4}(c_o^4 - c_i^4) = 20.627$ in^4 $\qquad r = \sqrt{\frac{I}{A}} = 2.0247$ in

Steel: $\quad E = 29000$ ksi $\qquad C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(29000)}{36}} = 126.10$

(a) $L_e = 18$ ft $= 216$ in $\qquad L_e/r = 106.68 < C_c \qquad \frac{L_e/r}{C_c} = 0.84601$

$F.S. = \frac{5}{3} + \frac{3}{8}(0.84601) - \frac{1}{8}(0.84601)^3 = 1.9082$

$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L/r}{C_c}\right)^2\right] = \frac{36}{1.9082}\left[1 - \frac{1}{2}(0.84601)^2\right] = 12.11$ ksi

$P_{all} = \sigma_{all} A = (12.11)(5.0316) = 61.0$ kips ◄

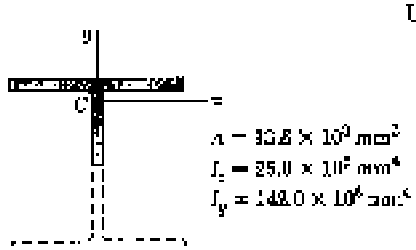
(b) $L_e = 26$ ft $= 312$ in $\qquad L_e/r = 154.097 > C_c$

$\sigma_{all} = \frac{\pi^2 E}{1.92(L/r)^2} = \frac{\pi^2(29000)}{(1.92)(154.097)^2} = 6.28$ ksi

$P_{all} = \sigma_{all} A = (6.28)(5.0316) = 31.6$ kips ◄

PROBLEM 10.60

10.60 A column is made from half of a W360 × 216 rolled-steel shape, with the geometric properties as shown. Using allowable stress design, determine the allowable centric load if the effective length of the column is (*a*) 4.0 m, (*b*) 6.5 m. Use $\sigma_Y = 345$ MPa and $E = 200$ GPa.



SOLUTION

$r = \sqrt{\frac{I_{min}}{A}} = \sqrt{\frac{26.0 \times 10^6}{13.8 \times 10^3}} = 43.406$ mm $= 43.406 \times 10^{-3}$ m

$A = 13.8 \times 10^{-3}$ m^2

Steel $\quad C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(200 \times 10^9)}{345 \times 10^6}} = 106.97$

(a) $L_e = 4.0$ m $\qquad \frac{L_e}{r} = 92.153 < C_c \qquad \frac{L_e/r}{C_c} = 0.86149$

$F.S. = \frac{5}{3} + \frac{3}{8}(0.86149) - \frac{1}{8}(0.86149)^3 = 1.9098$

$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L/r}{C_c}\right)^2\right] = \frac{345 \times 10^6}{1.9098}\left[1 - \frac{1}{2}(0.86149)^2\right] = 113.61 \times 10^6$ Pa

$P_{all} = \sigma_{all} A = (113.61 \times 10^6)(13.8 \times 10^{-3}) = 1568 \times 10^3$ N $= 1568$ kN ◄

(b) $L_e = 6.5$ m $\qquad \frac{L_e}{r} = 149.75 > C_c$

$\sigma_{all} = \frac{\pi^2 E}{1.92(L/r)^2} = \frac{\pi^2(200 \times 10^9)}{(1.92)(149.75)^2} = 45.845 \times 10^6$ Pa

$P_{all} = \sigma_{all} A = (45.845 \times 10^6)(13.8 \times 10^{-3}) = 633 \times 10^3$ N $= 633$ kN ◄

PROBLEM 10.61

10.61 A 3.5-m effective length column is made of sawn lumber with a 114 × 140-mm cross section. Knowing that for the grade of wood used the adjusted allowable stress for compression parallel to the grain $\sigma_C = 7.6$ MPa and $E = 10$ GPa, determine the maximum allowable centric load for the column.

SOLUTION

Sawn lumber: $c = 0.8$, $\quad \sigma_C = 7.6$ MPa $\quad K_{ce} = 0.3 \quad E = 10000$ MPa

$A = (114)(140) = 15960\ \text{mm}^2 = 15960 \times 10^{-6}\ \text{m}^2$

$d = 114\ \text{mm} = 114 \times 10^{-3}\ \text{m}$

$L/d = 3.5/114 \times 10^{-3} = 30.70$

$\sigma_{CE} = \dfrac{K_{ce}E}{(L/d)^2} = \dfrac{(0.3)(10000)}{(30.70)^2} = 3.1827\ \text{MPa} \qquad \dfrac{\sigma_{CE}}{\sigma_C} = 0.41878$

$u = \dfrac{1 + \sigma_{CE}/\sigma_C}{2c} = \dfrac{1.41878}{(2)(0.8)} = 0.88673 \qquad v = \dfrac{\sigma_{CE}/\sigma_C}{c} = 0.523475$

$C_P = u - \sqrt{u^2 - v} = 0.37408$

$\sigma_{all} = \sigma_C C_P = (7.6)(0.37408) = 2.84\ \text{MPa}$

$P_{all} = \sigma_{all} A = (2.84 \times 10^6)(15960 \times 10^{-6}) = 45.4 \times 10^3\ \text{N} = 45.4\ \text{kN}$ ◄

PROBLEM 10.62

10.62 A sawn lumber column with a 7.5 × 5.5-in. cross section has a 18-ft effective length. Knowing that for the grade of wood used the adjusted allowable stress for compression parallel to the grain is $\sigma_C = 1220$ psi and that $E = 1.3 \times 10^6$ psi, determine the maximum allowable centric load for the column.

SOLUTION

Sawn lumber: $c = 0.8$, $\quad \sigma_C = 1220$ psi $\quad E = 1.3 \times 10^6$ psi $\quad K_{ce} = 0.3$

$A = (7.5)(5.5) = 41.25\ \text{in}^2 \qquad d = 5.5\ \text{in} \qquad L = 18\ \text{ft} = 216\ \text{in}$

$L/d = 216/5.5 = 39.273$

$\sigma_{CE} = \dfrac{K_{ce}E}{(L/d)^2} = \dfrac{(0.3)(1.3 \times 10^6)}{(39.273)^2} = 252.86\ \text{psi} \qquad \dfrac{\sigma_{CE}}{\sigma_C} = 0.20726$

$u = \dfrac{1 + \sigma_{CE}/\sigma_C}{2c} = \dfrac{1.20726}{(2)(0.8)} = 0.754537 \qquad v = \dfrac{\sigma_{CE}/\sigma_C}{c} = 0.259075$

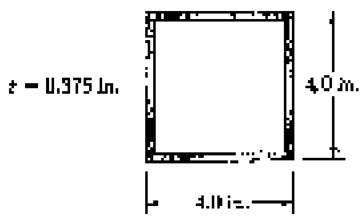
$C_P = u - \sqrt{u^2 - v} = 0.197535$

$\sigma_{all} = \sigma_C C_P = (1220)(0.197535) = 241.0\ \text{psi}$

$P_{all} = \sigma_{all} A = (241.0)(41.25) = 9.94 \times 10^3\ \text{lb} = 9.94\ \text{kips}$ ◄

PROBLEM 10.63

10.63 A compression member has the cross section shown and an effective length of 5 ft. Knowing that the aluminum alloy used is 2014-T6, determine the allowable centric load.



SOLUTION

$b_o = 4.0 \qquad b_i = b_o - 2t = 3.25 \text{ in.}$

$A = (4.0)^2 - (3.25)^2 = 5.4375 \text{ in}^2$

$I = \frac{1}{12}[(4.0)^4 - (3.25)^4] = 12.036 \text{ in}^4$

$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{12.036}{5.4375}} = 1.488 \text{ in.} \qquad L = 5 \text{ ft} = 60 \text{ in}$

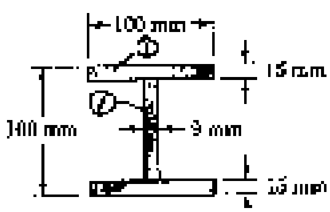
$\frac{L}{r} = \frac{60}{1.488} = 40.33 < 55$ for 2014-T6 aluminum alloy

$\sigma_{all} = 30.7 - 0.23(L/r) = 30.7 - (0.23)(40.33) = 21.42 \text{ ksi}$

$P_{all} = \sigma_{all} A = (21.42)(5.4375) = 116.5 \text{ kips}$

PROBLEM 10.64

10.64 A compression member has the cross section shown and an effective length of 1.55 m. Knowing that the aluminum alloy used is 6061-T6, determine the allowable centric load.



SOLUTION

$I_{x1} = \frac{1}{12}(100)(15)^3 + (100)(15)(42.5)^2 = 2.7375 \times 10^6 \text{ mm}^4$

$I_{x2} = \frac{1}{12}(9)(70)^3 = 257.25 \times 10^3 \text{ mm}^4$

$I_x = 2I_{x1} + I_{x2} = 5.73225 \times 10^6 \text{ mm}^4$

$I_y = 2\left[\frac{1}{12}(15)(100)^3\right] + \frac{1}{12}(70)(9)^3 = 2.50425 \times 10^6 \text{ mm}^4$

$A = 2(100)(15) + (9)(70) = 3630 \text{ mm}^2 = 3630 \times 10^{-6} \text{ m}^2$

$r = \sqrt{\frac{I_{min}}{A}} = \sqrt{\frac{2.50425 \times 10^6}{3630}} = 26.265 \text{ mm} = 26.265 \times 10^{-3} \text{ m}$

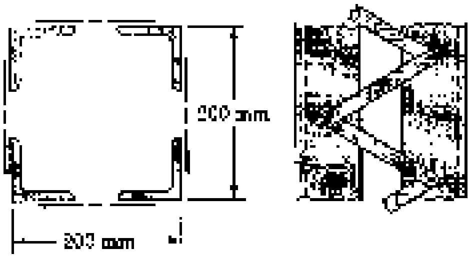
$L_e = 1.55 \text{ m} \qquad L_e/r = 59.01 < 66 \quad$ (6061-T6 aluminum)

$\sigma_{all} = 139 - 0.868(L/r) = 139 - (0.868)(59.01) = 87.78 \text{ MPa}$

$P_{all} = \sigma_{all} A = (87.78 \times 10^6)(3630 \times 10^{-6}) = 319 \times 10^3 \text{ N} = 319 \text{ kN}$

PROBLEM 10.65

10.65 A column of 6.4-m effective length is obtained by connecting four 89 × 89 × 9.5-mm steel angles with lacing bars as shown. Using allowable stress design, determine the allowable centric load for the column. Use $\sigma_Y = 345$ MPa and $E = 200$ GPa.



SOLUTION

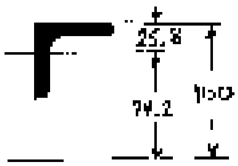
Steel: $C_c = \sqrt{\dfrac{2\pi^2 E}{\sigma_Y}} = \sqrt{\dfrac{2\pi^2(200\times10^9)}{345\times10^6}} = 106.97$

89 × 89 × 9.5 mm angle

$A_L = 1600 \text{ mm}^2$

$x = 25.8 \text{ mm}$

$I_x = 1.19\times10^6 \text{ mm}^4$



$d = 100 - x = 74.2 \text{ mm}$

$I = 4(Ad^2 + I_x) = 4[(1600)(74.2)^2 + 1.19\times10^6]$

$= 39.996\times10^6 \text{ mm}^4$

$A = 4A_L = 6400 \text{ mm}^2 = 6400\times10^{-6} \text{ m}^2$

$r = \sqrt{\dfrac{I}{A}} = 79.053 \text{ mm} = 79.053\times10^{-3} \text{ m}$

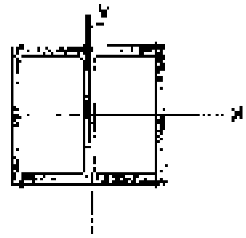
$\dfrac{L_e}{r} = \dfrac{6.4}{79.053\times10^{-3}} = 80.958 < C_c \qquad \dfrac{L_e/r}{C_c} = 0.75683$

$F.S. = \frac{5}{3} + \frac{3}{8}(0.75683) - \frac{1}{8}(0.75683)^3 = 1.8963$

$\sigma_{all} = \dfrac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\dfrac{L/r}{C_c}\right)^2\right] = \dfrac{345\times10^6}{1.8963}\left[1 - \frac{1}{2}(0.75683)^2\right] = 129.83\times10^6 \text{ Pa}$

$P_{all} = \sigma_{all}A = (129.83\times10^6)(6400\times10^{-6}) = 831\times10^3 \text{ N} = 831 \text{ kN}$

PROBLEM 10.68



10.68 A column of 23-ft effective length is obtained by welding two $\frac{3}{8}$-in. steel plates to a W10 × 33 rolled-steel shape as shown. Using allowable stress design, determine the allowable centric load for the column. Use $\sigma_Y = 50$ ksi and $E = 29 \times 10^6$ psi.

SOLUTION

For W10 × 33 $\quad A = 9.71\ \text{in}^2,\quad d = 9.73\ \text{in}\quad b_f = 7.960\ \text{in.}$
$I_x = 170\ \text{in}^4,\quad I_y = 36.6\ \text{in}^4$

For column: $A = 9.71 + (2)(\frac{3}{8})(9.73) = 17.0075\ \text{in}^2$

$I_x = 170 + (2)\frac{1}{12}(\frac{3}{8})(9.73)^3 = 227.57\ \text{in}^4$

$I_y = 36.6 + (2)\left[(\frac{3}{8})(9.73)(\frac{7.960}{2} + \frac{3}{16})^2 + \frac{1}{12}(9.73)(\frac{3}{8})^3\right]$
$= 36.6 + (2)[63.37 + 0.043] = 163.43\ \text{in}^4$

$$r = \sqrt{\frac{I_{min}}{A}} = \sqrt{\frac{163.43}{17.0075}} = 3.100\ \text{in.}$$

Steel: $C_c = \sqrt{\dfrac{2\pi^2 E}{\sigma_Y}} = \sqrt{\dfrac{2\pi^2(29000)}{50}} = 107.00$

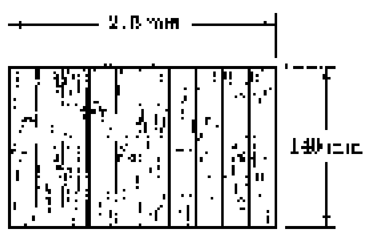
$L_e = 23\ \text{ft} = 276\ \text{in}\qquad \dfrac{L_e}{r} = \dfrac{276}{3.100} = 89.03 < C_c\qquad \dfrac{L_e/r}{C_c} = 0.83208$

$$F.S. = \frac{5}{3} + \frac{3}{8}(0.83208) - \frac{1}{8}(0.83208)^3 = 1.9067$$

$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L/r}{C_c}\right)^2\right] = \frac{50}{1.9067}\left[1 - \frac{1}{2}(0.83208)^2\right] = 17.145\ \text{ksi}$$

$$P_{all} = \sigma_{all}A = (17.145)(17.0075) = 292\ \text{kips}$$

PROBLEM 10.69



10.69 A rectangular column with a 4.4-m effective length is made of glued laminated wood. Knowing that for the grade of wood used the adjusted allowable stress for compression parallel to the grain is $\sigma_C = 8.3$ MPa and that $E = 10$ GPa, determine the maximum allowable centric load for the column.

SOLUTION

Glued laminated column $\quad c = 0.9,\quad K_{cE} = 0.418$

$\sigma_C = 8.3\ \text{MPa}\qquad E = 10000\ \text{MPa}$

$A = (216)(140) = 30240\ \text{mm}^2 = 30240 \times 10^{-6}\ \text{m}^2$

$d = 140\ \text{mm} = 0.140\ \text{m}\qquad L = 4.4\ \text{m}\qquad L/d = 31.429$

$$\sigma_{CE} = \frac{K_{cE}E}{(L/d)^2} = \frac{(0.418)(10000)}{(31.429)^2} = 4.2318\ \text{MPa}\qquad \sigma_{CE}/\sigma_C = 0.50986$$

$$u = \frac{1 + \sigma_{CE}/\sigma_C}{2c} = \frac{1.50986}{(2)(0.9)} = 0.838811\qquad v = \frac{\sigma_{CE}/\sigma_C}{c} = 0.566511$$

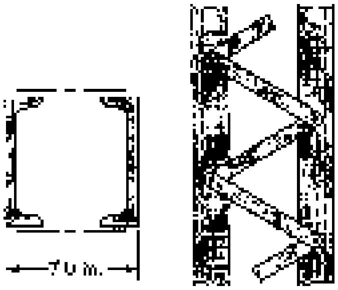
$$C_P = u - \sqrt{u^2 - v} = 0.46801$$

$$\sigma_{all} = \sigma_C C_P = (8.3)(0.46801) = 3.8845\ \text{MPa}$$

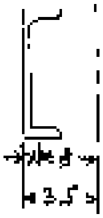
$$P_{all} = \sigma_{all}A = (3.8845 \times 10^6)(30240 \times 10^{-6}) = 117.5 \times 10^3\ \text{N} = 117.5\ \text{kN}$$

PROBLEM 10.66

10.66 A column of 21-ft effective length is obtained by connecting two C10 × 20 steel channels with lacing bars as shown. Using allowable stress design, determine the allowable centric load for the column. Use $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi.



SOLUTION



C10 × 20 $\quad A = 5.88\ \text{in}^2 \quad \bar{x} = 0.606$ in
$I_x = 78.9\ \text{in}^4 \quad I_y = 2.81\ \text{in}^4$
$d = 3.5 - \bar{x} = 2.894$ in

For the column: $A = (2)(5.88) = 11.76\ \text{in}^2$
$I_x = (2)(78.9) = 157.8\ \text{in}^4$
$I_y = 2[2.81 + (5.88)(2.894)^2] = 104.11\ \text{in}^4$

$$r = \sqrt{\frac{I_{min}}{A}} = \sqrt{\frac{104.11}{11.76}} = 2.975\ \text{in} \qquad L_e = 21\ \text{ft} = 252\ \text{in.}$$

$$\frac{L_e}{r} = 84.69 \qquad C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(29000)}{36}} = 126.10$$

$$\frac{L_e}{r} < C_c \qquad \frac{L_e/r}{C_c} = 0.67165$$

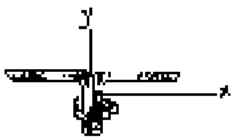
$$F.S. = \frac{5}{3} + \frac{3}{8}(0.67165) - \frac{1}{8}(0.67165)^3 = 1.8807$$

$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L_e/r}{C_c}\right)^2\right] = \frac{36}{1.8807}\left[1 - \frac{1}{2}(0.67165)^2\right] = 14.82\ \text{ksi}$$

$$P_{all} = \sigma_{all} A = (14.82)(11.76) = 174.3\ \text{kips}$$

PROBLEM 10.67

10.67 A compression member of 2.3-m effective length is obtained by bolting together two 127 × 76 × 12.7-mm steel angles as shown. Using allowable stress design, determine the allowable centric load for the column. Use $\sigma_Y = 250$ MPa and $E = 200$ GPa.



SOLUTION

Steel: $\quad C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(200 \times 10^9)}{250 \times 10^6}} = 125.66$

L 127 × 76 × 12.7 mm $\quad$ Table gives $\quad A = 2420\ \text{mm}^2, \quad I_x = 3.93 \times 10^6\ \text{mm}^4$
$\bar{y} = 44.4$ mm, $\quad I_y = 1.06 \times 10^6\ \text{mm}^4, \quad \bar{x} = 19.0$ mm, $\quad r_y =$

For column $\quad I_x = 2(I_y)_{angle} = (2)(1.06 \times 10^6) = 2.12 \times 10^6\ \text{mm}^4$
$I_y > I_x \quad \therefore \quad I_{min} = I_x = 2.12 \times 10^6\ \text{mm}^4 = 2.12 \times 10^{-6}\ \text{m}^4$
$A = 2A_s = 4840\ \text{mm}^2 = 4840 \times 10^{-6}\ \text{m}^2$
$r = \sqrt{\frac{I_{min}}{A}} = \sqrt{\frac{2.12 \times 10^{-6}}{4840 \times 10^{-6}}} = 20.93 \times 10^{-3}$ m

$$\frac{L_e}{r} = \frac{2.3}{20.93 \times 10^{-3}} = 109.90 < C_c \qquad \frac{L_e/r}{C_c} = 0.87455$$

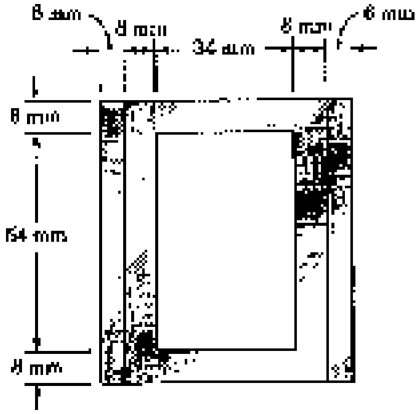
$$F.S. = \frac{5}{3} + \frac{3}{8}(0.87455) - \frac{1}{8}(0.87455)^3 = 1.9110$$

$$\sigma_{all} = \frac{\sigma_Y}{1.9110}\left[1 - \frac{1}{2}\left(\frac{L_e/r}{C_c}\right)^2\right] = \frac{250 \times 10^6}{1.9110}\left[1 - \frac{1}{2}(0.87455)^2\right] = 80.79 \times 10^6\ \text{Pa}$$

$$P_{all} = \sigma_{all} A = (80.79 \times 10^6)(4840 \times 10^{-6}) = 391 \times 10^3\ \text{N} = 391\ \text{kN}$$

PROBLEM 10.70

10.70 An aluminum structural tube is reinforced by riveting two plates to it as shown for use as a column of 1.7-m effective length. Knowing that all material is aluminum alloy 2014-T6, determine the maximum allowable centric load.



SOLUTION

$b_o = 6 + 8 + 34 + 8 + 6 = 62$ mm

$b_i = 34$ mm

$h_o = 8 + 54 + 8 = 70$ mm

$h_i = 54$ mm

$A = b_o h_o - b_i h_i = (62)(70) - (34)(54)$
$= 2.504 \times 10^3 \text{ mm}^2 = 2.504 \times 10^{-3} \text{ m}^2$

$I_x = \frac{1}{12}\left[b_o h_o^3 - b_i h_i^3\right] = \frac{1}{12}\left[(62)(70)^3 - (34)(54)^3\right]$
$= 1.32602 \times 10^6 \text{ mm}^4$

$I_y = \frac{1}{12}\left[h_o b_o^3 - h_i b_i^3\right] = \frac{1}{12}\left[(70)(62)^3 - (54)(34)^3\right] = 1.21337 \times 10^6 \text{ mm}^4 = I_{min}$

$r = \sqrt{\frac{I_{min}}{A}} = \sqrt{\frac{1.21337 \times 10^6}{2.504 \times 10^3}} = 22.013 \text{ mm} = 22.013 \times 10^{-3} \text{ m} \qquad L = 1.7 \text{ m}$

$\frac{L}{r} = \frac{1.7}{22.013 \times 10^{-3}} = 77.23 > 55$ for aluminum alloy 2014-T6

$\sigma_{all} = \frac{372 \times 10^3}{(L/r)^2} = \frac{372 \times 10^3}{77.23^2} = 62.37 \text{ MPa}$

$P_{all} = \sigma_{all} A = (62.37 \times 10^6)(2.504 \times 10^{-3}) = 156.2 \times 10^3 \text{ N} = 156.2 \text{ kN}$ ◀

PROBLEM 10.71

10.71 A 280-kN centric load is applied to the column shown, that is free at its top *A* and fixed at its base *B*. Using aluminum alloy 2014-T6, select the smallest square cross section that can be used.

SOLUTION

$$L_e = 2L = (2)(0.30) = 0.60 \text{ m}$$

$$A = b^2 \qquad I = \frac{1}{12}b^4 \qquad r = \sqrt{\frac{I}{A}} = \frac{b}{\sqrt{12}}$$

$$\frac{L}{r} = \frac{0.60\sqrt{12}}{b} = \frac{2.0785}{b}$$

2014-T6 aluminum alloy

Assume $\frac{L}{r} < 55$ $\qquad \sigma_{all} = 212 - 1.585(L/r) = 212 - (1.585)(2.0785/b)$

$$= \left(212 - \frac{3.294}{b}\right) \text{ MPa} = \left[212 - \frac{3.294}{b}\right](10^6) \text{ Pa}$$

$$P_{all} = \sigma_{all} A = \left[212\, b^2 - 3.294\, b\right](10^6) = 280 \times 10^3$$

$$212 b^2 - 3.294 b - 280 \times 10^{-3} = 0$$

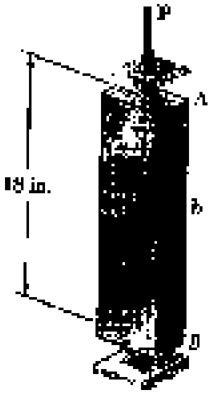
$$b = \frac{3.294 + \sqrt{(3.294)^2 + (4)(212)(280 \times 10^{-3})}}{(2)(212)} = 44.9 \times 10^{-3} \text{ m}$$

$$\frac{L}{r} = \frac{2.0785}{b} = \frac{2.0785}{44.9 \times 10^{-3}} = 46.26 < 55$$

Answer: $b = 44.9 \times 10^{-3}$ m $= 44.9$ mm

PROBLEM 10.72

10.72 A 16-kip centric load must be supported by an aluminum column as shown. Using the aluminum alloy 6061-T6, determine the minimum dimension b that can be used.



SOLUTION

$L_e = L = 18$ in $\qquad A = 2b^2 \qquad I_{min} = \frac{1}{12}(2b)(b)^3 = \frac{1}{6}b^4$

$r = \sqrt{\frac{I_{min}}{A}} = \frac{b}{\sqrt{12}} \qquad \frac{L}{r} = \frac{18\sqrt{12}}{b} = \frac{62.354}{b}$

6061-T aluminum alloy. $\quad$ Assume $\frac{L}{r} < 66$

$\sigma_{all} = 20.2 - 0.126(L/r) = 20.2 - (0.126)\frac{62.354}{b}$

$= 20.2 - \frac{7.8566}{b}$ ksi

$P_{all} = \sigma_{all} A = (20.2 - \frac{7.8566}{b})(2b^2) = 40.4\,b^2 - 15.713\,b$ kip

$40.4\,b^2 - 15.713\,b = 16 \qquad b = \frac{15.713 + \sqrt{(15.713)^2 + (4)(40.4)(16)}}{(2)(40.4)} = 0.853$ in

$\frac{L}{r} = \frac{62.354}{b} = \frac{62.354}{0.853} = 73.09 > 66$ $\quad$ Assumption not verified.

Assume $\frac{L}{r} > 66 \qquad \sigma_{all} = \frac{51000}{(L/r)^2} = \frac{51000\,b^2}{(62.354)^2} = 13.117\,b^2$ ksi

$P_{all} = \sigma_{all} A = (13.117\,b^2)(2b^2) = 26.234\,b^4 = 16$ kips

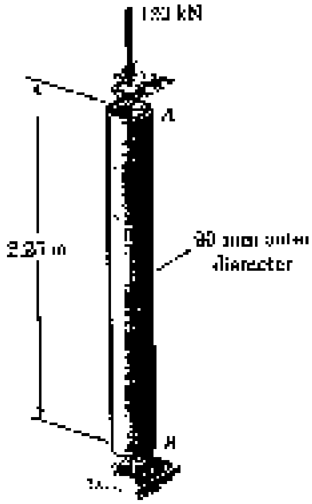
$b = \sqrt[4]{\frac{16}{26.234}} = 0.884$ in.

$\frac{L}{r} = \frac{62.354}{0.884} = 70.56 > 66$ $\quad$ Assumption verified

$b = 0.884$ in.

PROBLEM 10.73

10.73 An aluminum tube of 90-mm outer diameter is to carry a centric load of 120 kN. Knowing that the stock of tubes available for use are made of alloy 2014-T6 and with wall thickness in increments of 3 mm from 6 mm to 15 mm, determine the lightest tube that can be used.



SOLUTION

$L = 2250\text{ mm}, \quad P = 120 \times 10^3\text{ N} \qquad r_o = 45\text{ mm}$

$r_i = r_o - t \qquad A = \pi(r_o^2 - r_i^2) \qquad I = \frac{\pi}{4}(r_o^4 - r_i^4)$

$r = \sqrt{I/A}$

For 2014-T6 aluminum alloy

$\sigma_{all} = 212 - 1.585\,(L/r)\text{ MPa} \quad \text{if } L/r < 55$

$\sigma_{all} = \dfrac{372 \times 10^3}{(L/r)^2}\text{ MPa} \quad \text{if } L/r > 55$

$P_{all} = \sigma_{all} A$

Calculate P_{all} for each thickness.

	t (mm)	r_i (mm)	A (mm^2)	I (10^6 mm^4)	r (mm)	L/r	σ_{all} (MPa)	P_{all} (kN)	
	6	39	1583	1.404	29.78	75.56	65.16	103.1	
→	9	36	2290	1.901	28.82	78.08	61.01	139.7	←
	12	33	2941	2.289	27.90	80.65	57.20	168.2	
	15	30	3534	2.584	27.04	83.20	53.74	189.9	

Since P_{all} must be greater than 120 kN, use $t = 9$ mm ◄

PROBLEM 10.74

10.74 A 18-kip centric load is applied to a rectangular sawn lumber column of 22-ft effective length. Using sawn lumber for which the adjusted allowable stress for compression parallel to the grain is $\sigma_C = 1050$ psi and knowing that $E = 10 \times 10^6$ psi, determine the smallest cross section that can be used for the column if $b = 2d$.

SOLUTION

Sawn lumber $\quad c = 0.8 \qquad K_{cE} = 0.3$

$\sigma_C = 1050$ psi $\qquad E = 10 \times 10^6$ psi

$A = 2d^2 \qquad L = 22 \text{ ft} = 264 \qquad L/d = \dfrac{264}{d}$

Assumed $C_P = 0.5$

$$\sigma_{all} = \sigma_C C_P = (1050)(0.5) = 525 \text{ psi}$$

$$P_{all} = \sigma_{all} A = 2\sigma_{all} d^2$$

$$d = \sqrt{\frac{P_{all}}{2\sigma_{all}}} = \sqrt{\frac{18\,000}{2\sigma_{all}}} = \frac{94.868}{\sqrt{\sigma_{all}}} = 4.14 \text{ in.}$$

$L/d = 63.76 \qquad \sigma_{CE} = \dfrac{K_{cE} E}{(L/d)^2} = \dfrac{(0.3)(10 \times 10^6)}{(L/d)^2} = \dfrac{3 \times 10^6}{(L/d)^2} \quad 737.9$ psi

$\sigma_{CE}/\sigma_C = 0.7028$

Checked $C_P = \dfrac{1 + \sigma_{CE}/\sigma_C}{2c} - \sqrt{\left(\dfrac{1 + (\sigma_{CE}/\sigma_C)}{2c}\right)^2 - \dfrac{\sigma_{CE}/\sigma_C}{c}} \quad 0.5601$

Results of similar trials are summarized in the table below.

Assumed C_P	σ_{all} (psi)	d (in)	L/d	σ_{CE} (psi)	σ_{CE}/σ_C	Checked C_P	ΔC_P
0.5	525	4.14	63.76	737.9	0.7028	0.5601	0.0601
0.56	588	3.91	67.48	658.8	0.6275	0.5169	−0.0431
0.535	561.75	4.00	66.00	688.7	0.6559	0.5337	−0.0013
0.5348	561.0	4.005	65.92	690.4	0.6575	0.5346	≈ 0

Answer $\quad d = 4.01$ in.

PROBLEM 10.77

10.77 A column of 5.6-m effective length must carry a centric load of 2750 kN. Knowing that $\sigma_Y = 250$ MPa and $E = 200$ GPa, use allowable stress design to select the wide-flange shape of 360-mm nominal depth that should be used.

SOLUTION

$$P < \frac{\sigma_Y A}{F.S.}$$

$$A > \frac{(F.S.)P}{\sigma_Y} = \frac{(5/3)(2750\times10^3)}{250\times10^6} = 18.33\times10^{-3}\ \text{m}^2 = 18330\ \text{mm}^2$$

$$P < \frac{\pi^2 EI}{1.92 L^2}$$

$$I > \frac{1.92 P L^2}{\pi^2 E} = \frac{(1.92)(2750\times10^3)(5.6)^2}{\pi^2(200\times10^9)} = 83.9\times10^{-6}\ \text{m}^4 = 83.9\times10^6\ \text{mm}^4$$

Try W 360×216 $\quad A = 27600\ \text{mm}^2 = 27600\times10^{-6}\ \text{m}^2$ ok.

$I_{min} = 283\times10^6\ \text{mm}^4$ ok.

$r_y = 101\ \text{mm} = 101\times10^{-3}\ \text{m}$

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(200\times10^9)}{250\times10^6}} = 125.66$$

$$\frac{L_e}{r} = \frac{5.6}{101\times10^{-3}} = 55.45 < C_c \qquad \frac{L_e/r}{C_c} = 0.44123$$

$$F.S. = \frac{5}{3} + \frac{3}{8}(0.44123) - \frac{1}{8}(0.44123)^3 = 1.8214$$

$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L/r}{C_c}\right)^2\right] = \frac{250\times10^6}{1.8214}\left[1 - \frac{1}{2}(0.44123)^2\right] = 123.9\times10^6\ \text{Pa}$$

$$P_{all} = \sigma_{all}A = (123.9\times10^6)(27600\times10^{-6}) = 3420\times10^3\ \text{N} = 3420\ \text{kN}$$

$3420\ \text{kN} > 2750\ \text{kN}$ $\qquad$ Use W 360×216

PROBLEM 10.78

10.78 A column of 4.6-m effective length must carry a centric load of 525 kN. Knowing that $\sigma_Y = 345$ MPa and $E = 200$ GPa, use allowable stress design to select the wide-flange shape of 200-mm nominal depth that should be used.

SOLUTION

$$P < \frac{\sigma_Y A}{F.S.}$$

$$A > \frac{(F.S.)P}{\sigma_Y} = \frac{(5/3)(525\times10^3)}{345\times10^6} = 2.54\times10^{-3}\ \text{m}^2 = 2540\ \text{mm}^2$$

$$P < \frac{\pi^2 EI}{1.92 L_e^2}$$

$$I > \frac{1.92\,PL_e^2}{\pi^2 E} = \frac{(1.92)(525\times10^3)(4.6)^2}{\pi^2(200\times10^9)} = 10.89\times10^{-6}\ \text{m}^4 = 10.89\times10^6\ \text{mm}^4$$

Try W 200 × 46.1 $\quad A = 5860\ \text{mm}^2,\quad I_{min} = 15.3\times10^6\ \text{mm}^4,\quad r = 51.1\times10^{-3}\ \text{m}$

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(200\times10^9)}{345\times10^6}} = 106.97$$

$$\frac{L_e}{r} = \frac{4.6}{51.1\times10^{-3}} = 90.02 < C_c \qquad \frac{L_e/r}{C_c} = 0.84154$$

$$F.S. = \frac{5}{3} + \frac{3}{8}(0.84154) - \frac{1}{8}(0.84154)^3 = 1.9077$$

$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L/r}{C_c}\right)^2\right] = \frac{345\times10^6}{1.9077}\left[1 - \frac{1}{2}(0.84154)^2\right] = 116.8\times10^6\ \text{Pa}$$

$$P_{all} = \sigma_{all} A = (116.8\times10^6)(5860\times10^{-6}) = 684\ \text{kN} > 525\ \text{kN}$$

Use W 200 × 46.1 ◄

PROBLEM 10.79

10.79 A column of 22.5-ft effective length must carry a centric load of 288 kips. Using allowable stress design, select the wide-flange shape of 14-in. nominal depth that should be used. Use $\sigma_Y = 50$ ksi and $E = 29 \times 10^6$ psi.

SOLUTION

$$P < \frac{\sigma_Y A}{F.S.} \qquad A > \frac{(F.S.)P}{\sigma_Y} = \frac{(5/3)(288)}{50} = 9.6 \text{ in}^2$$

$$L_e = 22.5 \text{ ft} = 270 \text{ in} \qquad E = 29 \times 10^6 \text{ psi} = 29000 \text{ ksi}$$

$$P < \frac{\pi^2 EI}{1.92 L_e^2} \qquad I > \frac{1.92 P L_e^2}{\pi^2 E} = \frac{(1.92)(288)(270)^2}{\pi^2 (29000)} = 140.8 \text{ in}^4$$

Try W 14 × 82 $\quad A = 24.1 \text{ in}^2, \quad I_{min} = 148 \text{ in}^4, \quad r = 2.48$ in

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(29000)}{50}} = 107.00$$

$$\frac{L_e}{r} = \frac{270}{2.48} = 108.87 > 107.00$$

$$\sigma_{all} = \frac{\pi^2 E}{1.92 (L/r)^2} = \frac{\pi^2 (29000)}{(1.92)(108.87)^2} = 12.58 \text{ ksi}$$

$$P_{all} = \sigma_{all} A = (12.58)(24.1) = 303 \text{ kips} > 288 \text{ kips}$$

Use W 14 × 82

PROBLEM 10.80

10.80 A column of 17-ft effective length must carry a centric load of 235 kips. Using allowable stress design, select the wide-flange shape of 10-in. nominal depth that should be used. Use $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi.

SOLUTION

$$P < \frac{\sigma_Y A}{F.S.} \qquad A > \frac{(F.S.)P}{\sigma_Y} = \frac{(5/3)(235)}{36} = 10.88 \text{ in}^2$$

$$L_e = 17 \text{ ft} = 204 \text{ in} \qquad E = 29 \times 10^6 \text{ psi} = 29000 \text{ ksi}$$

$$P < \frac{\pi^2 EI}{1.92 L^2} \qquad I > \frac{1.92 P L_e^2}{\pi^2 E} = \frac{(1.92)(235)(204)^2}{\pi^2 (29000)} = 65.6 \text{ in}^4$$

Try W 10 × 54 $\quad A = 15.8 \text{ in}^2 \quad I_y = 103 \text{ in}^4 \quad r = 2.56$ in

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2 (29000)}{36}} = 126.10$$

$$\frac{L_e}{r} = \frac{204}{2.56} = 79.69 < C_c \qquad \frac{L_e/r}{C_c} = \frac{79.69}{126.10} = 0.63194$$

$$F.S. = \frac{5}{3} + \frac{3}{8}(0.63194) - \frac{1}{8}(0.63194)^3 = 1.8721$$

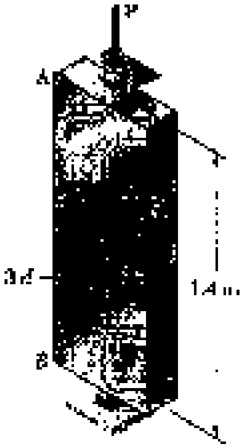
$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L/r}{C_c}\right)^2\right] = \frac{36}{1.8721}\left[1 - \frac{1}{2}(0.63194)^2\right] = 15.39 \text{ ksi}$$

$$P_{all} = \sigma_{all} A = (15.39)(15.8) = 243 \text{ kips} > 235 \text{ kips}$$

Use W 10 × 54

PROBLEM 10.81

10.81 A centric load P must be supported by the steel bar AB. Using allowable stress design, determine the smallest dimension d of the cross section that can be used when (a) $P = 108$ kN, (b) $P = 166$ kN. Use $\sigma_Y = 250$ MPa and $E = 200$ GPa.



SOLUTION

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(200\times10^9)}{250}} = 125.66$$

$$L_e = L = 1.4 \text{ m}$$

$$A = (3d)(d) = 3d^2$$

$$I = \frac{1}{12}(3d)(d)^3 = \frac{1}{4}d^4$$

$$r = \sqrt{\frac{I}{A}} = \frac{d}{\sqrt{12}} = 0.288675\ d$$

(a) $P = 108\times10^3$ N $\qquad$ Assume $\frac{L_e}{r} > C_c$

$$P_{all} = \frac{\pi^2 E I}{1.92\ L_e^2} \qquad I = \frac{(1.92)\,P_{all}\,L_e^2}{\pi^2 E} = \frac{1}{4}d^4$$

$$d^4 = \frac{(4)(1.92)P L_e^2}{\pi^2 E} = \frac{(4)(1.92)(108\times10^3)(1.4)^2}{\pi^2(200\times10^9)} = 823.59\times10^{-9}\ \text{m}^4$$

$$d = 30.125\times10^{-3}\text{ m} \qquad r = 8.696\times10^{-3}\text{ m}$$

$$\frac{L_e}{r} = \frac{1.4}{8.696\times10^{-3}} = 160.99 > 125.66\ \checkmark \qquad d = 30.1\text{ mm}$$

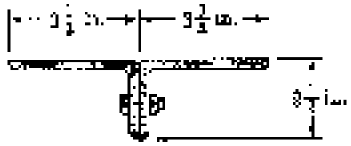
(b) $P = 166\times10^3$ N $\qquad$ Assume $\frac{L_e}{r} > C_c$

$$d^4 = \frac{(4)(1.92)P L_e^2}{\pi^2 E} = \frac{(4)(1.92)(166\times10^3)(1.4)^2}{\pi^2(200\times10^9)} = 1.26588\times10^{-6}\ \text{m}^4$$

$$d = 33.543\times10^{-3}\text{ m} \qquad r = 9.68295\times10^{-3}$$

$$\frac{L_e}{r} = \frac{1.4}{9.68295\times10^{-3}} = 144.58 > 125.66\ \checkmark \qquad d = 33.5\text{ mm}$$

PROBLEM 10.82



10.82 Two $3\frac{1}{2} \times 2\frac{1}{2}$-in. angles are bolted together as shown for use as a column of 8-ft effective length to carry a centric load of 41 kips. Knowing that the angles available have thicknesses of $\frac{1}{4}$ in., $\frac{3}{8}$ in., and $\frac{1}{2}$ in., use allowable stress design to determine the lightest angles that can be used. Use $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi.

SOLUTION

Steel: $E = 29000$ ksi $\qquad L_e = 8$ ft $= 96$ in.

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(29000)}{36}} = 126.10$$

Try L $3\frac{1}{2} \times 2\frac{1}{2} \times \frac{3}{8}$ in.

$$A = (2)(2.11) = 4.22 \text{ in}^2$$

$$I_x = (2)(1.09) = 2.18 \text{ in}^4 < I_y$$

$$r = \sqrt{\frac{I_x}{A}} = 0.719 \text{ in.}$$

$$\frac{L_e}{r} = \frac{96}{0.719} = 133.52 > C_c$$

$$\sigma_{all} = \frac{\pi^2 E}{1.92(L/r)^2} = \frac{\pi^2(29000)}{1.92(133.52)^2} = 8.36 \text{ ksi}$$

$$P_{all} = \sigma_{all} A = (8.36)(4.22) = 35.3 \text{ kips} < 41 \text{ kips} \qquad \text{Do not use.}$$

Try L $3\frac{1}{2} \times 2\frac{1}{2} \times \frac{1}{2}$ in

$$A = (2)(2.75) = 5.50 \text{ in}^2$$

$$r = 0.704 \text{ in}$$

$$\frac{L_e}{r} = \frac{96}{0.704} = 136.36 > C_c$$

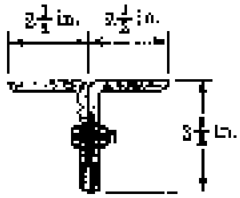
$$\sigma_{all} = \frac{\pi^2 E}{1.92(L/r)^2} = \frac{\pi^2(29000)}{(1.92)(136.36)^2} = 8.02 \text{ ksi}$$

$$P_{all} = \sigma_{all} A = (8.02)(5.50) = 44.1 \text{ kips} > 41 \text{ kips}$$

Use L $3\frac{1}{2} \times 2\frac{1}{2} \times \frac{1}{2}$ in.

PROBLEM 10.83

10.83 Two $3\frac{1}{2} \times 2\frac{1}{2}$ -in. angles are bolted together as shown for use as a column of 6-ft effective length to carry a centric load of 54 kips. Knowing that the angles available have thicknesses of $\frac{1}{4}$ in., $\frac{3}{8}$ in., and $\frac{1}{2}$ in., use allowable stress design to determine the lightest angles that can be used. Use $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi.



SOLUTION

Steel: $E = 29000$ ksi $\qquad L_e = 6$ ft $= 72$ in

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(29000)}{36}} = 126.10$$

Try L $3\frac{1}{2} \times 2\frac{1}{2} \times \frac{3}{8}$ in.

$$A = (2)(2.11) = 4.22 \text{ in}^2$$
$$I_x = (2)(2.56) = 5.12 \text{ in}^4$$
$$I_y = 2\left[1.09 + (2.11)(0.660)^2\right] = 4.018 \text{ in}^4 = I_{min}$$
$$r = \sqrt{\frac{I_{min}}{A}} = \sqrt{\frac{4.018}{4.22}} = 0.9758$$

$$\frac{L_e}{r} = \frac{72}{0.9758} = 73.78 < C_c \qquad \frac{L_e/r}{C_c} = \frac{73.78}{126.10} = 0.58509$$

$$F.S. = \frac{5}{3} + \frac{3}{8}(0.58509) - \frac{1}{8}(0.58509)^3 = 1.8610$$

$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L/r}{C_c}\right)^2\right] = \frac{36}{1.8610}\left[1 - \frac{1}{2}(0.58509)^2\right] = 16.03 \text{ ksi}$$

$$P_{all} = \sigma_{all} A = (16.03)(4.22) = 67.7 \text{ kips} > 54 \text{ kips} \quad \text{(allowed)}$$

Try L $3\frac{1}{2} \times 2\frac{1}{2} \times \frac{1}{4}$ in.

$$A = (2)(1.44) = 2.88 \text{ in}^2$$
$$I_x = (2)(1.80) = 3.60 \text{ in}^4$$
$$I_y = (2)\left[0.777 + (1.44)(0.614)^2\right] = 2.6397 \text{ in}^4 = I_{min}$$
$$r = \sqrt{\frac{I_{min}}{A}} = \sqrt{\frac{2.6397}{2.88}} = 0.95738 \text{ in.}$$

$$\frac{L_e}{r} = \frac{72}{0.95738} = 75.205 < C_c \qquad \frac{L/r}{C_c} = \frac{75.205}{126.10} = 0.59633$$

$$F.S. = \frac{5}{3} + \frac{3}{8}(0.59633) - \frac{1}{8}(0.59633)^3 = 1.8658$$

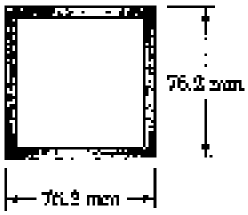
$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L/r}{C_c}\right)^2\right] = \frac{36}{1.8658}\left[1 - \frac{1}{2}(0.59633)^2\right] = 15.88 \text{ ksi}$$

$$P_{all} = \sigma_{all} A = (15.88)(2.88) = 45.7 \text{ kips} < 54 \text{ kips} \quad \text{Do not use}$$

Use L $3\frac{1}{2} \times 2\frac{1}{2} \times \frac{3}{8}$ in.

PROBLEM 10.84

10.84 A square structural tube having the cross section shown is used as a column of 3.1-m effective length to carry a centric load of 129 kN. Knowing that the tubes available for use are made with wall thicknesses of 3.2 mm, 4.8 mm, 6.4 mm, and 7.9 mm, use allowable stress design to determine the lightest tube that can be used. Use $\sigma_Y = 250$ MPa and $E = 200$ GPa.



SOLUTION

$b_o = 76.2$ mm $\quad b_i = b_o - 2t \quad A = b_o^2 - b_i^2$

$I = \frac{1}{12}(b_o^4 - b_i^4)$

$L_e = 3.1$ m $\quad P = 129$ kN $\quad b_o = 76.2$ mm

Steel: $C_c = \sqrt{\dfrac{2\pi^2 E}{\sigma_Y}} = \sqrt{\dfrac{2\pi^2(200\times10^9)}{250\times10^6}} = 125.66$

Try $t = 4.8$ mm $\quad b_i = 76.2 - 9.6 = 66.6$ mm

$A = (76.2)^2 - (66.6)^2 = 1.37088\times10^3\ \text{mm}^2 = 1.37088\times10^{-3}\ \text{m}^2$

$I = \frac{1}{12}[(76.2)^4 - (66.6)^4] = 1.17005\times10^6\ \text{mm}^4 = 1.17005\times10^{-6}\ \text{m}^4$

$r = \sqrt{\dfrac{I}{A}} = 29.21\times10^{-3}$ m

$\dfrac{L_e}{r} = \dfrac{3.1}{29.21\times10^{-3}} = 106.11 < C_c \qquad \dfrac{L_e/r}{C_c} = 0.84443$

$F.S. = \frac{5}{3} + \frac{3}{8}(0.84443) - \frac{1}{8}(0.84443)^3 = 1.9081$

$\sigma_{all} = \dfrac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L/r}{C_c}\right)^2\right] = \dfrac{250\times10^6}{1.9081}\left[1 - \frac{1}{2}(0.84443)^2\right] = 84.3\times10^6$ Pa

$P_{all} = \sigma_{all}A = (84.3\times10^6)(1.37088\times10^{-3}) = 115.6\times10^3$ N

$= 115.6$ kN < 129 kN $\qquad$ Do not use.

Try $t = 6.4$ mm $\quad b_i = 76.2 - 12.8 = 63.4$ mm

$A = (76.2)^2 - (63.4)^2 = 1.78688\times10^3\ \text{mm}^2 = 1.78688\times10^{-3}\ \text{m}^2$

$I = \frac{1}{12}[(76.2)^4 - (63.4)^4] = 1.46316\times10^6\ \text{mm}^4 = 1.46316\times10^{-6}\ \text{m}^4$

$r = \sqrt{\dfrac{I}{A}} = 28.615\times10^{-3}$ m

$\dfrac{L_e}{r} = \dfrac{3.1}{28.615\times10^{-3}} = 108.33 < C_c \qquad \dfrac{L_e/r}{C_c} = 0.86212$

$F.S. = \frac{5}{3} + \frac{3}{8}(0.86212) - \frac{1}{8}(0.86212)^3 = 1.9099$

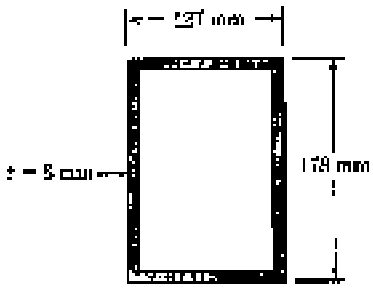
$\sigma_{all} = \dfrac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L/r}{C_c}\right)^2\right] = \dfrac{250\times10^6}{1.9099}\left[1 - \frac{1}{2}(0.86212)^2\right] = 82.25\times10^6$ Pa

$P_{all} = \sigma_{all}A = (82.25\times10^6)(1.78688\times10^{-3}) = 147.0\times10^3$ N

$= 147.0$ kN > 129 kN

Use $\quad t = 6.4$ mm

PROBLEM 10.85



10.85 A rectangular tube having the cross section shown is used as a column of 4.5-m effective length. Knowing that $\sigma_Y = 250$ MPa and $E = 200$ GPa, use load and resistance factor design to determine the largest centric live load that can be applied if the centric dead load is 140 kN. Use a dead load factor $\gamma_D = 1.2$, a live load factor $\gamma_L = 1.6$ and the resistance factor $\phi = 0.85$.

SOLUTION

$h_o = 127 \text{ mm} \quad b_o = 178 \text{ mm} \quad h_i = h_o - 2t = 111 \text{ mm}$

$b_i = b_o - 2t = 162 \text{ mm}$

$A = b_o h_o - b_i h_i = (178)(127) - (162)(111)$

$= 4624 \text{ mm}^2 = 4624 \times 10^{-6} \text{ m}^2$

$I = \frac{1}{12}\left[b_o h_o^3 - b_i h_i^3\right] = \frac{1}{12}\left[(178)(127)^3 - (162)(111)^3\right] = 11.9213 \times 10^6 \text{ mm}^4$

$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{11.9213 \times 10^6}{4624}} = 50.775 \text{ mm} = 50.775 \times 10^{-3} \text{ m}$

$\frac{L}{r} = \frac{4.5}{50.775 \times 10^{-3}} = 88.63$

$\lambda_c = \frac{L}{r\pi}\sqrt{\frac{\sigma_Y}{E}} = \frac{88.63}{\pi}\sqrt{\frac{250 \times 10^6}{200 \times 10^9}} = 0.9974 < 1.5$

$\lambda_c^2 = 0.9948$

$P_U = A(0.658)^{\lambda_c^2}\sigma_Y = (4624 \times 10^{-6})(0.658)^{0.9948}(250 \times 10^6) = 762.3 \times 10^3 \text{ N}$

$= 762.3 \text{ kN}$

$\gamma_D P_D + \gamma_L P_L = \phi P_U$

$(1.2)(140) + 1.6 P_L = (0.85)(762.3) \qquad P_L = 300 \text{ kN}$

PROBLEM 10.86

*10.86 A column with a 19.5-ft effective length supports a centric load, with ratio of dead to live load equal to 1.35. The dead load factor is $\gamma_D = 1.2$, the live load factor $\gamma_L = 1.6$, and the resistance factor $\phi = 0.85$. Use load and resistance factor design to determine the allowable centric dead and live loads if the column is made of the following rolled-steel shape: (a) W10 × 39, (b) W14 × 68. Use $E = 29 \times 10^6$ psi and $\sigma_Y = 50$ ksi.

SOLUTION

$L_e = 19.5 \text{ ft} = 234 \text{ in}$

(a) W10×39 $\quad A = 11.5 \text{ in}^2 \quad r_y = 1.98 \text{ in} \quad L_e/r_y = 118.18$

$$\lambda_c = \frac{L_e/r}{\pi}\sqrt{\frac{\sigma_Y}{E}} = \frac{118.18}{\pi}\sqrt{\frac{50}{29000}} = 1.5620 > 1.5$$

$$P_u = A\left(\frac{0.877}{\lambda_c^2}\right)\sigma_Y = \frac{(11.5)(0.877)(50)}{(1.5620)^2} = 206.67 \text{ kips}$$

$$\gamma_D P_D + \gamma_L P_L = \phi P_u$$

$$(1.2)(1.35 P_L) + 1.6 P_L = (0.85)(206.67)$$

$P_D = 73.7$ kips ◄

$P_L = 54.6$ kips ◄

(b) W 14×68 $\quad A = 20.0 \text{ in}^2 \quad r_y = 2.46 \text{ in} \quad L_e/r_y = 95.12$

$$\lambda_c = \frac{L_e/r}{\pi}\sqrt{\frac{\sigma_Y}{E}} = \frac{95.12}{\pi}\sqrt{\frac{50}{29000}} = 1.2572 < 1.5$$

$$\lambda_c^2 = 1.5806$$

$$P_u = A(0.658)^{\lambda_c^2}\sigma_Y = (20.0)(0.658)^{1.5806}(50) = 516 \text{ kips}$$

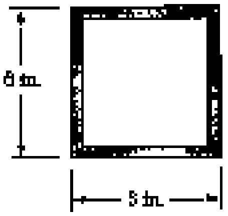
$$\gamma_D P_D + \gamma_L P_L = \phi P_u$$

$$(1.2)(1.35 P_L) + 1.6 P_L = (0.85)(516)$$

$P_D = 183.9$ kips ◄

$P_L = 136.2$ kips ◄

PROBLEM 10.87



*10.87 The structural tube having the cross section shown is used as a column of 15-ft effective length to carry a centric dead load of 51 kips and a centric live load of 58 kips. Knowing that the tubes available for use are made with wall thicknesses in increments of $\frac{1}{16}$ in. from $\frac{3}{16}$ in. to $\frac{3}{8}$ in., use load and resistance factor design to determine the lightest tube that can be used. Use $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi. The dead load factor $\gamma_D = 1.2$, the live load factor $\gamma_L = 1.6$ and the resistance factor $\phi = 0.85$.

SOLUTION

$L_e = 15 \text{ ft} = 180 \text{ in}$

$\gamma_D P_D + \gamma_L P_L = \phi P_U$

Required $P_U = \dfrac{\gamma_D P_D + \gamma_L P_L}{\phi} = \dfrac{(1.2)(51) + (1.6)(58)}{0.85} = 181.2$ kips

Try $t = \frac{1}{4}$ in. $= 0.25$ in $\quad b_o = 6.0$ in $\quad b_i = b_o - 2t = 5.5$ in

$A = b_o^2 - b_i^2 = (6)^2 - (5.5)^2 = 5.75 \text{ in}^2$

$I = \frac{1}{12}(b_o^4 - b_i^4) = \frac{1}{12}[(6)^4 - (5.5)^4] = 31.74 \text{ in}^4$

$r = \sqrt{\dfrac{I}{A}} = \sqrt{\dfrac{31.74}{5.75}} = 2.3496$ in $\qquad \dfrac{L_e}{r} = \dfrac{180}{2.3496} = 76.61$

$\lambda_c = \dfrac{L_e/r}{\pi}\sqrt{\dfrac{\sigma_Y}{E}} = \dfrac{76.61}{\pi}\sqrt{\dfrac{36}{29000}} = 0.85916 < 1.5 \qquad \lambda_c^2 = 0.73815$

$P_U = A(0.658)^{\lambda_c^2}\sigma_Y = (5.75)(0.658)^{0.73815}(36) = 152.0$ kips < 181.2 kips

Thickness is too small.

Since P_U is approximately proportional to thickness, the required thickness is approximately

$\dfrac{t_{req}}{0.25} \approx \dfrac{P_{U(req)}}{152} = \dfrac{181.18}{152} \qquad t_{req} \approx 0.296$ in

Try $t = \frac{5}{16}$ in $= 0.3125$ in, $\qquad b_i = 5.375$

$A = 7.1094 \text{ in}^2$, $I = 38.44 \text{ in}^4$, $\quad r = 2.3254$ in $\qquad \dfrac{L_e}{r} = 77.41$

$\lambda_c = \dfrac{77.41}{\pi}\sqrt{\dfrac{36}{29000}} = 0.86811 < 1.5 \qquad \lambda_c^2 = 0.75361$

$P_U = (7.1094)(0.658)^{0.75361}(36) = 186.7$ kips > 181.2 kips

Use $t = \frac{5}{16}$ in.

PROBLEM 10.88

*10.88 A column of 5.5-m effective length must carry a centric dead load of 310 kN and a centric live load of 375 kN. Knowing that $\sigma_Y = 250$ MPa and $E = 200$ GPa, use load and resistance factor design to select the wide-flange shape of 310-mm nominal depth that should be used. The dead load factor $\gamma_D = 1.2$, the live load factor $\gamma_L = 1.6$ and the resistance factor $\phi = 0.85$.

SOLUTION

$$\gamma_D P_D + \gamma_L P_L = \phi P_u$$

Required $P_u = \dfrac{\gamma_D P_D + \gamma_L P_L}{\phi} = \dfrac{(1.2)(310) + (1.6)(375)}{0.85} = 1143$ kN

Preliminary calculations

$P_u = \sigma_Y A \quad \therefore A > \dfrac{P_u}{\sigma_Y} = \dfrac{1143 \times 10^3}{250 \times 10^6} = 4572 \times 10^{-3}\ \text{m}^2 = 4572\ \text{mm}^2$

$P_u < \dfrac{\pi^2 EI}{L_e^2} \quad \therefore I > \dfrac{P_u L_e^2}{\pi^2 E} = \dfrac{(1143 \times 10^3)(5.5)^2}{\pi^2 (200 \times 10^9)} = 17.52 \times 10^{-6}\ \text{m}^4 = 17.52 \times 10^6\ \text{mm}^4$

Try W 310×60 $\quad A = 7590\ \text{mm}^2 = 7590 \times 10^{-6}\ \text{m}^2$

$I_y = 18.3 \times 10^6\ \text{mm}^4, \quad r_y = 49.1\ \text{mm} = 49.1 \times 10^{-3}\ \text{m}$

$$\lambda_c = \frac{L_e}{\pi r}\sqrt{\frac{\sigma_Y}{E}} = \frac{5.5}{\pi(49.1 \times 10^{-3})}\sqrt{\frac{250 \times 10^6}{200 \times 10^9}} = 1.2606 < 1.5$$

$$\lambda_c^2 = 1.5892$$

$$P_u = A(0.658)^{\lambda_c^2}\sigma_Y = (7590 \times 10^{-6})(0.658)^{1.5892}(250 \times 10^6)$$

$= 975 \times 10^3$ N $= 975$ kN < 1143 kN

Too light. Do not use.

Try W310 × 74 $\quad A = 9480\ \text{mm}^2 = 9480 \times 10^{-6}\ \text{m}^2$

$r_y = 49.7\ \text{mm} = 49.7 \times 10^{-3}\ \text{m}$

$\lambda_c = \dfrac{5.5}{\pi(49.7 \times 10^{-3})}\sqrt{\dfrac{250 \times 10^6}{200 \times 10^9}} = 1.2454 \qquad \lambda_c^2 = 1.5510$

$P_u = (9480 \times 10^{-6})(0.658)^{1.5510}(250 \times 10^6) = 1238 \times 10^3$ N

$= 1238$ kN > 1143 kN

Use W 310×74

PROBLEM 10.89

10.89 A sawn lumber column with a 125-mm-square cross section and a 3.6-m effective length is made of a grade of wood that has an adjusted allowable stress for compression parallel to the grain $\sigma_C = 9.2$ MPa and a modulus of elasticity $E = 12$ GPa. Using the allowable-stress method, determine the maximum load **P** that can be safely supported with an eccentricity of 50 mm.

SOLUTION

$d = 125 \text{ mm} = 0.125 \text{ m} \qquad A = d^2 = 15.625 \times 10^{-3} \text{ m}^2 \qquad \frac{L}{d} = \frac{3.6}{0.125} = 28.8$

$\sigma_C = 9.2 \text{ MPa}, \quad E = 12000 \text{ MPa}, \quad \text{sawn lumber: } c = 0.8, \quad K_{cE} = 0.300$

$\sigma_{CE} = \frac{K_{cE} E}{(L/d)^2} = \frac{(0.300)(12000)}{(28.8)^2} = 4.34 \text{ MPa} \qquad \sigma_{CE}/\sigma_C = 0.47177$

$C_P = \frac{1 + (\sigma_{CE}/\sigma_C)}{2c} - \sqrt{\left(\frac{1 + \sigma_{CE}/\sigma_C}{2c}\right)^2 - \frac{\sigma_{CE}/\sigma_C}{c}} = 0.41347$

$\sigma_{all} = \sigma_C C_P = (9.2)(0.41347) = 3.804 \text{ MPa} \qquad e = 50 \text{ mm} = 0.050 \text{ m}$

$I = \frac{1}{12} d^4 = \frac{1}{12}(0.125)^4 = 20.345 \times 10^{-6} \text{ m}^4 \qquad c = \frac{1}{2} d = 0.0625 \text{ m}$

$\frac{P}{A} + \frac{Pec}{I} \le \sigma_{all} \qquad \left(\frac{1}{A} + \frac{ec}{I}\right) P \le \sigma_{all}$

$P \le \frac{\sigma_{all}}{\frac{1}{A} + \frac{ec}{I}} = \frac{3.804 \times 10^6}{\frac{1}{15.625 \times 10^{-3}} + \frac{(0.050)(0.0625)}{20.345 \times 10^{-6}}} = 17.48 \times 10^3 \text{ N}$

$P = 17.48 \text{ kN}$ ◄

PROBLEM 10.90

10.89 A sawn lumber column with a 125-mm-square cross section and a 3.6-m effective length is made of a grade of wood that has an adjusted allowable stress for compression parallel to the grain $\sigma_C = 9.2$ MPa and a modulus of elasticity $E = 12$ GPa. Using the allowable-stress method, determine the maximum load **P** that can be safely supported with an eccentricity of 50 mm.

10.90 Solve Prob. 10.89 using the interaction method and an allowable stress in bending of 12.8 MPa.

SOLUTION

$d = 125 \text{ mm} = 0.125 \text{ m} \qquad A = d^2 = 15.625 \times 10^{-3} \text{ m}^2 \qquad \frac{L}{d} = \frac{3.6}{0.125} = 28.8$

$\sigma_C = 9.2 \text{ MPa} \qquad E = 12000 \text{ MPa} \qquad \text{sawn lumber: } c = 0.8,\ K_{cE} = 0.300$

$$\sigma_{CE} = \frac{K_{cE}E}{(L/d)^2} = \frac{(0.300)(12000)}{(28.8)^2} = 4.34 \text{ MPa} \qquad \sigma_{CE}/\sigma_C = 0.47177$$

$$C_P = \frac{1+(\sigma_{CE}/\sigma_C)}{2c} - \sqrt{\left(\frac{1+\sigma_{CE}/\sigma_C}{2c}\right)^2 - \frac{\sigma_{CE}/\sigma_C}{c}} = 0.41347$$

$$\sigma_{all,c} = \sigma_C C_P = (9.2)(0.41347) = 3.804 \text{ MPa} \qquad e = 50 \text{ mm} = 0.050 \text{ m}$$

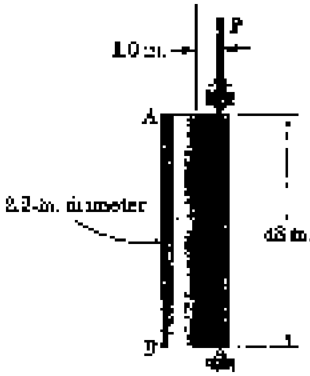
$$I = \frac{1}{12}d^4 = \frac{1}{12}(0.125)^4 = 20.345 \times 10^{-6} \text{ m}^4 \qquad c = \frac{1}{2}d = 0.0625 \text{ m}$$

$$\frac{P}{A\sigma_{all,c}} + \frac{Pec}{I\sigma_{all,B}} \le 1$$

$$P \le \frac{1}{\frac{1}{A\sigma_{all,c}} + \frac{ec}{I\sigma_{all,B}}} = \frac{1}{\frac{1}{(15.625\times10^{-3})(3.804\times10^6)} + \frac{(0.050)(0.0625)}{(20.345\times10^{-6})(12.8\times10^6)}}$$

$$= 34.7 \times 10^3 \text{ N} \qquad 34.7 \text{ kN}$$

PROBLEM 10.91



10.91 An eccentric load is applied at a point 1 in. from the geometric axis of a 2.2-in.-diameter rod made of a steel for which $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi. Using the allowable-stress method, determine the allowable load **P**.

SOLUTION

$c = \frac{1}{2}d = 1.1 \text{ in} \qquad A = \pi c^2 = 3.8013 \text{ in}^2$

$I = \frac{\pi}{4}c^4 = 1.1499 \text{ in}^4 \qquad r = \sqrt{\frac{I}{A}} = 0.550 \text{ in}$

$L_e = 48 \text{ in} \qquad L_e/r = 48/0.550 = 87.2724$

$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(29000)}{36}} = 126.10 > L_e/r$

$\frac{L_e/r}{C_c} = 0.6921$

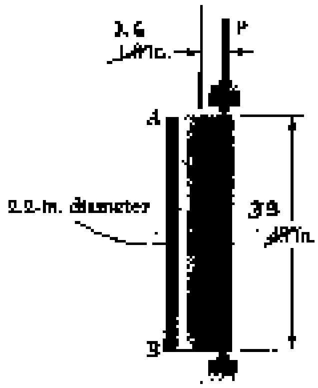
$F.S. = \frac{5}{3} + \frac{3}{8}(0.6921) - \frac{1}{8}(0.6921)^3 = 1.8848$

$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L_e/r}{C_c}\right)^2\right] = \frac{36}{1.8848}\left[1 - \frac{1}{2}(0.6921)^2\right] = 14.526 \text{ ksi}$

$\frac{P_{all}}{A} + \frac{P_{all}ec}{I} = \sigma_{all} \qquad \left(\frac{1}{A} + \frac{ec}{I}\right)P_{all} = \sigma_{all} \qquad P_{all} = \sigma_{all}\left[\frac{1}{A} + \frac{ec}{I}\right]^{-1}$

$P_{all} = (14.526)\left[\frac{1}{3.8013} + \frac{(1.0)(1.1)}{1.1499}\right]^{-1} = 11.91 \text{ kips}$

PROBLEM 10.92



10.91 An eccentric load is applied at a point 1 in. from the geometric axis of a 2.2-in.-diameter rod made of a steel for which $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi. Using the allowable-stress method, determine the allowable load **P**.

10.92 Solve Prob. 10.91, assuming that the load is applied at a point 1.6 in. from the geometric axis and that the effective length is 33 in.

SOLUTION

$c = \frac{1}{2}d = 1.1 \text{ in.} \qquad A = \pi c^2 = 3.8013 \text{ in}^2$

$I = \frac{\pi}{4}c^4 = 1.1499 \text{ in}^4 \qquad r = \sqrt{\frac{I}{A}} = 0.550 \text{ in}$

$L_e = 33 \text{ in} \qquad L_e/r = 33/0.550 = 60$

$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(29000)}{36}} = 126.10$

$\frac{L_e/r}{C_c} = 0.4758 \qquad F.S. = \frac{5}{3} + \frac{3}{8}(0.4758) - \frac{1}{8}(0.4758)^3 = 1.8316$

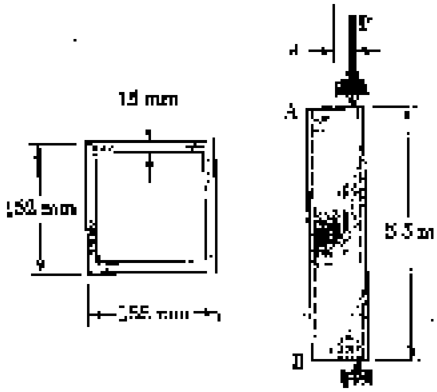
$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L_e/r}{C_c}\right)^2\right] = \frac{36}{1.8316}\left[1 - \frac{1}{2}(0.4758)^2\right] = 17.430 \text{ ksi}$

$\frac{P_{all}}{A} + \frac{P_{all}ec}{I} = \sigma_{all} \qquad \left(\frac{1}{A} + \frac{ec}{I}\right)P_{all} = \sigma_{all} \qquad P_{all} = \sigma_{all}\left[\frac{1}{A} + \frac{ec}{I}\right]^{-1}$

$P_{all} = (17.430)\left[\frac{1}{3.8013} + \frac{(1.6)(1.1)}{1.1499}\right]^{-1} = 9.72 \text{ kips}$

PROBLEM 10.93

10.93 A column of 5.5-m effective length is made of the aluminum alloy 2014-T6 for which the allowable stress in bending is 220 MPa. Using the interaction method, determine the allowable load P, knowing that when the eccentricity is (a) $e = 0$, (b) $e = 40$ mm.



SOLUTION

$b_o = 152$ mm $\qquad b_i = b_o - 2t = 122$ mm

$A = b_o^2 - b_i^2 = 8220$ mm^2 $= 8220 \times 10^{-6}$ m^2

$I = \frac{1}{12}(b_o^4 - b_i^4) = 26.02 \times 10^6$ mm^4

$r = \sqrt{\frac{I}{A}} = 56.26$ mm $= 56.26 \times 10^{-3}$ m

$\frac{L}{r} = \frac{5.5}{56.26 \times 10^{-3}} = 97.76 > 55$

$\sigma_{all,c} = \frac{372 \times 10^3}{(L/r)^2} = \frac{372 \times 10^3}{(97.76)^2} = 38.92$ MPa for centric loading

$$\frac{P}{A\sigma_{all,c}} + \frac{Pec}{I\sigma_{all,b}} = 1$$

(a) $e = 0$ $\qquad P = A\sigma_{all,c} = (8220 \times 10^{-6})(38.92 \times 10^6) = 320 \times 10^3$ N $= 320$ kN

(b) $e = 40 \times 10^{-3}$ m $\qquad c = \frac{1}{2}(152) = 76$ mm $= 76 \times 10^{-3}$ m

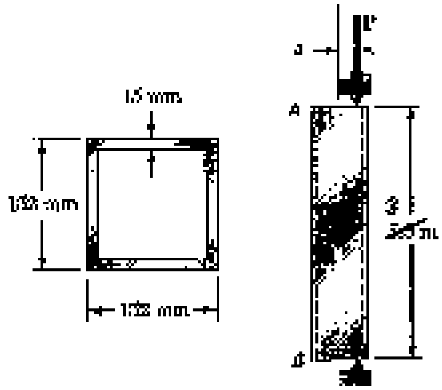
$$\frac{P}{(8220 \times 10^{-6})(38.92 \times 10^6)} + \frac{P(40 \times 10^{-3})(76 \times 10^{-3})}{(26.02 \times 10^{-6})(220 \times 10^6)} = 3.6568 \times 10^{-6} P = 1$$

$P = 273 \times 10^3$ N $= 273$ kN

PROBLEM 10.94

10.93 A column of 5.5-m effective length is made of the aluminum alloy 2014-T6 for which the allowable stress in bending is 220 MPa. Using the interaction method, determine the allowable load **P**, knowing that when the eccentricity is (*a*) $e = 0$, (*b*) $e = 40$ mm.

10.94 Solve Prob. 10.93, assuming that the effective length of a column is 3.0 m.



SOLUTION

$b_o = 152$ mm $\qquad b_i = b_o - 2t = 122$ mm

$A = b_o^2 - b_i^2 = 8220 \text{ mm}^2 = 8200 \times 10^{-6} \text{ m}^2$

$I = \frac{1}{12}(b_o^4 - b_i^4) = 26.02 \times 10^6 \text{ mm}^4$

$r = \sqrt{\frac{I}{A}} = 56.26 \text{ mm} = 56.26 \times 10^{-3} \text{ m}$

$\frac{L}{r} = \frac{3.0}{56.26 \times 10^{-3}} = 53.32 <$

$\sigma_{all,c} = 212 - 1.585(L/r) = 212 - (1.585)(53.32) = 127.5$ MPa

$$\frac{P}{A\sigma_{all,c}} + \frac{Pec}{I\sigma_{all,b}} = 1$$

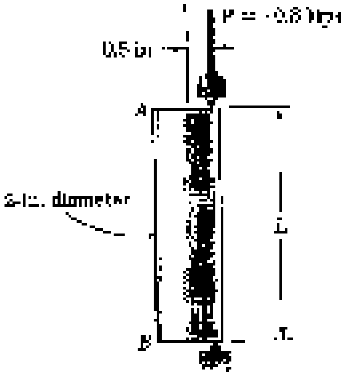
(a) $e = 0 \qquad P = A\sigma_{all} = (8220 \times 10^{-6})(127.5 \times 10^6) = 1048 \times 10^3 \text{ N} = 1048 \text{ kN}$ ◄

(b) $e = 40 \text{ mm} = 40 \times 10^{-3} \text{ m} \qquad c = (\frac{1}{2})(152) = 76 \text{ mm} = 76 \times 10^{-3} \text{ m}$

$$\frac{P}{(8220 \times 10^{-6})(127.5 \times 10^6)} + \frac{P(40 \times 10^{-3})(76 \times 10^{-3})}{(26.02 \times 10^{-6})(220 \times 10^6)} = 1.4852 \times 10^{-6} P = 1$$

$P = 673 \times 10^3 \text{ N} = 673 \text{ kN}$ ◄

PROBLEM 10.95



10.95 An eccentric load $P = 10.8$ kips is applied at a point 0.8 in. from the geometric axis of a 2-in.-diameter rod made of the aluminum alloy 6061-T6. Using the interaction method and an allowable stress in bending of 21 ksi, determine the largest allowable effective length L that can be used.

SOLUTION

$c = \frac{1}{2}d = 1.0$ in $\qquad A = \pi c^2 = 3.1416$ in^2

$I = \frac{\pi}{4}c^4 = 0.7854$ in^4 $\qquad r = \sqrt{\frac{I}{A}} = 0.5$ in.

$e = 0.8$ in $\qquad \sigma_{all,b} = 21$ ksi

$$\frac{P}{A\sigma_{all,c}} + \frac{Pec}{I\sigma_{all,b}} = 1 \qquad \frac{P}{A\sigma_{all,c}} = 1 - \frac{Pec}{I\sigma_{all,b}}$$

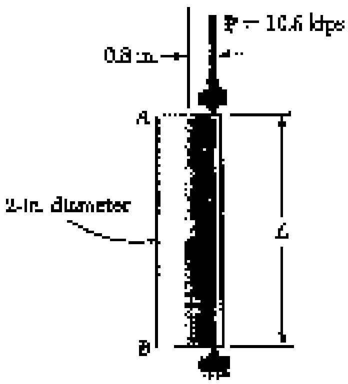
$$\frac{1}{\sigma_{all,c}} = \frac{A}{P}\left(1 - \frac{Pec}{I\sigma_{all,b}}\right) = \frac{3.1416}{10.8}\left[1 - \frac{(10.8)(0.8)(1.0)}{(0.7854)(21)}\right] = 0.1385 \text{ ksi}^{-1}$$

$\sigma_{all,c} = 7.22$ ksi $\qquad$ Assume $L/r > 66$

$$\sigma_{all,c} = \frac{51000}{(L/r)^2} \qquad \frac{L}{r} = \sqrt{\frac{51000}{\sigma_{all,c}}} = 84.05 > 66$$

$$L = 84.05\,r = (84.05)(0.5) = 42.0 \text{ in.}$$

PROBLEM 10.96



10.95 An eccentric load $P = 10.8$ kips is applied at a point 0.8 in. from the geometric axis of a 2-in.-diameter rod made of the aluminum alloy 6061-T6. Using the interaction method and an allowable stress in bending of 21 ksi, determine the largest allowable effective length L that can be used.

10.96 Solve Prob. 10.95, assuming that the aluminum alloy used is 2014-T6 and that the allowable stress in bending is 26 ksi.

SOLUTION

$c = \frac{1}{2}d = 1.0$ in. $\qquad A = \pi c^2 = 3.1416$ in^2

$I = \frac{\pi}{4}c^4 = 0.7854$ in^4 $\qquad r = \sqrt{\frac{I}{A}} = 0.5$ in.

$e = 0.8$ in. $\qquad \sigma_{all,b} = 26$ ksi

$$\frac{P}{A\sigma_{all,c}} + \frac{Pec}{I\sigma_{all,b}} = 1 \qquad \frac{P}{A\sigma_{all,c}} = 1 - \frac{Pec}{I\sigma_{all,b}}$$

$$\frac{1}{\sigma_{all,c}} = \frac{A}{P}\left(1 - \frac{Pec}{I\sigma_{all,b}}\right) = \frac{3.1416}{10.8}\left[1 - \frac{(10.8)(0.8)(1.0)}{(0.7854)(26)}\right] = 0.1678 \text{ ksi}^{-1}$$

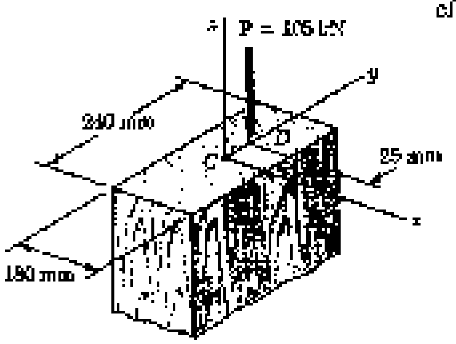
$\sigma_{all,c} = 5.96$ ksi $\qquad$ Assume $L/r > 55$

$$\sigma_{all,c} = \frac{54000}{(L/r)^2} \qquad \frac{L}{r} = \sqrt{\frac{54000}{\sigma_{all,c}}} = \sqrt{\frac{54000}{5.96}} = 95.19 > 55$$

$$L = 95.19\,r = (95.19)(0.5) = 47.6 \text{ in.}$$

PROBLEM 10.97

10.97 A rectangular column is made of sawn lumber that has an adjusted allowable stress for compression parallel to the grain $\sigma_C = 8.3$ MPa and a modulus of elasticity $E = 11.1$ GPa. Using the allowable-stress method, determine the largest allowable effective length L that can used.



SOLUTION

$d = 180$ mm $= 0.180$ m $b = 240$ mm $= 0.240$ m

$A = bd = 43.2 \times 10^{-3}$ m^2 $E = 11100$ MPa

$I_x = \frac{1}{12} d b^3 = \frac{1}{12}(0.180)(0.240)^3 = 207.36 \times 10^{-6}$ m^4

$e = 25$ mm $= 0.025$ m $c = \frac{b}{2} = 0.120$ m

$$\frac{P}{A} + \frac{Pec}{I_x} \le \sigma_{all} \qquad \sigma_{all} \ge \frac{105 \times 10^3}{43.2 \times 10^{-3}} + \frac{(105 \times 10^3)(0.025)(0.120)}{207.36 \times 10^{-6}} = 3.9496 \times 10^6 \text{ Pa}$$
$$= 3.9496 \text{ MPa}$$

$$C_P = \frac{\sigma_{all}}{\sigma_C} = \frac{3.9496}{8.3} = 0.47586 = y \qquad \text{Let } x = \sigma_{CE}/\sigma_C$$

$$y = \frac{1+x}{c} - \sqrt{\left(\frac{1+x}{2c}\right)^2 - \frac{x}{c}} \qquad \text{where } c = 0.8 \text{ for sawn lumber}$$

$$\frac{1+x}{2c} - y = \sqrt{\left(\frac{1+x}{2c}\right)^2 - \frac{x}{c}}$$

$$\left(\frac{1+x}{2c} - y\right)^2 = \left(\frac{1+x}{2c}\right)^2 - \frac{x}{c}$$

$$\left(\frac{1+x}{2c}\right)^2 - \frac{1+x}{c}y + y^2 = \left(\frac{1+x}{2c}\right)^2 - \frac{x}{c}$$

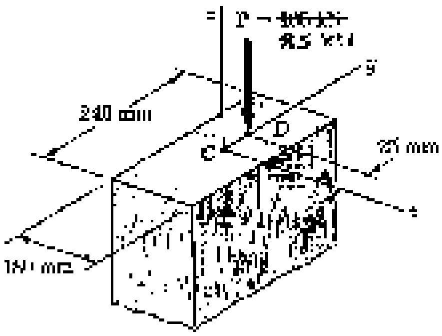
$$x = y\,\frac{1-cy}{1-y} = (0.47586)\frac{1-(0.8)(0.47586)}{1-0.47586} = 0.56227$$

$$\sigma_{CE} = \sigma_C(0.56227) = (8.3)(0.56227) = 4.6668 \text{ MPa}$$

$$\sigma_{CE} = \frac{K_{CE}E}{(L/d)^2} \qquad L^2 = \frac{K_{CE}Ed^2}{\sigma_{CE}} \qquad \text{where } K_{CE} = 0.300$$

$$L = d\sqrt{\frac{K_{CE}E}{\sigma_{CE}}} = 0.180\sqrt{\frac{(0.300)(11100)}{4.6668}} = 4.81 \text{ m}$$

PROBLEM 10.98



10.97 A rectangular column is made of sawn lumber that has an adjusted allowable stress for compression parallel to the grain $\sigma_C = 8.3$ MPa and a modulus of elasticity $E = 11.1$ GPa. Using the allowable-stress method, determine the largest allowable effective length L that can used.

10.98 Solve Prob. 10.97, assuming that $P = 85$ kN.

SOLUTION

$d = 180 \text{ mm} = 0.180 \text{ m} \qquad b = 240 \text{ mm} = 0.240 \text{ m}$

$A = bd = 43.2\times10^{-3} \text{ m}^2 \qquad E = 11100 \text{ MPa}$

$I_x = \frac{1}{12} d b^3 = \frac{1}{12}(0.180)(0.240)^3 = 207.36\times10^{-6} \text{ m}^4$

$e = 25 \text{ mm} = 0.025 \text{ m} \qquad c = \frac{b}{2} = 0.120 \text{ m}$

$\frac{P}{A} + \frac{Pec}{I} \le \sigma_{all} \qquad \sigma_{all} \ge \frac{85\times10^3}{43.2\times10^{-3}} + \frac{(85\times10^3)(0.025)(0.120)}{207.36\times10^{-6}} = 3.1496\times10^6 \text{ Pa}$

$= \qquad \text{MPa}$

$C_P = \frac{\sigma_{all}}{\sigma_C} = \frac{3.1473}{8.3} = 0.38522 = y \qquad \text{Let } x = \sigma_{CE}/\sigma_C$

$y = \frac{1+x}{2c} - \sqrt{\left(\frac{1+x}{2c}\right)^2 - \frac{x}{c}} \qquad \text{where } c = 0.8 \text{ for sawn lumber}$

$\frac{1+x}{2c} - y = \sqrt{\left(\frac{1+x}{2c}\right)^2 - \frac{x}{c}}$

$\cancel{\left(\frac{1+x}{2c}\right)^2} - y\left(\frac{1+x}{c}\right) + y^2 = \cancel{\left(\frac{1+x}{2c}\right)^2} - \frac{x}{c}$

$x = y\frac{(1-cy)}{1-y} = (0.38522)\frac{1-(0.8)(0.38522)}{1-0.38522} = 0.43350$

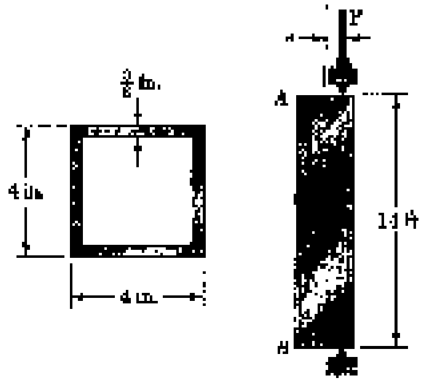
$\sigma_{ce} = \sigma_C(0.43350) = (8.3)(0.43350) = 3.598 \text{ MPa}$

$\sigma_{CE} = \frac{K_{CE}E}{(L/d)^2} \qquad L^2 = \frac{K_{CE}Ed^2}{\sigma_{CE}} \qquad \text{where } K_{CE} = 0.300$

$L = d\sqrt{\frac{K_{CE}E}{\sigma_{CE}}} = (0.180)\sqrt{\frac{(0.300)(11100)}{3.598}} = 5.48 \text{ m}$

PROBLEM 10.99

10.99 A column of 14-ft effective length consists of a section of steel tubing having the cross section shown. Using the allowable-stress method, determine the maximum allowable eccentricity e if (a) $P = 55$ kips, (b) $P = 35$ kips. Use $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi.



SOLUTION

Steel: $\sigma_Y = 36$ ksi $\quad E = 29000$ ksi

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(29000)}{36}} = 126.10$$

$b_o = 4.0$ in $\quad b_i = b_o - 2t = 3.25$ in $\quad c = 2.0$ in

$A = b_o^2 - b_i^2 = 5.4375 \text{ in}^2 \quad I = \frac{1}{12}(b_o^4 - b_i^4) = 12.036 \text{ in}^4$

$r = \sqrt{\frac{I}{A}} = 1.4878$ in $\quad L_e = 14$ ft $= 168$ in

$L_e/r = 112.92 < C_c \qquad \frac{L_e/r}{C_c} = 0.89547$

$$F.S. = \frac{5}{3} + \frac{3}{8}(0.89547) - \frac{1}{8}(0.89547)^3 = 1.9127$$

$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L_e/r}{C_c}\right)^2\right] = \frac{36}{1.9127}\left[1 - \frac{1}{2}(0.89547)^2\right] = 11.275 \text{ ksi}$$

$$\frac{P_{all}}{A} + \frac{P_{all}ec}{I} = \sigma_{all} \qquad \frac{P_{all}ec}{I} = \sigma_{all} - \frac{P_{all}}{A} \qquad e = \frac{I}{cP_{all}}\left(\sigma_{all} - \frac{P_{all}}{A}\right)$$

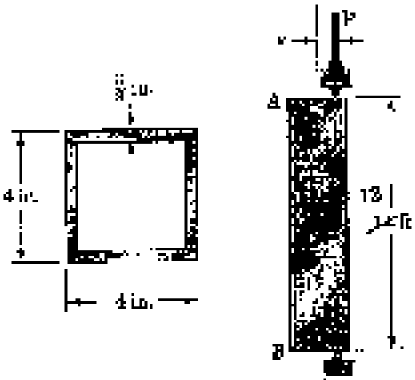
(a) $P_{all} = 55$ kips

$$e = \frac{12.036}{(2.0)(55)}\left[11.275 - \frac{55}{5.4375}\right] = 0.127 \text{ in}$$ ◀

(b) $P_{all} = 35$ kips

$$e = \frac{12.036}{(2.0)(35)}\left[11.275 - \frac{35}{5.4375}\right] = 0.832 \text{ in.}$$ ◀

PROBLEM 10.100



10.99 A column of 14-ft effective length consists of a section of steel tubing having the cross section shown. Using the allowable-stress method, determine the maximum allowable eccentricity e if (a) $P = 55$ kips, (b) $P = 35$ kips. Use $\sigma_Y = 36$ ksi and $E = 29 \times 10^6$ psi.

10.100 Solve Prob. 11.99, assuming that the effective length of the column is increased to 18 ft and that (a) $P = 28$ kips, (b) $P = 18$ kips.

SOLUTION

Steel: $\sigma_Y = 36$ ksi $\qquad E = 29000$ ksi

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(29000)}{36}} = 126.10$$

$b_o = 4.0$ in $\qquad b_i = b_o - 2t = 3.25$ in $\qquad c = 2.0$ in.

$A = b_o^2 - b_i^2 = 5.4375\ \text{in}^2 \qquad I = \frac{1}{12}(b_o^4 - b_i^4) = 12.036\ \text{in}^4$

$r = \sqrt{\frac{I}{A}} = 1.4878 \qquad L_e = 18\ \text{ft} = 216\ \text{in} \qquad L_e/r = 145.18 > C_c$

$$\sigma_{all} = \frac{\pi^2 E}{1.92(L_e/r)^2} = \frac{\pi^2(29000)}{(1.92)(145.18)^2} = 7.0726\ \text{ksi}$$

$$\frac{P_{all}}{A} + \frac{P_{all}ec}{I} = \sigma_{all} \qquad \frac{P_{all}ec}{I} = \sigma_{all} - \frac{P_{all}}{A} \qquad e = \frac{I}{cP_{all}}\left(\sigma_{all} - \frac{P_{all}}{A}\right)$$

(a) $P_{all} = 28$ kips

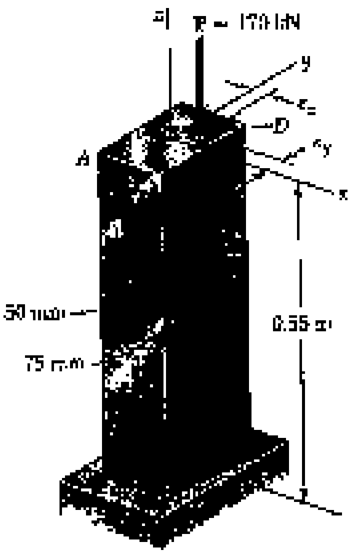
$$e = \frac{12.036}{(2.0)(28)}\left[7.0726 - \frac{28}{5.4375}\right] = 0.419\ \text{in.}$$

(b) $P_{all} = 18$ kips

$$e = \frac{12.036}{(2.0)(18)}\left[7.0726 - \frac{18}{5.4375}\right] = 1.258\ \text{in}$$

PROBLEM 10.101

10.101 The compression member AB is made of a steel for which $\sigma_Y = 250$ MPa and $E = 200$ GPa. It is free at its top A and fixed at its base B. Using the allowable-stress method, determine the largest allowable eccentricity e_x, knowing that (a) $e_y = 0$, (b) $e_y = 8$ mm.



SOLUTION

Steel: $\sigma_Y = 250$ MPa $\quad E = 200000$ MPa

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2\, 200000}{250}} = 125.66 \text{ MPa}$$

$$A = (75\times10^{-3})(50\times10^{-3}) = 3750\times10^{-6}\ \text{m}^2$$

$$I_y = \tfrac{1}{12}(75\times10^{-3})(50\times10^{-3})^3 = 781.25\times10^{-9}\ \text{m}^4$$

$$r_y = \sqrt{\frac{I_y}{A}} = 14.434\times10^{-3}\ \text{m} = r_{min}$$

$$I_x = \tfrac{1}{12}(50\times10^{-3})(75\times10^{-3})^3 = 1.7578\times10^{-6}\ \text{m}^4$$

$$r_x = \sqrt{\frac{I_x}{A}} = 21.651\times10^{-6}\ \text{m}$$

$L_e = 2L = (2)(0.55) = 1.10$ m $\qquad L_e/r_{min} = 1.10/14.434\times10^{-3} = 76.21 < C_c$

$$\frac{L_e/r_{min}}{C_c} = \frac{76.21}{125.66} = 0.6065 \qquad F.S. = \frac{5}{3} + \frac{3}{8}(0.6065) - \frac{1}{8}(0.6065)^3 = 1.8662$$

$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L_e/r_{min}}{C_c}\right)^2\right] = \frac{250}{1.8662}\left[1 - \tfrac{1}{2}(0.6065)^2\right] = 109.32 \text{ MPa}$$

$$\frac{P}{A} + \frac{Pe_x}{S_y} + \frac{Pe_y}{S_x} = \sigma_{all} \qquad \frac{Pe_x}{S_y} = \sigma_{all} - \frac{P}{A} - \frac{Pe_y}{S_x}$$

$$e_x = \frac{S_y}{P}\left[\sigma_{all} - \frac{P}{A} - \frac{Pe_y}{S_x}\right] = S_y\left[\frac{\sigma_{all}}{P} - \frac{1}{A} - \frac{e_y}{S_x}\right]$$

$$S_y = \frac{I_y}{x} = \frac{781.25\times10^{-9}}{25\times10^{-3}} = 31.25\times10^{-6}\ \text{m}^3$$

$$S_x = \frac{I_x}{y} = \frac{1.7578\times10^{-6}}{37.5\times10^{-3}} = 46.875\times10^{-6}\ \text{m}^3$$

$$P = 170\times10^3\ \text{N}$$

(a) $e_y = 0$

$$e_x = 31.25\times10^{-6}\left[\frac{109.32\times10^6}{170\times10^3} - \frac{1}{3750\times10^{-6}} - 0\right]$$

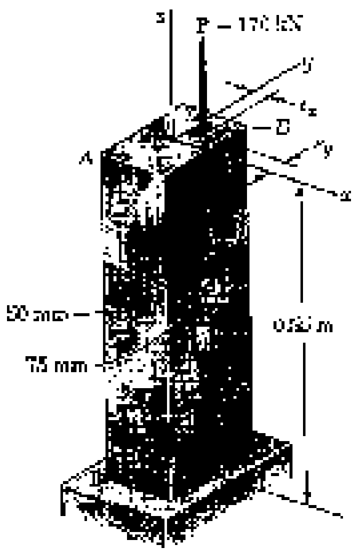
$$= 11.76\times10^{-5}\ \text{m} = 11.76 \text{ mm}$$

(b) $e_y = 8\times10^{-3}$ m

$$e_x = 31.25\times10^{-6}\left[\frac{109.32\times10^6}{170\times10^3} - \frac{1}{3750\times10^{-6}} - \frac{8\times10^{-3}}{46.875\times10^{-6}}\right]$$

$$= 6.43\times10^{-5}\ \text{m} = 6.43 \text{ mm}$$

PROBLEM 10.102



10.102 The compression member AB is made of a steel for which $\sigma_Y = 250$ MPa and $E = 200$ GPa. It is free at its top A and fixed at its base B. Using the interaction method with an allowable bending stress equal to 120 MPa and knowing that the eccentricities e_x and e_y are equal, determine the largest allowable common value.

SOLUTION

Steel: $\sigma_Y = 250$ MPa $\quad E = 200000$ MPa

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(200000)}{250}} = 125.66$$

$$A = (75\times10^{-3})(50\times10^{-3}) = 3750\times10^{-6}\ \text{m}^2$$

$$I_y = \tfrac{1}{12}(75\times10^{-3})(50\times10^{-3})^3 = 781.25\times10^{-9}\ \text{m}^4$$

$$r_y = \sqrt{\frac{I_y}{A}} = 14.434\times10^{-3}\ \text{m}$$

$$I_x = \tfrac{1}{12}(50\times10^{-3})(75\times10^{-3})^3 = 1.7578\times10^{-6}\ \text{m}^4$$

$$r_x = \sqrt{\frac{I_x}{A}} = 21.651\times10^{-3}\ \text{m}$$

$$L_e = 2L = (2)(0.55) = 1.10\ \text{m} \qquad L_e/r_{min} = 1.10/14.434\times10^{-3} = 76.21 < C_c$$

$$\frac{L_e/r_{min}}{C_c} = \frac{76.21}{125.66} = 0.6065 \qquad \text{F.S.} = \frac{5}{3} + \frac{3}{8}(0.6065) - \frac{1}{8}(0.6065)^3 = 1.8662$$

$$\sigma_{all\,(centric)} = \frac{\sigma_Y}{\text{F.S.}}\left[1 - \frac{1}{2}\left(\frac{L_e/r_{min}}{C_c}\right)^2\right] = \frac{250}{1.8662}\left[1 - \tfrac{1}{2}(0.6065)^2\right] = 109.32\ \text{MPa}$$

$$\sigma_{all\,(bending)} = 120\ \text{MPa}$$

$$\frac{P}{A\sigma_{all\,(centric)}} + \frac{Pe_y y}{I_x\sigma_{all\,(bending)}} + \frac{Pe_x x}{I_y\sigma_{all\,(bending)}} = 1 \qquad \text{with } e_x = e_y$$

$$\frac{P}{\sigma_{all\,(bending)}}\left(\frac{y}{I_x} + \frac{x}{I_y}\right)e = 1 - \frac{P}{A\sigma_{all\,(centric)}}$$

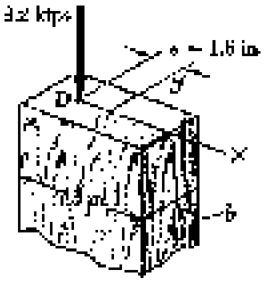
$$\frac{170\times10^3}{120\times10^6}\left(\frac{37.5\times10^{-3}}{1.7578\times10^{-6}} + \frac{25\times10^{-3}}{781.25\times10^{-9}}\right)e = 1 - \frac{170\times10^3}{(3750\times10^{-6})(109.32\times10^6)}$$

$$75.556\,e = 1 - 0.41468$$

$$e = 7.75\times10^{-3}\ \text{m} = 7.75\ \text{mm}$$

PROBLEM 10.103

10.103 A sawn lumber column of rectangular cross section has a 7.2-ft effective length and supports a 9.2 kip load as shown. The sizes available for use have b equal to 3.5 in., 5.5 in., 7.5 in. and 9.5 in. The grade of wood has an adjusted allowable stress for compression parallel to the grain $\sigma_C = 1180$ psi and $E = 1.2 \times 10^6$ psi. Use the allowable-stress method to determine the lightest section that can be used.



SOLUTION

Sawn lumber: $\sigma_C = 1180$ psi $\quad E = 1.2 \times 10^6$ psi

$c = 0.8 \quad K_{cE} = 0.300$

$L_E = 7.2$ ft $= 86.4$ in

$$\frac{P_{all}}{A} + \frac{P_{all}ec}{I_x} = \sigma_{all} \qquad P_{all} = \frac{\sigma_{all}}{\frac{1}{A} + \frac{ec}{I}}$$

$e = 1.6$ in $\qquad c = \frac{1}{2}(7.5) = 3.75$ in. $\qquad A = 7.5\ b$

$$I_x = \frac{1}{12}b(7.5)^3 = 35.156\ b$$

$$\frac{1}{\frac{1}{A} + \frac{ec}{I_x}} = \frac{1}{\frac{1}{7.5b} + \frac{(1.6)(3.75)}{35.156\ b}} = 3.2895\ b \qquad P_{all} = 3.2895\ b\sigma_{all}$$

$d = 7.5$ in. or b, whichever is smaller.

$$\sigma_{cE} = \frac{K_{cE}E}{(L/d)^2} = \frac{K_{cE}Ed^2}{L^2} = \frac{(0.300)(1.2\times10^6)d^2}{(86.4)^2} = 48.225\ d^2 \quad \text{(psi)}$$

$$\sigma_{cE}/\sigma_C = (48.225 d^2)/1180 = 0.04087\ d^2$$

$$C_P = \frac{1+\sigma_{cE}/\sigma_C}{2c} - \sqrt{\left(\frac{1+\sigma_{cE}/\sigma_C}{2c}\right)^2 - \frac{\sigma_{cE}/\sigma_C}{c}}$$

$$\sigma_{all} = \sigma_C C_P = 1180\ C_P$$

$$P_{all} = (3.2895)\ b\ (1180\ C_P) = 3882\ b\ C_P \quad \text{(lb.)}$$

Calculate P_{all} for all four values of b. See table below.

b (in)	d (in)	σ_{cE}/σ_C	C_P	P_{all} (lb.)
3.5	3.5	0.5007	0.4341	5900
5.5	5.5	1.2363	0.7588	16200
7.5	7.5	2.299	0.8862	25900
9.5	7.5	2.299	0.8882	32800

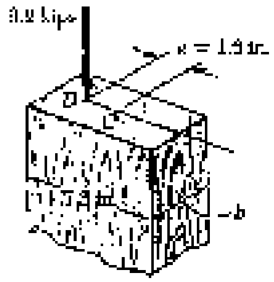
$P = 9200$ lb.

Use $b = 5.5$ in.

PROBLEM 10.104

10.103 A sawn lumber column of rectangular cross section has a 7.2-ft effective length and supports a 9.2 kip load as shown. The sizes available for use have b equal to 3.5 in., 5.5 in., 7.5 in. and 9.5 in. The grade of wood has an adjusted allowable stress for compression parallel to the grain $\sigma_C = 1180$ psi and $E = 1.2 \times 10^6$ psi. Use the allowable-stress method to determine the lightest section that can be used.

10.104 Solve Prob. 10.103, assuming that $e = 3.2$ in.



SOLUTION

Sawn lumber: $\sigma_C = 1180$ psi $\quad E = 1.2 \times 10^6$ psi

$c = 0.8 \quad K_{cE} = 0.300$

$L_e = 7.2$ ft $= 86.4$ in.

$$\frac{P_{all}}{A} + \frac{P_{all}\,ec}{I_x} = \sigma_{all} \qquad P_{all} = \frac{\sigma_{all}}{\frac{1}{A} + \frac{ec}{I_x}}$$

$e = 3.2$ in $\qquad c = \frac{1}{2}(7.5) = 3.75$ in $\qquad A = 7.5\,b$

$I_x = \frac{1}{12} b (7.5)^3 = 35.156\,b$

$$\frac{1}{\frac{1}{A} + \frac{ec}{I_x}} = \frac{1}{\frac{1}{7.5\,b} + \frac{(3.2)(3.75)}{35.156\,b}} = 2.1067\,b \qquad P_{all} = 2.1067\,b\,\sigma_{all}$$

$d = 7.5$ in. or b, whichever is smaller

$$\sigma_{cE} = \frac{K_{cE}E}{(L/d)^2} = \frac{K_{cE}Ed^2}{L^2} = \frac{(0.300)(1.2 \times 10^6)d^2}{(86.4)^2} = 48.225\,d^2 \text{ (psi)}$$

$\sigma_{cE}/\sigma_C = 48.225\,d^2/1180 = 0.04087\,d^2$

$$C_p = \frac{1 + \sigma_{cE}/\sigma_C}{2c} - \sqrt{\left(\frac{1 + \sigma_{cE}/\sigma_C}{2c}\right)^2 - \frac{\sigma_{cE}/\sigma_C}{c}}$$

$\sigma_{all} = \sigma_C\,C_p = 1180\,C_p$

$P_{all} = (2.1067)\,b\,(1180\,C_p) = 2486\,b\,C_p$

Calculate P_{all} for all four values of b. See table below.

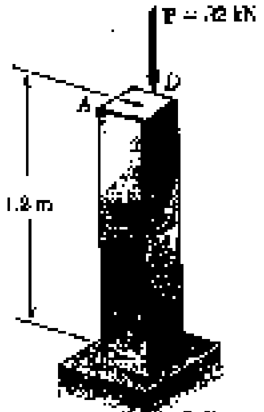
b (in.)	d (in)	σ_{cE}/σ_C	C_p	P_{all} (lb)
3.5	3.5	0.5007	0.4341	3780
5.5	5.5	1.2363	0.7588	10370
7.5	7.5	2.299	0.8882	16560
9.5	7.5	2.299	0.8882	20100

$P = 9200$ lb.

Use $b = 5.5$ in.

PROBLEM 10.105

10.105 A 32-kN vertical load **P** is applied at the midpoint of one edge of the square cross section of the aluminum compression member AB that is free at its top A and fixed at its base B. Knowing that the alloy used is 6061-T6, use the allowable-stress method to determine the smallest allowable dimension d.



$L_e = 2L = 2.4$ m

SOLUTION

$$A = d^2 \qquad I = \tfrac{1}{12}d^4 \qquad r = \sqrt{\frac{I}{A}} = \frac{1}{\sqrt{12}}d \qquad c = \tfrac{1}{2}d \qquad e = \tfrac{1}{2}d$$

$$\frac{P}{A} + \frac{Pec}{I} = \frac{P}{d^2} + \frac{P(\frac{1}{2}d)(\frac{1}{2}d)}{\frac{1}{12}d^4} = \frac{4P}{d^2} = \sigma_{all}$$

Assume $L/r > 66$ $\qquad \sigma_{all} = \dfrac{B}{(L/r)^2} \qquad B = 351\times10^9$ Pa

$$\frac{Br^2}{L^2} = \frac{Bd^2}{12L^2} = \frac{4P}{d^2} \qquad d^4 = \frac{48PL^2}{B}$$

$$d = \sqrt[4]{\frac{48PL^2}{B}} = \sqrt[4]{\frac{(48)(32\times10^3)(2.4)^2}{351\times10^9}} = 70.9\times10^{-3}\text{ m}$$

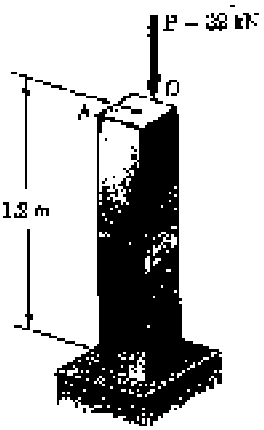
$$r = \frac{70.9\times10^{-3}}{\sqrt{12}} = 20.45\times10^{-3}\text{ m} \qquad \frac{L_e}{r} = \frac{2.4}{20.45\times10^{-3}} = 117.3 > 66$$

answer $\qquad d = 70.9$ mm

PROBLEM 10.106

10.105 A 32-kN vertical load **P** is applied at the midpoint of one edge of the square cross section of the aluminum compression member AB that is free at its top A and fixed at its base B. Knowing that the alloy used is 6061-T6, use the allowable-stress method to determine the smallest allowable dimension d.

10.106 Solve Prob. 10.105, assuming that the vertical load **P** is applied at a corner of the square cross section of the compression member AB.



$L_e = 2L = 2.4$ m

SOLUTION

$$A = d^2, \qquad I = \tfrac{1}{12}d^4, \qquad r = \sqrt{\frac{I}{A}} = \frac{d}{\sqrt{12}} \qquad x = y = \tfrac{1}{2}d$$

$$e_x = e_y = \tfrac{1}{2}d$$

$$\frac{P}{A} + \frac{Pe_x x}{I_y} + \frac{Pe_y y}{I_x} = \frac{P}{d^2} + \frac{P(\frac{1}{2}d)(\frac{1}{2}d)}{\frac{1}{12}d^4} + \frac{P(\frac{1}{2}d)(\frac{1}{2}d)}{\frac{1}{12}d^4}$$

$$= \frac{7P}{d^2} = \sigma_{all}$$

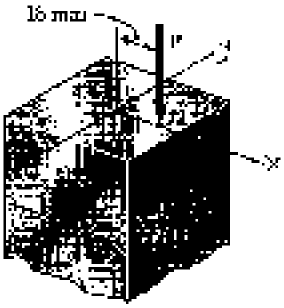
Assume $L/r > 66$ $\qquad \sigma_{all} = \dfrac{B}{(L_e/r)^2} \qquad B = 351\times10^9$ Pa

$$\frac{Br^2}{L_e^2} = \frac{Bd^2}{12L_e^2} = \frac{7P}{d^2} \qquad d^4 = \frac{84PL_e^2}{B}$$

$$d = \sqrt[4]{\frac{84PL_e^2}{B}} = \sqrt[4]{\frac{(84)(32\times10^3)(2.4)^2}{351\times10^9}} = 81.5\times10^{-3}\text{ m} \qquad d = 81.5\text{ mm}$$

$$r = \frac{d}{\sqrt{12}} = 23.5\times10^{-3}\text{ m} \qquad \frac{L_e}{r} = \frac{2.4}{23.5\times10^{-3}} = 102.0 > 66$$

PROBLEM 10.107



10.107 A compression member made of steel has a 720-mm effective length and must support the 198-kN load P as shown. For the material used $\sigma_Y = 250$ MPa and $E = 200$ GPa. Using the interaction method with an allowable bending stress equal to 150 MPa, determine the smallest dimension d of the cross section that can be used.

SOLUTION

Using dimensions in meters

$A = 40 \times 10^{-3}\, d \qquad L_e = 720 \text{ mm} = 0.720 \text{ m}$

$I_x = \frac{1}{12}(40 \times 10^{-3})^3 d = 5.3333 \times 10^{-6}\, d$

$I_y = \frac{1}{12}(40 \times 10^{-3})\, d^3 = 3.3333 \times 10^{-3}\, d^3$

$|x| = \frac{d}{2}, \quad |y| = 20 \text{ mm} = 0.020 \text{ m} \qquad |e_x| = 18 \text{ mm} = 18 \times 10^{-3} \text{ m}$

Steel: $\sigma_Y = 250$ MPa $\quad E = 200000$ MPa $\qquad C_c = \sqrt{\dfrac{2\pi^2 E}{\sigma_Y}}$

$$C_c = \sqrt{\frac{2\pi^2(200000)}{250}} = 125.66$$

Assume $d > 40 \text{ mm} = 40 \times 10^{-3}$ m. Then $I_{min} = I_x$

$r = \sqrt{\dfrac{I_x}{A}} = \sqrt{\dfrac{5.3333 \times 10^{-6}\, d}{40 \times 10^{-3}\, d}} = 11.547 \times 10^{-3} \text{ m}, \quad \dfrac{L_e}{r} = 62.35 < C_c$

$\dfrac{L_e/r}{C_c} = 0.49621 \qquad F.S. = \frac{5}{3} + \frac{3}{8}(0.49621) - \frac{1}{8}(0.49621)^3 = 1.83747$

$\sigma_{all,centric} = \dfrac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\dfrac{L_e/r}{C_c}\right)^2\right] = \dfrac{250}{1.83747}\left[1 - \frac{1}{2}(0.49621)^2\right] = 119.31 \text{ MPa}$

$\sigma_{all,bending} = 150$ MPa

$$\frac{P}{A\sigma_{all,centric}} + \frac{P e_x x}{I_y \sigma_{all,bending}} = 1$$

$$\frac{198 \times 10^3}{(40 \times 10^{-3} d)(119.31 \times 10^6)} + \frac{(198 \times 10^3)(18 \times 10^{-3})(\frac{1}{2}d)}{(3.3333 \times 10^{-3} d^3)(150 \times 10^6)} = 1$$

$$\frac{41.489 \times 10^{-3}}{d} + \frac{3.5640 \times 10^{-3}}{d^2} = 1$$

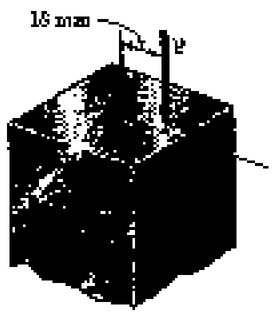
$$d^2 - 41.489 \times 10^{-3}\, d - 3.5640 \times 10^{-3} = 0$$

$$d = \frac{1}{2}\left\{41.489 \times 10^{-3} + \sqrt{(41.489 \times 10^{-3})^2 + (4)(3.5640 \times 10^{-3})}\right\}$$

$$= 83.9 \times 10^{-3} \text{ m} \quad > 40 \times 10^{-3} \text{ m}$$

$d = 83.9$ mm

PROBLEM 10.108



10.107 A compression member made of steel has a 720-mm effective length and must support the 198-kN load **P** as shown. For the material used $\sigma_Y = 250$ MPa and $E = 200$ GPa. Using the interaction method with an allowable bending stress equal to 150 MPa, determine the smallest dimension d of the cross section that can be used.

10.108 Solve Prob. 10.107, assuming that the effective length is 1.62 m and that the magnitude P of the eccentric load is 128 kN.

SOLUTION

Using dimensions in meters

$A = 40 \times 10^{-3}\, d \qquad L_e = 1.62 \text{ m}$

$I_x = \frac{1}{12}(40\times10^{-3})^3 d = 5.3333 \times 10^{-6}\, d$

$I_y = \frac{1}{12}(40\times10^{-3}) d^3 = 3.3333 \times 10^{-3}\, d^3$

$|x| = \frac{1}{2} d, \qquad |y| = 20 \text{ mm} = 20\times10^{-3} \text{ m} \qquad |e_x| = 18 \text{ mm} = 18 \times 10^{-3} \text{ m}$

Steel: $\sigma_Y = 250$ MPa $\quad E = 200000$ MPa

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(200000)}{250}} = 125.66$$

Assume $d > 40 \text{ mm} = 40\times10^{-3}$ m $\qquad$ Then $I_{min} = I_x$

$$r = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{5.3333\times10^{-6}\, d}{40\times10^{-3}\, d}} = 11.547\times10^{-3} \text{ m} \qquad \frac{L_e}{r} = 140.29 > C_c$$

$$\sigma_{all,centric} = \frac{\pi^2 E}{1.92(L_e/r)^2} = \frac{\pi^2(200000)}{(1.92)(140.29)^2} = 52.236 \text{ MPa} \qquad \sigma_{all,bending} = 150 \text{ MPa}$$

$$\frac{P}{A\,\sigma_{all,centric}} + \frac{P e_x x}{I_y\, \sigma_{all,bending}} = 1$$

$$\frac{128\times10^3}{(40\times10^{-3}\, d)(52.236\times10^6)} + \frac{(128\times10^3)(18\times10^{-3})(\frac{1}{2}d)}{(3.3333\times10^{-3}\, d^3)(150\times10^6)} = 1$$

$$\frac{61.260\times10^{-3}}{d} + \frac{2.304\times10^{-3}}{d^2} = 1$$

$$d^2 - 61.260\times10^{-3}\, d - 2.304\times10^{-3} = 0$$

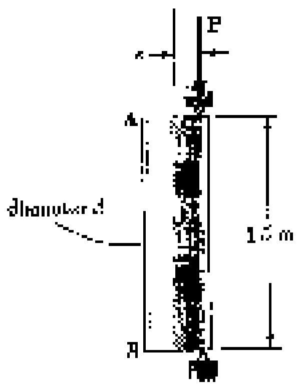
$$d = \frac{1}{2}\left\{61.260\times10^{-3} + \sqrt{(61.260\times10^{-3})^2 + (4)(2.304\times10^{-3})}\right\}$$

$$= 87.6\times10^{-3} \text{ m} > 40\times10^{-3} \text{ m}$$

$d = 87.6$ mm ◄

PROBLEM 10.109

10.109 The eccentric load P has a magnitude of 85 kN and is applied at a point located at a distance $e = 30$ mm from the geometric axis of a rod made of the aluminum alloy 6016-T6. Use the interaction method with a 140-MPa allowable stress in bending to determine the smallest diameter d that can be used.



SOLUTION

Assume $L/r > 66$ $\qquad \sigma_{all} = \frac{B}{(L/r)^2}$ $\qquad B = 351 \times 10^9$ Pa

$$c = \frac{d}{2} \qquad A = \pi c^2 = \frac{\pi}{4} d^2 \qquad I = \frac{\pi}{4} c^4 = \frac{\pi}{64} d^4$$

$$r = \sqrt{\frac{I}{A}} = \frac{1}{4} d$$

$$\frac{P}{A \sigma_{all,centric}} + \frac{Pec}{I \sigma_{all,bending}} = 1$$

$$\frac{PL^2}{AB r^2} + \frac{Ped}{2I \sigma_{all,bending}} = 1$$

$$\frac{64 PL^2}{\pi B d^4} + \frac{32 Ped}{\pi \sigma_{all,bending}} = 1$$

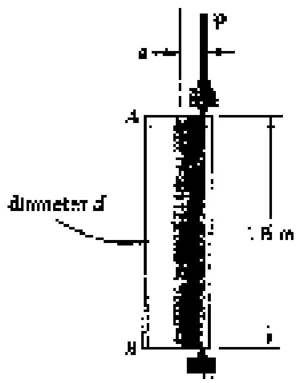
$$\frac{(64)(85 \times 10^3)(1.5)^2}{\pi (351 \times 10^9) d^4} + \frac{(32)(85 \times 10^3)(30 \times 10^{-3})}{\pi (140 \times 10^6) d^3} = 1 \qquad \text{Let } x = \frac{1}{d}$$

$$11.1 \times 10^{-6} x^4 + 185.53 \times 10^{-6} x^3 = 1 \qquad \text{Solving } x = 14.2725 \text{ m}^{-1}$$

$$d = \frac{1}{x} = 70.0 \times 10^{-3} \text{ m} \qquad d = 70.0 \text{ mm}$$

$$r = \frac{d}{4} = 17.50 \times 10^{-3} \text{ m} \qquad \frac{L}{r} = \frac{1.5}{17.5 \times 10^{-3}} = 85.7 > 66$$

PROBLEM 10.110



10.109 The eccentric load **P** has a magnitude of 85 kN and is applied at a point located at a distance $e = 30$ mm from the geometric axis of a rod made of the aluminum alloy 6016-T6. Use the interaction method with a 140-MPa allowable stress in bending to determine the smallest diameter d that can be used.

10.110 Solve Prob. 10.109, using the allowable-stress method and assuming that the aluminum alloy used is 2014-T6.

SOLUTION

Assume $L/r > 55$ $\qquad \sigma_{all} = \dfrac{B}{(L/r)^2}$ $\qquad B = 372 \times 10^9$ Pa

$c = \dfrac{d}{2} \qquad A = \pi c^2 = \dfrac{\pi}{4} d^2 \qquad I = \dfrac{\pi}{4} c^4 = \dfrac{\pi}{64} d^4$

$r = \sqrt{\dfrac{I}{A}} = \dfrac{1}{4} d$

$$\frac{P}{A} + \frac{Pec}{I} = \sigma_{all} = \frac{Br^2}{L^2}$$

$$\frac{PL^2}{ABr^2} + \frac{PL^2 e \frac{1}{2}d}{IBr^2} = 1 \qquad \frac{64 PL^2}{\pi d^4 B} + \frac{32 PL^2}{\pi d^3 B} = 1 \qquad \text{Let } x = \frac{1}{d}$$

$$\frac{64 PL^2}{\pi B} x^4 + \frac{(16)(64) PL^2 e}{2\pi B} x^5 = 1$$

$$\frac{(64)(85 \times 10^3)(1.5)^2}{\pi (372 \times 10^9)} x^4 + \frac{(16)(64)(85 \times 10^3)(1.5)^2 (30 \times 10^{-3})}{2\pi (372 \times 10^9)} = 1$$

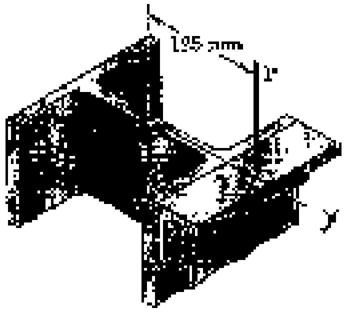
$$10.473 \times 10^{-6} x^4 + 2.5136 \times 10^{-6} x^5 = 1 \qquad x = 12.441 \text{ m}^{-1}$$

$$d = \frac{1}{x} = 80.4 \times 10^{-3} \text{ m} \qquad d = 80.4 \text{ mm}$$

$$r = \frac{d}{4} = 20.1 \times 10^{-3} \text{ m} \qquad \frac{L}{r} = \frac{1.5}{20.1 \times 10^{-3}} = 74.5 > 55$$

PROBLEM 10.111

10.111 A steel compression member of 5.8-m effective length is to support a 296-kN eccentric load **P**. Using the interaction method, select the wide-flange shape of 200-mm nominal depth that should be used. Use $E = 200$ GPa, $\sigma_Y = 250$ MPa and $\sigma_{all} = 150$ MPa in bending.



SOLUTION

Steel: $E = 200000$ MPa $\quad \sigma_Y = 250$ MPa

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = 125.66$$

$L_e = 5.8$ m $\qquad \sigma_{all,\,bending} = 150$ MPa

For 200 mm nominal depth wide flange section

$r_x \approx 88$ mm $= 88\times10^{-3}$ m, $\quad y \simeq \frac{210}{2} = 105$ mm $= 105\times10^{-3}$ m

$r_y \approx 48$ mm $= 48\times10^{-3}$ m $\qquad \frac{L}{r_y} \approx \frac{5.8}{48\times10^{-3}} = 121 \qquad \frac{L/r_y}{C_c} \approx 0.96$

$$F.S. \approx \frac{5}{3} + \frac{3}{8}(0.96) - \frac{1}{8}(0.96)^3 = 1.916$$

$$\sigma_{all} \approx \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L/r}{C_c}\right)^2\right] = \frac{250}{1.916}\left[1 - \frac{1}{2}(0.96)^2\right] = 70 \text{ MPa}$$

$$\frac{P}{A\,\sigma_{all,\,centric}} + \frac{P e_y\, y}{I_x\,\sigma_{all,\,bending}} = \frac{1}{A}\left[\frac{P}{\sigma_{all,\,centric}} + \frac{P e_y\, y}{r_x^2\,\sigma_{all,\,bending}}\right] = 1$$

$$A = \frac{P}{\sigma_{all,\,centric}} + \frac{P e_y\, y}{r_x^2\,\sigma_{all,\,bending}}$$

$$= \frac{296\times10^3}{70\times10^6} + \frac{(296\times10^3)(125\times10^{-3})(105\times10^{-3})}{(88\times10^{-3})^2(150\times10^6)} = 7573\times10^{-6} \text{ m}^2$$

$$= 7573 \text{ mm}^2$$

Try W 200 × 59 $\qquad A = 7560\times10^{-6}$ m², $\quad y = 105\times10^{-3}$ m, $\quad I_x = 61.1\times10^{-6}$ m⁴

$r_y = 51.9\times10^{-3}$ m, $\quad L_e/r_y = 111.75 < C_c$

$\frac{L_e/r_y}{C_c} = 0.8893 \qquad F.S. = 1.9122 \qquad \sigma_{all,\,centric} = 79.04$ MPa

$$\frac{P}{A\,\sigma_{all,\,centric}} + \frac{P e_y\, y}{I_x\,\sigma_{all,\,bending}}$$

$$= \frac{296\times10^3}{(7560\times10^{-6})(79.04\times10^6)} + \frac{(296\times10^3)(125\times10^{-3})(105\times10^{-3})}{(61.1\times10^{-6})(150\times10^6)}$$

$$= 0.4954 + 0.4239 = 0.9193 < 1 \quad \text{(allowed)}$$

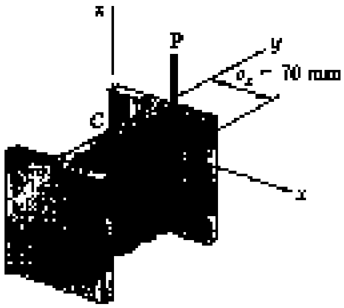
Trying W 200 × 52 leads to

$$\frac{P}{A\,\sigma_{all,\,centric}} + \frac{P e_y\, y}{I_x\,\sigma_{all,\,bending}} = 1.047 > 1 \quad \text{(not allowed)}$$

Use W 200 × 59

PROBLEM 10.112

10.112 A steel column of 7.2-m effective length is to support an 83-kN eccentric load P at a point D located on the x axis as shown. Using the allowable-stress method, select the wide-flange shape of 250-mm nominal depth that should be used. Use $E = 200$ GPa, $\sigma_Y = 250$ MPa.



SOLUTION

Steel: $E = 200000$ MPa $\quad \sigma_Y = 250$ MPa

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(200000)}{250}} = 125.66$$

$L_e = 7.2$ m

Try W 250 × 49.1 $\quad A = 6250 \times 10^{-6}$

$b_f = 202 \times 10^{-3}$ m, $\quad c = 101 \times 10^{-3}$ m, $\quad I_y = 15.1 \times 10^{-6}$ m^4, $\quad r_y = 49.2 \times 10^{-3}$ m

$$\frac{L_e}{r_y} = \frac{7.2}{49.2 \times 10^{-3}} = 146.34 > C_c$$

$$\sigma_{all} = \frac{\pi^2 E}{1.92 (L/r_y)^2} = \frac{\pi^2(200000)}{(1.92)(146.34)^2} = 48.01 \text{ MPa}$$

$$\frac{P}{A} + \frac{P e_x c}{I_y} = \frac{83 \times 10^3}{6250 \times 10^{-6}} + \frac{(83 \times 10^3)(70 \times 10^{-3})(101 \times 10^{-3})}{15.1 \times 10^{-6}}$$

$= 13.28 \times 10^6 + 38.86 \times 10^6 = 52.14$ MPa > 48.01 MPa

(not allowed)

Required area $\quad A \approx \left(\frac{52.14}{48.01}\right)(6250 \text{ mm}^2) = 6788 \text{ mm}^2$

Try W 250 × 58

$$\frac{L_e}{r_y} = \frac{7.2}{50.3 \times 10^{-3}} = 143.14 \qquad \sigma_{all} = \frac{\pi^2(200000)}{(1.92)(143.14)^2} = 50.18 \text{ MPa}$$

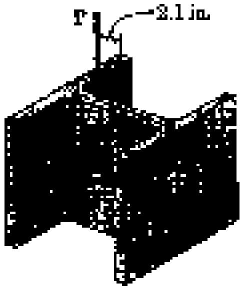
$$\frac{P}{A} + \frac{P e_x c}{I_y} = \frac{83 \times 10^3}{7420 \times 10^{-6}} + \frac{(83 \times 10^3)(70 \times 10^{-3})(101.5 \times 10^{-3})}{18.8 \times 10^{-6}}$$

$= 11.19 \times 10^6 + 31.37 \times 10^6 = 42.56$ MPa < 50.18 MPa

Use W250 × 58

PROBLEM 10.113

10.113 A steel column of 21-ft effective length must carry a load of 82 kips with an eccentricity of 2.1 in. as shown. Using the interaction method, select the wide-flange shape of 12-in. nominal depth that should be used. Use $E = 29 \times 10^6$ psi, $\sigma_Y = 36$ ksi, and $\sigma_{all} = 22$ ksi in bending.



SOLUTION

Steel: $E = 29000$ ksi $\quad C_c = \sqrt{\dfrac{2\pi^2 E}{\sigma_Y}} = \sqrt{\dfrac{2\pi^2(29000)}{36}} = 126.10$

$L_e = 21$ ft $= 252$ in.

Try W 12 × 35 $\quad r_y = 1.54$ in $\quad \dfrac{L_e}{r_y} = 163.64 > C_c$

$$\sigma_{all,centric} = \frac{\pi^2 E}{1.92(L/r)^2} = \frac{\pi^2(29000)}{(1.92)(163.64)^2} = 5.57 \text{ ksi}$$

$$\frac{P}{A\sigma_{all,centric}} + \frac{Pec}{I_x\sigma_{all,bending}} = \frac{82}{(10.3)(5.57)} + \frac{(82)(2.1)(\frac{1}{2}\cdot 12.50)}{(285)(22)}$$

$$= 1.429 + 0.172 = 1.601 \quad \text{(not allowed)}$$

Approximate required $A = (1.596)(10.3) = 16.4 \text{ in}^2$

Try W 12 × 50 $\quad r_y = 1.96$ in. $\quad \dfrac{L_e}{r_y} = 128.57 > C_c$

$$\sigma_{all,centric} = \frac{\pi^2 E}{1.92(L_e/r)^2} = \frac{\pi^2(29000)}{(1.92)(128.57)^2} = 9.02 \text{ ksi}$$

$$\frac{P}{A\,\sigma_{all,centric}} + \frac{Pec}{I_x\,\sigma_{all,bending}} = \frac{82}{(14.7)(9.02)} + \frac{(82)(2.1)(\frac{1}{2}\cdot 12.19)}{(394)(22)}$$

$$= 0.618 + 0.121 = 0.739 \quad \text{(allowed)}$$

Try W 12 × 40 $\quad r_y = 1.93$ in $\quad \dfrac{L_e}{r_y} = \dfrac{252}{1.93} = 130.57 > C_c$

$$\sigma_{all,centric} = \frac{\pi^2 E}{(1.92)(L_e/r_y)^2} = \frac{\pi^2(29000)}{(1.92)(130.57)^2} = 8.74 \text{ ksi}$$

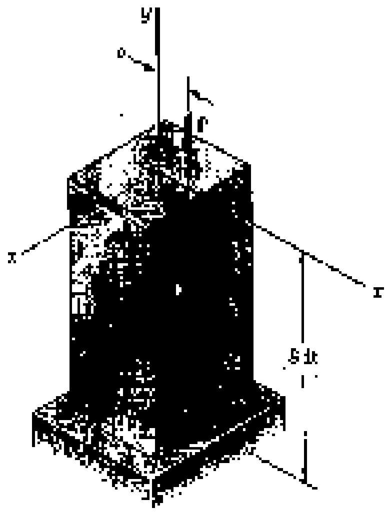
$$\frac{P}{A\,\sigma_{all,centric}} + \frac{Pec}{I_x\,\sigma_{all,bending}} = \frac{82}{(11.8)(8.74)} + \frac{(82)(2.1)(\frac{1}{2}\cdot 11.94)}{(310)(22)}$$

$$= 0.795 + 0.151 = 0.946 \quad \text{(allowed)}$$

Use W 12 × 40

PROBLEM 10.114

10.114 A 43-kip axial load **P** is applied to the rolled-steel column *BC* at a point on the *x* axis at a distance *e* = 2.5 in. from the geometric axis of the column. Using the allowable-stress method, select the wide-flange shape of 8-in. nominal depth that should be used. Use $E = 29 \times 10^6$ psi and $\sigma_Y = 36$ ksi.



SOLUTION

Steel: $E = 29000$ ksi $\quad \sigma_Y = 36$ ksi

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(29000)}{36}} = 126.10$$

$L = 8$ ft $= 96$ in. $\quad L_e = 2L = 192$ in

Try $\underline{W\,8\times31}$: $r_y = 2.02$ in, $\frac{L_e}{r_y} = 95.05 < C_c$

$$\frac{L_e/r_y}{C_c} = 0.754$$

$$F.S. = \frac{5}{3} + \frac{3}{8}(0.754) - \frac{1}{8}(0.754)^3 = 1.896$$

$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L_e/r_y}{C_c}\right)^2\right] = \frac{36}{1.896}\left[1 - \frac{1}{2}(0.754)^2\right] = 13.59 \text{ ksi}$$

$$\frac{P}{A} + \frac{Pec}{I_y} = \frac{43}{9.13} + \frac{(43)(2.5)(\frac{1}{2}\cdot 7.995)}{37.1} = 4.71 + 11.58 = 16.29 \text{ ksi} > 13.59 \text{ ksi}$$

(not allowed)

Approximate required area $\left(\frac{16.29}{13.59}\right)(9.13) = 10.9$ in^2

Try $\underline{W\,8\times35}$ $\quad r_y = 2.03 \quad \frac{L_e}{r_y} = 94.58 < C_c \quad \frac{L_e/r_y}{C_c} = 0.750$

$F.S. = 1.895 \quad \sigma_{all} = 13.65$ ksi

$$\frac{P}{A} + \frac{Pec}{I_y} = \frac{43}{10.3} + \frac{(43)(2.5)(\frac{1}{2}\cdot 8.020)}{42.6} = 14.29 \text{ ksi} > 13.65 \text{ ksi}$$ (not allowed)

Try $\underline{W8\times40}$ $\quad r_y = 2.04 \quad \frac{L_e}{r_y} = 94.12 < C_c \quad \frac{L_e/r_y}{C_c} = 0.746$

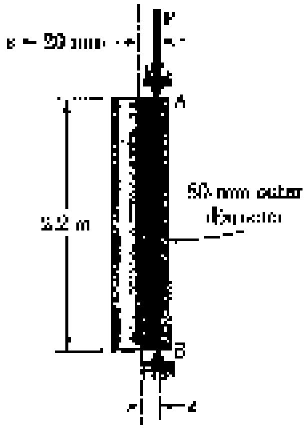
$F.S. = 1.895 \quad \sigma_{all} = 13.71$ ksi

$$\frac{P}{A} + \frac{Pec}{I_y} = \frac{43}{11.7} + \frac{(43)(2.5)(\frac{1}{2}\cdot 8.07)}{49.1} = 12.51 \text{ ksi} < 13.71 \text{ ksi}$$ (allowed)

Use W 8×40

PROBLEM 10.115

10.115 A steel tube of 80-mm outer diameter is to carry a 93-kN load **P** with an eccentricity of 20 mm. The tubes available for use are made with wall thicknesses in increments of 3 mm from 6 mm to 15 mm. Using the allowable-stress method, determine the lightest tube that can be used. Assume $E = 200$ GPa, $\sigma_Y = 250$ MPa.



SOLUTION

$r_o = \frac{1}{2} d_o = 40$ mm ; $r_i = r_o - t$

$A = \pi (r_o^2 - r_i^2)$, $\quad I = \frac{\pi}{4}(r_o^4 - r_i^4)$ $\quad r = \sqrt{\frac{I}{A}}$

t (mm)	r_i (mm)	A (mm²)	I (10^6 mm⁴)	r (mm)
3	37	726	0.539	27.24
6	34	1395	0.961	26.25
9	31	2007	1.285	25.31
12	28	2564	1.528	24.41
15	25	3063	1.704	23.59

$L_e = 2.2$ m

$P = 93 \times 10^3$ N

Steel: $E = 200000$ MPa $\quad C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(200000)}{250}} = 125.66$

Try $t = 9$ mm $\quad \frac{L_e}{r} = \frac{2.2}{25.31 \times 10^{-3}} = 86.92 < C_c \quad \frac{L_e/r}{C_c} = 0.6917$

$$F.S. = \frac{5}{3} + \frac{3}{8}(0.6917) - \frac{1}{8}(0.6917)^3 = 1.885$$

$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L_e/r}{C_c}\right)^2\right] = \frac{250}{1.885}\left[1 - \frac{1}{2}(0.6917)^2\right] = 100.9 \text{ MPa}$$

$$\frac{P}{A} + \frac{Pec}{I} = \frac{93 \times 10^3}{2007 \times 10^{-6}} + \frac{(93 \times 10^3)(20 \times 10^{-3})(40 \times 10^{-3})}{1.285 \times 10^{-6}} = 104.2 \text{ MPa} > 100.9 \text{ MPa}$$

(not allowed)

Approximate required area $\left(\frac{104.2}{100.9}\right)(2007 \times 10^{-6}) = 2073 \times 10^{-6}$ m² $= 2073$ mm²

For $t = 12$ mm $\quad \frac{L_e}{r} = \frac{2.2}{24.41 \times 10^{-3}} = 90.12 < C_c \quad \frac{L_e/r}{C_c} = 0.7172$

F.S. $= 1.890$ $\qquad \sigma_{all} = 98.3$ MPa

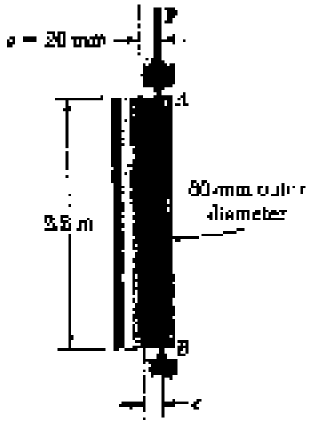
$$\frac{P}{A} + \frac{Pec}{I} = \frac{93 \times 10^3}{2564 \times 10^{-6}} + \frac{(93 \times 10^3)(20 \times 10^{-3})(40 \times 10^{-3})}{1.528 \times 10^{-6}} = 85.0 \text{ MPa} < 98.3 \text{ MPa}$$

Use $t = 12$ mm ◄

10.115 A steel tube of 80-mm outer diameter is to carry a 93-kN load **P** with an eccentricity of 20 mm. The tubes available for use are made with wall thicknesses in increments of 3 mm from 6 mm to 15 mm. Using the allowable-stress method, determine the lightest tube that can be used. Assume $E = 200$ GPa, $\sigma_Y = 250$ MPa.

10.116 Solve Prob. 10.115, using the interaction method with $P = 165$ kN, $e = 15$ mm, and an allowable stress in bending of 150 MPa.

PROBLEM 10.116



SOLUTION

$r_o = \frac{1}{2} d_o = 40$ mm $\qquad r_i = r_o - t$

$A = \pi(r_o^2 - r_i^2) \qquad I = \frac{\pi}{4}(r_o^4 - r_i^4) \qquad r = \sqrt{\frac{I}{A}}$

t mm	r_i mm	A mm^2	I $10^6 mm^4$	r mm
3	37	726	0.539	27.24
6	34	[illegible]	[illegible]	[illegible]
9	31	2007	1.285	25.31
12	28	2564	1.528	24.41
15	25	3063	1.704	23.59

$L_e = 2.2$ m

$P = 165 \times 10^3$ N

$\sigma_{all,\,bending} = 150$ MPa

Steel: $E = 200000$ MPa $\qquad C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(200000)}{250}} = 125.66$

Try $\underline{t = 9 \text{ mm}}$ $\qquad \frac{L_e}{r} = \frac{2.2}{25.31 \times 10^{-3}} = 86.92 < C_c \qquad \frac{L_e/r}{C_c} = 0.6917$

$$F.S. = \frac{5}{3} + \frac{3}{8}(0.6917) - \frac{1}{8}(0.6917)^3 = 1.885$$

$$\sigma_{all,\,centric} = \frac{\sigma_Y}{F.S.}\left[1 - \frac{1}{2}\left(\frac{L_e/r}{C_c}\right)^2\right] = \frac{250}{1.885}\left[1 - \frac{1}{2}(0.6917)^2\right] = 100.9 \text{ MPa}$$

$$\frac{P}{A\sigma_{all,\,centric}} + \frac{Pec}{I\sigma_{all,\,bending}} = \frac{165 \times 10^3}{(2007 \times 10^{-6})(100.9 \times 10^6)} + \frac{(165 \times 10^3)(15 \times 10^{-3})(40 \times 10^{-3})}{(1.285 \times 10^{-6})(150 \times 10^6)}$$

$$= 0.815 + 0.514 = 1.329 > 1 \quad \text{(not allowed)}$$

Approximate required area $\qquad A = (1.329)(2007) = 2667$ mm^2

For $\underline{t = 12 \text{ mm}}$ $\qquad \frac{L_e}{r} = \frac{2.2}{24.41 \times 10^{-3}} = 90.12 < C_c \qquad \frac{L_e/r}{C_c} = 0.7172$

$F.S. = 1.890 \qquad \sigma_{all,\,centric} = 98.3$ MPa

$$\frac{P}{A\sigma_{all,\,centric}} + \frac{Pec}{I\sigma_{all,\,bending}} = \frac{165 \times 10^3}{(2564 \times 10^{-6})(98.3 \times 10^6)} + \frac{(165 \times 10^3)(15 \times 10^{-3})(40 \times 10^{-3})}{(1.528 \times 10^{-6})(150 \times 10^6)}$$

$$= 0.655 + 0.432 = 1.087 > 1 \quad \text{(not allowed)}$$

Try $\underline{t = 15 \text{ mm}}$ $\qquad \frac{L_e}{r} = \frac{2.2}{23.59 \times 10^{-3}} = 93.26 < C_c \qquad (L_e/r)/C_c = 0.7422$

$F.S. = 1.894 \qquad \sigma_{all,\,centric} = 95.64$ MPa

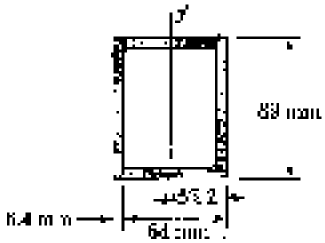
$$\frac{P}{A\sigma_{all,\,centric}} + \frac{Pec}{I\sigma_{all,\,bending}} = \frac{165 \times 10^3}{(3063 \times 10^{-6})(95.64 \times 10^6)} + \frac{(165 \times 10^3)(15 \times 10^{-3})(40 \times 10^{-3})}{(1.704 \times 10^{-6})(150 \times 10^6)}$$

$$= 0.563 + 0.387 = 0.950 < 1 \quad \text{(allowed)}$$

Use $t = 15$ mm

PROBLEM 10.117

10.117 A column of 3.5-m effective length is made by welding together two 89 × 64 × 6.4 mm angles as shown. Using $E = 200$ GPa, determine the allowable centric load if a factor of safety of 2.8 is required.



SOLUTION

One angle $\quad x = 15.8$ mm

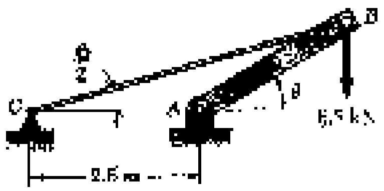
$$I_y = \bar{I}_y + A(35.2 - 15.8)^2$$
$$= 0.333 \times 10^6 + (938)(19.4)^2$$
$$= 0.686 \times 10^6 \text{ mm}^4$$

Two angles $\quad I_y = (2)(0.686 \times 10^6) = 1.372 \times 10^6 \text{ mm}^4 = 1.372 \times 10^{-6} \text{ m}^4$

$$P_{all} = \frac{P_{cr}}{F.S.} = \frac{\pi^2 E I_y}{(F.S.) L_e^2} = \frac{\pi^2 (200 \times 10^9)(1.372 \times 10^{-6})}{(2.8)(3.5)^2} = 79.0 \times 10^3 \text{ N}$$
$$= 79.0 \text{ kN}$$

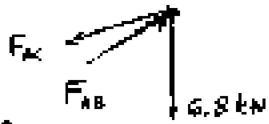
PROBLEM 10.118

10.118 Member AB consists of a single C130 × 10.4 steel channel of length 2.5 m. Knowing that the pins at A and B pass through the centroid of the cross section of the channel, determine the factor of safety for the load shown with respect to buckling in the plane of the figure when $\theta = 30°$. Use Euler's formula with $E = 200$ GPa.



SOLUTION

Since $AB = 2.5$ m, triangle ABC is isosceles.



$$\Sigma F_x = 0$$

$$-F_{AC}\cos 15° + F_{AB}\cos 30° = 0$$

$$F_{AC} = \frac{F_{AB}\cos 30°}{\cos 15°}$$

$$+\uparrow \Sigma F_y = 0$$

$$F_{AB}\sin 30° - F_{AC}\sin 15° - 6.8 = 0$$

$$F_{AB}\left(\sin 30° - \frac{\sin 15°\cos 30°}{\cos 15°}\right) = 0.26795\,F_{AB} = 6.8 \qquad F_{AB} = 25.378\ \text{kN}$$

C 130 × 10.4 $\qquad I_{min} = 0.229\times10^{6}\ \text{mm}^4 = 0.229\times10^{-6}\ \text{m}^4$

$$P_{cr} = \frac{\pi^2 E I_{min}}{L_{AB}^2} = \frac{\pi^2(200\times10^9)(0.229\times10^{-6})}{(2.5)^2} = 72.324\times10^3\ \text{N} = 72.324\ \text{kN}$$

$$F.S. = \frac{P_{cr}}{F_{AB}} = \frac{72.324}{25.378} = 2.85 \quad \blacktriangleleft$$

PROBLEM 10.119

10.119 Supports A and B of the pin-ended column shown are at a fixed distance L from each other. Knowing that at a temperature T_0 the force in the column is zero and that buckling occurs when the temperature is $T_1 = T_0 + \Delta T$, express ΔT in terms of b, L, and the coefficient of thermal expansion α.

SOLUTION

Let P be the compressive force in the column.

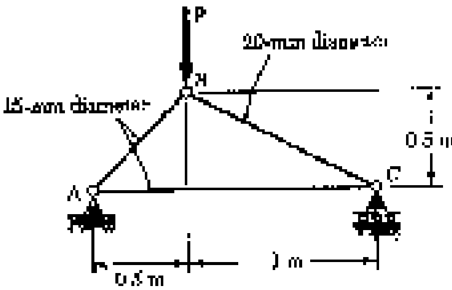
$$L\alpha(\Delta T) - \frac{PL}{EA} = 0 \qquad P = EA\alpha(\Delta T)$$

$$P_{cr} = \frac{\pi^2 E I}{L^2} = P = EA\alpha(\Delta T)$$

$$\Delta T = \frac{\pi^2 E I}{L^2 E A \alpha} = \frac{\pi^2 E b^4/12}{L^2 E b^2} = \frac{\pi^2 b^2}{12 L^2 \alpha} \quad \blacktriangleleft$$

PROBLEM 10.120

10.120 Knowing that a factor of safety of 2.6 is required, determine the largest load **P** that can be applied to the structure shown. Use $E = 200$ GPa and consider only buckling in the plane of the structure.



SOLUTION

BC: $L_{BC} = \sqrt{1^2 + 0.5^2} = 1.1180$ m

$I = \frac{\pi}{64} d_{BC}^4 = \frac{\pi}{64}(20)^4 = 7.854 \times 10^3\ mm^4$
$= 7.854 \times 10^{-9}\ m^4$

$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (200 \times 10^9)(7.854 \times 10^{-9})}{(1.1180)^2}$
$= 12.403 \times 10^3\ N = 12.403\ kN$

$F_{BC,all} = \frac{P_{cr}}{F.S.} = \frac{12.403}{2.6} = 4.770\ kN$

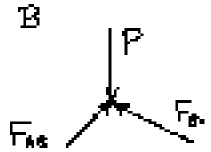
AB. $L_{AB} = \sqrt{0.5^2 + 0.5^2} = 0.70711$ m

$I = \frac{\pi}{64} d^4 = \frac{\pi}{64}(15)^4 = 2.485 \times 10^3\ mm^4 = 2.485 \times 10^{-9}\ m^4$

$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (200 \times 10^9)(2.485 \times 10^{-9})}{(0.70711)^2} = 9.8106 \times 10^3\ N = 9.8106\ kN$

$F_{AB,all} = \frac{P_{cr}}{F.S.} = \frac{9.8106}{2.6} = 3.773\ kN$

Joint B



$\overset{+}{\rightarrow} \Sigma F_x = 0 \qquad \frac{0.5}{0.70711} F_{AB} - \frac{1.0}{1.1180} F_{BC} = 0$

$F_{BC} = 0.79057\ F_{AB}$

$+\uparrow \Sigma F_y = 0 \qquad \frac{.5}{0.70711} F_{AB} + \frac{0.5}{1.1180} F_{BC} - P = 0$

$0.70711\ F_{AB} + (0.44721)(0.79057 F_{AB}) - P = 0$

$P = 1.06066\ F_{AB}$

$P = (1.06066)\frac{F_{BC}}{0.79057} = 1.3416\ F_{BC}$

Allowable value for P.

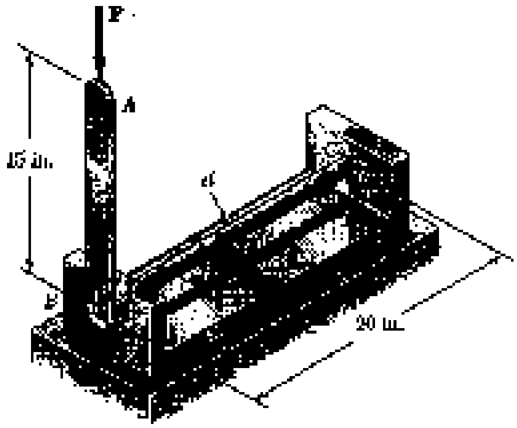
$P = 1.06066\ F_{AB,all} = (1.06066)(3.773) = 4.00\ kN$

$P = 1.3416\ F_{BC,all} = (1.3416)(4.770) = 6.40\ kN$

$P_{all} = 4.00\ kN$

PROBLEM 10.121

10.121 The steel rod BC is attached to the rigid bar AB and to the fixed support at C. Knowing that $G = 11.2 \times 10^6$ psi, determine the diameter of rod BC for which the critical load P_{cr} of the system is 80 lb.



SOLUTION

Look at torsion spring BC

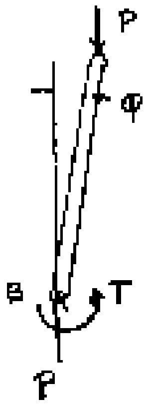
$$\varphi = \frac{TL}{GJ} \qquad T = \frac{GJ}{L}\varphi = K\varphi$$

$$G = 11.2 \times 10^6 \text{ psi}$$

$$J = \frac{\pi}{2}c^4 = \frac{\pi}{2}\left(\frac{d}{2}\right)^4 = \frac{\pi d^4}{32}$$

$$L = 20 \text{ in}$$

$$K = \frac{(11.2 \times 10^6)\pi d^4}{(20)(32)} = 54978\, d^4$$



$$\circlearrowleft \Sigma M_B = 0$$

$$T - P\ell \sin\varphi = 0$$

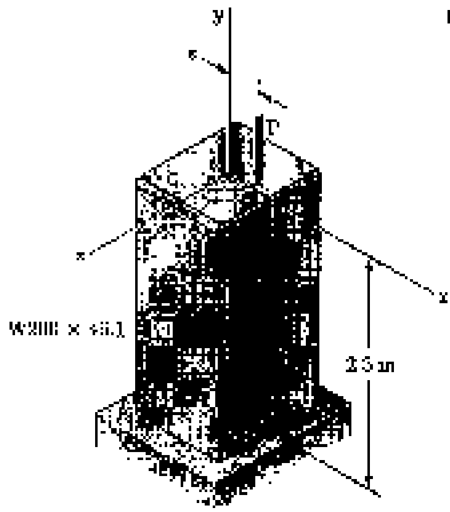
$$K\varphi - P\ell\sin\varphi = 0$$

$$P = \frac{K\varphi}{\ell \sin\varphi} \qquad P_{cr} = \frac{K}{\ell}$$

$$K = 54978\, d^4 = P_{cr}\ell = (80)(15) = 1200$$

$$d = \sqrt[4]{\frac{1200}{54978}} = 0.384 \text{ in.}$$

PROBLEM 10.122



10.122 An axial load P of magnitude 560 kN is applied at a point on the x axis at a distance $e = 8$ mm from the geometric axis of the W 200 × 46.1 rolled-steel column BC. Using $E = 200$ GPa, determine (*a*) the horizontal deflection of end C, (*b*) the maximum stress in the column.

SOLUTION

$$L_e = 2L = (2)(2.3) = 4.6 \text{ m} \qquad e = 8 \times 10^{-3} \text{ m}$$

$$\text{W } 200 \times 46.1 \qquad A = 5860 \text{ mm}^2 = 5860 \times 10^{-6} \text{ m}^2$$

$$I_y = 15.3 \times 10^6 \text{ mm}^4 = 15.3 \times 10^{-6} \text{ m}^4$$

$$P_{cr} = \frac{\pi^2 E I}{L_e^2} = \frac{\pi^2 (200 \times 10^9)(15.3 \times 10^{-6})}{(4.6)^2}$$

$$= 1.42727 \times 10^6 \text{ N}$$

$$\frac{P}{P_{cr}} = \frac{560 \times 10^3}{1.42727 \times 10^6} = 0.39236$$

$$y_m = e\left[\sec \frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}} - 1\right] = (8 \times 10^{-3})\left[\sec\left(\frac{\pi}{2}\sqrt{0.39236}\right) - 1\right]$$

$$= (8 \times 10^{-3})\left[\sec(0.98393) - 1\right] = (8 \times 10^{-3})\left[1.8058 - 1\right]$$

$$= 6.447 \times 10^{-3} \text{ m} \qquad = 6.45 \text{ mm}$$

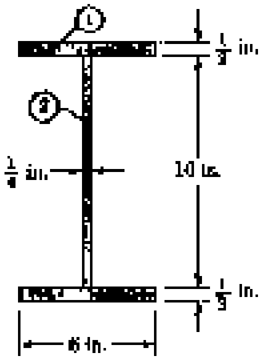
$$M_{max} = P(y_m + e) = (560 \times 10^3)(8 \times 10^{-3} + 6.447 \times 10^{-3}) = 8.090 \times 10^3 \text{ N-m}$$

$$S_y = 151 \times 10^3 \text{ mm}^3 = 151 \times 10^{-6} \text{ m}^3$$

$$\sigma_{max} = \frac{P}{A} + \frac{M}{S_y} = \frac{560 \times 10^3}{5860 \times 10^{-6}} + \frac{8.090 \times 10^3}{151 \times 10^{-6}} = 149.1 \times 10^6 \text{ Pa} = 149.1 \text{ MPa}$$

PROBLEM 10.123

10.123 A column with the cross section shown has a 13.5-ft effective length. Knowing that $\sigma_Y = 36$ ksi, and $E = 29 \times 10^6$ psi, use the AISC allowable stress design formulas to determine the largest centric load that can be applied to the column.



SOLUTION

$$A = 2A_1 + A_2 = (2)(\tfrac{1}{2})(6) + (10)(\tfrac{1}{4}) = 8.5 \text{ in}^2$$

$$I_y = 2I_1 + I_2 = (2)(\tfrac{1}{12})(\tfrac{1}{2})(6)^3 + (\tfrac{1}{12})(10)(\tfrac{1}{4})^3 = 18.013 \text{ in}^4$$

$$r_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{18.013}{8.5}} = 1.4557 \text{ in.}$$

$$L_e = 13.5 \text{ ft} = 162 \text{ in} \qquad \frac{L_e}{r} = 111.29 < C_c$$

Steel: $E = 29000$ ksi, $\sigma_Y = 36$ ksi $\qquad C_c = \sqrt{\dfrac{2\pi^2 E}{\sigma_Y}} = \sqrt{\dfrac{2\pi^2(29000)}{36}} = 126.10$

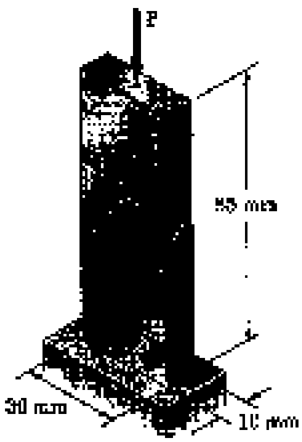
$$\frac{L_e/r}{C_c} = 0.8826 \qquad F.S. = \tfrac{5}{3} + \tfrac{3}{8}(0.8826) - \tfrac{1}{8}(0.8826)^3 = 1.912$$

$$\sigma_{all} = \frac{\sigma_Y}{F.S.}\left[1 - \tfrac{1}{2}\left(\frac{L_e/r}{C_c}\right)^2\right] = \frac{36}{1.912}\left[1 - \tfrac{1}{2}(0.8826)^2\right] = 11.49 \text{ ksi}$$

$$P_{all} = \sigma_{all} A = (11.49)(8.5) = 97.7 \text{ kips}$$

PROBLEM 10.125

10.125 Bar AB is free at its end A and fixed at its base B. Determine the allowable centric load P if the aluminum alloy is (a) 6061-T6, (b) 2014-T6.



SOLUTION

$$A = (30)(10) = 300 \text{ mm}^2 = 300 \times 10^{-6} \text{ m}^2$$

$$I_{min} = \frac{1}{12}(30)(10)^3 = 2.50 \times 10^3 \text{ mm}^4$$

$$r_{min} = \sqrt{\frac{I}{A}} = \sqrt{\frac{2.50 \times 10^3}{300}} = 2.887 \text{ mm}$$

$$L_e = 2L = (2)(85) = 170 \text{ mm} \qquad \frac{L_e}{r_m} = 58.88$$

(a) 6061-T6 $\quad L/r < 66$

$$\sigma_{all} = 139 - 0.868(L/r) = 139 - (0.868)(58.88)$$
$$= 87.9 \text{ MPa}$$

$$P_{all} = \sigma_{all} A = (87.9 \times 10^6)(300 \times 10^{-6}) = 26.4 \times 10^3 \text{ N}$$
$$= 26.4 \text{ kN}$$

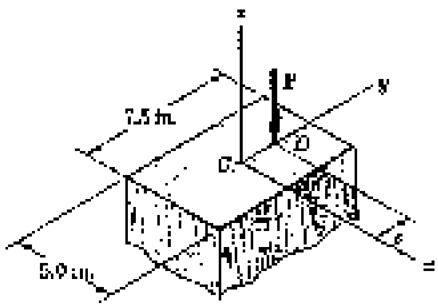
(b) 2014-T6 $\quad L/r > 55$

$$\sigma_{all} = \frac{372 \times 10^3}{(L/r)^2} = \frac{372 \times 10^3}{(58.88)^2} = 107.3 \text{ MPa}$$

$$P_{all} = \sigma_{all} A = (107.3 \times 10^6)(300 \times 10^{-6}) = 32.2 \times 10^3 \text{ N} = 32.2 \text{ kN}$$

PROBLEM 10.126

10.126 A sawn lumber column of 5.0 × 7.5-in. cross section has an effective length of 8.5 ft. The grade of wood used has an adjusted allowable stress for compression parallel to the grain $\sigma_C = 1180$ psi and a modulus of elasticity $E = 1.2 \times 10^6$ psi. Using the allowable-stress method, determine the largest eccentric load P that can be applied when (*a*) $e = 0.5$ in., (*b*) $e = 1.0$ in.



SOLUTION

Sawn lumber: $\sigma_C = 1180$ psi $\quad E = 1.2 \times 10^6$ psi

$c = 0.8 \quad K_{cE} = 0.300$

$L_e = 8.5$ ft $= 102$ in

$b = 7.5$ in, $\quad d = 5.0$ in, $\quad c = \frac{b}{2} = 3.75$ in

$A = bd = (7.5)(5.0) = 37.5$ in^2 $\qquad I_y = \frac{1}{12}(5.0)(7.5)^3 = 175.78$ in^4

$$\sigma_{CE} = \frac{K_{cE}E}{(L/d)^2} = \frac{K_{cE}Ed^2}{L^2} = \frac{(0.300)(1.2\times10^6)(5.0)^2}{(102)^2} = 865 \text{ psi}$$

$$\sigma_{CE}/\sigma_C = 865/1180 = 0.7331$$

$$C_P = \frac{1+\sigma_{CE}/\sigma_C}{2c} - \sqrt{\left(\frac{1+\sigma_{CE}/\sigma_C}{2c}\right)^2 - \frac{\sigma_{CE}/\sigma_C}{c}} = 0.5763$$

$$\sigma_{all} = \sigma_C C_P = (1180)(0.5763) = 680 \text{ psi}$$

$$\frac{P_{all}}{A} + \frac{P_{all}ec}{I_y} = \sigma_{all}$$

$$P_{all} = \frac{\sigma_{all}}{\frac{1}{A} + \frac{ec}{I_y}}$$

(a) $e = 0.5$ in

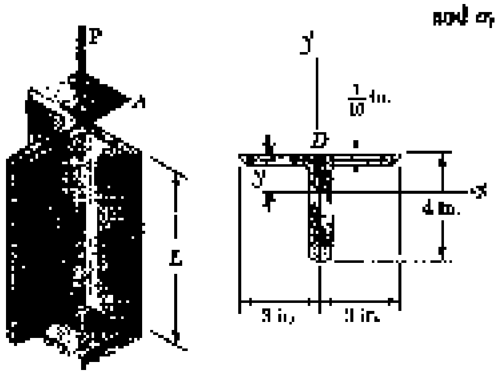
$$P_{all} = \frac{680}{\frac{1}{37.5} + \frac{(0.5)(3.75)}{175.78}} = 18210 \text{ lb} = 18.21 \text{ kips}$$

(b) $e = 1.0$ in

$$P_{all} = \frac{680}{\frac{1}{37.5} + \frac{(1.0)(3.75)}{175.78}} = 14170 \text{ lb} = 14.17 \text{ kips}$$

PROBLEM 10.127

10.127 Two $4 \times 3 \times \frac{3}{8}$-in. steel angles are welded together to form the column AB. An axial load **P** of magnitude 14 kips is applied at point D. Using the allowable-stress method, determine the largest allowable length L. Assume $E = 29 \times 10^6$ psi and $\sigma_Y = 36$ ksi.



SOLUTION

One angle $L\ 4 \times 3 \times \frac{3}{8}$ $\qquad A = 2.48\ \text{in}^2$

$I_x = 3.96\ \text{in}^4,\ S_x = 1.46\ \text{in}^3,\ r_x = 1.26\ \text{in},\ y = 1.28\ \text{in.}$

$I_y = 1.92\ \text{in}^4,\ \qquad r_y = 0.879\ \text{in},\ x = 0.782\ \text{in}$

Two angles $\qquad A = (2)(2.48) = 4.96\ \text{in}^2$

$I_x = (2)(3.96) = 7.92\ \text{in}^4,\qquad S_x = (2)(1.46) = 2.92\ \text{in}^3,\ r_x = 1.26,\ y = 1.28\ \text{in.}$

$$I_y = 2\left[I_{y1} + Ax^2\right] = (2)\left[1.92 + (2.48)(0.782)^2\right] = 6.873\ \text{in}^4 = I_{min}$$

$$r_{min} = \sqrt{\frac{I_{min}}{A}} = 1.177\ \text{in.} \qquad e = y - \frac{3}{16} = 1.28 - \frac{3}{16} = 1.0925\ \text{in}$$

$$P = 14\ \text{kips} \qquad \sigma_{all} = \frac{P}{A} + \frac{Pey}{I_x} = \frac{14}{4.96} + \frac{(14)(1.0925)(1.28)}{7.92} = 5.294\ \text{ksi}$$

$$E = 29000\ \text{ksi} \qquad C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}} = \sqrt{\frac{2\pi^2(29000)}{36}} = 126.1$$

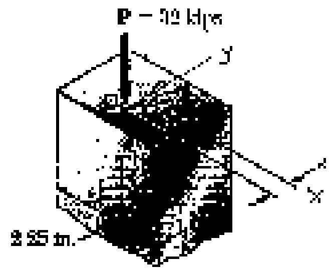
$$\text{Assume } \frac{L_e}{r} > C_c \qquad \sigma_{all} = \frac{\pi^2 E}{1.92\,(L/r_{min})^2} \qquad \left(\frac{L}{r_{min}}\right)^2 = \frac{\pi^2 E}{1.92\,\sigma_{all}}$$

$$\frac{L}{r_{min}} = \sqrt{\frac{\pi^2 E}{1.92\,\sigma_{all}}} = \sqrt{\frac{\pi^2(29000)}{(1.92)(5.294)}} = 167.8 > C_c$$

$$L = 167.8\ r_{min} = (167.8)(1.177) = 197.5\ \text{in.} = 16.46\ \text{ft}$$

PROBLEM 10.128

10.128 A compression member of rectangular cross section has an effective length of 36 in. and is made of the aluminum alloy 2014-T6 for which the allowable stress in bending is 24 ksi. Using the interaction method, determine the smallest dimension d of the cross section that can be used when $e = 0.4$ in.



SOLUTION

$A = 2.25\,d \qquad c = \frac{1}{2}d \qquad e = 0.4 \text{ in} \qquad L_e = 36 \text{ in}$

$\sigma_{all,b} = 24 \text{ ksi} \qquad P = 32 \text{ kips}$

$I_x = \frac{1}{12}(2.25)d^3 \qquad r_x = \frac{d}{\sqrt{12}}$

Assume $r_x = r_{min}$, i.e. $d < 2.25$

$L/r_{min} = \sqrt{12}\,L_e/d$

Assume $L/r_{min} > 55$. $\quad \sigma_{all,c} = \frac{54000}{(L/r_x)^2} = \frac{54000\,d^2}{12\,L_e^2} = \frac{54000}{(12)(36)^2}d^2 = 3.47222\,d^2$

$$\frac{P}{A\sigma_{all,c}} + \frac{Pec}{I\sigma_{all,b}} = \frac{32}{(2.25d)(3.47222d^2)} + \frac{(12)(32)(0.4)(\frac{1}{2}d)}{(2.25d^3)(24)} = 1$$

$\frac{4.096}{d^3} + \frac{1.42222}{d^2} = 1 \qquad$ Let $x = \frac{1}{d} \qquad 4.096x^3 + 1.42222x^2 = 1$

Solving for x, $\quad x = 0.528118, \quad d = \frac{1}{x} = 1.894$ in. < 2.25 in.

$L/r_x = (\sqrt{12})(36)/1.894 = 65.8 > 55 \qquad\qquad d = 1.894$ in.

PROBLEM 10.C1

10.C1 A solid steel rod having an effective length of 500 mm is to be used as a compression strut to carry a centric load P. For the grade of steel used E = 200 GPa and σ_Y = 245 MPa. Knowing that a factor of safety of 2.8 is required and using Euler's formula, write a computer program and use it to calculate the allowable centric load P_{all} for values of the radius of the rod from 6 mm to 24 mm, using 2-mm increments.

SOLUTION

ENTER RADIUS RAD, EFFECTIVE LENGTH L_e AND FACTOR OF SAFETY FS

COMPUTE RADIUS OF GYRATION

$$A = \pi\, RAD^2$$

$$I = \frac{1}{4}\pi\, RAD^4$$

$$r = \sqrt{\frac{I}{A}}$$

DETERMINE ALLOWABLE CENTRIC LOAD

CRITICAL STRESS:

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2}$$

LET σ EQUAL SMALLER OF σ_{cr} AND σ_Y

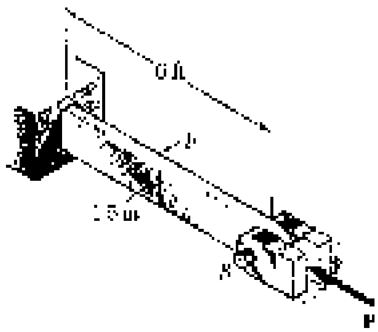
$$P_{all} = \frac{\sigma A}{FS}$$

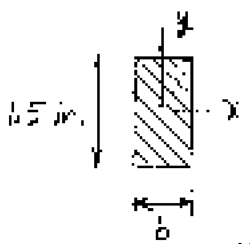
PROGRAM OUTPUT

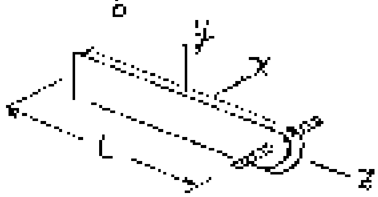
Radius of rod m	Critical stress MPa	Allowable load kN
.006	71.1	2.87
.008	126.3	9.07
.010	197.4	22.15
- - - - - -	- - - - - -	- - - - - -
.012	284.2	39.58
.014	386.9	53.88
.016	505.3	70.37
.018	639.5	89.06
.020	789.6	109.96
.022	955.4	133.05
.024	1137.0	158.34

Below the dashed line we have:
critical stress > yield strength

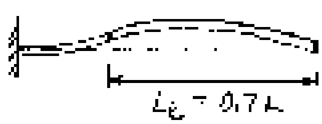
PROBLEM 10.C2



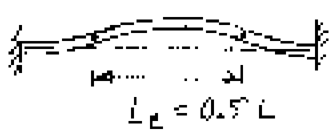




BUCKLING IN XY PLANE



BUCKLING IN XZ PLANE



10.C2 An aluminum bar is fixed at end *A* and supported at end *B* so that it is free to rotate about a horizontal axis through the pin. Rotation about a vertical axis at end *B* is prevented by the brackets. Knowing that $E = 10.1 \times 10^6$ psi, use Euler's formula with a factor of safety of 2.5 to determine the allowable centric load **P** for values of *b* from 0.75 in. to 1.5 in., using 0.125-in. increments.

SOLUTION

ENTER E, LENGTH L AND FACTOR OF SAFETY FS

FOR $b = 0.75$ TO 1.5 WITH 0.125 INCREMENTS

COMPUTE RADIUS OF GYRATION

$$A = 1.5\,b$$

$$I_x = \frac{1}{12}\,b\,1.5^3 \qquad r_x = \sqrt{\frac{I_x}{A}}$$

$$I_y = \frac{1}{12}\,b^3 \qquad r_y = \sqrt{\frac{I_y}{A}}$$

COMPUTE CRITICAL STRESSES

$$(\sigma_{cr})_x = \frac{\pi^2 E}{(0.7L/r_x)^2}$$

$$(\sigma_{cr})_y = \frac{\pi^2 E}{(0.5L/r_y)^2}$$

LET σ_{cr} EQUAL SMALLER STRESS

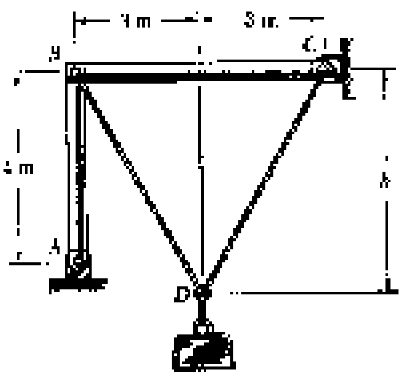
COMPUTE ALLOWABLE CENTRIC LOAD

$$P_{all} = \frac{\sigma_{cr}\,A}{FS}$$

PROGRAM OUTPUT

b in.	Critical stress x axis ksi	Critical stress y axis ksi	Allowable load kips
.750	7.358	3.6	1.62
.875	7.358	4.9	2.59
1.000	7.358	6.4	3.85
1.125	7.358	8.1	4.97
1.250	7.358	10.0	5.52
1.375	7.358	12.1	6.07
1.500	7.358	14.4	6.62

PROBLEM 10.C3



10.C3 The pin-ended members AB and BC consist of sections of aluminum pipe of 120-mm outer diameter and 10-mm wall thickness. Knowing that a factor of safety of 3.5 is required, determine the mass m of the largest block that can be supported by the cable arrangement shown for values of h from 4 m to 8 m, using 0.25-m increments. Use $E = 70$ GPa and consider only buckling in the plane of the structure.

SOLUTION

COMPUTE MOMENT OF INERTIA

$$I = \frac{\pi}{4}\left(0.06^4 - 0.05^4\right)$$

FOR $h = 4$ TO 8 USING 0.25 INCREMENTS

COMPUTE ALLOWABLE LOADS FOR MEMBERS

$$(F_{AB})_{cr} = \frac{\pi^2 EI}{3.5(4)^2}\ ;\quad (F_{BC})_{cr} = \frac{\pi^2 EI}{3.5(6)^2}$$

DETERMINE ALLOWABLE W

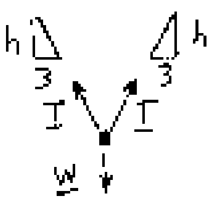
$$(W_{all})_1 = 2\,(F_{AB})_{cr}\ ;\quad (W_{all})_2 = \frac{h}{1.5}\,(F_{BC})_{cr}$$

W_{all} EQUALS SMALLER VALUE

COMPUTE MASS m

$$m = \frac{W_{all}}{9.81}$$

JOINT D:

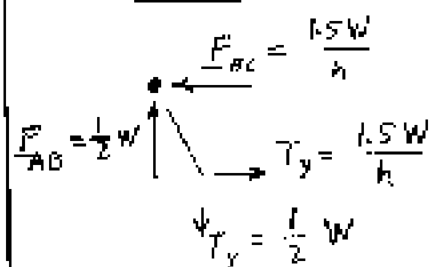


$\Sigma F_y = 0$ YIELDS

$$T_y = \frac{1}{2}W$$

$$\frac{T_x}{T_y} = \frac{3}{h} \quad \text{YIELDS} \quad T_x = \frac{1.5W}{h}$$

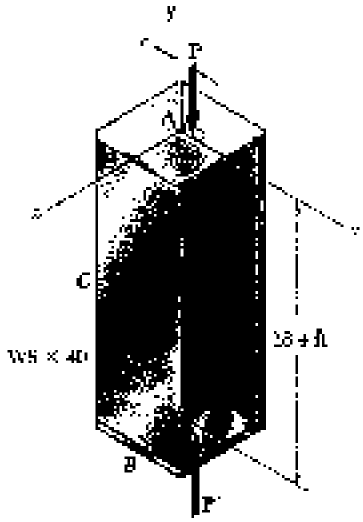
JOINT B:



PROGRAM OUTPUT

h m	Weight critical stress AB kN	Weight critical stress BC kN	mass kg
4.00	455.11	269.7	7854.80
4.25	455.11	286.6	8345.80
4.50	455.11	303.4	8836.74
4.75	455.11	320.3	9327.66
5.00	455.11	337.1	9818.59
5.25	455.11	354.0	10309.52
5.50	455.11	370.9	10800.45
5.75	455.11	387.7	11291.38
6.00	455.11	404.5	11782.31
6.25	455.11	421.4	12273.24
6.50	455.11	438.3	12764.17
6.75	455.11	455.1	13255.10
7.00	455.11	472.0	13255.10
7.25	455.11	488.8	13255.10
7.50	455.11	505.7	13255.10
7.75	455.11	522.5	13255.10
8.00	455.11	539.4	13255.10

PROBLEM 10.C4



10.C4 An axial load P is applied at a point located on the x axis at a distance $e = 0.5$ in. from the geometric axis of the W8 × 40 rolled-steel column AB. Using $E = 29 \times 10^6$ psi, write a computer program and use it to calculate for values of P from 25 to 75 kips, using 5-kip increments, (*a*) the horizontal deflection at the midpoint C, (*b*) the maximum stress in the column.

SOLUTION

ENTER LENGTH L, ECCENTRICITY e

ENTER PROPERTIES A, I_y, r_y, b_f

COMPUTE CRITICAL LOAD

$$P_{cr} = \frac{\pi^2 E I_y}{L^2}$$

FOR P = 25 TO 75 IN INCREMENTS OF 5

COMPUTE HORIZONTAL DEFLECTION AT C

$$y_c = e\left(\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1.0\right)$$

COMPUTE MAXIMUM STRESS

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{e\,b_f}{2\,r_y^2}\sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)$$

PROGRAM OUTPUT

load kip	maximum deflection in.	maximum stress kips
25.0	.059	3.29
30.0	.072	3.99
35.0	.086	4.69
40.0	.100	5.41
45.0	.115	6.14
50.0	.130	6.80
55.0	.146	7.65
60.0	.163	8.43
65.0	.191	9.22
70.0	.199	10.04
75.0	.219	10.88

PROBLEM 10.C5

10.C5 A column of effective length L is made from a rolled-steel shape and carries a centric axial load **P**. The yield strength for the grade of steel used is denoted by σ_Y, the modulus of elasticity by E, the cross-sectional area of the selected shape by A, and its smallest radius of gyration by r. Using the AISC design formulas for allowable stress design, write a computer program that can be used with either SI or U.S. customary units to determine the allowable load **P**. Use this program to solve (*a*) Prob. 10.57, (*b*) Prob. 10.58, (*c*) Prob. 10.60.

SOLUTION

ENTER L, E, σ_Y

ENTER PROPERTIES A, r_y

DETERMINE ALLOWABLE STRESS

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}}$$

IF $L/r_y \geq C_c$

$$\sigma_{all} = \frac{\pi^2 E}{1.92 (L/r_y)^2}$$

IF $L/r_y < C_c$

$$FS = \frac{5}{3} + \frac{3}{8}\left(\frac{L/r_y}{C_c}\right) - \frac{1}{8}\left(\frac{L/r_y}{C_c}\right)^3$$

$$\sigma_{all} = \frac{\sigma_Y}{FS}\left(1 - \frac{(L/r_y)^2}{2\, C_c^2}\right)$$

CALCULATE ALLOWABLE LOAD:

$$P_{all} = \sigma_{all} A$$

CONTINUED

PROBLEM 10.C5 CONTINUED

PROGRAM OUTPUT

```
Problem 10.57 (a)

   Effective Length =   6.50  m
                  A =   6250.0 mm**2
                 ry =   49.2 mm
     Yield strength =   250.0 MPa
                  E =   200 GPa
   ------------------------------------------------
   Allowable centroid load: P =  368.139 kN

Problem 10.57 (b)

   Effective Length =   6.50  m
                  A =   10200.0 mm**2
                 ry =   65.0 mm
     Yield strength =   250.0 MPa
                  E =   200 GPa
   ------------------------------------------------
   Allowable centroid load: P =  915.148 kN

Problem 10.58 (a)

   Effective Length =   21.00  ft
                  A =   9.130 in**2
                 ry =   2.020 in.
     Yield strength =   36.0 ksi
                  E =   29000 ksi
   ------------------------------------------------
   Allowable centroid load: P =  87.566 kips

Problem 10.58 (b)

   Effective Length =   21.00  ft
                  A =   9.130 in**2
                 ry =   2.020 in.
     Yield strength =   50.0 ksi
                  E =   29000 ksi
   ------------------------------------------------
   Allowable centroid load: P =  97.457 kips

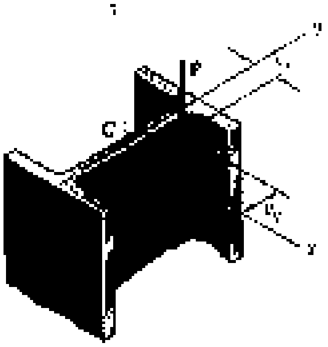
Problem 10.60 (a)

   Effective Length =   4.00  m
                  A =   13800.0 mm**2
                 ry =   43.4 mm
     Yield strength =   345.0 MPa
                  E =   200 GPa
   ------------------------------------------------
   Allowable centroid load: P = 1567.839 kN

Problem 10.60 (b)

   Effective Length =   8.50  m
                  A =   13800.0 mm**2
                 ry =   43.4 mm
     Yield strength =   345.0 MPa
                  E =   200 GPa
   ------------------------------------------------
   Allowable centroid load: P =  632.667 kN
```

PROBLEM 10.C6



10.C6 A column of effective length L is made from a rolled-steel shape and is loaded eccentrically as shown. The yield strength of the grade of steel used is denoted by σ_Y, the allowable stress in bending by σ_{all}, the modulus of elasticity by E, the cross-sectional area of the selected shape by A, and its smallest radius of gyration by r. Write a computer program that can be used with either SI or U.S. customary units to determine the allowable load P, using either the allowable-stress method or the interaction method. Use this program to check the given answer for (a) Prob. 10.111, (b) Prob. 10.112, (c) Prob. 10.113.

SOLUTION

ENTER L, E, σ_Y, $(\sigma_{all})_{bending}$, e_x, e_y

ENTER PROPERTIES A, S_x, S_y, r_y

DETERMINE ALLOWABLE STRESS

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_Y}}$$

IF $L/r_y \geq C_c$

$$\sigma_{all} = \frac{\pi^2 E}{1.92\,(L/r)^2}$$

IF $L/r_y < C_c$

$$FS = \frac{5}{3} + \frac{3}{8}\left(\frac{L/r_y}{C_c}\right) - \frac{1}{8}\left(\frac{L/r_y}{C_c}\right)^3$$

$$\sigma_{all} = \frac{\sigma_Y}{FS}\left(1 - \frac{(L/r_y)^2}{2\,C_c^2}\right)$$

FOR ALLOWABLE-STRESS METHOD

$$COEF = \frac{1}{A} + \frac{e_x}{S_x} + \frac{e_y}{S_y}$$

$$P_{all} = \frac{\sigma_{all}}{COEF}$$

FOR INTERACTION METHOD

$$COEF = \frac{1}{A\,\sigma_{all}} + \frac{(e_x/S_x) + (e_y/S_y)}{(\sigma_{all})_{bending}}$$

$$P_{all} = \frac{1.0}{COEF}$$

CONTINUED

PROBLEM 10.C6 CONTINUED

PROGRAM OUTPUT

```
Problem 10.111

  Effective Length = 3.40 m
                 A = 7560.0 mm**2
                ry = 51.900 mm
                Sx = 582000.0 mm**3
    Yield strength = 250.0 MPa
                 E = 200 GPa
  ---------------------------------------------
  Using Interaction Method
  Allowable load: P = 322.022 kN
  ---------------------------------------------

Problem 10.112

  Effective Length = 7.20 m
                 A = 7420.0 mm**2
                ry = 50.300 mm
                Sy = 403000.0 mm**3
    Yield strength = 250.0 MPa
                 E = 200 GPa
  ---------------------------------------------
  Using Allowable-Stress Method
  Allowable load: P = 97.701 kN
  ---------------------------------------------

Problem 10.113

  Effective Length = 21.00 ft
                 A = 11.900 in**2
                ry = 1.935 in.
                Sx = 51.95 in**3
    Yield strength = 36.0 ksi
                 E = 29 x 10^3 ksi
  ---------------------------------------------
  Using Interaction Method
  Allowable load: P = 86.722 kips
  ---------------------------------------------
```

CHAPTER 11

PROBLEM 11.1

11.1 Determine the modulus of resilience for each of the following grades of structural steel:

(a) ASTM A709 Grade 50: $\sigma_Y = 50$ ksi
(b) ASTM A913 Grade 65: $\sigma_Y = 65$ ksi
(c) ASTM A709 Grade 100: $\sigma_Y = 100$ ksi

SOLUTION

Structural steel $E = 29 \times 10^6$ psi for all three steels given.

(a) $\sigma_Y = 50$ ksi $= 50 \times 10^3$ psi

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{(50 \times 10^3)^2}{(2)(29 \times 10^6)} = 43.1 \text{ in-lb/in}^3$$

(b) $\sigma_Y = 65$ ksi $= 65 \times 10^3$ psi

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{(65 \times 10^3)^2}{(2)(29 \times 10^6)} = 72.8 \text{ in-lb/in}^3$$

(c) $\sigma_Y = 100$ ksi $= 100 \times 10^3$ psi

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{(100 \times 10^3)^2}{(2)(29 \times 10^6)} = 172.4 \text{ in-lb/in}^3$$

PROBLEM 11.2

11.2 Determine the modulus of resilience for each of the following aluminum alloys:

(a) 1100-H14: $E = 70$ GPa, $\sigma_Y = 55$ MPa
(b) 2014-T6: $E = 72$ GPa, $\sigma_Y = 220$ MPa
(c) 6061-T6: $E = 69$ GPa, $\sigma_Y = 140$ MPa

SOLUTION

Aluminum alloys

(a) $E = 70 \times 10^9$ Pa, $\sigma_Y = 55 \times 10^6$ Pa

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{(55 \times 10^6)^2}{(2)(70 \times 10^9)} = 21.6 \times 10^3 \text{ N}\cdot\text{m/m}^3 = 21.6 \text{ kJ/m}^3$$

(b) $E = 72 \times 10^9$ Pa, $\sigma_Y = 220 \times 10^6$ Pa

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{(220 \times 10^6)^2}{(2)(72 \times 10^9)} = 336 \times 10^3 \text{ N}\cdot\text{m/m}^3 = 336 \text{ kJ/m}^3$$

(c) $E = 69 \times 10^9$ Pa, $\sigma_Y = 140 \times 10^6$ Pa

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{(140 \times 10^6)^2}{(2)(69 \times 10^9)} = 142.0 \times 10^3 \text{ N}\cdot\text{m/m}^3 = 142.0 \text{ kJ/m}^3$$

PROBLEM 11.3

11.3 Determine the modulus of resilience for each of the following metals:
(a) Stainless steel AISI 302 (annealed): $E = 190$ GPa, $\sigma_Y = 260$ MPa
(b) Stainless steel AISI 302 (cold-rolled): $E = 190$ GPa, $\sigma_Y = 520$ MPa
(c) Malleable cast iron: $E = 165$ GPa, $\sigma_Y = 230$ MPa

SOLUTION

(a) $E = 190 \times 10^9$ Pa, $\sigma_Y = 260 \times 10^6$ Pa

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{(260 \times 10^6)^2}{(2)(190 \times 10^9)} = 177.9 \times 10^3 \text{ N·m/m}^3 = 177.9 \text{ kJ/m}^3$$

(b) $E = 190 \times 10^9$ Pa, $\sigma_Y = 520 \times 10^6$ Pa

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{(520 \times 10^6)^2}{(2)(190 \times 10^9)} = 712 \times 10^3 \text{ N·m/m}^3 = 712 \text{ kJ/m}^3$$

(c) $E = 165 \times 10^9$ Pa, $\sigma_Y = 230 \times 10^6$ Pa

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{(230 \times 10^6)^2}{(2)(165 \times 10^9)} = 160.3 \times 10^3 \text{ N·m/m}^3 = 160.3 \text{ kJ/m}^3$$

PROBLEM 11.4

11.4 Determine the modulus of resilience for each of the following alloys:
(a) Titanium: $E = 16.5 \times 10^6$ psi; $\sigma_Y = 120$ ksi
(b) Magnesium: $E = 6.5 \times 10^6$ psi; $\sigma_Y = 29$ ksi
(c) Cupronickel (annealed): $E = 20 \times 10^6$ psi; $\sigma_Y = 16$ ksi

SOLUTION

(a) $E = 16.5 \times 10^6$ psi, $\sigma_Y = 120 \times 10^3$ psi

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{(120 \times 10^3)^2}{(2)(16.5 \times 10^6)} = 436 \text{ in·lb/in}^3$$

(b) $E = 6.5 \times 10^6$ psi, $\sigma_Y = 29 \times 10^3$ psi

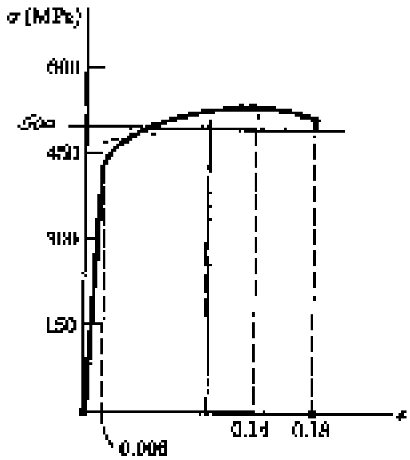
$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{(29 \times 10^3)^2}{(2)(6.5 \times 10^6)} = 64.7 \text{ in·lb/in}^3$$

(c) $E = 20 \times 10^6$ psi, $\sigma_Y = 16 \times 10^3$ psi

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{(16 \times 10^3)^2}{(2)(20 \times 10^6)} = 6.40 \text{ in·lb/in}^3$$

PROBLEM 11.5

11.5 The stress-strain diagram shown has been drawn from data obtained during a tensile test of an aluminum alloy. Using $E = 72$ GPa, (*a*) determine the modulus of resilience of the alloy, (*b*) the modulus of toughness of the alloy.



SOLUTION

(a) $\sigma_Y = E\varepsilon_Y$

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{1}{2}E\varepsilon_Y^2 = \frac{1}{2}(72\times10^9)(0.006)^2$$

$$= 1296\times10^3 \text{ N}\cdot\text{m}/\text{m}^3 = 1296 \text{ kJ/m}^3$$

(b) Modulus of toughness = total area under the stress-strain curve

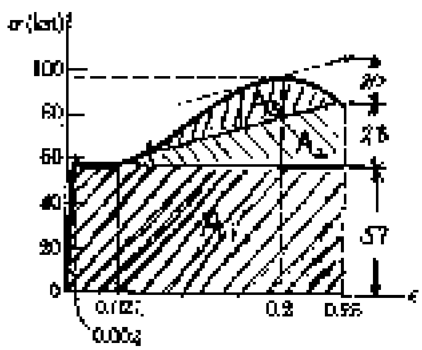
The average ordinate of the stress-strain curve is 500 MPa = 500×10^6 N/m²

The area under the curve is $A = (500\times10^6)(0.18) = 90\times10^6$ N/m²

modulus of toughness $= 90\times10^6$ J/m³ $= 90$ MJ/m³

PROBLEM 11.6

11.6 The stress-strain diagram shown has been drawn from data obtained during the tensile test of a specimen of structural steel. Using $E = 29\times10^6$ psi, (*a*) determine the modulus of resilience of the steel, (*b*) determine the modulus of toughness of the steel.



SOLUTION

(a) $\sigma_Y = E\varepsilon_Y$

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{1}{2}E\varepsilon_Y^2 = \frac{1}{2}(29\times10^6)(0.002)^2$$

$$= 58.0 \text{ in}\cdot\text{lb/in}^3$$

(b) Modulus of toughness = total area under the stress-strain curve

$A_1 = (57)(0.25 - 0.002) = 14.14$ kips/in² $= 14.14$ in·kip/in³

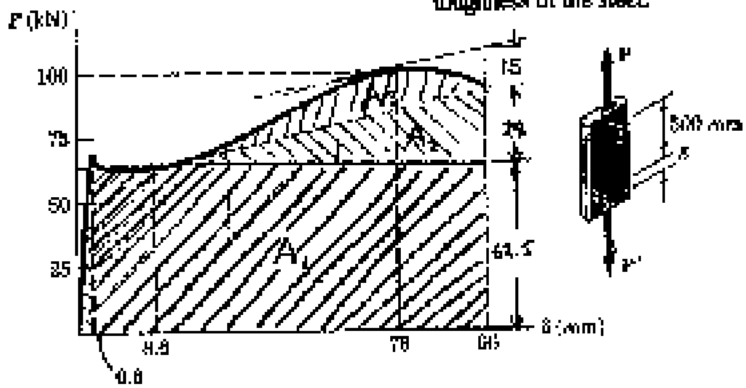
$A_2 = \frac{1}{2}(28)(0.25 - 0.021) = 3.21$ kips/in² $= 3.21$ in·kip/in³

$A_3 = \frac{2}{3}(20)(0.25 - 0.075) = 2.33$ kips/in² $= 2.33$ in·kip/in³

modulus of toughness $= u_Y + A_1 + A_2 + A_3 \approx 20$ in·kip/in³

PROBLEM 11.7

11.7 The load-deformation diagram shown has been drawn from data obtained during the tensile test of a specimen of structural steel. Knowing that the cross-sectional area of the specimen is 250 mm² and that the deformation was measured using a 500-mm gage length, determine (*a*) the modulus of resilience of the steel, (*b*) the modulus of toughness of the steel.



SOLUTION

Assuming that yielding occurs at $P = 62.5$ kN and $\delta = 0.6$ mm

$U_Y = \frac{1}{2}(62.5\times10^3)(0.6\times10^{-3})$

$= 18.75$ N·m

$= 18.75$ J

Volume of stressed material $V = AL = (250)(500) = 125\times10^3$ mm³

$= 125\times10^{-6}$ m³

$u_Y = \dfrac{U_Y}{V} = \dfrac{18.75}{125\times10^{-6}} = 150\times10^3 = 150$ kJ/m³

$A_1 = (62.5\times10^3)(96\times10^{-3}) = 6\times10^3$ N·m $= 6\times10^3$ J

$A_2 = \frac{1}{2}(28\times10^3)(96-8.6)\times10^{-3} = 1.22\times10^3$ N·m $= 1.22\times10^3$ J

$A_3 = \frac{2}{3}(15\times10^3)(61\times10^{-3}) = 0.61\times10^3$ N·m $= 0.61\times10^3$ J

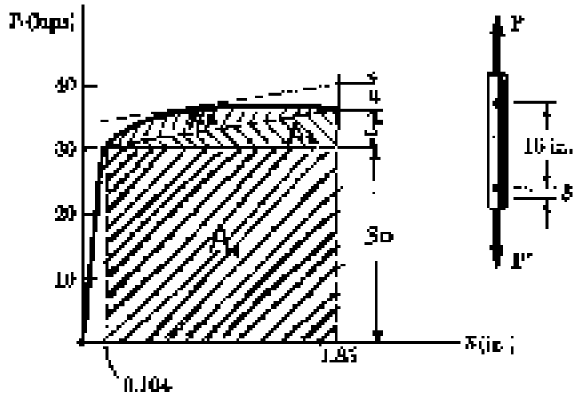
Total energy $U = U_Y + A_1 + A_2 + A_3 = 7.85\times10^3$ J

modulus of toughness $= \dfrac{U}{V} = \dfrac{7.85\times10^3}{125\times10^{-6}} = 63\times10^6$ J/m³

$= 63$ MJ/m³

PROBLEM 11.8

11.8 The load-deformation diagram shown has been drawn from data obtained during the tensile test of a 0.75-in.-diameter rod of an aluminum alloy. Knowing that the deformation was measured using a 16-in. gage length, determine (*a*) the modulus of resilience of the alloy, (*b*) modulus of toughness of the alloy.



SOLUTION

Volume of stressed material involved in the measurement.

$$V = \frac{\pi}{4} d^2 L$$

$$= \frac{\pi}{4}(0.75)^2(16) = 7.0686 \text{ in}^3$$

(a) Modulus of resilience.

$$P_Y = 30 \text{ kips}, \quad \delta_Y = 0.104 \text{ in.}$$

$$U_Y = \frac{1}{2} P_Y \delta_Y = \frac{1}{2}(30)(0.104) = 1.56 \text{ in.}\cdot\text{kip} = 1560 \text{ in.}\cdot\text{lb.}$$

modulus of resilience $\quad u_Y = \dfrac{U_Y}{V} = \dfrac{1560}{7.0686} = 221 \text{ in}\cdot\text{lb/in}^3$ ◄

(b) modulus of toughness

$$A_1 = (30)(1.85 - 0.104) = 52.38 \text{ kip}\cdot\text{in} = 52380 \text{ in}\cdot\text{lb/in}^3$$

$$A_2 = \frac{1}{2}(5)(1.85 - 0.104) = 4.365 \text{ kip}\cdot\text{in} = 4365 \text{ in}\cdot\text{lb/in}^3$$

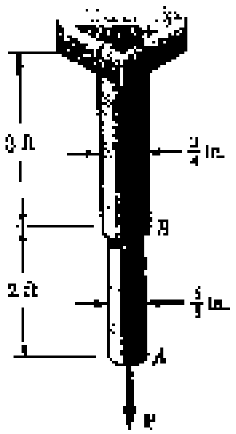
$$A_3 = \frac{2}{3}(4)(1.85 - 0.104) = 4.656 \text{ kip}\cdot\text{in} = 4656 \text{ in}\cdot\text{lb/in}$$

$$U = U_Y + A_1 + A_2 + A_3 = 62961 \text{ in}\cdot\text{lb/in}^3$$

modulus of toughness $\quad \dfrac{U}{V} = \dfrac{62961}{7.0686} = 8900 \text{ in}\cdot\text{lb/in}^3$ ◄

PROBLEM 11.9

11.9 Using $E = 29 \times 10^6$ psi, determine (*a*) the strain energy of the steel rod *ABC* when $P = 8$ kips, (*b*) the corresponding strain-energy density in portions *AB* and *BC* of the rod.



SOLUTION

$P = 8$ kips , $E = 29 \times 10^3$ ksi

$A = \frac{\pi}{4} d^2$, $V = AL$, $\sigma = \frac{P}{A}$, $u = \frac{\sigma^2}{2E}$

$U = uV$

Portion	d in.	L in.	A in^2	V in^3	σ ksi	u $in \cdot kip/in^3$	U $in \cdot kip$
AB	0.625	24	0.3068	7.365	26.08	11.72×10^{-3}	86.32×10^{-3}
BC	0.75	36	0.4418	15.904	18.11	5.65×10^{-3}	89.92×10^{-3}
Σ							176.24×10^{-3}

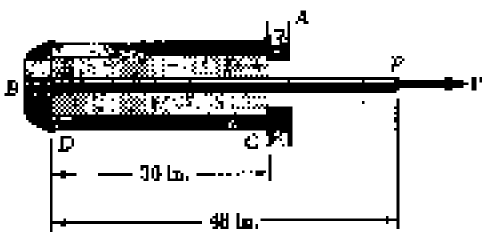
(a) $U = 176.2 \times 10^{-3}$ in$\cdot$kip $= 176.2$ in$\cdot$lb ◄

(b) In AB $u = 11.72 \times 10^{-3}$ in$\cdot$kip/in^3 $= 11.72$ in$\cdot$lb/in^3 ◄

In BC $u = 5.65 \times 10^{-3}$ in$\cdot$kip/in^3 $= 5.65$ in$\cdot$lb/in^3 ◄

PROBLEM 11.10

11.10 A 30-in. length of aluminum pipe of cross-sectional area 1.85 in² is welded to a fixed support *A* and to a rigid cap *B*. The steel rod *EF*, of 0.75-in. diameter, is welded to cap *B*. Knowing that the modulus of elasticity is 29×10^6 psi for steel and 10.6×10^6 for aluminum, determine (*a*) the total strain energy of the system when $P = 10$ kips, (*b*) the corresponding strain-energy density in the pipe *CD* and in the rod *EF*.



SOLUTION

For EF: $A = \frac{\pi}{4} d^2 = 0.4418$ in^2

$$CD: \quad U_{CD} = \frac{P^2 L}{2EA} = \frac{(10 \times 10^3)^2 (30)}{(2)(10.6 \times 10^6)(1.85)} = 76.49 \text{ in}\cdot\text{lb}$$

$$EF: \quad U_{EF} = \frac{P^2 L}{2EA} = \frac{(10 \times 10^3)^2 (48)}{(2)(29 \times 10^6)(0.4418)} = 187.33 \text{ in}\cdot\text{lb}$$

$$\text{Total:} \quad U = U_{CD} + U_{EF} = 264 \text{ in}\cdot\text{lb} \quad ◄$$

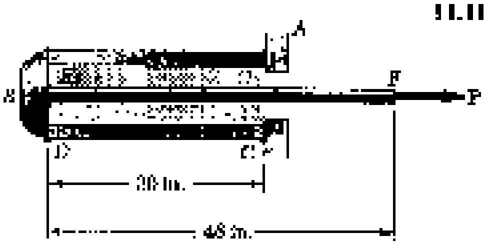
$$CD: \quad \sigma = -\frac{10000}{1.85} = -5405 \text{ psi}, \quad u = \frac{\sigma^2}{2E} = \frac{(-5405)^2}{(2)(10.6 \times 10^6)} = 1.378 \text{ in}\cdot\text{lb/in}^3 \quad ◄$$

$$EF: \quad \sigma = \frac{10000}{0.4418} = 22635 \text{ psi}, \quad u = \frac{\sigma^2}{2E} = \frac{22635^2}{(2)(29 \times 10^6)} = 8.83 \text{ in}\cdot\text{lb/in}^3 \quad ◄$$

PROBLEM 11.11

11.10 A 30-in. length of aluminum pipe of cross-sectional area 1.85 in² is welded to a fixed support A and to a rigid cap B. The steel rod EF, of 0.75-in. diameter, is welded to cap B. Knowing that the modulus of elasticity is 29×10^6 psi for steel and 10.6×10^6 for aluminum, determine (a) the total strain energy of the system when $P = 10$ kips, (b) the corresponding strain-energy density in the pipe CD and in the rod EF.

11.11 Solve Prob. 11.10, when $P = 8$ kips.



SOLUTION

For EF: $\quad A = \frac{\pi}{4} d^2 = 0.4418 \text{ in}^2$

$$CD: \quad U_{CD} = \frac{P^2 L}{2EA} = \frac{(-8000)^2(30)}{(2)(10.6\times10^6)(1.85)} = 48.95 \text{ in}\cdot\text{lb}$$

$$EF: \quad U_{EF} = \frac{P^2 L}{2EA} = \frac{(8000)^2(48)}{(2)(29\times10^6)(0.4418)} = 119.89 \text{ in}\cdot\text{lb}$$

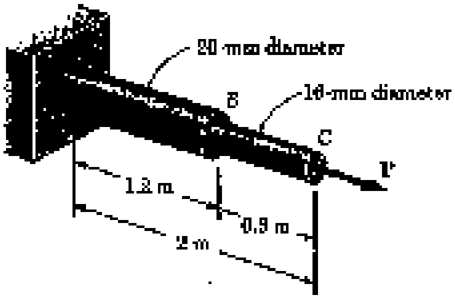
$$\text{Total} \quad U = U_{CD} + U_{EF} = 168.8 \text{ in}\cdot\text{lb}$$

$$CD: \quad \sigma = -\frac{8000}{1.85} = -4324 \text{ psi}, \quad u = \frac{\sigma^2}{2E} = \frac{(-4324)^2}{(2)(10.6\times10^6)} = 0.882 \text{ in}\cdot\text{lb/in}^3$$

$$EF: \quad \sigma = \frac{8000}{0.4418} = 18108 \text{ psi}, \quad u = \frac{\sigma^2}{2E} = \frac{(18108)^2}{(2)(29\times10^6)} = 5.65 \text{ in}\cdot\text{lb/in}^3$$

PROBLEM 11.12

11.12 Using $E = 200$ GPa, determine (a) the strain energy of the steel rod ABC when $P = 25$ kN, (b) the corresponding strain-energy density of portions AB and BC of the rod.



SOLUTION

$$A_{AB} = \frac{\pi}{4}(20)^2 = 314.16 \text{ mm}^2 = 314.16 \times 10^{-6} \text{ m}^2$$

$$A_{BC} = \frac{\pi}{4}(16)^2 = 201.06 \text{ mm}^2 = 201.06 \times 10^{-6} \text{ m}^2$$

$$P = 25 \times 10^3 \text{ N}$$

$$U = \Sigma \frac{P^2 L}{2EA} = \frac{(25\times10^3)^2(1.2)}{(2)(200\times10^9)(314.16\times10^{-6})} + \frac{(25\times10^3)^2(0.8)}{(2)(200\times10^9)(201.06\times10^{-6})}$$

(a) $U = 5.968 + 6.213 = 12.18 \text{ N·m} = 12.18 \text{ J}$

(b) $$\sigma_{AB} = \frac{P}{A_{AB}} = \frac{25\times10^3}{314.16\times10^{-6}} = 79.58\times10^6 \text{ Pa}$$

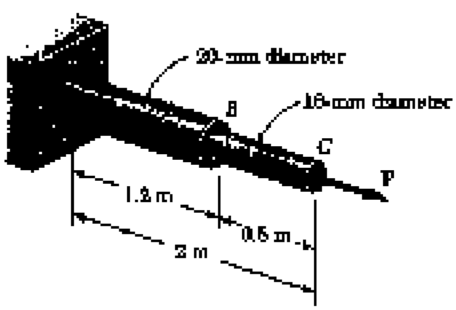
$$u_{AB} = \frac{\sigma_{AB}^2}{2E} = \frac{(79.58\times10^6)^2}{(2)(200\times10^9)} = 15.83\times10^3 = 15.83 \text{ kJ/m}^3$$

$$\sigma_{BC} = \frac{P}{A_{BC}} = \frac{25\times10^3}{201.16\times10^{-6}} = 124.28\times10^6 \text{ Pa}$$

$$u_{BC} = \frac{\sigma_{BC}^2}{2E} = \frac{(124.28\times10^6)^2}{(2)(200\times10^9)} = 38.6\times10^3 = 38.6 \text{ kJ/m}^3$$

PROBLEM 11.13

11.13 The steel rod ABC is made of a steel for which the yield strength is $\sigma_Y = 250$ MPa and the modulus of elasticity is $E = 200$ GPa. Determine, for the loading shown, the maximum strain energy that can be acquired by the rod without causing any permanent deformation.



SOLUTION

$$A_{AB} = \frac{\pi}{4}(20)^2 = 314.16 \text{ mm}^2 = 314.16\times10^{-6} \text{ m}^2$$

$$A_{BC} = \frac{\pi}{4}(16)^2 = 201.06 \text{ mm}^2 = 201.06\times10^{-6} \text{ m}^2$$

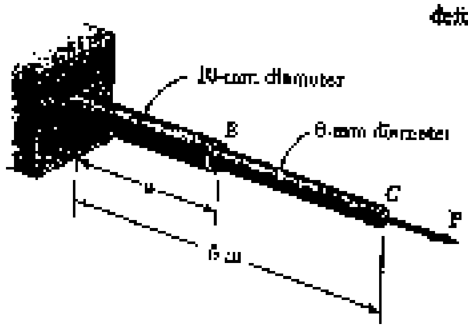
$$P = \sigma_Y A_{min} = (250\times10^6)(201.06\times10^{-6}) = 50.265\times10^3 \text{ Pa}$$

$$U = \Sigma \frac{P^2 L}{2EA} = \frac{(50265)^2(1.2)}{(2)(200\times10^9)(314.16\times10^{-6})} + \frac{(50265)^2(0.8)}{(2)(200\times10^9)(201.06\times10^{-6})}$$

$$= 24.13 + 25.13 = 49.3 \text{ J}$$

PROBLEM 11.14

11.14 The steel rods AB and BC are made of a steel for which the yield strength is $\sigma_Y = 300$ MPa and the modulus of elasticity is $E = 200$ GPa. Determine the maximum strain energy that can be acquired by the assembly without causing any permanent deformation when the length a of rod AB is (a) 2 m, (b) 4 m.



SOLUTION

$A_{AB} = \frac{\pi}{4}(10)^2 = 78.54 \text{ mm}^2 = 78.54 \times 10^{-6} \text{ m}^2$

$A_{BC} = \frac{\pi}{4}(6)^2 = 28.274 \text{ mm}^2 = 28.274 \times 10^{-6} \text{ m}^2$

$P = \sigma_Y A_{min} = (300 \times 10^6)(28.274 \times 10^{-6})$
$= 8.4822 \times 10^3 \text{ N}$

$U = \Sigma \frac{P^2 L}{2EA}$

(a) $a = 2\text{m}$ $\quad L - a = 6 - 2 = 4\text{m}$

$$U = \frac{(8.4822 \times 10^3)^2 (2)}{(2)(200 \times 10^9)(78.54 \times 10^{-6})} + \frac{(8.4822 \times 10^3)^2 (4)}{(2)(200 \times 10^9)(28.274 \times 10^{-6})}$$

$= 4.5803 + 25.4466 = 30.0 \text{ N}\cdot\text{m} = 30.0 \text{ J}$ ◄

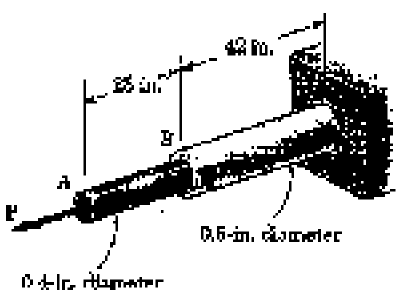
(b) $a = 4\text{m}$ $\quad L - a = 6 - 4 = 2\text{m}$

$$U = \frac{(8.4822 \times 10^3)^2 (4)}{(2)(200 \times 10^9)(78.54 \times 10^{-6})} + \frac{(8.4822 \times 10^3)^2 (2)}{(2)(200 \times 10^9)(28.274 \times 10^{-6})}$$

$= 9.1606 + 12.7233 = 21.9 \text{ N}\cdot\text{m} = 21.9 \text{ J}$ ◄

PROBLEM 11.15

11.15 Rod AB is made of a steel for which the yield strength is $\sigma_Y = 65$ ksi and the modulus of elasticity is $E = 29 \times 10^6$ psi; rod BC is made of an aluminum alloy for which $\sigma_Y = 40$ ksi and $E = 10.6 \times 10^6$ psi. Determine the maximum strain energy that can be acquired by the composite rod ABC without causing permanent deformation.



SOLUTION

$A_{AB} = \frac{\pi}{4}(0.4)^2 = 0.12566 \text{ in}^2 \quad E = 29000 \text{ ksi}$

$A_{BC} = \frac{\pi}{4}(0.6)^2 = 0.28274 \text{ in}^2 \quad E = 10600 \text{ ksi}$

$P_{all} = \sigma_Y A$ for each part

AB: $P_{all} = (65)(0.12566) = 8.1679 \text{ kips}$

BC: $P_{all} = (40)(0.28274) = 11.3096 \text{ kips}$

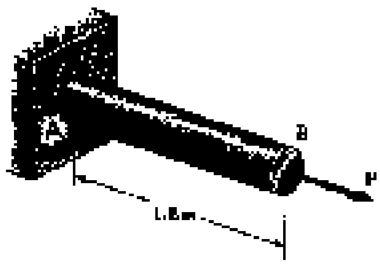
Use smaller value $\quad P = 8.1679 \text{ kips}$

$$U = \Sigma \frac{P^2 L}{2EA} = \frac{(8.1679)^2 (28)}{(2)(29000)(0.12566)} + \frac{(8.1679)^2 (42)}{(2)(10600)(0.28274)}$$

$= 256.3 \times 10^{-3} + 467.5 \times 10^{-3} = 724 \times 10^{-3} \text{ in}\cdot\text{kip} = 724 \text{ in}\cdot\text{lb}$ ◄

PROBLEM 11.16

11.16 Rod AB is made of a steel for which the yield strength is $\sigma_Y = 300$ MPa and the modulus of elasticity is $E = 200$ GPa. Knowing that a strain energy of 10 J must be acquired by the rod when the axial load P is applied, determine the diameter of the rod for which the factor of safety with respect to permanent deformation is six.



SOLUTION

For factor of safety of six on the energy

$$U_Y = (6)(10) = 60 \text{ J}$$

$$u_Y = \frac{\sigma_Y^2}{2E} = \frac{(300\times10^6)^2}{(2)(200\times10^9)} = 225\times10^3 \text{ J/m}^3$$

$$U_Y = A L u_Y$$

$$A = \frac{U_Y}{L u_Y} = \frac{60}{(1.8)(225\times10^3)} = 148.148\times10^{-6} \text{ m}^2$$

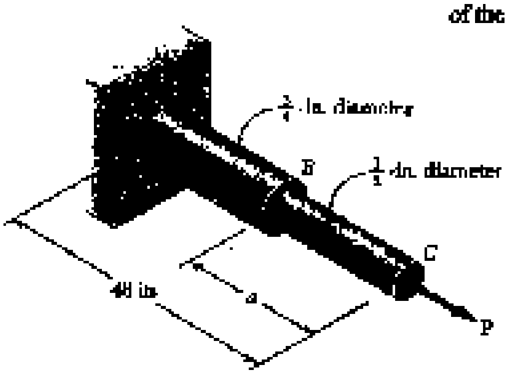
$$A = \frac{\pi}{4} d^2$$

$$d = \sqrt{\frac{4A}{\pi}} = \sqrt{\frac{(4)(148.148\times10^{-6})}{\pi}} = 13.73\times10^{-3} \text{ m}$$

$$= 13.73 \text{ mm}$$

PROBLEM 11.17

11.17 The rod ABC is made of a steel for which the yield strength is $\sigma_Y = 65$ ksi and the modulus of elasticity is $E = 29\times10^6$ psi. Knowing that a strain energy of 90 in · lb must be acquired by the rod as the axial load P is applied, determine the factor of safety of the rod with respect to permanent deformation when $a = 18$ in.



SOLUTION

$$A_{AB} = \frac{\pi}{4}\left(\frac{3}{4}\right)^2 = 0.4418 \text{ in}^2$$

$$A_{BC} = \frac{\pi}{4}\left(\frac{1}{2}\right)^2 = 0.19635 \text{ in}^2$$

$$P_Y = \sigma_Y A_{min} = (65000)(0.19635) = 12763 \text{ lb.}$$

$$U_Y = \sum \frac{P_Y^2 L}{2EA}$$

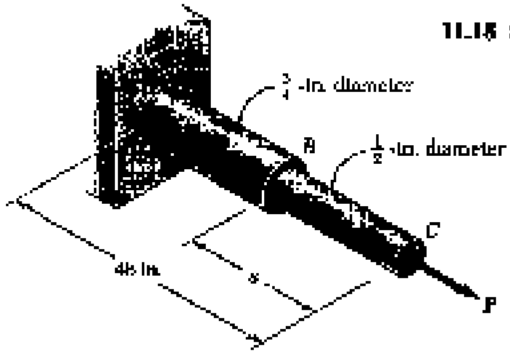
$$U_Y = \frac{(12763)^2(48-18)}{(2)(29\times10^6)(0.4418)} + \frac{(12763)^2(18)}{(2)(29\times10^6)(0.19635)} = 448 \text{ in}\cdot\text{lb.}$$

$$F.S. = \frac{U_Y}{U_{design}} = \frac{448}{90} = 4.98$$

PROBLEM 11.18

11.17 The rod ABC is made of a steel for which the yield strength is $\sigma_Y = 65$ ksi and the modulus of elasticity is $E = 29 \times 10^6$ psi. Knowing that a strain energy of 90 in · lb must be acquired by the rod as the axial load **P** is applied, determine the factor of safety of the rod with respect to permanent deformation when $a = 18$ in.

11.18 Solve Prob. 11.17, assuming that $a = 30$ in.



SOLUTION

$$A_{AB} = \frac{\pi}{4}\left(\frac{3}{4}\right)^2 = 0.4418 \ \text{in}^2$$

$$A_{BC} = \frac{\pi}{4}\left(\frac{1}{2}\right)^2 = 0.19635 \ \text{in}^2$$

$$P_Y = \sigma_Y A_{min} = (65000)(0.19635) = 12763 \ \text{lb.}$$

$$U_Y = \sum \frac{P_Y^2 L}{2EA} = \frac{(12763)^2(48-30)}{(2)(29\times10^6)(0.4418)} + \frac{(12763)^2(30)}{(2)(29\times10^6)(0.19635)} = 543.5 \ \text{in}\cdot\text{lb}$$

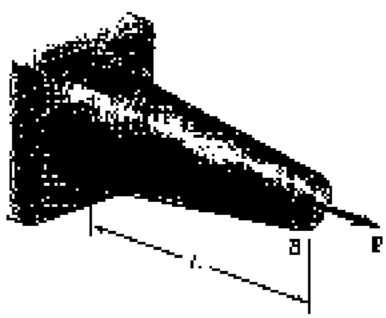
$$F.S. = \frac{U_Y}{U_{design}} = \frac{543.5}{90} = 6.04$$

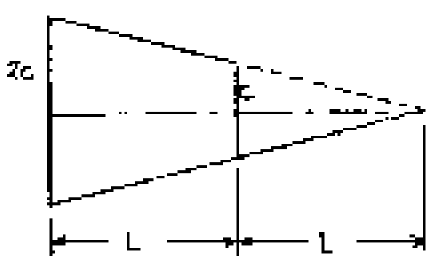
PROBLEM 11.19

11.19 Show by integration that the strain energy of the tapered rod AB is

$$U = \frac{1}{4}\frac{P^2 L}{EA_{min}}$$

where A_{min} is the cross-sectional area at end B.





SOLUTION

$$\text{radius} \quad r = \frac{cx}{L} \qquad A_{min} = \pi c^2$$

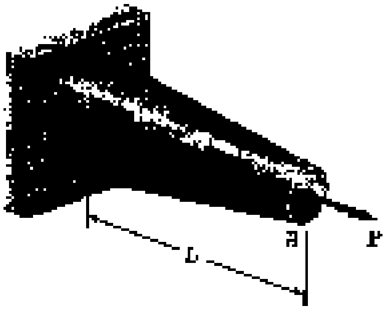
$$A = \pi r^2 = \frac{\pi c^2}{L^2} x^2$$

$$U = \int_L^{2L} \frac{P^2\,dx}{2EA} = \frac{P^2}{2E}\int_L^{2L} \frac{L^2}{\pi c^2}\frac{dx}{x^2}$$

$$= \frac{P^2 L^2}{2E\pi c^2}\left(-\frac{1}{x}\right)\Big|_L^{2L}$$

$$= \frac{P^2 L^2}{2EA_{min}}\left(-\frac{1}{2L}+\frac{1}{L}\right) = \frac{P^2 L}{4EA_{min}}$$

PROBLEM 11.20

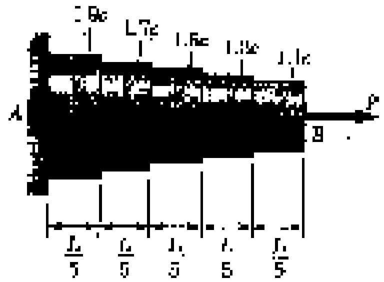


11.19 Show by integration that the strain energy of the tapered rod AB is

$$U = \frac{1}{4}\frac{P^2 L}{EA_{min}}$$

where A_{min} is the cross-sectional area at end B.

11.20 Solve Prob. 11.19, using the stepped rod shown as an approximation of the tapered rod. What is the percentage error in the answer obtained?



SOLUTION

$$A_i = \pi r_i^2 \qquad A_{min} = \pi c^2$$

$$U = \sum \frac{P^2 l_i}{2EA_i} = \frac{P^2(L/5)}{2E}\sum \frac{1}{A_i}$$

$$= \frac{P^2 L}{10\pi E}\sum \frac{1}{r_i^2}$$

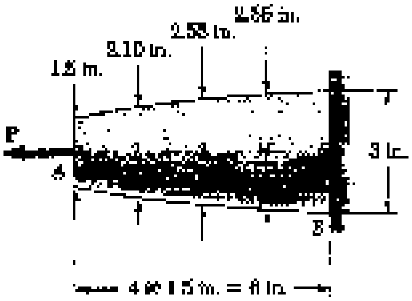
$$= \frac{P^2 L}{10\pi E}\left\{\frac{1}{(1.9c)^2} + \frac{1}{(1.7c)^2} + \frac{1}{(1.5c)^2} + \frac{1}{(1.3c)^2} + \frac{1}{(1.1c)^2}\right\}$$

$$= \frac{P^2 L}{10E(\pi c^2)}\{2.4856\} = 0.24856\,\frac{P^2 L}{EA_{min}}$$

$$\%\ \text{error} = \frac{0.24856 - 0.25}{0.25} \times 100\% = -0.575\ \%$$

PROBLEM 11.21

11.21 Using $E = 10.6 \times 10^6$ psi, determine by approximate means the maximum strain energy that can be acquired by the aluminum rod shown if the allowable normal stress is $\sigma_{all} = 22$ ksi.



SOLUTION

$$A_{min} = \frac{\pi}{4}(1.5)^2 = 1.7671 \text{ in}^2$$

$$\sigma_{all} = 22000 \text{ psi}$$

$$P_{all} = \sigma_{all} A_{min} = 38877 \text{ lb.}$$

$$U = \int \frac{P^2 dx}{2EA} = \frac{P^2}{2E}\int \frac{dx}{\frac{\pi}{4}d^2} = \frac{2P^2}{\pi E}\int \frac{dx}{d^2}$$

Use Simpson's rule to compute the integral $\qquad h = 0.15$ in

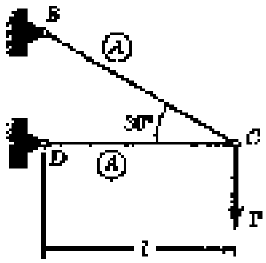
Section	d (in)	$1/d^2$ (in^{-2})	multiplier	$m(1/d^2)$ (in^{-2})
1	1.50	0.4444	1	0.4444
2	2.10	0.22675	4	0.9070
3	2.55	0.15379	2	0.3076
4	2.85	0.12311	4	0.4924
5	3.00	0.11111	1	0.1111
Σ				2.2625

$$\int \frac{dx}{d^2} = \frac{h}{3}\sum m\left(\frac{1}{d^2}\right) = \frac{1.5}{3}(2.2625) = 1.13125 \text{ in}^{-1}$$

$$U = \frac{(2)(38877)^2(1.13125)}{\pi(10.6 \times 10^6)} = 102.7 \text{ in}\cdot\text{lb.}$$

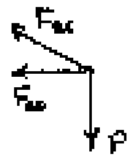
PROBLEM 11.22

11.22 In the truss shown, all members are made of the same material and have the uniform cross-sectional area indicated. Determine the strain energy of the truss when the load **P** is applied.



SOLUTION

Joint C



$+\uparrow \Sigma F_y = 0$

$\frac{1}{2}F_{BC} - P = 0 \qquad F_{BC} = 2P$

$+\rightarrow \Sigma F_x = 0$

$-F_{CD} - \frac{\sqrt{3}}{2}F_{BC} = 0 \qquad F_{BC} = -\sqrt{3}\,P$

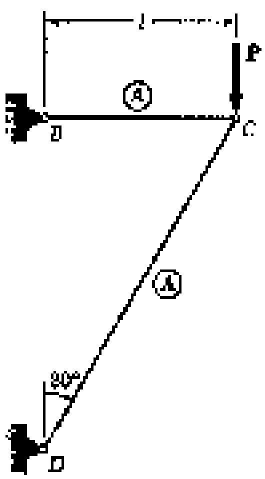
$$U = \Sigma \frac{F^2L}{2EA} = \frac{1}{2E}\Sigma\frac{F^2L}{A}$$

Member	F	L	A	F^2L/A
BC	$2P$	$\frac{2}{\sqrt{3}}\ell$	A	$\frac{8}{\sqrt{3}}P^2\ell/A$
CD	$-\sqrt{3}P$	ℓ	A	$3P^2\ell/A$
Σ				$7.62P^2\ell/A$

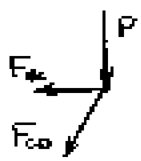
$$U = \frac{1}{2E}\left(7.62\frac{P^2\ell}{A}\right) = 3.81\frac{P^2\ell}{EA}$$

PROBLEM 11.23

11.23 In the truss shown, all members are made of the same material and have the uniform cross-sectional area indicated. Determine the strain energy of the truss when the load **P** is applied.



SOLUTION



$+\uparrow \Sigma F_y = 0$

$-\frac{\sqrt{3}}{2}F_{CD} - P = 0 \qquad F_{CD} = -\frac{2}{\sqrt{3}}P$

$+\rightarrow \Sigma F_x = 0$

$-F_{BC} - \frac{1}{2}F_{CD} = 0 \qquad F_{BC} = \frac{1}{\sqrt{3}}P$

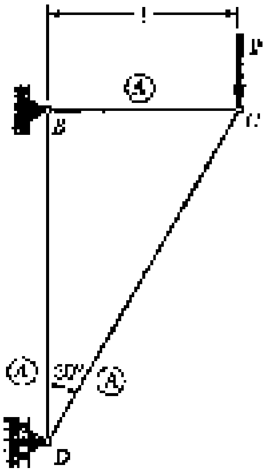
$$U = \Sigma\frac{F^2L}{2EA} = \frac{1}{2E}\Sigma\frac{F^2L}{A}$$

Member	F	L	A	F^2L/A
BC	$\frac{1}{\sqrt{3}}P$	ℓ	A	$\frac{1}{3}P^2\ell/A$
CD	$-\frac{2}{\sqrt{3}}P$	2ℓ	A	$\frac{8}{3}P^2\ell/A$
Σ				$3P^2\ell/A$

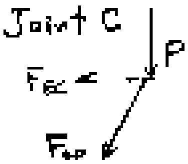
$$U = \frac{1}{2E}\left(3\frac{P^2\ell}{A}\right) = 1.5\frac{P^2\ell}{EA}$$

PROBLEM 11.24

11.24 In the truss shown, all members are made of the same material and have the uniform cross-sectional area indicated. Determine the strain energy of the truss when the load **P** is applied.



Joint C



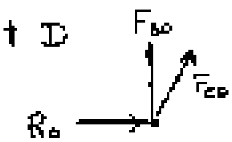
$+\uparrow \Sigma F_y = 0 \qquad -\frac{\sqrt{3}}{2} F_{CD} - P = 0$

$F_{CD} = -\frac{2}{\sqrt{3}} P$

$\xrightarrow{+} \Sigma F_x = 0 \qquad -F_{BC} - \frac{1}{2} F_{CD} = 0$

$F_{BC} = \frac{1}{\sqrt{3}} P$

Joint D



$+\uparrow \Sigma F_y = 0$

$F_{BD} + \frac{\sqrt{3}}{2} F_{CD} = 0$

$F_{BD} = P$

Member	F	L	A	F^2L/A
BC	$\frac{1}{\sqrt{3}} P$	l	A	$\frac{1}{3} P^2 l / A$
CD	$-\frac{2}{\sqrt{3}} P$	$2l$	A	$\frac{8}{3} P^2 l / A$
BD	P	$\sqrt{3} l$	A	$\sqrt{3} P^2 l / A$
Σ				$4.732 P^2 l / A$

$U = \Sigma \frac{1}{2} \frac{F^2 L}{EA}$

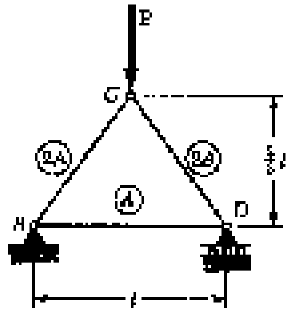
$= \frac{1}{2E} \Sigma \frac{F^2 L}{A}$

$= \frac{1}{2E} \left(4.732 \frac{P^2 l}{A} \right)$

$= 2.37 \frac{P^2 l}{EA}$

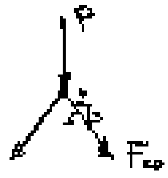
PROBLEM 11.25

11.25 In the truss shown, all members are made of the same material and have the uniform cross-sectional area indicated. Determine the strain energy of the truss when the load P is applied.



SOLUTION

Joint C



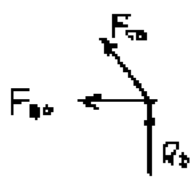
$\xrightarrow{+} \Sigma F_x = 0 \qquad \frac{3}{5}F_{CB} - \frac{3}{5}F_{CD} = 0$

$F_{CB} = F_{CD}$

$+\uparrow \Sigma F_y = 0 \qquad -P - 2\cdot\frac{4}{5}F_{CD} = 0$

$F_{CB} = F_{CD} = -\frac{5}{8}P$

Joint D



$\xrightarrow{+} \Sigma F_x = 0$

$-F_{BD} - \frac{3}{5}F_{CD} = 0$

$F_{BD} = -\frac{3}{5}\cdot\frac{5}{8}P = \frac{3}{8}P$

$U = \Sigma \frac{F^2L}{2EA} = \frac{1}{2E}\Sigma\frac{F^2L}{A}$

Member	F	L	A	F^2L/A
CB	$-\frac{5}{8}P$	$\frac{5}{6}l$	2A	$\frac{125}{768}P^2l/A$
CD	$-\frac{5}{8}P$	$\frac{5}{6}l$	2A	$\frac{125}{768}P^2l/A$
BD	$\frac{3}{8}P$	l	A	$\frac{9}{64}P^2l/A$
Σ				$\frac{179}{384}P^2l/A$

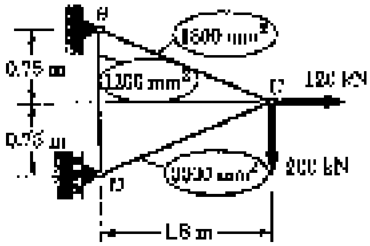
$= \frac{1}{2E}\left(\frac{179}{384}\frac{P^2l}{A}\right)$

$= \frac{179}{768}\frac{P^2l}{EA}$

$= 0.233\frac{P^2l}{EA}$

PROBLEM 11.26

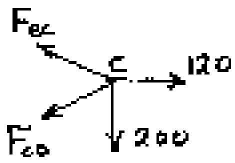
11.26 In the truss shown, all members are made of aluminum and have the uniform cross-sectional area indicated. Using $E = 72$ GPa, determine the strain energy of the truss for the loading shown.



SOLUTION

$$l_{BC} = l_{CD} = \sqrt{1.8^2 + 0.75^2} = 1.95 \text{ m}$$

Joint C



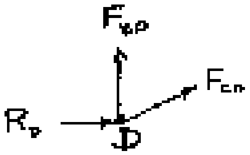
$$\xrightarrow{+} \Sigma F_x = 0 \quad -\frac{1.8}{1.95}F_{BC} - \frac{1.8}{1.95}F_{CD} + 120 = 0 \qquad (1)$$

$$+\uparrow \Sigma F_y = 0 \quad \frac{0.75}{1.95}F_{BC} + \frac{0.75}{1.95}F_{CD} - 200 = 0 \qquad (2)$$

Solving (1) and (2) simultaneously,

$$F_{BC} = 325 \text{ kN} \qquad F_{CD} = -195 \text{ kN}$$

Joint D



$$+\uparrow \Sigma F_y = 0 \quad F_{BD} + \frac{0.75}{1.95}F_{CD} = 0 \qquad F_{BD} = 75 \text{ kN}$$

$$U = \Sigma \frac{F^2L}{2EA} = \frac{1}{2E}\Sigma\frac{F^2L}{A}$$

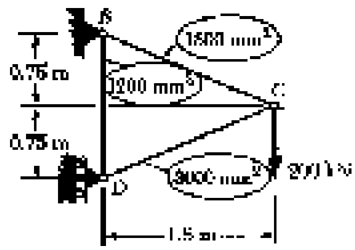
Member	F (kN)	L (m)	A (10^{-6} m²)	F^2L/A (N²/m)
BC	325	1.95	1800	114.43×10^{12}
BD	75	1.5	1200	7.03×10^{12}
CD	−195	1.95	3000	24.72×10^{12}
Σ				146.18×10^{12}

$$U = \frac{146.18 \times 10^{12}}{(2)(72 \times 10^9)} = 1.015 \times 10^3 \text{ N}\cdot\text{m} = 1015 \text{ J}$$

PROBLEM 11.27

11.27 In the truss shown, all members are made of aluminum and have the uniform cross-sectional area indicated. Using $E = 72$ GPa, determine the strain energy of the truss for the loading shown.

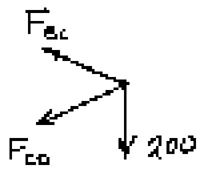
11.27 Solve Prob. 11.26, assuming that the 120-kN load is removed.



SOLUTION

$$\ell_{BC} = \ell_{CD} = \sqrt{1.8^2 + 0.75^2} = 1.95 \text{ m}$$

Joint C



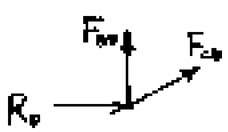
$$\stackrel{+}{\rightarrow} \Sigma F_x = 0 \qquad -\frac{1.8}{1.95} F_{BC} - \frac{1.8}{1.95} F_{CD} = 0$$

$$+\uparrow \Sigma F_y = 0 \qquad \frac{0.75}{1.95} F_{BC} - \frac{0.75}{1.95} F_{CD} - 200 = 0$$

Solving (1) and (2) simultaneously,

$$F_{BC} = 260 \text{ kN} \qquad F_{CD} = -260 \text{ kN}$$

Joint D



$$+\uparrow \Sigma F_y = 0 \qquad F_{BD} + \frac{0.75}{1.95} F_{CD} = 0 \qquad F_{BD} = 100 \text{ kN}$$

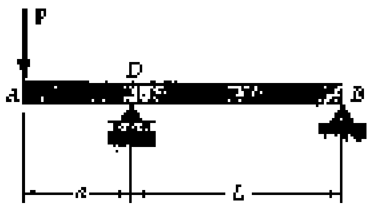
$$U = \Sigma \frac{F^2 L}{2EA} = \frac{1}{2E} \Sigma \frac{F^2 L}{A}$$

Member	F (kN)	L (m)	A (10^{-6} m²)	F^2L/A (N²/m)
BC	260	1.95	1800	73.23×10^{12}
BD	100	1.5	1200	12.50×10^{12}
CD	−260	1.95	3000	43.94×10^{12}
Σ				129.67×10^{12}

$$U = \frac{129.67 \times 10^{12}}{(2)(72 \times 10^9)} = 900 \text{ N·m} = 900 \text{ J}$$

PROBLEM 11.28

11.28 Taking into account only the effect of normal stresses, determine the strain energy of the prismatic beam *AB* for the loading shown.

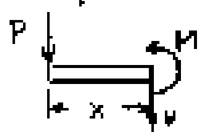


SOLUTION

$$\circlearrowleft \Sigma M_D = 0$$

$$aP + LR_B = 0 \qquad R_B = -\frac{aP}{L} = \frac{aP}{L}\downarrow$$

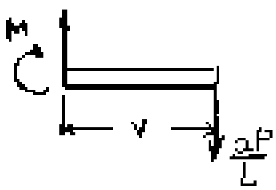
Over portion AD : $M = -Px$



$$U_{AD} = \int_0^a \frac{M^2}{2EI}dx = \frac{1}{2EI}\int_0^a P^2x^2dx = \frac{P^2}{2EI}\frac{x^3}{3}\Big|_0^a$$

$$= \frac{P^2a^3}{6EI}$$

Over portion DB : $M = -\frac{aP}{L}v$



$$U_{DB} = \int_0^L \frac{M^2}{2EI}dv = \frac{1}{2EI}\int_0^L \frac{a^2P^2}{L^2}v^2dv$$

$$= \frac{P^2a^2}{2EIL^2}\int_0^L v^2dv = \frac{P^2a^2}{2EIL^2}\frac{v^3}{3}\Big|_0^L = \frac{P^2a^2L}{6EI}$$

Total $U = U_{AD} + U_{DB} = \frac{P^2a^2}{6EI}(a+L)$

PROBLEM 11.29

11.29 Taking into account only the effect of normal stresses, determine the strain energy of the prismatic beam *AB* for the loading shown.

SOLUTION

Symmetric beam and loading $R_A = R_B$.

$$+\uparrow \Sigma F_y = 0 \qquad R_A + R_B - 2P = 0 \qquad R_A = R_B = P$$

Over portion AD : $M = R_A x = Px$

$$U_{AD} = \int_0^a \frac{M^2}{2EI}dx = \frac{P^2}{2EI}\int_0^a x^2dx = \frac{P^2}{2EI}\frac{x^3}{3}\Big|_0^a = \frac{P^2a^3}{6EI}$$

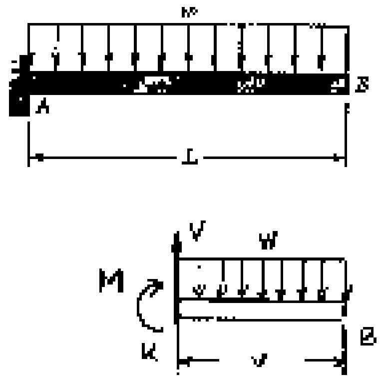
Over portion DE : $M = Pa$ $\qquad U_{DE} = \frac{P^2a^2(L-2a)}{2EI}$

Over portion EB : By symmetry $\qquad U_{EB} = U_{AD} = \frac{P^2a^3}{6EI}$

Total $U = U_{AD} + U_{DE} + U_{EB} = \frac{P^2a^2}{6EI}(3L - 4a)$

PROBLEM 11.30

11.30 Taking into account only the effect of normal stresses, determine the strain energy of the prismatic beam AB for the loading shown.



SOLUTION

$$+\circlearrowleft \Sigma M_K = 0 \qquad -M - (wv)\left(\frac{v}{2}\right) = 0$$

$$M = -\frac{1}{2}wv^2$$

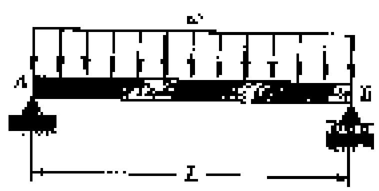
$$U = \int_0^L \frac{M^2}{2EI}\,dv = \frac{1}{2EI}\int_0^L \left(\frac{1}{2}wv^2\right)^2 dv$$

$$= \frac{w^2}{8EI}\int_0^L v^4\,dv = \frac{w^2}{8EI}\,\frac{v^5}{5}\Big|_0^L$$

$$= \frac{w^2L^5}{40EI}$$

PROBLEM 11.31

11.31 Taking into account only the effect of normal stresses, determine the strain energy of the prismatic beam AB for the loading shown.



SOLUTION

$$+\circlearrowleft \Sigma M_B = 0 \qquad -R_A L + (wL)\left(\frac{L}{2}\right) = 0 \qquad R_A = \frac{wL}{2}$$

Bending moment $\qquad M = R_A x - \frac{1}{2}wx^2$

$$= \frac{w}{2}(Lx - x^2)$$

$$U = \int_0^L \frac{M^2}{2EI}\,dx = \frac{w^2}{8EI}\int_0^L (Lx - x^2)^2\,dx$$

$$= \frac{w^2}{8EI}\int_0^L (L^2x^2 - 2Lx^3 + x^4)\,dx = \frac{w^2}{8EI}\left[\frac{L^2x^3}{3} - \frac{2Lx^4}{4} + \frac{x^5}{5}\right]_0^L$$

$$= \frac{w^2L^5}{8EI}\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right] = \frac{w^2L^5}{240EI}$$

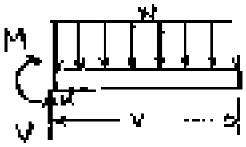
PROBLEM 11.32

11.32 Assuming that the prismatic beam AB has a rectangular cross section, show that for the given loading the maximum value of the strain-energy density in the beam is

$$u_{max} = 15\frac{U}{V}$$

where U is the strain energy of the beam and V is its volume.

SOLUTION



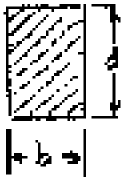
$$+\circlearrowleft \Sigma M_K = 0 \qquad -M - (wv)\frac{v}{2} = 0$$

$$M = -\frac{1}{2}wv^2$$

$$U = \int_0^L \frac{M^2}{2EI}\,dv = \frac{1}{2EI}\int_0^L\left(\frac{1}{2}wv^2\right)^2 dv = \frac{w^2}{8EI}\int_0^L v^4\,dv = \frac{w^2}{8EI}\,\frac{v^5}{5}\Big|_0^L = \frac{w^2L^5}{40EI}$$

$$M_{max} = \frac{1}{2}wL^2 \qquad \sigma_{max} = \frac{M_{max}c}{I}$$

$$u_{max} = \frac{\sigma_{max}^2}{2E} = \frac{M_{max}^2c^2}{2EI^2} = \frac{\frac{1}{4}w^2L^4c^2}{2EI^2} = \frac{w^2L^4c^2}{8EI^2}$$



$$\frac{U}{u_{max}} = \frac{LI}{5c^2} = \frac{L\left(\frac{1}{12}bd^3\right)}{5\left(\frac{d}{2}\right)^2} = \frac{1}{15}Lbd = \frac{1}{15}V$$

$$u_{max} = 15\frac{U}{V}$$

PROBLEM 11.33

11.33 Assuming that the prismatic beam AB has a rectangular cross section, show that for the given loading the maximum value of the strain-energy density in the beam is

$$u_{max} = \frac{45}{8}\frac{U}{V}$$

where U is the strain energy of the beam and V is its volume.

SOLUTION

$+\circlearrowleft \; M_B = 0 \qquad -R_A L + (wL)\frac{L}{2} = 0 \qquad R_A = \frac{1}{2}wL$

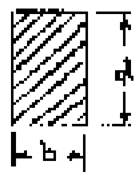
$$M = R_A x - \frac{1}{2}wx^2 = \frac{1}{2}w(Lx - x^2)$$

$$U = \int_0^L \frac{M^2}{2EI}dx = \frac{w^2}{8EI}\int_0^L (L^2x^2 - 2Lx^3 + x^4)dx = \frac{w^2}{8EI}\left[\frac{L^2x^3}{3} - \frac{2Lx^4}{4} + \frac{x^5}{5}\right]_0^L$$

$$= \frac{w^2L^5}{8EI}\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right] = \frac{w^2L^5}{240EI}$$

$$M_{max} = \frac{1}{2}w\left[L\cdot\frac{L}{2} - \left(\frac{L}{2}\right)^2\right] = \frac{1}{8}wL^2 \qquad \sigma_{max} = \frac{M_{max}c}{I} = \frac{wL^2c}{8I}$$

$$u_{max} = \frac{\sigma_{max}^2}{2E} = \frac{w^2L^4c^2}{128EI^2}$$

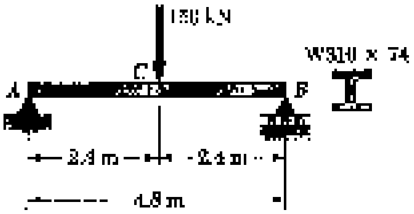


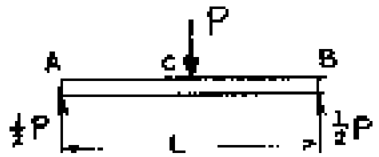
$$\frac{U}{u_{max}} = \frac{8LI}{15c^2} = \frac{8L\left(\frac{1}{12}bd^3\right)}{15\left(\frac{d}{2}\right)^2} = \frac{8}{45}Lbd = \frac{8}{45}V$$

$$u_{max} = \frac{45}{8}\frac{U}{V}$$

PROBLEM 11.34

11.34 Using $E = 200$ GPa, determine the strain energy due to bending for the steel beam and loading shown.





SOLUTION

Over portion AC $\qquad M = \frac{1}{2}Px$

$$U_{AC} = \int_0^{\frac{L}{2}} \frac{M^2}{2EI}\,dx = \frac{P^2}{8EI}\int_0^{\frac{L}{2}} x^2\,dx$$

$$= \frac{P^2}{8EI}\,\frac{x^3}{3}\Big|_0^{\frac{L}{2}} = \frac{P^2L^3}{192EI}$$

By symmetry $\quad U_{CB} = U_{AC} = \frac{P^2L^3}{192EI}$

Total: $\quad U = U_{AC} + U_{CB} = \frac{P^2L^3}{96EI}$

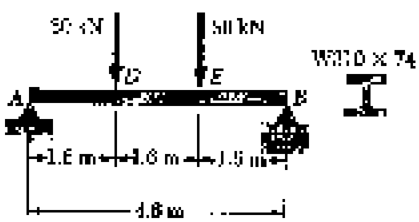
Data: $P = 160\times10^3$ N, $L = 4.8$ m, $E = 200\times10^9$ Pa

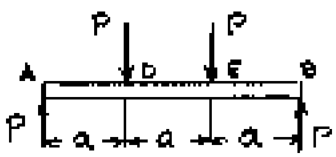
$I = 165\times10^6\ \text{mm}^4 = 165\times10^{-6}\ \text{m}^4$

$$U = \frac{(160\times10^3)^2(4.8)^3}{(96)(200\times10^9)(165\times10^{-6})} = 894\ \text{N}\cdot\text{m} = 894\ \text{J}$$

PROBLEM 11.35

11.35 Using $E = 200$ GPa, determine the strain energy due to bending for the steel beam and loading shown.





SOLUTION

Over portion AD: $\qquad M = Px$

$$U_{AD} = \int_0^a \frac{M^2}{2EI}\,dx = \frac{1}{2EI}\int_0^a (Px)^2\,dx$$

$$= \frac{P^2}{2EI}\,\frac{x^3}{3}\Big|_0^a = \frac{P^2a^3}{6EI}$$

Over portion DE: $\qquad M = Pa$

$$U_{DE} = \frac{(Pa)^2a}{2EI} = \frac{P^2a^3}{2EI}$$

By symmetry $\quad U_{EB} = U_{AD} = \frac{P^2a^3}{6EI}$

$$U = U_{AD} + U_{DE} + U_{EB} = \frac{5}{6}\,\frac{P^2a^3}{EI}$$

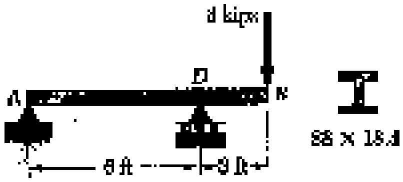
Data: $P = 80\times10^3$ N, $a = 1.6$ m, $E = 200\times10^9$ Pa

$I = 165\times10^6\ \text{mm}^4 = 165\times10^{-6}\ \text{m}^4$

$$U = \frac{5}{6}\,\frac{(80\times10^3)^2(1.6)^3}{(200\times10^9)(165\times10^{-6})} = 662\ \text{N}\cdot\text{m} = 662\ \text{J}$$

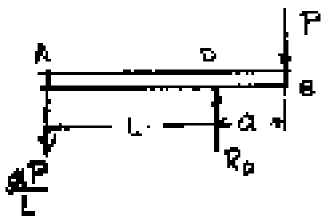
PROBLEM 11.36

11.36 Using $E = 29 \times 10^6$ psi, determine the strain energy due to bending for the steel beam and loading shown.

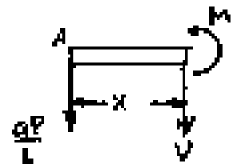


SOLUTION

$$\circlearrowleft \Sigma M_D = 0 \qquad -R_A L - aP = 0 \qquad R_A = \frac{aP}{L} \downarrow$$



Over portion AD $\qquad M = -\frac{aP}{L}x$

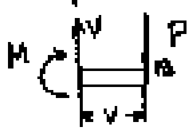


$$U_{AD} = \int_0^L \frac{M^2}{2EI}dx = \frac{1}{2EI}\int_0^L \left(\frac{aP}{L}x\right)^2 dx$$

$$= \frac{P^2a^2}{2EIL^2}\int_0^L x^2\,dx$$

$$= \frac{P^2a^2L}{6EI}$$

Over portion DB $\qquad M = -Pv$



$$U_{DB} = \int_0^a \frac{M^2}{2EI}dv = \frac{1}{2EI}\int_0^a P^2x^2\,dx = \frac{P^2a^3}{6EI}$$

Total: $\quad U = U_{AD} + U_{DB} = \frac{P^2a^2}{6EI}(a + L)$

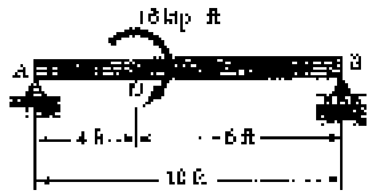
Data: $P = 8000$ lb, $L = 6$ ft $= 72$ in, $a = 3$ ft $= 36$ in, $E = 29 \times 10^6$ psi

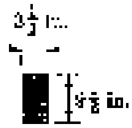
$I = 57.6$ in^4

$$U = \frac{(8000)^2(36)^2(72 + 36)}{(6)(29 \times 10^6)(57.6)} = 894 \text{ in}\cdot\text{lb}$$

PROBLEM 11.37

11.37 Using $E = 1.8 \times 10^6$ psi, determine the strain energy due to bending for the timber beam and loading shown.

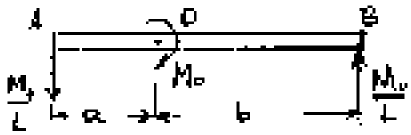




SOLUTION

$\circlearrowleft \Sigma M_A = 0 \qquad -M_o + R_B L = 0 \qquad R_B = \frac{M_o}{L} \uparrow$

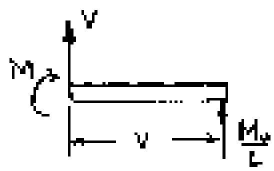
$R_A = \frac{M_o}{L} \downarrow$



Over portion AD $\quad M = \frac{M_o x}{L}$

$$U_{AD} = \int_0^a \frac{M^2}{2EI} dx = \frac{M_o^2}{2EIL^2} \int_0^a x^2 dx$$

$$= \frac{M_o^2 a^3}{6EIL^2}$$



Over portion DB $\quad M = \frac{M_o}{L} v$

$$U_{DB} = \int_0^b \frac{M^2}{2EI} dx = \frac{M_o^2}{2EIL^2} \int_0^b x^2 dx = \frac{M_o^2 b^3}{6EIL^2}$$

Total $\quad U = U_{AD} + U_{DB} = \dfrac{M_o^2 (a^3 + b^3)}{6EIL^2}$

Data: $\quad M_o = 16$ kip · ft, $\quad a = 4$ ft, $\quad b = 6$ ft, $\quad L = 10$ ft

$E = 1.8 \times 10^3$ ksi

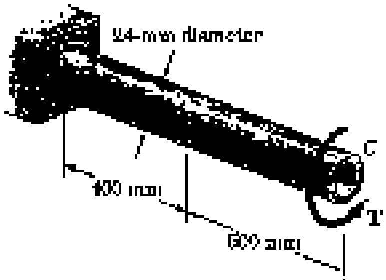
$I = \frac{1}{12}(3\frac{1}{2})(9\frac{3}{8})^3 = 250.07 \text{ in}^4$

$EI = (1.8 \times 10^3)(250.07) = 450.13 \times 10^3 \text{ kip} \cdot \text{in}^2 = 3126 \text{ kip} \cdot \text{ft}^2$

$$U = \frac{(16)^2 (4^3 + 6^3)}{(6)(3126)(10)^2} = 38.2 \times 10^{-3} \text{ kip} \cdot \text{ft} = 38.2 \text{ ft} \cdot \text{lb}$$

$$= 458 \text{ in} \cdot \text{lb}$$

PROBLEM 11.38



11.38 Rod AC is made of aluminum ($G = 73$ GPa) and is subjected to a torque **T** applied at end C. Knowing that portion BC of the rod is hollow and has an inside diameter of 16 mm, determine the strain energy of the rod for a maximum shearing stress of 120 MPa.

SOLUTION

$$c_o = \frac{d_o}{2} = 12 \text{ mm}, \qquad c_i = \frac{d_i}{2} = 8 \text{ mm}$$

$$J_{AB} = \frac{\pi}{2} c_o^4 = \frac{\pi}{2}(12)^4 = 32.572 \times 10^3 \text{ mm}^4 = 32.572 \times 10^{-9} \text{ m}^4$$

$$J_{BC} = \frac{\pi}{2}(c_o^4 - c_i^4) = \frac{\pi}{2}(12^4 - 8^4) = 26.138 \times 10^3 \text{ mm}^4$$
$$= 26.138 \times 10^{-9} \text{ m}^4$$

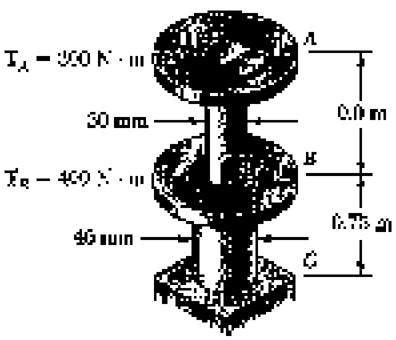
$$\tau_m = \frac{Tc}{J_{min}} \qquad T = \frac{J_{min}\tau_m}{c} = \frac{(26.138 \times 10^{-9})(120 \times 10^6)}{12 \times 10^{-3}} = 261.38 \text{ N}\cdot\text{m}$$

$$U_{AB} = \frac{T^2 L_{AB}}{2GJ_{AB}} = \frac{(261.38)^2(400 \times 10^{-3})}{(2)(73 \times 10^9)(32.572 \times 10^{-9})} = 5.747 \text{ J}$$

$$U_{BC} = \frac{T^2 L_{BC}}{2GJ_{BC}} = \frac{(261.38)^2(500 \times 10^{-3})}{(2)(73 \times 10^9)(26.138 \times 10^{-9})} = 8.951 \text{ J}$$

$$\text{Total} \quad U = U_{AB} + U_{BC} = 14.70 \text{ J}$$

PROBLEM 11.39



11.39 In the assembly shown torques $\mathbf{T}_A$ and $\mathbf{T}_B$ are exerted on disks A and B respectively. Knowing that both shafts are solid and made of aluminum ($G = 73$ GPa), determine the total energy acquired by the assembly.

SOLUTION

Over portion AB

$$T_{AB} = T_A = 300 \text{ N}\cdot\text{m}$$

$$J_{AB} = \frac{\pi}{2} c^4 = \frac{\pi}{2}\left(\frac{30}{2}\right)^4 = 79.52 \times 10^3 \text{ mm}^4 = 79.52 \times 10^{-9} \text{ m}^4$$

$$L_{AB} = 0.9 \text{ m}$$

$$U_{AB} = \frac{T_{AB}^2 L_{AB}}{2GJ_{AB}} = \frac{(300)^2(0.9)}{(2)(73 \times 10^9)(79.52 \times 10^{-9})}$$
$$= 6.977 \text{ J}$$

Over portion BC: $T_{BC} = T_A + T_B = 300 + 400 = 700$ N·m, $L_{BC} = 0.75$ m

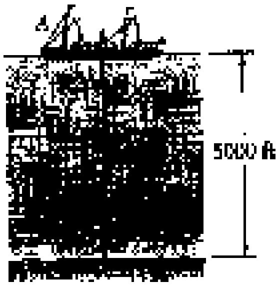
$$J_{BC} = \frac{\pi}{2}\left(\frac{46}{2}\right)^4 = 439.57 \times 10^3 \text{ mm}^4 = 439.57 \times 10^{-9} \text{ m}^4$$

$$U_{BC} = \frac{T_{BC}^2 L_{BC}}{2GJ_{BC}} = \frac{(700)^2(0.75)}{(2)(73 \times 10^9)(439.57 \times 10^{-9})} = 5.726 \text{ J}$$

$$\text{Total} \quad U = U_{AB} + U_{BC} = 6.977 + 5.726 = 12.70 \text{ J}$$

PROBLEM 11.40

11.40 The ship at A has just started to drill for oil on the ocean floor at a depth of 5000 ft. The steel drill pipe has an outside diameter of 8 in. and a uniform wall thickness of 0.5 in. Knowing that the top of the drill pipe rotates through two complete revolutions before the drill bit at B starts to operate and using $G = 11.2 \times 10^6$ psi, determine the maximum strain energy acquired by the drill pipe.



SOLUTION

$$\varphi = (2)(2\pi) = 4\pi \text{ rad}$$

$$L = 5000 \text{ ft} = 60 \times 10^3 \text{ in}$$

$$c_o = \frac{d_o}{2} = 4 \text{ in} \qquad c_i = c_o - t = 3.5 \text{ in.}$$

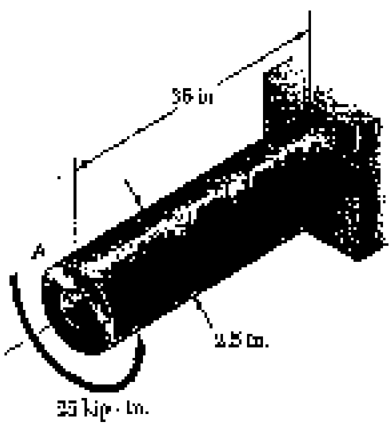
$$J = \frac{\pi}{2}(c_o^4 - c_i^4) = 166.406 \text{ in}^4$$

$$\varphi = \frac{TL}{GJ} \qquad T = \frac{GJ\varphi}{L} \qquad U = \frac{T^2 L}{2GJ} = \left(\frac{GJ\varphi}{L}\right)^2 \frac{L}{2GJ} = \frac{GJ\varphi^2}{2L}$$

$$U = \frac{(11.2\times10^6)(166.406)(4\pi)^2}{(2)(60\times10^3)} = 2.45 \times 10^6 \text{ in}\cdot\text{lb}$$

PROBLEM 11.41

11.41. The design specifications for the steel shaft AB require that the shaft acquire a strain energy of 300 in·lb as the 25-kip·in. torque is applied. Using $G = 11.2 \times 10^6$ psi, determine (*a*) the largest inside diameter of the shaft that can be used, (*b*) the corresponding maximum shearing stress in the shaft.



SOLUTION

$$U = 300 \text{ in}\cdot\text{lb} \qquad T = 25 \text{ kip}\cdot\text{in} = 25\times10^3 \text{ lb}\cdot\text{in}$$

$$L = 36 \text{ in.}$$

$$U = \frac{T^2 L}{2GJ}$$

$$J = \frac{T^2 L}{2GU} = \frac{(25\times10^3)^2(36)}{(2)(11.2\times10^6)(300)} = 3.3482 \text{ in}^4$$

$$\text{But} \quad J = \frac{\pi}{2}\left[\left(\frac{d_o}{2}\right)^4 - \left(\frac{d_i}{2}\right)^4\right] = \frac{\pi}{32}(d_o^4 - d_i^4)$$

$$d_i^4 = d_o^4 - \frac{32}{\pi}J = 2.5^4 - \frac{32}{\pi}(3.3482) = 4.95787 \text{ in}^4$$

$$d_i = 1.492 \text{ in.}$$

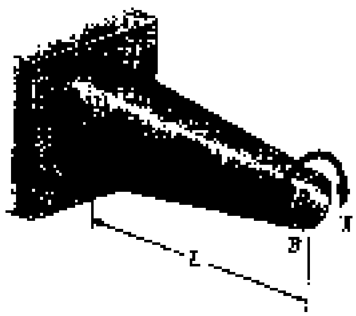
$$\tau = \frac{Tc_o}{J} = \frac{(25\times10^3)(1.25)}{3.3482} = 9.33\times10^3 \text{ psi} = 9.33 \text{ ksi}$$

PROBLEM 11.42

11.42 Show by integration that the strain energy in the tapered rod AB is

$$U = \frac{7}{48}\frac{T^2 L}{GJ_{min}}$$

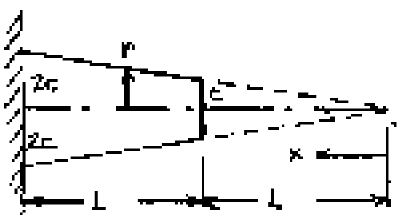
where J_{min} is the polar moment of inertia of the rod at end B.



SOLUTION

$$r = \frac{cx}{L}$$

$$J = \frac{\pi}{2}r^4 = \frac{\pi}{2}\frac{c^4}{L^4}x^4, \quad J_{min} = \frac{\pi}{2}c^4$$



$$U = \int_L^{2L}\frac{T^2\,dx}{2GJ} = \int_L^{2L}\frac{T^2\,dx}{2G\left(\frac{\pi}{2}\frac{c^4}{L^4}x^4\right)}$$

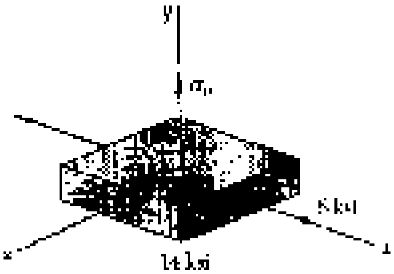
$$= \frac{T^2L^4}{2GJ_{min}}\int_L^{2L}\frac{dx}{x^4}$$

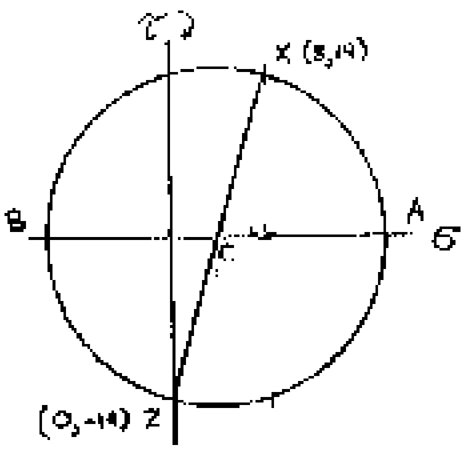
$$= \frac{T^2L^4}{2GJ_{min}}\left(-\frac{1}{3x^3}\right)\Bigg|_L^{2L}$$

$$U = \frac{T^2L^4}{2GJ_{min}}\left(-\frac{1}{3(2L)^3} + \frac{1}{3L^3}\right) = \frac{7}{48}\frac{T^2L}{GJ_{min}}$$

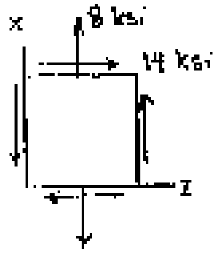
PROBLEM 11.43

11.43 The state of stress shown occurs in a machine component made of a grade of steel for which $\sigma_Y = 65$ ksi. Using the maximum-distortion-energy criterion, determine the range of values of σ_y for which the factor of safety associated with the yield strength is equal to or larger than 2.2.





SOLUTION



$$\sigma_{ave} = \frac{1}{2}(0+8) = 4 \text{ ksi}$$

$$\frac{\sigma_x - \sigma_z}{2} = \frac{8-0}{2} = 4 \text{ ksi}$$

$$\tau_{xz} = 14 \text{ ksi}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} = \sqrt{4^2 + 14^2} = 14.56 \text{ ksi}$$

$$\sigma_a = \sigma_{ave} + R = 18.56 \text{ ksi}$$

$$\sigma_b = \sigma_{ave} - R = -10.56 \text{ ksi}$$

$$\sigma_c = \sigma_y$$

$$(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2 = 2\left(\frac{\sigma_Y}{F.S.}\right)^2$$

$$(18.56 + 10.56)^2 + (-10.56 - \sigma_y)^2 + (\sigma_y - 18.56)^2 = 2\left(\frac{65}{2.2}\right)^2$$

$$847.97 + (111.51 + 21.12\sigma_y + \sigma_y^2) + (\sigma_y^2 - 37.12\sigma_y + 344.47) = 1745.87$$

$$2\sigma_y^2 - 16\sigma_y - 441.92 = 0$$

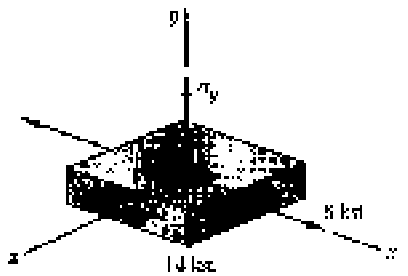
$$\sigma_y = \frac{16 \pm \sqrt{16^2 + (4)(2)(441.92)}}{(2)(2)} = 4 \pm 15.39$$

$$\sigma_y = 19.39 \text{ ksi}, \quad -11.39 \text{ ksi}$$

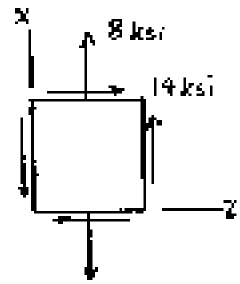
$$-11.39 \text{ ksi} \le \sigma_y \le 19.39 \text{ ksi}$$

PROBLEM 11.44

11.44 The state of stress shown occurs in a machine component made of a grade of steel for which $\sigma_Y = 65$ ksi. Using the maximum-distortion-energy criterion, determine the factor of safety associated with the yield strength when (*a*) $\sigma_y = +16$ ksi, (*b*) $\sigma_y = -16$ ksi.



SOLUTION



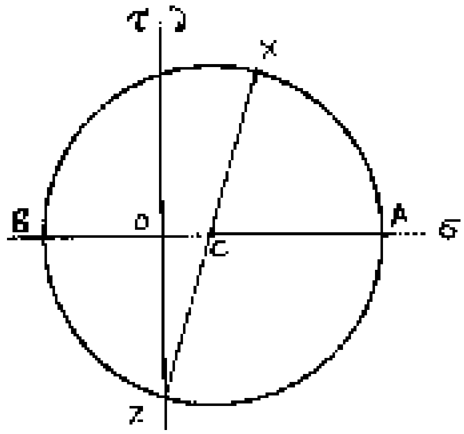
$\sigma_{ave} = \frac{1}{2}(0 + 8) = 4$ ksi

$\frac{\sigma_x - \sigma_z}{2} = \frac{8 - 0}{2} = 4$ ksi

$\tau_{xz} = 14$ ksi

$R = \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2}$

$= \sqrt{4^2 + 14^2} = 14.56$ ksi



$\sigma_a = \sigma_{ave} + R = 18.56$

$\sigma_b = \sigma_{ave} - R = -10.56$

$\sigma_c = \sigma_y$

$$(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2 = 2\left(\frac{\sigma_Y}{F.S.}\right)^2$$

(a) $\sigma_c = \sigma_y = 16$ ksi

$$(18.56 + 10.56)^2 + (-10.56 - 16)^2 + (16 - 18.56)^2 = 2\left(\frac{65}{F.S.}\right)^2$$

$$847.97 + 705.43 + 6.55 = \frac{8450}{(F.S.)^2} \qquad F.S. = 2.33$$

(b) $\sigma_c = \sigma_y = -16$ ksi

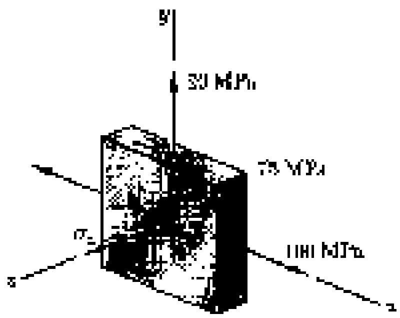
$$(18.56 + 10.56)^2 + (-10.56 + 16)^2 + (-16 - 18.56)^2 = 2\left(\frac{65}{F.S.}\right)^2$$

$$847.97 + 29.59 + 1194.39 = \frac{8450}{(F.S.)^2}$$

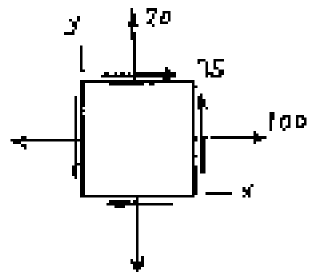
$F.S. = 2.02$

PROBLEM 11.45

11.45 The state of stress shown occurs in a machine component made of a brass for which $\sigma_Y = 160$ MPa. Using the maximum-distortion-energy criterion, determine whether yield occurs when (*a*) $\sigma_z = +45$ MPa, (*b*) $\sigma_z = -45$ MPa.



SOLUTION

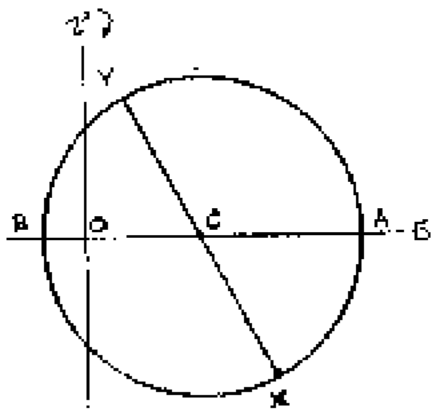


$$\sigma_{ave} = \tfrac{1}{2}(100 + 20) = 60 \text{ MPa}$$

$$\frac{\sigma_x - \sigma_y}{2} = \frac{100 - 20}{2} = 40 \text{ MPa}$$

$$\tau_{xy} = 75 \text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{40^2 + 75^2} = 85 \text{ MPa}$$



$$\sigma_a = \sigma_{ave} + R = 145 \text{ MPa}$$

$$\sigma_b = \sigma_{ave} - R = -25 \text{ MPa}$$

$$\sigma_c = \sigma_z$$

$$(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2 \overset{?}{\leq} 2\sigma_Y^2$$

(a) $\sigma_c = \sigma_z = +45$ MPa

$$(145 + 25)^2 + (-25 - 45)^2 + (45 - 145)^2 \overset{?}{<} 2(160)^2 = 51200$$

$$28900 + 4900 + 10000 = 43800 < 51200$$ (No yield)

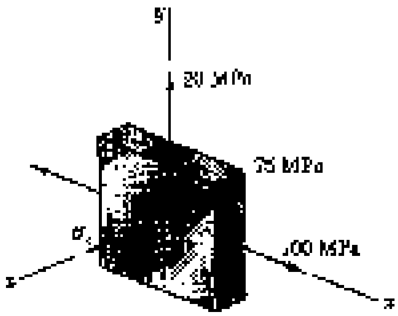
(b) $\sigma_c = \sigma_z = -45$ MPa

$$(145 + 25)^2 + (-25 + 45)^2 + (-45 - 145)^2 \overset{?}{<} 51200$$

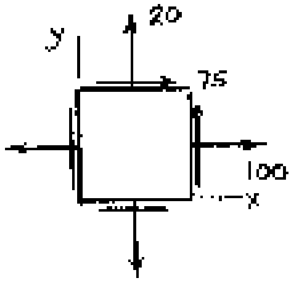
$$28900 + 400 + 36100 = 65400 > 51200$$ (Yield occurs)

PROBLEM 11.46

11.46 The state of stress shown occurs in a machine component made of a brass for which $\sigma_Y = 160$ MPa. Using the maximum-distortion-energy criterion, determine the range of values of σ_z for which yield does not occur.



SOLUTION

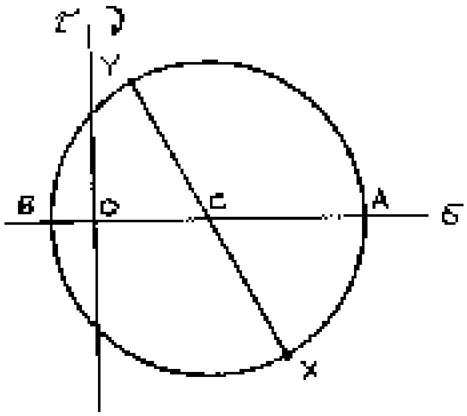


$$\sigma_{ave} = \tfrac{1}{2}(100+20) = 60 \text{ MPa}$$

$$\frac{\sigma_x - \sigma_y}{2} = \frac{100-20}{2} = 40 \text{ MPa}$$

$$\tau_{xy} = 75 \text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{40^2 + 75^2} = 85 \text{ MPa}$$



$$\sigma_a = \sigma_{ave} + R = 145 \text{ MPa}$$

$$\sigma_b = \sigma_{ave} - R = -25 \text{ MPa}$$

$$\sigma_c = \sigma_z$$

$$(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2 = 2\sigma_Y^2$$

$$(145+25)^2 + (-25-\sigma_z)^2 + (\sigma_z - 145)^2 = (2)(160)^2$$

$$28900 + (625 + 50\sigma_z + \sigma_z^2) + (\sigma_z^2 - 290\sigma_z + 21025 = 51200$$

$$2\sigma_z^2 - 240\sigma_z - 650 = 0$$

$$\sigma_z = \frac{240 \pm \sqrt{240^2 + (4)(2)(650)}}{(2)(2)} = 60 \pm 62.65$$

$$\sigma_z = 122.65 \text{ MPa}, \quad -2.65 \text{ MPa}$$

$$-2.65 \text{ MPa} < \sigma_z < 122.65 \text{ MPa}$$

PROBLEM 11.47

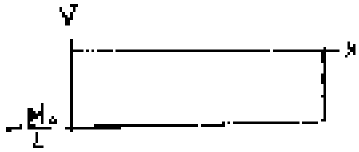
11.47 Determine the strain energy of the prismatic beam AB, taking into account the effect of both normal and shearing stresses.

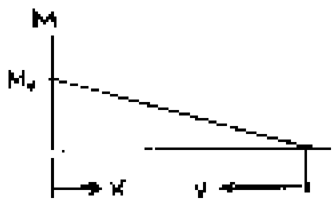
SOLUTION

Reactions $\quad R_A = \frac{M_0}{L}\downarrow, \quad R_B = \frac{M_0}{L}\uparrow$

Shear: $\quad V = -\frac{M_0}{L}$

Bending moment: $\quad M = \frac{M_0}{L}v$





For bending

$$U_1 = \int_0^L \frac{M^2}{2EI}dv = \frac{M_0^2}{2EIL^2}\int_0^L v^2\,dv = \frac{M_0^2 L^3}{6EIL^2}$$

$$= \frac{M_0^2 L}{6EI}$$

For shear

$$\tau_{xy} = \frac{3}{2}\frac{V}{A}\left(1-\frac{y^2}{c^2}\right) \qquad c = \frac{1}{2}d$$

$$u = \frac{\tau_{xy}^2}{2G} = \frac{9V^2}{8GA^2}\left(1-\frac{y^2}{c^2}\right)^2 = \frac{9M_0^2}{8G(bd)^2L^2}\left(1-2\frac{y^2}{c^2}+\frac{y^4}{c^4}\right)$$

$$U_2 = \int u\,dV = \int_0^L\int_{-c}^{c} u\, b\,dy\,dx = \frac{9M_0^2 b}{8Gb^2d^2L^2}\int_0^L\int_{-c}^{c}\left(1-2\frac{y^2}{c^2}+\frac{y^4}{c^4}\right)dy\,dx$$

$$= \frac{9M_0^2}{8Gbd^2L^2}\int_0^L\left(y-\frac{2}{3}\frac{y^3}{c^2}+\frac{1}{5}\frac{y^5}{c^4}\right)\Big|_{-c}^{c}dx = \frac{9M_0^2}{8Gbd^2L^2}\int_0^L\left(2c-\frac{4}{3}c+\frac{2}{5}c\right)dx$$

$$\frac{9M_0^2}{8Gbd^2L^2}\left(\frac{16}{15}c\right)L = \frac{6}{5}\frac{M_0^2 c}{Gbd^2L} = \frac{3}{5}\frac{M_0^2}{GbdL}$$

Total $\quad U = U_1 + U_2 = \frac{M_0^2 L}{6EI} + \frac{3}{5}\frac{M_0^2}{GbdL}$

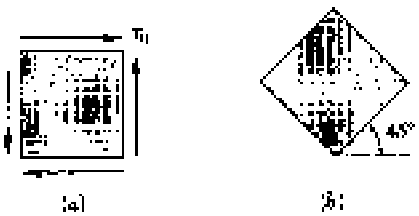
With $\quad I = \frac{1}{12}bd^3$

$$U = \frac{2M_0^2 L}{Ebd^3} + \frac{3}{5}\frac{M_0^2}{GbdL} = \frac{2M_0^2 L}{Ebd^3}\left\{1+\frac{3}{10}\cdot\frac{E}{G}\frac{d^2}{L^2}\right\}$$

PROBLEM 11.48

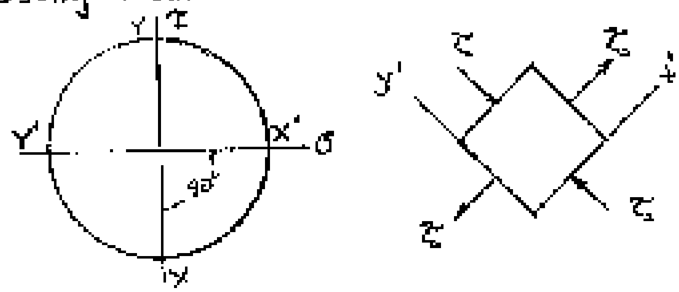
11.48 For the state of stress shown in Fig. *a*, determine the stresses in an element oriented as shown in Fig. *b*. Compare the strain energy density in the given state first by using Fig. *a* and then by using Fig. *b*. Equating the two results obtained, show that

$$G = \frac{E}{2(1+\nu)}$$



SOLUTION

Using Mohr's circle



(a) $\sigma_x = 0, \quad \sigma_y = 0, \quad \tau_{xy} = \tau_0$

$$u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y) + \frac{1}{2G}\tau_{xy}^2 = \frac{\tau_0^2}{2G}$$

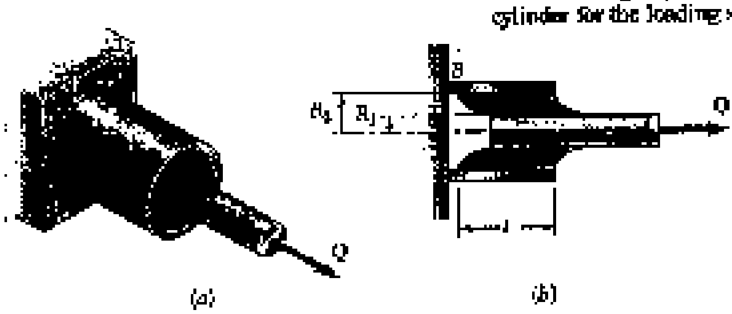
(b) $\sigma_{x'} = \tau_0, \quad \sigma_{y'} = -\tau_0, \quad \tau_{x'y'} = 0$

$$u = \frac{1}{2E}(\sigma_{x'}^2 + \sigma_{y'}^2 - 2\nu\sigma_{x'}\sigma_{y'}) + \frac{1}{2G}\tau_{x'y'}^2 = \frac{(2+2\nu)\tau_0^2}{2E}$$

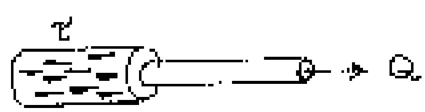
Equate $\quad \dfrac{\tau_0^2}{2G} = \dfrac{(2+2\nu)\tau_0^2}{2E} \qquad G = \dfrac{E}{2(1+\nu)}$

PROBLEM 11.49

***11.49** A vibration isolation support is made by bonding a rod *A*, of radius R_1, and a tube *B*, of inner radius R_2 to a hollow rubber cylinder. Denoting by *G* the modulus of rigidity of the rubber, determine the strain energy of the hollow rubber cylinder for the loading shown.



SOLUTION



$$+\!\rightarrow \Sigma F_x = 0 \qquad -\tau(2\pi r L) + Q = 0$$

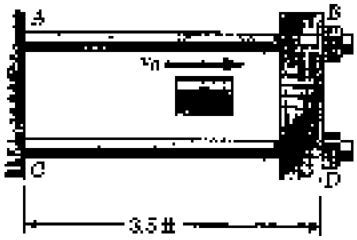
$$\tau = \frac{Q}{2\pi r L}$$

$$u = \frac{\tau^2}{2G} = \frac{Q^2}{8\pi^2 r^2 L^2 G}$$

$$U = \int u\, dV = \frac{Q^2}{8\pi^2 G L^2}\int \frac{dV}{r^2} = \frac{Q^2}{8\pi^2 G L^2}\int_0^L\int_{R_1}^{R_2} \frac{2\pi r\, dr}{r^2}\, dx$$

$$= \frac{Q^2}{4\pi G L^2}\int_0^L\int_{R_1}^{R_2}\frac{dr}{r}\, dx = \frac{Q^2}{4\pi G L^2}\int_0^L \left(\ln r \Big|_{R_1}^{R_2}\right) dx = \frac{Q^2}{4\pi G L}\ln\frac{R_2}{R_1}$$

PROBLEM 11.50



11.50 The cylindrical block E has a speed $v_0 = 16$ ft/s when it strikes squarely the yoke BD that is attached to the $\frac{7}{8}$-in.-diameter rods AB and CD. Knowing that the rods are made of a steel for which $\sigma_Y = 50$ ksi and $E = 29 \times 10^6$ psi, determine the weight of the block E for which the factor of safety is five with respect to permanent deformation of the rods.

SOLUTION

At the onset of yielding the force in each rod is

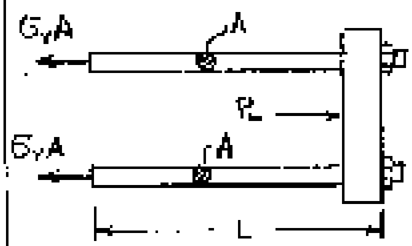
$$F = \sigma_Y A$$

Corresponding strain energy.

$$U_{AB} = \frac{F_{AB}^2 L_{AB}}{2 E A_{AB}} = \frac{\sigma_Y^2 A^2 L}{2E A} = \frac{\sigma_Y^2 A L}{2E}$$

$$U_{CD} = \text{same} = \frac{\sigma_Y^2 A L}{2E}$$

$$U_m = U_{AB} + U_{CD} = \frac{\sigma_Y^2 A L}{E}$$



$$U_m = \left(\tfrac{1}{2} m v_0^2\right)(F.S.) = \left(\tfrac{1}{2} \frac{W}{g} v_0^2\right)(F.S.)$$

Solving for W:
$$W = \frac{2g\,U_m}{v_0^2 (F.S.)} = \frac{2g\,\sigma_Y^2 A L}{v_0^2 (F.S.)\,E}$$

Data: $g = 32.17\ \text{ft/sec}^2 = 386\ \text{in/sec}^2$, $\sigma_Y = 50 \times 10^3$ psi,

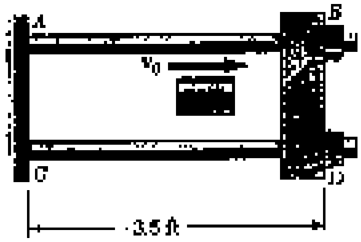
$A = \frac{\pi}{4} d^2 = \frac{\pi}{4}\left(\frac{7}{8}\right)^2 = 0.60132\ \text{in}^2$ $\qquad E = 29 \times 10^6$ psi

$L = 3.5\ \text{ft} = 42\ \text{in}$ $\qquad F.S. = 5$

$v_0 = 16\ \text{ft/sec} = 192\ \text{in/sec}$

$$W = \frac{(2)(386)(50\times10^3)^2(0.60132)(42)}{(192)^2(5)(29\times10^6)} = 9.12\ \text{lb.}$$

PROBLEM 11.51



11.51 The 18-lb cylindrical block A has a horizontal velocity v_0 when it strikes squarely the yoke BD that is attached to the $\frac{7}{8}$-in.-diameter rods AB and CD. Knowing that the rods are made of a steel for which $\sigma_Y = 50$ ksi and $E = 29 \times 10^6$ psi, determine the maximum allowable speed v_0 if the rods are not to be permanently deformed.

SOLUTION

At the onset of yielding the force in each rod is

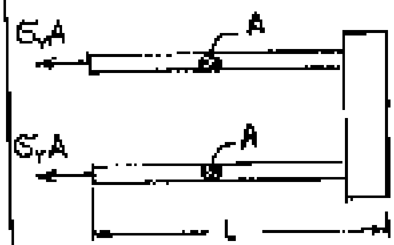
$$F = \sigma_Y A$$

Corresponding strain energy

$$U_{AB} = \frac{F_{AB}^2 L_{AB}}{2 E A_{AB}} = \frac{\sigma_Y^2 A^2 L}{2EA} = \frac{\sigma_Y^2 AL}{2E}$$

$$U_{CD} = \text{same} = \frac{\sigma_Y^2 AL}{2E}$$

$$\text{Total} \quad U_m = U_{AB} + U_{CD} = \frac{\sigma_Y^2 AL}{E}$$



$$U_m = \frac{1}{2} m v_0^2 = \frac{1}{2}\frac{W}{g} v_0^2$$

Solving for v_0^2 $\qquad v_0^2 = \frac{2g U_m}{W} = \frac{2g\sigma_Y^2 AL}{EW}$

$$v_0 = \sqrt{\frac{2g\sigma_Y^2 AL}{EW}}$$

Data: $\quad g = 32.17 \text{ ft/sec}^2 = 386 \text{ in/sec}^2, \qquad \sigma_Y = 50 \times 10^3 \text{ psi}$

$A = \frac{\pi}{4} d^2 = \frac{\pi}{4}\left(\frac{7}{8}\right)^2 = 0.60132 \text{ in}^2, \qquad E = 29 \times 10^6 \text{ psi}$

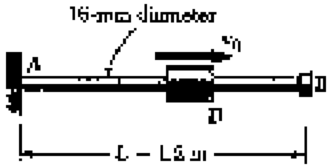
$L = 3.5 \text{ ft} = 42 \text{ in} \qquad W = 18 \text{ lb.}$

$$v_0 = \sqrt{\frac{(2)(386)(50\times10^3)^2(0.60132)(42)}{(29\times10^6)(18)}} = 305.6 \text{ in/sec}$$

$$= 25.5 \text{ ft/sec}$$

PROBLEM 11.52

11.52 The uniform rod AB is made of a brass for which $\sigma_Y = 125$ MPa and $E = 105$ GPa. Collar D moves along the rod and has a speed $v_0 = 3$ m/s as it strikes a small plate attached to end B of the rod. Using a factor of safety of four, determine the largest allowable mass of the collar if the rod is not to be permanently deformed.



SOLUTION

At onset of yielding $P_m = \sigma_Y A$

$\sigma_Y = 125 \times 10^6$ Pa

$A = \frac{\pi}{4}(16)^2 = 201.06$ mm^2 $= 201.06 \times 10^{-6}$ m^2

$P_m = 25133$ N

Corresponding strain energy $U_m = \dfrac{P_m^2 L}{2EA} = \dfrac{(25133)^2(1.2)}{(2)(105 \times 10^9)(201.06 \times 10^{-6})}$

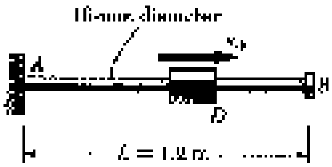
$= 17.953$ J

Kinetic energy times safety factor $= \frac{1}{2} m v_0^2 (F.S.) = 2 m v_0^2$

$2 m v_0^2 = U_m$, $\quad m = \dfrac{U_m}{2 v_0^2} = \dfrac{17.953}{(2)(3)^2} = 0.997$ kg ◀

PROBLEM 11.53

11.52 The uniform rod AB is made of a brass for which $\sigma_Y = 125$ MPa and $E = 105$ GPa. Collar D moves along the rod and has a speed $v_0 = 3$ m/s as it strikes a small plate attached to end B of the rod. Using a factor of safety of four, determine the largest allowable mass of the collar if the rod is not to be permanently deformed.



11.53 Solve Prob. 11.52, assuming that the length of the brass rod is increased from 1.2 m to 2.4 m.

SOLUTION

At onset of yielding $P_m = \sigma_Y A$ $\qquad \sigma_Y = 125 \times 10^6$ Pa

$A = \frac{\pi}{4} d^2 = \frac{\pi}{4}(16) = 201.06$ mm^2 $= 201.06 \times 10^{-6}$ m^2

$P_m = 25133$ N

Corresponding strain energy $U_m = \dfrac{P_m^2 L}{2EA} = \dfrac{(25133)^2 (2.4)}{(2)(105 \times 10^9)(201.06 \times 10^{-6})}$

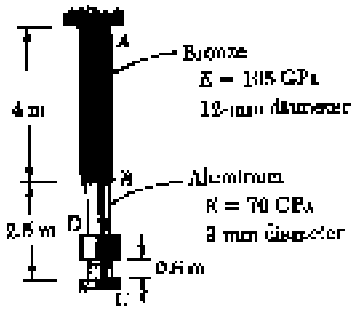
$= 35.906$ J

Kinetic energy time safety factor $= \frac{1}{2} m v_0^2 (4) = 2 m v_0^2$

$2 m v_0^2 = U_m$ $\qquad m = \dfrac{U_m}{2 v_0^2} = \dfrac{35.906}{(2)(3)^2} = 1.995$ kg ◀

PROBLEM 11.54

11.54 Collar D is released from rest in the position shown and is stopped by a small plate attached at end C of the vertical rod ABC. Determine the mass of the collar for which the maximum normal stress in portion BC is 125 MPa.



SOLUTION

Portion BC: $\sigma_m = 125 \times 10^6$ Pa

$A_{BC} = \frac{\pi}{4}(9)^2 = 63.617 \text{ mm}^2 = 63.617 \times 10^{-6} \text{ m}^2$

$P_m = \sigma_m A_{BC} = 7952$ N

Corresponding strain energy

$$U_{BC} = \frac{P_m^2 L_{BC}}{2E_{BC} A_{BC}} = \frac{(7952)^2 (2.5)}{(2)(70\times10^9)(63.617\times10^{-6})} = 17.750 \text{ J}$$

$$A_{AB} = \frac{\pi}{4}(12)^2 = 113.907 \text{ mm}^2 = 113.907 \times 10^{-6} \text{ m}^2$$

$$U_{AB} = \frac{P_m^2 L_{AB}}{2E_{AB} A_{AB}} = \frac{(7952)^2 (4)}{(2)(105\times10^9)(113.907\times10^{-6})} = 10.574 \text{ J}$$

$$U_m = U_{BC} + U_{AB} = 28.324 \text{ J}$$

Corresponding elongation Δ_m $\qquad \frac{1}{2} P_m \Delta_m = U_m$

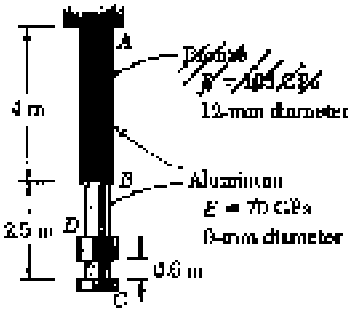
$$\Delta_m = \frac{2U_m}{P_m} = \frac{(2)(28.324)}{7952} = 7.12 \times 10^{-3} \text{ m}$$

Falling distance $\quad h = 0.6 + 7.12\times10^{-3} = 0.60712$ m.

Work of weight $= U_m \qquad Wh = mgh = U_m$

$$m = \frac{U_m}{gh} = \frac{28.324}{(9.81)(0.60712)} = 4.76 \text{ kg}$$

PROBLEM 11.55



11.54 Collar D is released from rest in the position shown and is stopped by a small plate attached at end C of the vertical rod ABC. Determine the mass of the collar for which the maximum normal stress in portion BC is 125 MPa.

11.55 Solve Prob. 11.54, assuming that both portions of rod ABC are made of aluminum.

SOLUTION

Portion BC: $\sigma_m = 125 \times 10^6$ Pa

$$A_{BC} = \frac{\pi}{4}(9)^2 = 63.617 \text{ mm}^2 = 63.617 \times 10^{-6} \text{ m}^2$$

$$P_m = \sigma_m A_{BC} = 7952 \text{ N}$$

Corresponding strain energy

$$U_{BC} = \frac{P_m^2 L_{BC}}{2EA_{BC}} = \frac{(7952)^2(2.5)}{(2)(70 \times 10^9)(63.617 \times 10^{-6})} = 17.750 \text{ J}$$

$$A_{AB} = \frac{\pi}{4}(12)^2 = 113.907 \text{ mm}^2 = 113.907 \times 10^{-6} \text{ m}^2$$

$$U_{AB} = \frac{P_m^2 L_{AB}}{2EA_{AB}} = \frac{(7952)^2(4)}{(2)(70 \times 10^9)(113.907 \times 10^{-6})} = 15.861 \text{ J}$$

Total $\quad U_m = U_{BC} + U_{AB} = 33.611 \text{ J}$

Corresponding elongation $\Delta_m \qquad \frac{1}{2}P_m\Delta_m = U_m$

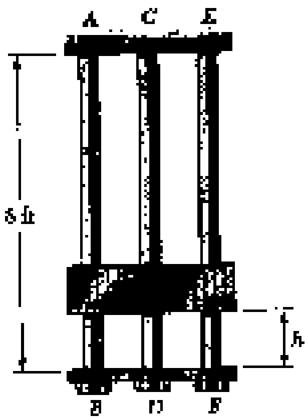
$$\Delta_m = \frac{2U_m}{P_m} = \frac{(2)(33.611)}{7952} = 8.45 \times 10^{-3} \text{ m}$$

Falling distance $\qquad h = 0.6 + \Delta_m = 0.60845 \text{ m}$

Work of weight $= U_m \qquad Wh = mgh = U_m$

$$m = \frac{U_m}{gh} = \frac{33.611}{(9.81)(0.60845)} = 5.63 \text{ kg}$$

PROBLEM 11.56



11.56 The 100-lb collar G is released from rest in the position shown and is stopped by plate BDF that is attached to the $\frac{7}{8}$-in.-diameter steel rod CD and to the $\frac{5}{8}$-in.-diameter steel rods AB and EF. Knowing that for the grade of steel used $\sigma_{all} = 24$ ksi and $E = 29 \times 10^6$ psi, determine the largest allowable distance h.

SOLUTION

Let Δ_m be the elongation

$$\Delta_m = \frac{\sigma_{AB} L}{E} = \frac{\sigma_{CD} L}{E} = \frac{\sigma_{EF} L}{E}$$

$$\sigma_{AB} = \sigma_{CD} = \sigma_{EF} = 24 \times 10^3 \text{ psi}$$

$$L = 8 \text{ ft} = 96 \text{ in}$$

$$\Delta_m = \frac{(24 \times 10^3)(96)}{29 \times 10^6} = 79.448 \times 10^{-3} \text{ in.}$$

For each rod $U = \dfrac{F_m^2 L}{2EA} = \dfrac{(EA\Delta_m/L)^2 L}{2EA} = \dfrac{EA\Delta_m^2}{2L}$

Rod CD: $A_{CD} = \frac{\pi}{4}\left(\frac{7}{8}\right)^2 = 0.60132 \text{ in}^2$

$$U_{CD} = \frac{(29 \times 10^6)(0.60132)(79.448 \times 10^{-3})^2}{(2)(96)} = 573.28 \text{ in}\cdot\text{lb}$$

Rods AB and EF: $A_{AB} = A_{EF} = \frac{\pi}{4}\left(\frac{5}{8}\right)^2 = 0.30680 \text{ in}^2$

$$U_{AB} = U_{EF} = \frac{(29 \times 10^6)(0.30680)(79.448 \times 10^{-3})^2}{(2)(96)} = 292.49 \text{ in}\cdot\text{lb}$$

Total $U_m = U_{AB} + U_{CD} + U_{EF} = 1158.27 \text{ in}\cdot\text{lb}$

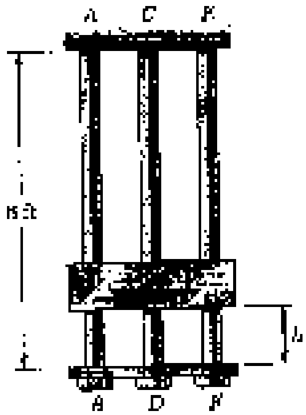
Falling distance is $h + \Delta_m$, $\quad W = 100 \text{ lb}$

$$W(h + \Delta_m) = U_m$$

$$h + \Delta_m = \frac{U_m}{W} = \frac{1158.27}{100} = 11.583 \text{ in.}$$

$$h = 11.583 - 79.448 \times 10^{-3} = 11.50 \text{ in.}$$

PROBLEM 11.57



11.56 The 100-lb collar G is released from rest in the position shown and is stopped by plate BDF that is attached to the $\frac{7}{8}$-in.-diameter steel rod CD and to the $\frac{5}{8}$-in.-diameter steel rods AB and EF. Knowing that for the grade of steel used $\sigma_{all} = 24$ ksi and $E = 29 \times 10^6$ psi, determine the largest allowable distance h.

11.57 Solve Prob. 11.56, assuming that the $\frac{7}{8}$-in.-diameter steel rod CD is replaced by a $\frac{7}{8}$-in.-diameter rod made of a grade of aluminum for which $\sigma_{all} = 20$ ksi and $E = 10.6 \times 10^6$ psi.

SOLUTION

Let Δ_m be the elongation. $\quad L = 8\text{ ft} = 96\text{ in}$

$$\Delta_m = \frac{\sigma_{AB} L}{E_{AB}} = \frac{\sigma_{CD} L}{E_{CD}} = \frac{\sigma_{EF} L}{E_{EF}}$$

If $\sigma_{AB} = 24 \times 10^3$ psi, $\quad \Delta_m = \dfrac{(24 \times 10^3)(96)}{29 \times 10^6} = 79.448 \times 10^{-3}$ in.

If $\sigma_{CD} = 20 \times 10^3$ psi, $\quad \Delta_m = \dfrac{(20 \times 10^3)(96)}{10.6 \times 10^6} = 181.13 \times 10^{-3}$ in.

Smaller value governs $\quad \Delta_m = 79.448 \times 10^{-3}$ in

For each rod $\quad U = \dfrac{F^2 L}{2EA} = \dfrac{(EA\Delta_m/L)^2 L}{2EA} = \dfrac{EA\,\Delta_m^2}{2L}$

Rod CD: $\quad A_{CD} = \frac{\pi}{4}\left(\frac{7}{8}\right)^2 = 0.60132$ in^2, $\quad E_{CD} = 10.6 \times 10^6$ psi

$$U_{CD} = \frac{(10.6 \times 10^6)(0.60132)(79.448 \times 10^{-3})^2}{(2)(96)} = 209.54\text{ in-lb}$$

Rods AB and EF: $\quad A_{AB} = A_{EF} = \frac{\pi}{4}\left(\frac{5}{8}\right)^2 = 0.30680$ in^2

$$U_{AB} = U_{EF} = \frac{(29 \times 10^6)(0.30680)(79.448 \times 10^{-3})^2}{(2)(96)} = 292.49\text{ in-lb}$$

Total $\quad U_m = U_{AB} + U_{CD} + U_{EF} = 794.52$ in-lb.

Falling distance is $h + \Delta_m$ $\qquad W = 100$ lb.

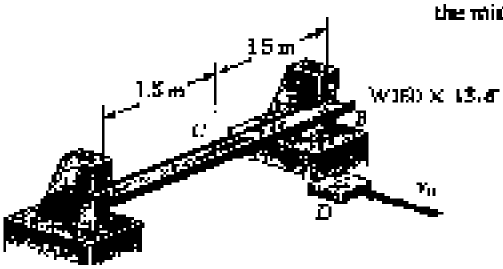
$$W(h + \Delta_m) = U_m$$

$$h + \Delta_m = \frac{U_m}{W} = \frac{794.52}{100} = 7.9452\text{ in}$$

$$h = 7.9452 - 79.448 \times 10^{-3} = 7.87\text{ in.}$$

PROBLEM 11.58

11.58 The steel beam AB is struck squarely at its midpoint C by a 45-kg block moving horizontally with a speed $v_0 = 2$ m/s. Using $E = 200$ GPa, determine (*a*) the equivalent static load, (*b*) the maximum normal stress in the beam, (*c*) the maximum deflection of the midpoint C of the beam.



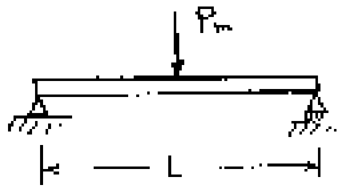
SOLUTION

From Appendix C, for W 150 × 13.5

$I_x = 6.87 \times 10^6 \text{ mm}^4 = 6.87 \times 10^{-6} \text{ m}^4$

$S_x = 91.6 \times 10^3 \text{ mm}^3 = 91.6 \times 10^{-6} \text{ m}^3$

Kinetic energy $T = \frac{1}{2} m v_0^2 = \frac{1}{2}(45)(2)^2 = 90 \text{ J}$



From Appendix D, Case 4

$|y_m| = \dfrac{PL^3}{48EI}, \qquad M_{max} = \dfrac{PL}{4}$

$$U = \frac{1}{2} P_m |y_m| = \frac{P_m^2 L^3}{96EI} = T$$

(a) $$P_m = \sqrt{\frac{96EIT}{L^3}} = \sqrt{\frac{(96)(200 \times 10^9)(6.87 \times 10^{-6})(90)}{(3.0)^3}} = 20.968 \times 10^3 \text{ N}$$
$$= 21.0 \text{ kN}$$

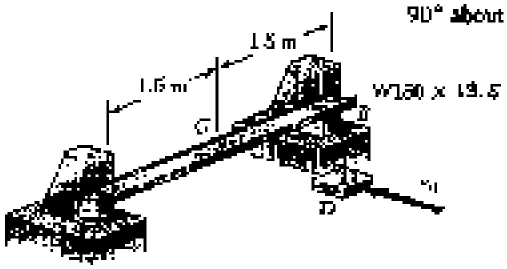
(b) $$\sigma_m = \frac{M_{max}}{S} = \frac{P_m L}{4S} = \frac{(20.968 \times 10^3)(3.0)}{(4)(91.6 \times 10^{-6})} = 171.7 \times 10^6 \text{ Pa}$$
$$= 171.7 \text{ MPa}$$

(c) $$|y_m| = \frac{2U}{P_m} = \frac{(2)(90)}{20.968 \times 10^3} = 8.58 \times 10^{-3} \text{ m} = 8.58 \text{ mm}$$

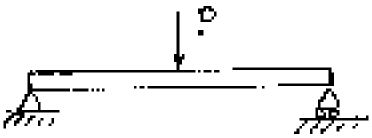
PROBLEM 11.59

11.58 The steel beam AB is struck squarely at its midpoint C by a 45-kg block moving horizontally with a speed $v_0 = 2$ m/s. Using $E = 200$ GPa, determine (*a*) the equivalent static load, (*b*) the maximum normal stress in the beam, (*c*) the maximum deflection of the midpoint C of the beam.

11.59 Solve Prob. 11.58, assuming that the W150 × 13.5 rolled-steel beam is rotated by 90° about its longitudinal axis so that its web is vertical.



SOLUTION



From Appendix C, for W 150 × 13.5

$I_y = 0.918 \times 10^6 \text{ mm}^4 = 0.918 \times 10^{-6} \text{ m}^4$

$S_y = 18.4 \times 10^3 \text{ mm}^3 = 18.4 \times 10^{-6} \text{ m}^3$

Kinetic energy $T = \frac{1}{2} m v_0^2 = \frac{1}{2}(45)(2)^2 = 90 \text{ J}$

From Appendix D, Case 4

$|y_m| = \dfrac{PL^3}{48EI} \qquad M_{max} = \dfrac{PL}{4}$

$$U = \frac{1}{2} P_m |y_m| = \frac{P_m^2 L^3}{96 EI} = T$$

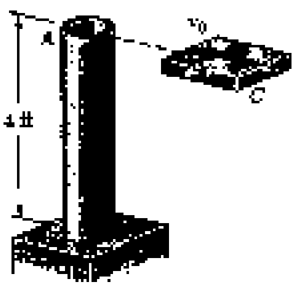
$$\text{(a)} \quad P_m = \sqrt{\frac{96 EIT}{L^3}} = \sqrt{\frac{(96)(200 \times 10^9)(0.918 \times 10^{-6})(90)}{(3.0)^3}} = 7.665 \times 10^3 \text{ N} = 7.67 \text{ kN}$$

$$\text{(b)} \quad \sigma_m = \frac{M_{max}}{S} = \frac{P_m L}{4S} = \frac{(7.665 \times 10^3)(3.0)}{(4)(18.4 \times 10^{-6})} = 312 \times 10^6 \text{ Pa} = 312 \text{ MPa}$$

$$\text{(c)} \quad |y_m| = \frac{2U}{P_m} = \frac{(2)(90)}{7.665 \times 10^3} = 23.5 \times 10^{-3} \text{ m} = 23.5 \text{ mm}$$

PROBLEM 11.60

11.60 The post AB consists of a steel pipe of 3.5-in outer diameter and 0.3-in. wall thickness. A 15-lb block C moving horizontally with a velocity v_0 hits the post squarely at A. Using $E = 29 \times 10^6$ psi, determine the largest speed v_0 for which the maximum normal stress in the pipe does not exceed 24 ksi.



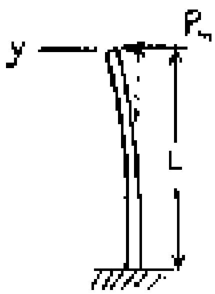
SOLUTION

$$c_o = \tfrac{1}{2} d_o = \tfrac{1}{2}(3.5) = 1.75 \text{ in}, \quad c_i = c_o - t = 1.75 - 0.3 = 1.45 \text{ in}$$

$$I = \frac{\pi}{4}(c_o^4 - c_i^4) = 3.8943 \text{ in}^4 \qquad \sigma_m = 24000 \text{ psi}$$

$$\sigma_m = \frac{M_m c}{I}, \quad M_m = \frac{I\sigma_m}{c} = \frac{(3.8943)(24000)}{1.75} = 53407 \text{ lb}\cdot\text{in}$$

$$P_m = \frac{M_m}{L} = \frac{53407}{48} = 1112.66 \text{ lb.}$$



By Appendix D, Case 1

$$y_m = \frac{P_m L^3}{3EI} = \frac{(1112.66)(48)^3}{(3)(29\times10^6)(3.8943)} = 0.36319 \text{ in}$$

$$U_m = \tfrac{1}{2} P_m y_m = \tfrac{1}{2}(1112.66)(0.36319) = 202.05 \text{ in}\cdot\text{lb.}$$

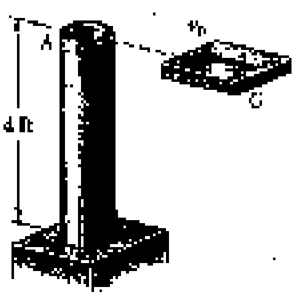
$$\frac{1}{2}\frac{W}{g} v_0^2 = U_m \qquad v_0^2 = \frac{2gU_m}{W} = \frac{(2)(386)(202.05)}{15} = 10399 \text{ in}^2/\text{sec}^2$$

$$v_0 = 102.0 \text{ in/sec} = 8.50 \text{ ft/sec}$$

PROBLEM 11.61

11.60 The post AB consists of a steel pipe of 3.5-in outer diameter and 0.3-in. wall thickness. A 15-lb block C moving horizontally with a velocity v_0 hits the post squarely at A. Using $E = 29 \times 10^6$ psi, determine the largest speed v_0 for which the maximum normal stress in the pipe does not exceed 24 ksi.

11.61 Solve Prob 11.60, assuming that the post AB consists of a solid steel rod of 3.5-in outer diameter.



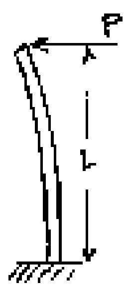
SOLUTION

$$c = \tfrac{1}{2} d = 1.75 \text{ in} \qquad I = \frac{\pi}{4} c^4 = 7.3662 \text{ in}^4$$

$$\sigma_m = 24000 \text{ psi} \qquad L = 4 \text{ ft} = 48 \text{ in}$$

$$\sigma_m = \frac{M_m c}{I}, \quad M_m = \frac{I\sigma_m}{c} = \frac{(7.3662)(24000)}{1.75} = 101022 \text{ lb}\cdot\text{in}$$

$$P_m = \frac{M_m}{L} = 2104.6 \text{ lb.}$$



By Appendix D, Case 1

$$y_m = \frac{P_m L^3}{3EI} = \frac{(2104.6)(48)^3}{(3)(29\times10^6)(7.3662)} = 0.36319 \text{ in}$$

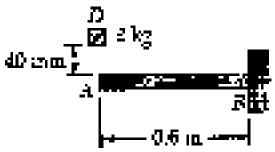
$$U_m = \tfrac{1}{2} P_m y_m = \tfrac{1}{2}(2104.6)(0.36319) = 382.19 \text{ in}\cdot\text{lb.}$$

$$\frac{1}{2}\frac{W}{g} v_0^2 = U_m, \quad v_0^2 = \frac{2gU_m}{W} = \frac{(2)(386)(382.19)}{15} = 19670 \text{ in}^2/\text{sec}^2$$

$$v_0 = 140.25 \text{ in/sec} = 11.69 \text{ ft/sec}$$

PROBLEM 11.62

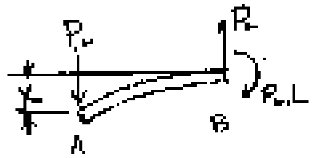
11.62 The 2-kg block D is dropped from the position shown onto the end of a 16-mm-diameter rod. Knowing that $E = 200$ GPa, determine (*a*) the maximum deflection of end A, (*b*) the maximum bending moment in the rod, (*c*) the maximum normal stress in the rod.



SOLUTION

$$I = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi}{4}\left(\frac{16}{2}\right)^4 = 3.2170\times10^3\ \text{mm}^4 = 3.2170\times10^{-9}\ \text{m}^4$$

$$c = \frac{d}{2} = 8\ \text{mm} = 8\times10^{-3}\ \text{m} \qquad L_{AB} = 0.6\ \text{m}$$



Appendix D, Case 1

$$y_m = \frac{P_m L_{AB}^3}{3EI} \qquad M_m = P_m L_{AB}$$

$$P_m = \frac{3EI}{L_{AB}^3}y_m = \frac{(3)(200\times10^9)(3.217\times10^{-9})}{(0.6)^3} = 8.9361\times10^3\ y_m$$

$$U_m = \frac{1}{2}P_m y_m = \frac{1}{2}(8.9361\times10^3)y_m^2 = 4.4681\times10^3\ y_m^2$$

Work of dropped weight $\quad mg(h+y_m) = (2)(9.81)(0.040+y_m)$

$$= 0.7848 + 19.62\ y_m$$

Equating work and energy

$$0.7848 + 19.62\ y_m = 4.4681\times10^3\ y_m^2$$

$$y_m^2 - 4.3911\times10^{-3}\ y_m - 175.645\times10^{-6} = 0$$

(a) $$y_m = \frac{1}{2}\left\{4.3911\times10^{-3} + \sqrt{(4.3911\times10^{-3})^2 + (4)(175.645\times10^{-6})}\right\}$$

$$= 15.629\times10^{-3}\ \text{m} = 15.63\ \text{mm}$$

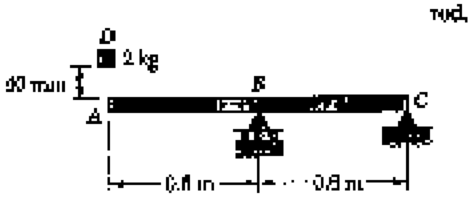
$$P_m = (8.9361\times10^3)(15.629\times10^{-3}) = 139.66\ \text{N}$$

(b) $$M_m = -P_m L_{AB} = -(139.66)(0.6) = -83.8\ \text{N}\cdot\text{m}$$

(c) $$\sigma_m = \frac{|M_m|c}{I} = \frac{(83.8)(8\times10^{-3})}{3.2170\times10^{-9}} = 208\times10^6\ \text{Pa} = 208\ \text{MPa}$$

PROBLEM 11.63

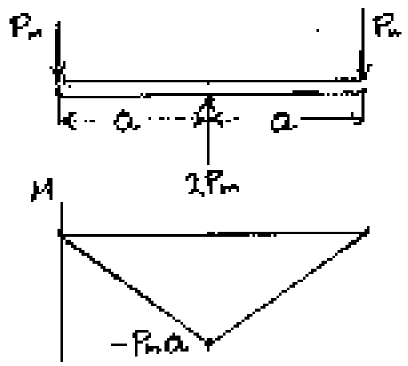
11.63 The 2-kg block D is dropped from the position shown onto the end of a 16-mm-diameter rod. Knowing that $E = 200$ GPa, determine (a) the maximum deflection of end A, (b) the maximum bending moment in the rod, (c) the maximum normal stress in the rod.



SOLUTION

$$I = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi}{4}\left(\frac{16}{2}\right)^4 = 3.2170\times10^3\ \text{mm}^4 = 3.2170\times10^{-9}\ \text{m}^4$$

$$c = \frac{d}{2} = 8\ \text{mm} = 8\times10^{-3}\ \text{m} \qquad a = 0.6\ \text{m}$$



Over AB $\qquad M = -P_m x \qquad M_m = -P_m a$

$$U_{AB} = \int_0^a \frac{P_m^2 x^2}{2EI}\,dx = \frac{P_m^2 a^3}{6EI}$$

$$= \frac{(0.6)^3}{(6)(200\times10^9)(3.2170\times10^{-9})} P_m^2$$

$$= 55.953\times10^{-6}\ P_m^2$$

By symmetry of bending moment diagram

$$U_{BC} = U_{AB} = 55.953\times10^{-6}\ P_m^2$$

$$U_m = U_{AB} + U_{BC} = 111.906\times10^{-6}\ P_m^2$$

$$\frac{1}{2}P_m y_m = U_m = 111.906\times10^{-6}\ P_m^2 \qquad P_m = 4.4681\times10^3\ y_m$$

$$U_m = \frac{1}{2}P_m y_m = 2.2340\times10^3\ y_m^2$$

Work of dropped weight $\qquad mg(h+y_m) = (2)(9.81)(0.040 + y_m)$

$$= 0.7848 + 19.62\ y_m$$

Equating work and energy

$$0.7848 + 19.62\ y_m = 2.2340\times10^3\ y_m^2$$

$$y_m^2 - 8.7825\times10^{-3}\ y_m - 351.298\times10^{-6} = 0$$

(a) $\quad y_m = \frac{1}{2}\left\{8.7825\times10^{-3} + \sqrt{(8.7282\times10^{-3})^2 + (4)(351.298\times10^{-6})}\right\}$

$$= 23.636\times10^{-3}\ \text{m} = 23.6\ \text{mm}$$

$$P_m = (4.4681\times10^3)(23.636\times10^{-3}) = 105.61\ \text{N}$$

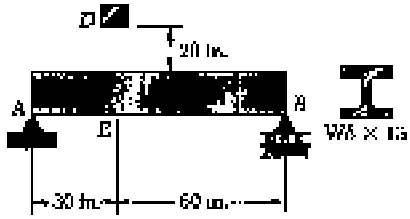
(b) $\quad M_m = -(105.61)(0.6) = -64.4\ \text{N}\cdot\text{m}$

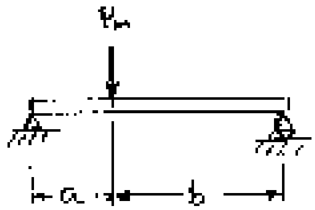
(c) $\quad \sigma_m = \frac{|M_m| c}{I} = \frac{(64.4)(8\times10^{-3})}{3.2170\times10^{-9}} = 157.6\times10^6\ \text{Pa}$

$$= 157.6\ \text{MPa}$$

PROBLEM 11.64

11.64 The 50-lb block D is dropped from a height of 20 in. onto the steel beam AB. Knowing that $E = 29 \times 10^6$ psi, determine (a) the maximum deflection at point E, (b) the maximum normal stress in the beam.





SOLUTION

$I_x = 39.6 \text{ in}^4, \quad S_x = 9.91 \text{ in}^3$

Appendix D, Case 5

$$y_E = \frac{P_m a^2 b^2}{3EIL} = \frac{(30)^2(60)^2 P_m}{(3)(29\times10^6)(39.6)(90)}$$

$$= 10.4493\times10^{-6}\, P_m$$

$$P_m = 95700\, y_E$$

$$U_m = \tfrac{1}{2} P_m y_E = 47850\, y_E^2$$

Work of falling weight $\qquad W(h + y_E) = 50(20 + y_E) = 1000 + 50 y_E$

Equating work and energy: $\qquad 1000 + 50 y_E = 47850\, y_E^2$

$$y_E^2 - 1.04493\times10^{-3} - 20.899\times10^{-3} = 0$$

(a) $\quad y_E = \frac{1}{2}\left\{ 1.04493\times10^{-3} + \sqrt{(1.04493\times10^{-3})^2 + (4)(20.899\times10^{-3})} \right\}$

$$= 0.1451 \text{ in}$$

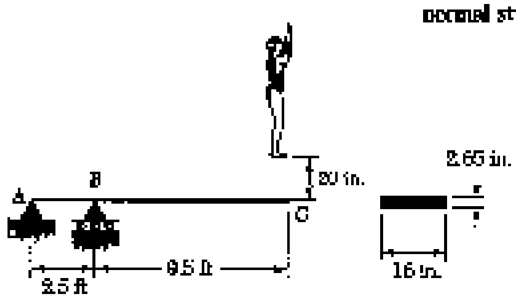
$$P_m = (95700)(0.1451) = 13885 \text{ lb.}$$

$$M_m = \frac{P_m ab}{L} = \frac{(13885)(30)(60)}{90} = 277.7\times10^3 \text{ lb-in.}$$

(b) $\quad \sigma_m = \frac{M_m}{S_x} = \frac{277.7\times10^3}{9.91} = 28.0\times10^3 \text{ psi} = 28.0 \text{ ksi}$

PROBLEM 11.65

11.65 A 160-lb diver jumps from a height of 20 in. onto end C of a diving board having the uniform cross section shown. Assuming that the diver's legs remain rigid and using $E = 1.8 \times 10^6$ psi, determine (*a*) the maximum deflection at point C, (*b*) the maximum normal stress in the board, (*c*) the equivalent static load.

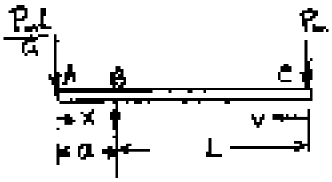


SOLUTION

$$I = \tfrac{1}{12}(16)(2.65)^3 = 24.813 \text{ in}^4$$

$$L = 9.5 \text{ ft} = 114 \text{ in}, \quad a = 2.5 \text{ ft} = 30 \text{ in}$$

$$c = \tfrac{1}{2}(2.65) = 1.325 \text{ in}$$

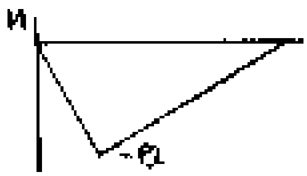


Over portion AB $\quad M = -\frac{P_m L}{a} x$

$$U_{AB} = \int_0^a \frac{M^2}{2EI}dx = \frac{P_m^2 L^2}{2EIa^2}\int_0^a x^2\,dx = \frac{P_m^2 L^2 a}{6EI}$$

Over portion BC $\quad M = -P_m v$

$$U_{BC} = \int_0^L \frac{M^2}{2EI}dv = \frac{P_m^2}{2EI}\int_0^L v^2\,dv = \frac{P_m^2 L^3}{6EI}$$



Total $\quad U = U_{AB} + U_{BC} = \frac{P_m^2 L^2 (a+L)}{6EI}$

$$\tfrac{1}{2}P_m y_m = U_m \qquad y_m = \frac{2U_m}{P_m} = \frac{P_m L^2(a+L)}{3EI}$$

$$P_m = \frac{3EI}{L^2(a+L)} y_m = \frac{(3)(1.8\times10^6)(24.813)}{(114)^2(114+30)} y_m = 71.598\, y_m$$

$$U_m = \tfrac{1}{2}P_m y_m = 35.799\, y_m^2$$

Work of weight $= W(h+y_m) = (160)(20+y_m) = 3200 + 160\,y_m$

Equating $\quad 3200 + 160\,y_m = 35.799\,y_m^2$

$$y_m^2 - 4.4694\,y_m - 89.388 = 0$$

(a) $\quad y_m = \frac{1}{2}\left\{4.4694 + \sqrt{4.4694^2 + (4)(89.388)}\right\} = 11.95 \text{ in}$

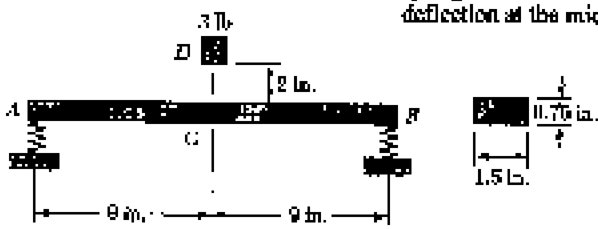
(c) $\quad P_m = (71.598)(11.95) = 856 \text{ lb}$

$$M_m = -(856)(114) = 97535 \text{ lb}\cdot\text{in}$$

(b) $\quad \sigma_m = \frac{|M_m|c}{I} = \frac{(97535)(1.325)}{24.813} = 5210 \text{ psi}$

PROBLEM 11.66

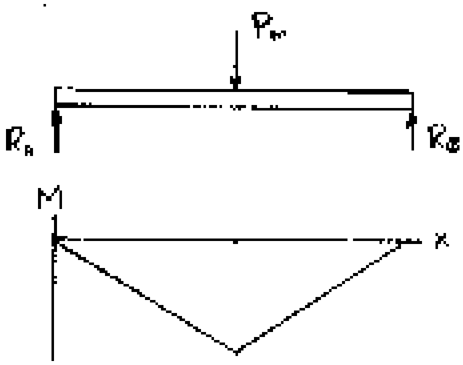
11.66 The 3-lb block D is released from rest in the position shown and strikes a steel bar AB having the uniform cross section shown. The bar is supported at each end by springs of constant 20 kips/in. Using $E = 29 \times 10^6$ psi, determine the maximum deflection at the midpoint of the bar.



SOLUTION

$k = 20$ kips/in $= 20 \times 10^3$ lb/in

$R_A = R_B = \frac{1}{2} P_m$



For spring A, $U_A = \frac{1}{2} R_A y_A = \frac{1}{2} \frac{R_A^2}{k} = \frac{1}{8} \frac{P_m^2}{k}$

For spring B, $U_B = \frac{1}{2} R_B y_B = \frac{1}{2} \frac{R_B^2}{k} = \frac{1}{8} \frac{P_m^2}{k}$

Portion AC of beam ACB $\quad M = \frac{1}{2} P_m x$

$$U_{AC} = \int_0^{L_{AC}} \frac{M^2}{2EI} dx = \frac{P_m^2}{8EI} \int_0^{L_{AC}} x^2 dx = \frac{P_m^2 L_{AC}^3}{24 EI}$$

Portion CB of beam

By symmetry $U_{CB} = U_{AC} = \frac{P_m^2 L_{AC}^3}{24 EI}$

Total $\quad U = U_A + U_B + U_{AC} + U_{CB} = \frac{P_m^2}{4k} + \frac{P_m^2 L_{AC}^3}{12 EI}$

$I = \frac{1}{12} bd^3 = \frac{1}{12}(1.5)(0.75)^3 = 52.734 \times 10^{-3}$ in^4

$$U = \left\{ \frac{1}{(4)(20 \times 10^3)} + \frac{(9)^3}{(12)(29 \times 10^6)(52.734 \times 10^{-3})} \right\} P_m^2 = 52.224 \times 10^{-6} \; P_m^2 = \frac{1}{2} P_m y_m$$

$y_m = \frac{2U}{P_m} = 104.448 \times 10^{-6} \; P_m \qquad P_m = 9.5741 \times 10^3 \; y_m$

$U = (52.224 \times 10^{-6})(9.5741 \times 10^3)^2 y_m^2 = 4.7871 \times 10^3 \; y_m^2$

Work of falling weight $\qquad W(h + y_m) = (3)(2 + y_m) = 6 + 3y_m$

Equating $\qquad 6 + 3y_m = 4.7871 \times 10^3 \; y_m^2$

$$y_m^2 - 626.69 \times 10^{-6} \; y_m - 1.25338 \times 10^{-3} = 0$$

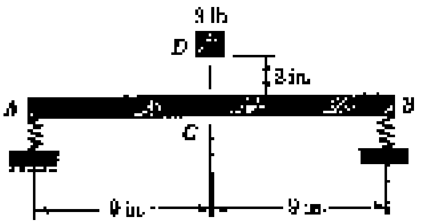
$$y_m = \frac{1}{2}\left\{ 626.69 \times 10^{-6} + \sqrt{(626.69 \times 10^{-6})^2 + (4)(1.25338 \times 10^{-3})} \right\}$$

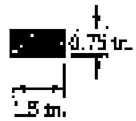
$$= 35.7 \times 10^{-3} \text{ in.} = 0.0357 \text{ in.}$$

PROBLEM 11.67

11.67 The 3-lb block D is released from rest in the position shown and strikes a steel bar AB having the uniform cross section shown. The bar is supported at each end by springs of constant 20 kips/in. Using $E = 29 \times 10^6$ psi, determine the maximum deflection at the midpoint of the bar.

11.67 Solve Prob 11.66, assuming that the constant of each spring is 40 kips/in.

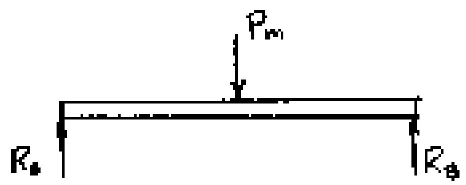




SOLUTION

$k = 40$ kips/in $= 40 \times 10^3$ lb/in.

$R_A = R_B = \frac{1}{2} P_m$



For spring A, $U_A = \frac{1}{2} R_A y_A = \frac{1}{2}\frac{R_A^2}{k} = \frac{1}{8}\frac{P_m^2}{k}$

For spring B, $U_B = \frac{1}{2} R_B y_B = \frac{1}{2}\frac{R_B^2}{k} = \frac{1}{8}\frac{P_m^2}{k}$

Portion AC of beam ACB $\qquad M = \frac{1}{2} P_m x$

$$U_{AC} = \int_0^{L_{AC}} \frac{M^2}{2EI}\,dx = \frac{P_m^2}{8EI}\int_0^{L_{AC}} x^2\,dx = \frac{P_m^2 L_{AC}^3}{24\,EI}$$

Portion CB of beam

By symmetry $\quad U_{CB} = U_{AC} = \dfrac{P_m^2 L_{AC}^3}{24\,EI}$

Total $\quad U = U_A + U_B + U_{AC} + U_{CB} = \dfrac{P_m^2}{4k} + \dfrac{P_m^2 L_{AC}^3}{12\,EI}$

$$I = \frac{1}{12} b d^3 = \frac{1}{12}(1.5)(0.75)^3 = 52.734 \times 10^{-3}\ \text{in}^4$$

$$U = \left\{\frac{1}{(4)(40\times10^3)} + \frac{(9)^3}{(12)(29\times10^6)(52.734\times10^{-3})}\right\}P_m^2 = 45.974\times10^{-6}\,P_m^2 = \frac{1}{2}P_m y_m$$

$$y_m = \frac{2U}{P_m} = 91.949\times10^{-6}\,P_m \qquad P_m = 10.8756\times10^3\,y_m$$

$$U = (45.974\times10^{-6})(10.8756\times10^3)^2 y_m^2 = 5.4378\times10^3\,y_m^2$$

Work of falling weight $\qquad W(h + y_m) = (3)(2 + y_m) = 6 + 3y_m$

Equating $\qquad 6 + 3y_m = 5.4378\times10^3\,y_m^2$

$$y_m^2 - 551.70\times10^{-6}\,y_m - 1.1034\times10^{-3} = 0$$

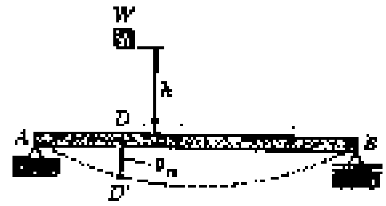
$$y_m = \frac{1}{2}\left\{551.70\times10^{-6} + \sqrt{(551.70\times10^{-6})^2 + (4)(1.1034\times10^{-3})}\right\}$$

$$= 33.5\times10^{-3}\ \text{in.} = 0.0335\ \text{in.}$$

PROBLEM 11.68

11.68 A block of weight W is placed in contact with a beam at some given point D and released. Show that the resulting maximum deflection at point D is twice as large as the deflection due to a static weight W applied at D.

SOLUTION



Consider dropping the weight from a height h above the beam. The work done by the weight is

$$\text{Work} = W(h + y_m)$$

Strain energy $U = \frac{1}{2} P_m y_m = \frac{1}{2} k y_m^2$

where k is the spring constant of the beam for loading at point D.

Equating work and energy $W(h + y_m) = \frac{1}{2} k y_m^2$

Setting $h = 0$, $W y_m = \frac{1}{2} k y_m^2$, $y_m = \frac{2W}{k}$

The static deflection at point D due to weight applied at D is

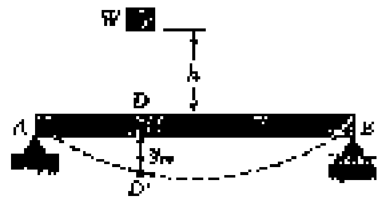
$\delta_{st} = \frac{W}{k}$ Thus $y_m = 2\delta_{st}$

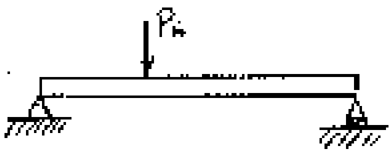
PROBLEM 11.69

11.69 A block of weight W is dropped from a height h onto the horizontal beam AB and hits it at point D. (*a*) Show that the maximum deflection y_m at point D can be expressed as

$$y_m = y_{st}\left(1 + \sqrt{1 + \frac{2h}{y_{st}}}\right)$$

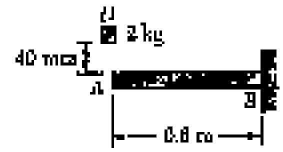
where y_{st} represents the deflection at D caused by a static load W applied at that point and where the quantity in parentheses is referred to as the *impact factor*. (*b*) Compute the impact factor for the beam and impact factor of Prob. 11.62.





11.62 The 2-kg block D is dropped from the position shown onto the end of a 16-mm-diameter rod. Knowing that $E = 200$ GPa, determine (*a*) the maximum deflection of end A, (*b*) the maximum bending moment in the rod, (*c*) the maximum normal stress in the rod.

SOLUTION



Work of falling weight $\qquad$ Work $= W(h + y_m)$

Strain energy $\qquad U = \frac{1}{2}P y_m = \frac{1}{2}k y_m^2$

where k is the spring constant for a load applied at point D.

Equating work and energy

$$W(h + y_m) = \tfrac{1}{2}k y_m^2$$

$$y_m^2 - \frac{2W}{k}y_m - \frac{2W}{k}h = 0$$

$$y_m^2 - 2y_{st}\,y_m - 2y_{st}\,h = 0 \qquad \text{where} \quad y_{st} = \frac{W}{k}$$

$$y_m = \frac{2y_{st} + \sqrt{4y_{st}^2 + 8y_{st}h}}{2} = y_{st}\left(1 + \sqrt{1 + \frac{2h}{y_{st}}}\right)$$

For Prob. 11.62 $\qquad W = mg = (2)(9.81) = 19.62$ N

$E = 200 \times 10^9$ Pa $\qquad I = \frac{\pi}{4}\left(\frac{16}{2}\right)^4 = 3.217 \times 10^3\ \text{mm}^4 = 3.217 \times 10^{-9}\ \text{m}^4$

$L = 0.6$ m $\qquad h = 40$ mm $= 40 \times 10^{-3}$ m

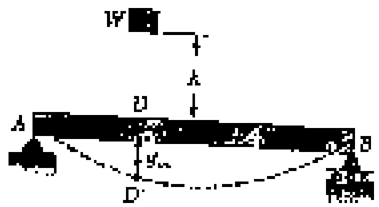
Using Appendix D Case 1 $\qquad y_{st} = \frac{WL^3}{3EI}$

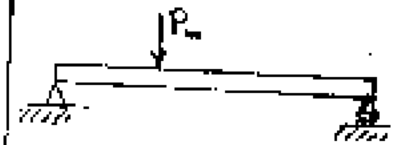
$$y_{st} = \frac{(19.62)(0.6)^3}{(3)(200 \times 10^9)(3.217 \times 10^{-9})} = 2.196 \times 10^{-3}\ \text{m}$$

$$\frac{2h}{y_{st}} = \frac{(2)(40 \times 10^{-3})}{2.196 \times 10^{-3}} = 36.44$$

$$\text{impact factor} = 1 + \sqrt{1 + 36.44} = 7.12$$

PROBLEM 11.70

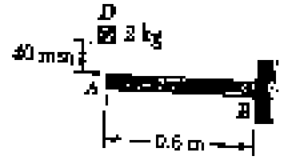




11.70 A block of weight W is dropped from a height h onto the horizontal beam AB and hits it at point D. (a) Denoting by y_m the exact value of the maximum deflection at D and by y'_m the value obtained by neglecting the effect of this deflection on the change in potential energy of the block, show that the absolute value of the relative error is $(y_m - y'_m)/y_m$ never exceeds $y'_m/2h$. (b) Check the result obtained in part a by solving part a of Prob. 11.62 without taking y'_m into account when determining the change in potential energy of the load, and comparing the answer obtained in this way with the exact answer to that problem.

11.62 The 2-kg block D is dropped from the position shown onto the end of a 16-mm-diameter rod. Knowing that $E = 200$ GPa, determine (a) the maximum deflection of end A, (b) the maximum bending moment in the rod, (c) the maximum normal stress in the rod.

SOLUTION



$U = \frac{1}{2} P_m y_m = \frac{1}{2} k y_m^2$ where k is the spring constant for a load at point D.

Work of falling weight: exact: Work $= W(h + y_m)$

approximate: Work $\approx Wh$

Equating work and energy: $\frac{1}{2} k y_m^2 = W(h + y_m)$ (1) exact

$\frac{1}{2} k y_m'^2 = Wh$ (2) approximate

where y'_m is the approximate value for y_m

Subtracting $\frac{1}{2} k (y_m^2 - y_m'^2) = W y_m$

$$y_m^2 - y_m'^2 = (y_m - y'_m)(y_m + y'_m) = \frac{2W}{k} y_m$$

Relative error $\frac{y_m - y'_m}{y_m} = \frac{2W}{k(y_m + y'_m)}$

But $\frac{2W}{k} = \frac{y_m'^2}{h}$ from equation (2)

(a) Relative error $= \frac{y_m - y'_m}{y_m} = \frac{y_m'^2}{h(y_m + y'_m)} < \frac{y'_m}{2h}$

(b) From the solution to Prob. 11.62 $y_m = 15.63$ mm

Approximate solution: $W = mg = (2)(9.81) = 19.62$ N

$E = 200 \times 10^9$ Pa $\quad I = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi}{4}\left(\frac{16}{2}\right)^4 = 3.217 \times 10^3 \text{ mm}^4 = 3.217 \times 10^{-9} \text{ m}^4$

$L = 0.6$ m, $\quad h = 40$ mm $= 40 \times 10^{-3}$ m

$$k = \frac{3EI}{L^3} = \frac{(3)(200 \times 10^9)(3.217 \times 10^{-9})}{(0.6)^3} = 8.936 \times 10^3 \text{ N/m}$$

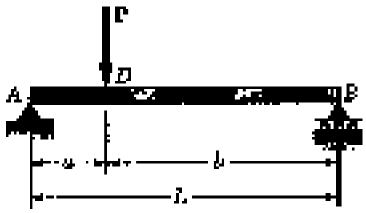
$$y_m'^2 = \frac{2Wh}{k} = \frac{(2)(19.62)(40 \times 10^{-3})}{8.936 \times 10^3} = 175.65 \times 10^{-6} \text{ m}^2$$

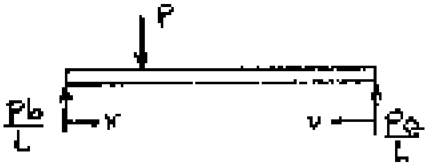
$y'_m = 13.25 \times 10^{-3}$ m $= 13.25$ mm

relative error $= \frac{15.63 - 13.25}{15.63} = 0.152$ ◀ $\quad \frac{y'_m}{2h} = 0.166$ ◀

PROBLEM 11.71

11.71 Using the method of work-energy, determine the deflection at point D caused by the load **P**.





SOLUTION

Reactions: $R_A = \dfrac{Pb}{L}$, $R_B = \dfrac{Pa}{L}$

Over AD $\quad M = R_A x = \dfrac{Pbx}{L}$

$$U_{AD} = \int_0^a \frac{M^2}{2EI}\,dx = \frac{P^2b^2}{2EIL^2}\int_0^a x^2\,dx = \frac{P^2b^2a^3}{6EIL^2}$$

Over DB $\quad M = R_B v = \dfrac{Pav}{L}$

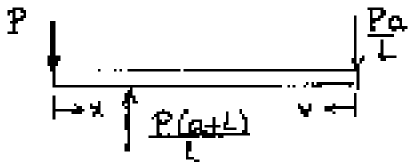
$$U_{DB} = \int_0^b \frac{M^2}{2EI}\,dv = \frac{P^2a^2}{2EIL^2}\int_0^b v^2\,dv = \frac{P^2a^2b^3}{6EIL^2}$$

$$\text{Total } U = U_{AD} + U_{DB} = \frac{P^2a^2b^2(a+b)}{6EIL^2} = \frac{P^2a^2b^2}{6EIL}$$

$$\tfrac{1}{2}P\delta_D = U \qquad \delta_D = \frac{2U}{P} = \frac{Pa^2b^2}{3EIL}\ \downarrow$$

PROBLEM 11.72

11.72 Using the method of work-energy, determine the deflection at point D caused by the load **P**.



SOLUTION

$\circlearrowleft \Sigma M_A = 0 \qquad Pa + R_B L = 0 \qquad R_B = -\dfrac{Pa}{L}$

Over portion DA $\quad M = -Px$

$$U_{DA} = \int_0^a \frac{M^2}{2EI}\,dx = \frac{P^2}{2EI}\int_0^a x^2\,dx = \frac{P^2a^3}{6EI}$$

Over portion AB $\quad M = -\dfrac{Pav}{L}$

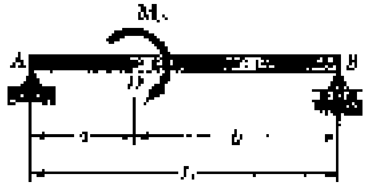
$$U_{AB} = \int_0^L \frac{M^2}{2EI}\,dv = \frac{P^2a^2}{2EIL^2}\int_0^L v^2\,dv = \frac{Pa^2L}{6EI}$$

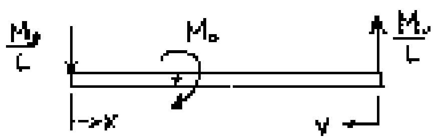
$$\text{Total } U = U_{DA} + U_{AB} = \frac{P^2a^2(a+L)}{6EI}$$

$$\tfrac{1}{2}P\delta_D = U \qquad \delta_D = \frac{2U}{P} = \frac{Pa^2(a+L)}{3EI}\ \downarrow$$

PROBLEM 11.73

11.73 Using the method of work-energy, determine the slope at point D caused by the couple M_0.





SOLUTION

Reactions $\quad R_A = \frac{M_0}{L}\downarrow \qquad R_B = \frac{M_0}{L}\uparrow$

Over portion AD $\quad M = -\frac{M_0 x}{L}$

$$U_{AD} = \int_0^a \frac{M^2}{2EI}\,dx = \frac{M_0^2}{2EIL^2}\int_0^a x^2\,dx$$

$$= \frac{M_0^2 a^3}{6EIL^2}$$

Over portion DB $\quad M = \frac{M_0 v}{L}$

$$U_{DB} = \int_0^b \frac{M^2}{2EI}\,dv = \frac{M_0^2}{2EIL^2}\int_0^b v^2\,dv$$

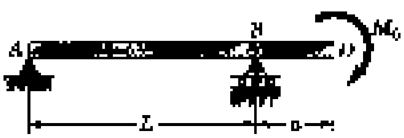
$$= \frac{M_0^2 b^3}{6EIL^2}$$

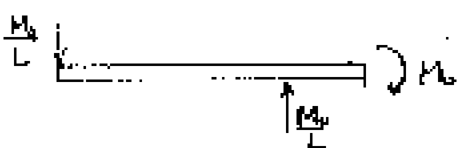
Total $\quad U = U_{AD} + U_{DB} = \frac{M_0^2(a^3+b^3)}{6EIL^2}$

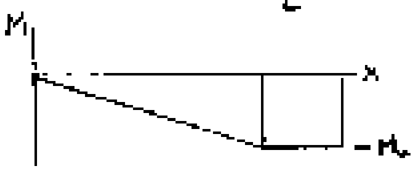
$\frac{1}{2}M_0\theta_D = U \qquad \theta_D = \frac{2U}{M_0} = \frac{M_0(a^3+b^3)}{3EIL^2}$

PROBLEM 11.74

11.74 Using the method of work-energy, determine the slope at point D caused by the couple M_0.







SOLUTION

Reactions $\quad R_A = \frac{M_0}{L}\downarrow \qquad R_B = \frac{M_0}{L}\uparrow$

Over portion AB $\quad M = -\frac{M_0 x}{L}$

$$U_{AB} = \int_0^L \frac{M^2}{2EI}\,dx = \frac{M_0^2}{2EIL^2}\int_0^L x^2\,dx$$

$$= \frac{M_0^2 L}{6EI}$$

Over portion BD $\quad M = -M_0$

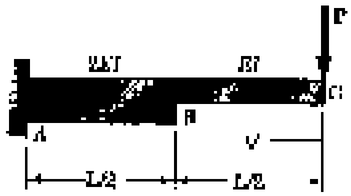
$$U_{BD} = \frac{M_0^2 a}{2EI}$$

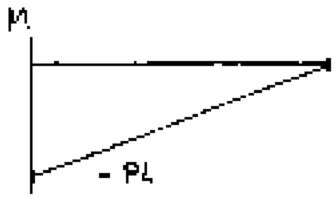
Total $\quad U = U_{AB} + U_{BD} = \frac{M_0^2(L+3a)}{6EI}$

$\frac{1}{2}M_0\theta_D = U \qquad \theta_D = \frac{2U}{M_0} = \frac{M_0(L+3a)}{3EI}$

PROBLEM 11.75

11.75 Using the method of work and energy, determine the deflection at point C caused by the load $\mathbf{P}$.





SOLUTION

Bending moment $\quad M = -Pv$

Over AB

$$U_{AB} = \int_{L/2}^{L} \frac{M^2}{4EI}\,dv = \frac{P^2}{4EI}\int_{L/2}^{L} v^2\,dv$$

$$= \frac{P^2}{12EI}\left[L^3 - \left(\tfrac{L}{2}\right)^3\right] = \frac{7}{96}\frac{P^2L^3}{EI}$$

Over BC

$$U_{BC} = \int_0^{L/2} \frac{M^2}{2EI}\,dv = \frac{P^2}{2EI}\int_0^{L/2} v^2\,dv$$

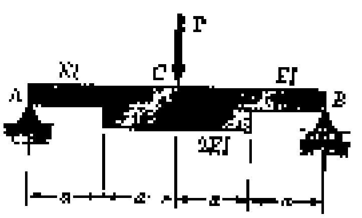
$$= \frac{1}{48}\frac{P^2L^3}{EI}$$

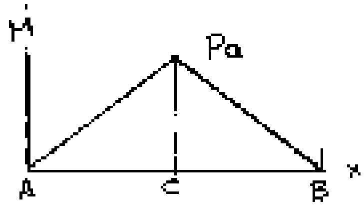
Total $\quad U = U_{AB} + U_{BC} = \dfrac{3}{32}\dfrac{P^2L^3}{EI}$

$\frac{1}{2}P\delta_C = U \qquad \delta_C = \dfrac{2U}{P} = \dfrac{3}{16}\dfrac{PL^3}{EI}\ \downarrow$

PROBLEM 11.76

11.76 Using the method of work and energy, determine the deflection at point C caused by the load $\mathbf{P}$.





SOLUTION

Symmetric beam and loading $\quad R_A = R_B = \frac{1}{2}P$

From A to C $\quad M = R_A x = \frac{1}{2}Px$

$$U_{AC} = \int_0^{a} \frac{M^2}{2EI}\,dx + \int_a^{2a} \frac{M^2}{4EI}\,dx$$

$$= \frac{P^2}{8EI}\int_0^{a} x^2\,dx + \frac{P^2}{16EI}\int_a^{2a} x^2\,dx$$

$$\frac{P^2a^3}{24EI} + \frac{P^2}{48EI}\left[(2a)^3 - a^3\right] = \frac{3}{16}\frac{P^2a^3}{EI}$$

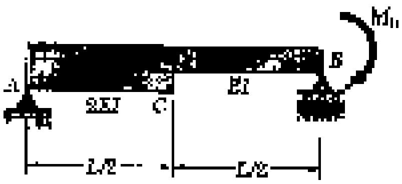
By symmetry $\quad U_{CB} = U_{AC} = \dfrac{3}{16}\dfrac{P^2a^3}{EI}$

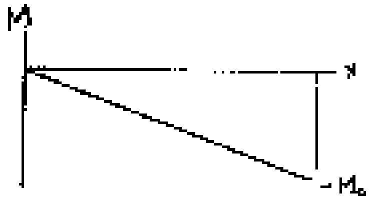
Total $\quad U = U_{AC} + U_{CB} = \dfrac{3}{8}\dfrac{P^2a^3}{EI}$

$\frac{1}{2}P\delta_C = U \qquad \delta_C = \dfrac{2U}{P} = \dfrac{3}{4}\dfrac{Pa^3}{EI}$

PROBLEM 11.77

11.77 Using the method of work and energy, determine the slope at point B caused by the couple $\mathbf{M}_0$.





SOLUTION

$$\circlearrowleft \Sigma M_B = 0 \qquad -R_A L - M_0 = 0 \qquad R_A = -\frac{M_0}{L}$$

$$M = R_A x = -\frac{M_0}{L}x$$

Over portion AC $\quad U_{AC} = \int_0^{L/2} \frac{M^2}{2(2EI)}\,dx$

$$U_{AC} = \frac{M_0^2}{4EIL^2}\int_0^{L/2} x^2\,dx = \frac{1}{96}\frac{M_0^2 L}{EI}$$

Over portion CB $\quad U_{CB} = \int_{L/2}^{L} \frac{M^2}{2EI}\,dx$

$$U_{CB} = \frac{M_0^2}{2EIL^2}\int_{L/2}^{L} x^2\,dx = \frac{M_0^2}{6EIL^2}\left[L^3-\left(\frac{L}{2}\right)^3\right]$$

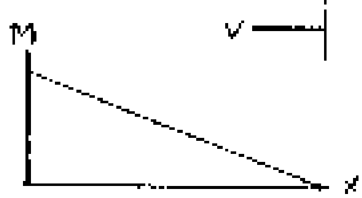
$$= \frac{7}{48}\frac{M_0^2 L}{EI}$$

Total $\quad U = U_{AC} + U_{CB} = \frac{5}{32}\frac{M_0^2 L}{EI}$

$$\frac{1}{2}M_0\theta_B = U \qquad \theta_B = \frac{2U}{M_0} = \frac{5}{16}\frac{M_0 L}{EI}$$

PROBLEM 11.78

11.78 Using the method of work-energy, determine the slope at point A caused by the couple $\mathbf{M}_0$.



SOLUTION

$$R_B = \frac{M_0}{L}$$

$$M = R_B v = \frac{M_0}{L}v$$

Over AC $\quad U_{AC} = \int_{L/2}^{L} \frac{M^2}{2(2EI)}\,dv$

$$U_{AC} = \frac{M_0^2}{4EIL^2}\int_{L/2}^{L} v^2\,dv = \frac{M_0^2}{12EIL^2}\left[L^3-\left(\frac{L}{2}\right)^3\right]$$

$$= \frac{7}{96}\frac{M_0^2 L}{EI}$$

Over CB $\quad U_{CB} = \int_0^{L/2} \frac{M^2}{2EI}\,dv$

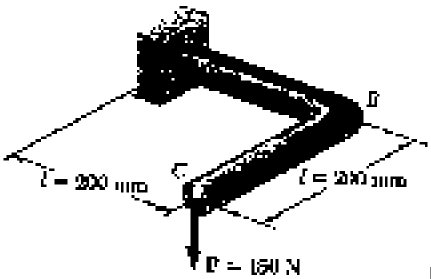
$$U_{CB} = \frac{M_0^2}{2EIL^2}\int_0^{L/2} v^2\,dv = \frac{1}{48}\frac{M_0^2 L}{EI}$$

Total $\quad U = U_{AC} + U_{CB} = \frac{3}{32}\frac{M_0^2 L}{EI}$

$$\frac{1}{2}M_0\theta_A = U \qquad \theta_A = \frac{2U}{M_0} = \frac{3}{16}\frac{M_0 L}{EI}$$

PROBLEM 11.79

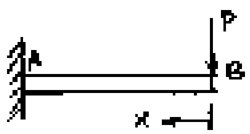
11.79 The 12-mm-diameter steel rod *ABC* has been bent into the shape shown. Knowing that $E = 200$ GPa and $G = 77.2$ GPa, determine the deflection of end *C* caused by the 150-N force.



SOLUTION

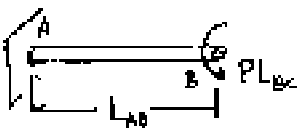
$$J = \frac{\pi}{2}c^4 = \frac{\pi}{2}\left(\frac{12}{2}\right)^4 = 2.0358 \times 10^3 \text{ mm}^4$$
$$= 2.0358 \times 10^{-9} \text{ m}^4$$

$$I = \tfrac{1}{2}J = 1.0179 \times 10^{-9} \text{ m}^4$$



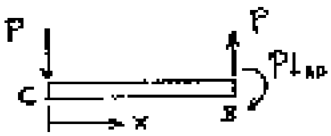
Portion AB: bending $M = -Px$

$$U_{AB,b} = \int_0^{L_{AB}} \frac{M^2}{2EI}dx = \frac{P^2}{2EI}\int_0^{L_{AB}} x^2\,dx$$
$$= \frac{P^2 L_{AB}^3}{6EI} = \frac{(150)^2(200 \times 10^{-3})^3}{(6)(200 \times 10^9)(1.0179 \times 10^{-9})}$$
$$= 0.14736 \text{ J}$$



torsion $T = PL_{BC}$

$$U_{AB,t} = \frac{T^2 L_{AB}}{2GJ} = \frac{P^2 L_{BC}^2 L_{AB}}{2GJ}$$
$$= \frac{(150)^2(200 \times 10^{-3})^2(200 \times 10^{-3})}{(2)(77.2 \times 10^9)(2.0358 \times 10^{-9})}$$
$$= 0.57265 \text{ J}$$



Portion BC: $M = -Px$

$$U_{BC} = \int_0^{L_{BC}} \frac{M^2}{2EI}dx = \frac{P^2}{2EI}\int_0^{L_{BC}} x^2\,dx = \frac{P^2 L_{BC}^3}{6EI}$$
$$= \frac{(150)^2(200 \times 10^{-3})^3}{(6)(200 \times 10^9)(1.0179 \times 10^{-9})} = 0.14736 \text{ J}$$

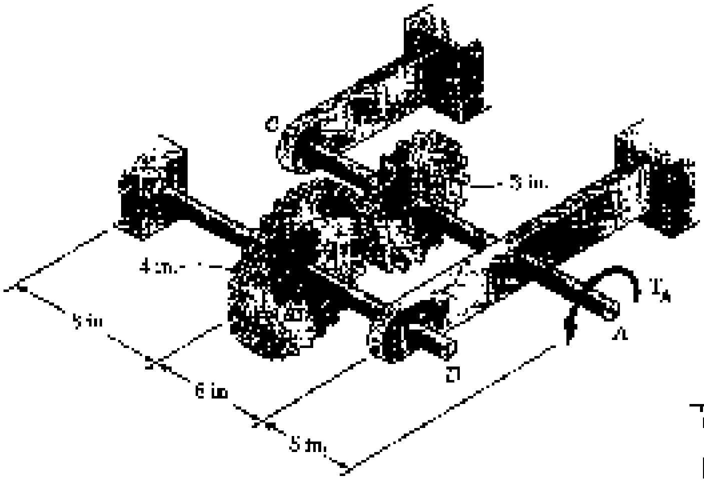
Total: $U = U_{AB,b} + U_{AB,t} + U_{BC} = 0.86737 \text{ J}$

Work-energy $\quad \frac{1}{2}P\delta = U \qquad \delta = \frac{2U}{P} = \frac{(2)(0.86737)}{150}$

$$= 11.57 \times 10^{-3} \text{ m} = 11.57 \text{ mm} \downarrow$$

PROBLEM 11.80

11.80 Two steel shafts, each of 0.75-in. diameter, are connected by the gears shown. Knowing that $G = 11.2 \times 10^6$ psi and that shaft DF is fixed at F, determine the angle through which end A rotates when a 750-lb·in. torque is applied at A. (Ignore the strain energy due to the bending of the shafts.)



SOLUTION

Work-energy equation

$$\frac{1}{2} T_A \varphi_A = U$$

$$\varphi_A = \frac{2U}{T_A}$$

Portion AB of shaft ABC:

$T_{AB} = T_A = 750$ lb-in

$L_{AB} = 5 + 6 = 11$ in

$$J_{AB} = \frac{\pi}{2}\left(\frac{d}{2}\right)^4 = \frac{\pi}{2}\left(\frac{0.75}{2}\right)^4 = 31.063 \times 10^{-3} \text{ in}^4$$

$$U_{AB} = \frac{T_{AB}^2 L_{AB}}{2GJ_{AB}} = \frac{(750)^2(11)}{(2)(11.2\times10^6)(31.063\times10^{-3})} = 8.892 \text{ in-lb}$$

Portion BC of shaft ABC: $U_{BC} = 0$

Gear B $\qquad F_{BE} = \frac{T_B}{r_B} = \frac{T_{AB}}{r_B} = \frac{750}{3} = 250$ lb

Gear E $\qquad T_E = r_E F_{BE} = (4)(250) = 1000$ lb-in

Portion DE of shaft DEF. $\qquad U_{DE} = 0$

Portion EF of shaft DEF. $\qquad T_{EF} = T_E = 1000$ lb-in

$L_{EF} = 8$ in $\qquad J_{EF} = \frac{\pi}{2}\left(\frac{d}{2}\right)^4 = 31.063 \times 10^{-3}$ in^4

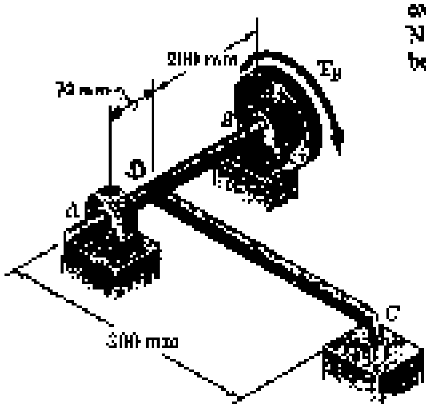
$$U_{EF} = \frac{T_{EF}^2 L_{EF}}{2GJ_{EF}} = \frac{(1000)^2(8)}{(2)(11.2\times10^6)(31.063\times10^{-3})} = 11.497 \text{ lb-in}$$

Total: $\qquad U = U_{AB} + U_{BC} + U_{DE} + U_{EF} = 20.389$ in-lb

$$\varphi_A = \frac{2U}{T_A} = \frac{(2)(20.389)}{750} = 54.4\times10^{-3} \text{ rad} = 3.12°$$

PROBLEM 11.81

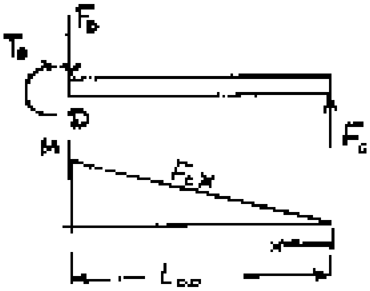
11.81 The 20-mm-diameter steel rod *CD* is welded to the 20-mm-diameter steel shaft *AB* as shown. End *C* of rod *CD* is touching the rigid surface shown when a couple T_B is applied to a disk attached to shaft *AB*. Knowing that the bearings are self-aligning and exert no couples on the shaft, determine the angle of rotation of the disk when $T_B = 400$ N·m. Use $E = 200$ GPa and $G = 77.2$ GPa. (Consider the strain energy due to both bending and twisting in shaft *AB* and to bending in arm *CD*.)



SOLUTION

$\circlearrowleft \Sigma M_{AB} = 0 \qquad r_{CD} F_C = T_B \qquad F_C = \dfrac{T_B}{r_{CD}}$

$F_C = \dfrac{400}{300 \times 10^{-3}} = 1333.3 \text{ N}, \qquad F_D = 1333.3 \text{ N}$



Bending of rod CD:

$I = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi}{4}\left(\frac{20}{2}\right)^4 = 7.854 \times 10^3 \text{ mm}^4 = 7.854 \times 10^{-9} \text{ m}^4$

$M = F_C x$

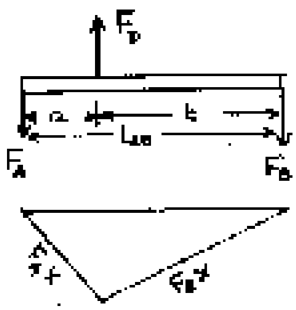
$U = \int_0^{L_{CD}} \dfrac{(F_C x)^2}{2EI} = \dfrac{F_C^2 L_{CD}^3}{6EI}$

$= \dfrac{(1333.3)^2 (300 \times 10^{-3})^3}{(6)(200 \times 10^9)(7.854 \times 10^{-9})} = 5.093 \text{ J}$

Bending of shaft ADB

$\circlearrowleft \Sigma M_B = 0 \qquad -F_A L_{AB} + F_D b = 0 \qquad F_A = \dfrac{F_D b}{L_{AB}}$

$\circlearrowleft \Sigma M_A = 0 \qquad +F_B L_{AB} - F_D b = 0 \qquad F_B = \dfrac{F_D a}{L_{AB}}$



$U = \dfrac{1}{2EI}\left\{\int_0^a \left(\dfrac{F_D b}{L_{AB}}\right)^2 dx + \int_0^b \left(\dfrac{F_D a}{L_{AB}}\right)^2 dx\right\} = \dfrac{F_D^2 a^2 b^2}{6EI L_{AB}}$

$I = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = 7.854 \times 10^3 \text{ mm}^4 = 7.854 \times 10^{-9} \text{ m}^4$

$L_{AB} = (270 \times 10^{-3}) \text{ m}$

$U = \dfrac{(1333.3)^2 (70 \times 10^{-3})^2 (200 \times 10^{-3})^2}{(6)(200 \times 10^9)(7.854 \times 10^{-9})(270 \times 10^{-3})} = 0.137 \text{ J}$

Torsion: Only portion DB carries torque. $\quad J = 2I = 15.708 \times 10^{-9} \text{ m}^4$

$U = \dfrac{T_B^2 L_{DB}}{2GJ} = \dfrac{(400)^2 (200 \times 10^{-3})}{(2)(77.2 \times 10^9)(15.708 \times 10^{-9})} = 13.194 \text{ J}$

Total: $\quad U = 5.093 + 0.137 + 13.194 = 18.424 \text{ J}$

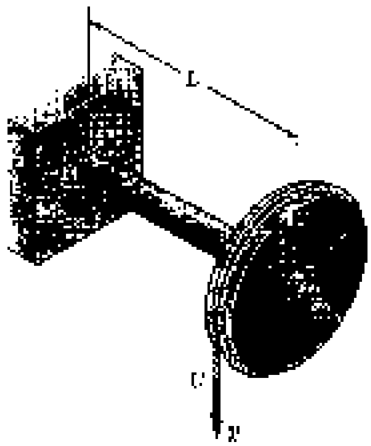
$\frac{1}{2} T_B \varphi_B = U$

$\varphi_B = \dfrac{2U}{T_B} = \dfrac{(2)(18.424)}{400} = 92.1 \times 10^{-3} \text{ rad}$

PROBLEM 11.82

11.82 A disk of radius a has been welded to end B of the solid steel shaft AB. A cable is then wrapped around the disk and a vertical force $\mathbf{P}$ is applied to end C of the cable. Knowing that the radius of the shaft is r and neglecting the deformations of the disk and of the cable, show that the deflection of point C caused by the application of $\mathbf{P}$ is

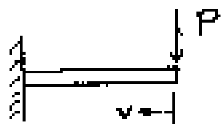
$$\delta_C = \frac{PL^3}{3EI}\left(1 + 1.5\frac{Ea^2}{GL^2}\right)$$



SOLUTION

Torsion: $T = Pa$

$$U_t = \frac{T^2L}{2GJ} = \frac{P^2a^2L}{2GJ}$$



Bending: $M = Pv$

$$U_b = \int_0^L \frac{M^2\,dv}{2EI} = \int_0^L \frac{P^2v^2\,dv}{2EI} = \frac{P^2L^3}{6EI}$$

Total $\quad U = \dfrac{P^2a^2L}{2GJ} + \dfrac{P^2L^3}{6EI} = \dfrac{1}{2}P\delta_c$

$$\delta_c = \frac{Pa^2L}{GJ} + \frac{PL^3}{3EI} = \frac{PL^3}{3EI}\left(1 + \frac{3EIa^2}{GJL^2}\right)$$

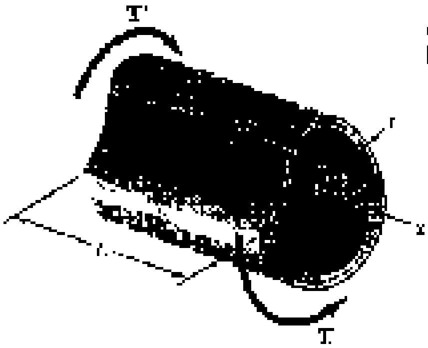
Since $J = 2I$ $\qquad \delta_c = \dfrac{PL^3}{3EI}\left(1 + \dfrac{3}{2}\dfrac{Ea^2}{GL^2}\right)$

PROBLEM 11.83

11.83 The thin-walled hollow cylindrical member AB has a noncircular cross section of nonuniform thickness. Using the expression given in Eq. (3.53) of Sec. 3.13, and the expression for the strain-energy density given in Eq. (11.19) of Sec. 11.4, show that the angle of twist of member AB is

$$\phi = \frac{TL}{4A^2G}\oint\frac{ds}{t}$$

where ds is an element of the centerline of wall cross section and A is the area enclosed by that centerline.



SOLUTION

From equation (3.53) $\qquad \tau = \dfrac{T}{2tA}$

Strain energy density

$$u = \frac{\tau^2}{2G} = \frac{T^2}{8Gt^2A^2}$$

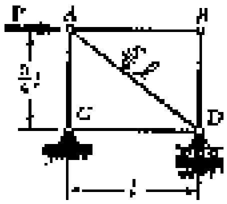
$$U = \int_0^L \oint u\,t\,ds\,dx$$

$$= \int_0^L \frac{T^2}{8GA^2}\oint\frac{ds}{t}\,dx = \frac{T^2L}{8GA^2}\oint\frac{ds}{t}$$

Work of torque $= \frac{1}{2}T\varphi = \dfrac{T^2L}{8GA^2}\oint\dfrac{ds}{t} \qquad \varphi = \dfrac{TL}{4GA^2}\oint\dfrac{ds}{t}$

PROBLEM 11.84

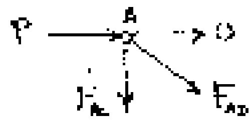
11.84 Each member of the truss shown has a uniform cross-sectional area A. Using the method of work and energy, determine the horizontal deflection of the point of application of the load $\mathbf{P}$.



SOLUTION

Members AB and BD are zero force members.

Joint A



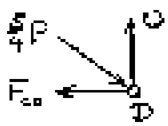
$\xrightarrow{+} \Sigma F_x = 0$

$\frac{4}{5} F_{AD} + P = 0 \qquad F_{AD} = -\frac{5}{4} P$

$+\uparrow \Sigma F_y = 0$

$-F_{AC} - \frac{3}{5} F_{AD} = 0 \qquad F_{AC} = \frac{3}{4} P$

Joint D $\quad \xrightarrow{+} \Sigma F_x = 0$



$\frac{4}{5} \cdot \frac{5}{4} P - F_{CD} = 0$

$F_{CD} = P$

Member	F	L	F^2L
AB	0	l	0
BD	0	$\frac{3}{4} l$	0
AD	$-\frac{5}{4} P$	$\frac{5}{4} l$	$\frac{125}{64} P^2 l$
CD	P	l	$P^2 l$
AC	$\frac{3}{4} P$	$\frac{3}{4} L$	$\frac{27}{64} P^2 l$
Σ			$\frac{27}{8} P^2 l$

$$U = \Sigma \frac{F^2 L}{2EA} = \frac{1}{2EA} \Sigma F^2 L$$

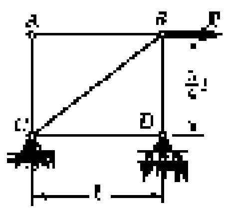
$$= \frac{27}{16} \frac{P^2 l}{EA}$$

Work of $P = \frac{1}{2} P \Delta = U$

$$\Delta = \frac{2U}{P} = \frac{27}{8} \frac{Pl}{EA} = 3.375 \frac{Pl}{EA} \longleftarrow$$

PROBLEM 11.85

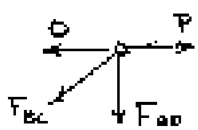
11.85 Each member of the truss shown has a uniform cross-sectional area A. Using the method of work and energy, determine the horizontal deflection of the point of application of the load $\mathbf{P}$.



SOLUTION

Members AB, AC, and CD are zero force members.

Joint B



$\xrightarrow{+} \Sigma F_x = 0$

$P - \frac{4}{5} F_{BC} = 0 \qquad F_{BC} = \frac{5}{4} P$

$+\uparrow \Sigma F_y = 0$

$-F_{BD} - \frac{3}{5} F_{BC} = 0 \qquad F_{BD} = -\frac{3}{4} P$

$$U = \Sigma \frac{F^2 L}{2EA} = \frac{1}{2EA} \Sigma F^2 L$$

$$= \frac{19}{16} \frac{P^2 l}{EA}$$

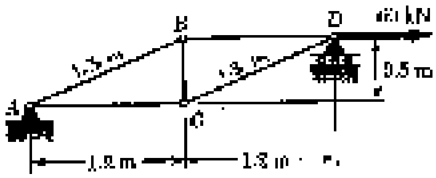
Member	F	L	F^2L
AB	0	l	0
AC	0	$\frac{3}{4} l$	0
CD	0	l	0
BC	$\frac{5}{4} P$	$\frac{5}{4} l$	$\frac{125}{64} P^2 l$
BD	$-\frac{3}{4} P$	$\frac{3}{4} l$	$\frac{27}{64} P^2 l$
Σ			$\frac{19}{8} P^2 l$

Work of $P = \frac{1}{2} P \Delta = U$

$$\Delta = \frac{2U}{P} = \frac{19}{8} \frac{Pl}{EA} = 2.375 \frac{Pl}{EA} \longleftarrow$$

PROBLEM 11.86

11.86 Each member of the truss shown is made of steel; the cross-sectional area of member BC is 800 mm² and for all other members the cross-sectional area is 400 mm². Using $E = 200$ GPa, determine the deflection of point D caused by the 60-kN load shown.

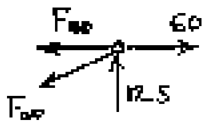


SOLUTION

Entire truss $\circlearrowright \Sigma M_A = 0$

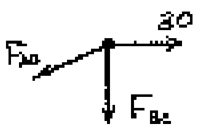
$2.4\, R_D - (0.5)(60) = 0 \qquad R_D = 12.5 \text{ kN}$

Joint D $\quad +\uparrow \Sigma F_y = 0 \qquad 12.5 - \frac{0.5}{1.3} F_{CD} = 0 \qquad F_{CD} = 32.5 \text{ kN}$



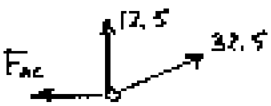
$\xrightarrow{+} \Sigma F_x = 0 \qquad 60 - F_{BD} - \frac{1.2}{1.3} F_{CD} = 0 \qquad F_{BD} = 30 \text{ kN}$

Joint B $\quad \xrightarrow{+} \Sigma F_x = 0 \qquad 30 - \frac{1.2}{1.3} F_{AB} = 0 \qquad F_{AB} = 32.5 \text{ kN}$



$+\uparrow \Sigma F_y = 0 \qquad -\frac{0.5}{1.3} F_{AB} + F_{BC} = 0 \qquad F_{BC} = 12.5 \text{ kN}$

Joint C $\quad \xrightarrow{+} \Sigma F_x = 0 \qquad -F_{AC} + \frac{1.2}{1.3}(32.5) = 0 \qquad F_{AC} = 30 \text{ kN}$



$$U = \Sigma \frac{F^2 L}{2EA} = \frac{1}{2E} \Sigma \frac{F^2 L}{A}$$

Member	F (kN)	L (m)	A (10^{-6} m²)	F^2L/A (N²/m)
CD	32.5	1.3	400	3.4328×10^{12}
BD	30	1.2	400	2.7×10^{12}
AB	32.5	1.3	400	3.4328×10^{12}
BC	12.5	0.5	800	0.0977×10^{12}
AC	30	1.2	400	2.7×10^{12}
				12.3633×10^{12}

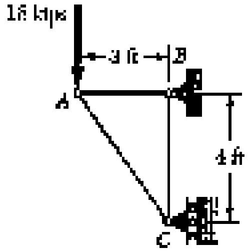
$$U = \frac{12.3633 \times 10^{12}}{(2)(200 \times 10^{9})} = 30.908 \text{ J}$$

$$\frac{1}{2} P \Delta = U \qquad \Delta = \frac{2U}{P} = \frac{(2)(30.908)}{60 \times 10^{3}} = 1.030 \times 10^{-3} \text{ m}$$

$$= 1.030 \text{ mm} \rightarrow$$

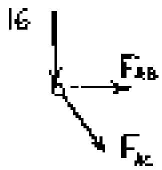
PROBLEM 11.87

11.87 Each member of the truss shown is made of steel and has a uniform cross-sectional area of 3 in². Using $E = 29 \times 10^6$ psi, determine the vertical deflection of the point of application of joint A caused by the 16-kip load.



SOLUTION

Joint A



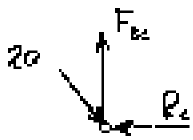
$+\uparrow \Sigma F_y = 0$

$-16 - \frac{4}{5} F_{AC} = 0 \qquad F_{AC} = -20 \text{ kips}$

$+\rightarrow \Sigma F_x = 0$

$\frac{3}{5} F_{AC} + F_{AB} = 0 \qquad F_{AB} = 12 \text{ kips}$

Joint C $\qquad +\uparrow \Sigma F_y = 0 \qquad F_{B} - \frac{4}{5}(20) = 0 \qquad F_{BC} = 16 \text{ kips}$



$$U = \Sigma \frac{F^2 L}{2EA} = \frac{1}{2EA} \Sigma F^2 L \qquad E = 29 \times 10^3 \text{ ksi}, \quad A = 3 \text{ in}^2$$

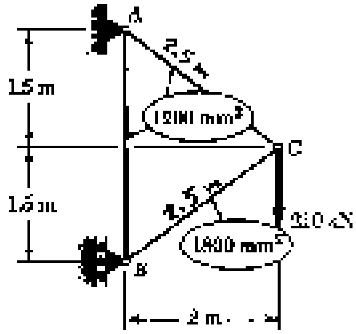
Member	F (kips)	L (in)	F^2L (kip²·in)
AB	12	36	5184
AC	-20	60	24000
BC	16	48	12288
			41472

$$U = \frac{41472}{(2)(29 \times 10^3)(3)} = 0.23834 \text{ kip}\cdot\text{in}$$

$$\frac{1}{2} P\Delta = U \qquad \Delta = \frac{2U}{P} = \frac{(2)(0.23834)}{16} = 0.0298 \text{ in.} \downarrow$$

PROBLEM 11.88

11.88 Members of the truss shown are made of steel and have the cross-sectional areas shown. Using $E = 200$ GPa, determine the vertical deflection of joint C caused by the application of the 210-kN load.



SOLUTION

Joint C

$\xrightarrow{+} \Sigma F_x = 0$

$-\frac{4}{5}F_{AC} - \frac{4}{5}F_{BC} = 0$

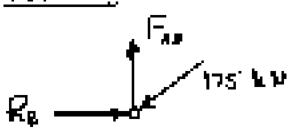
$+\uparrow \Sigma F_y = 0$

$\frac{3}{5}F_{AC} - \frac{3}{5}F_{BC} - 210 = 0$

Solving simultaneously

$F_{AC} = 175$ kN $\qquad F_{BC} = -175$ kN

Joint B



$+\uparrow \Sigma F_y = 0$

$F_{AB} - \left(\frac{3}{5}\right)(175) = 0$

$F_{AB} = 105$ kN

$$U_m = \Sigma \frac{F^2 L}{2EA}$$

Member	F (kN)	L (m)	A (10^{-6} m²)	F^2L/A (N²/m)
AB	105	3.0	1200	27.5625×10^{12}
AC	175	2.5	1200	63.8021×10^{12}
BC	−175	2.5	1800	42.5347×10^{12}
				133.8993×10^{12}

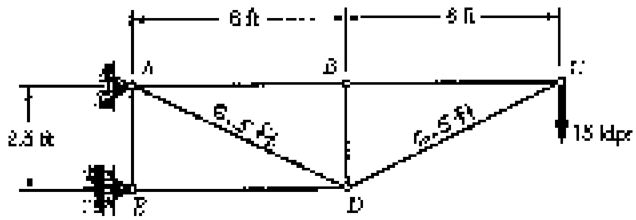
$$U_m = \frac{1}{2E}\Sigma\frac{F^2L}{A} = \frac{133.8993 \times 10^{12}}{(2)(200 \times 10^9)} = 334.75 \text{ J}$$

$$\frac{1}{2}P_m \Delta_m = U_m$$

$$\Delta_m = \frac{2U_m}{P_m} = \frac{(2)(334.75)}{210 \times 10^3} = 3.19 \times 10^{-3} \text{ m} = 3.19 \text{ mm}$$

PROBLEM 11.89

11.89 Each member of the truss shown is made of steel and has a uniform cross-sectional area of 5 in². Using $E = 29 \times 10^6$ psi, determine the vertical deflection of the point of application of joint C caused by the 15-kip load.



SOLUTION

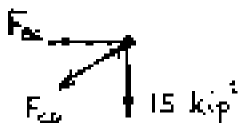
Members BD and AE are zero force members.

For entire truss $+\circlearrowleft M_A = 0$

$$2.5\,R_D - (12)(15) = 0$$
$$R_D = 72 \text{ kips}$$

For equilibrium of joint E $\quad F_{ED} = -R_D = -72$ kips

Joint C



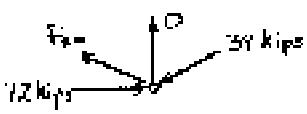
$+\uparrow \Sigma F_y = 0$

$$-\frac{2.5}{6.5} F_{CD} - 15 = 0$$
$$F_{CD} = -39 \text{ kips}$$

$+\!\rightarrow \Sigma F_x = 0$

$$-\frac{6}{6.5} F_{CD} - F_{BC} = 0$$
$$F_{BC} = 36 \text{ kips.}$$

Joint D



$+\!\rightarrow \Sigma F_x = 0$

$$72 - \frac{6}{6.5}(F_{AD} + 39) = 0$$
$$F_{AD} = 39 \text{ kips}$$

Joint B $\quad \Sigma F_x = 0$

$$-F_{AB} + F_{BC} = 0$$
$$F_{AB} = 36 \text{ kips}$$

Strain energy $\quad U_m = \Sigma \dfrac{F^2 L}{2EA} = \dfrac{1}{2EA} \Sigma F^2 L$

Member	F (kips)	L (in)	F^2L (kip²·in)
AB	36	72	93312
BC	36	72	93312
CD	-39	78	118638
DE	-72	72	373248
BD	0	30	0
AE	0	30	0
AD	39	78	118638
Σ			797148

Data: $E = 29 \times 10^3$ ksi

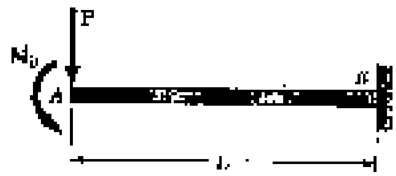
$A = 5$ in²

$$U_m = \frac{797148}{(2)(29 \times 10^3)(5)} = 2.7488 \text{ kip} \cdot \text{in}$$

$$\tfrac{1}{2} P_m \Delta_m = U \qquad \Delta_m = \frac{2U_m}{P_m} = \frac{(2)(2.7488)}{15} = 0.366 \text{ in.} \downarrow$$

PROBLEM 11.90

11.90 Using the information provided in Appendix D, compute the work of the loads as they are applied to the beam (*a*) if the load P is applied first, (*b*) if the couple M_0 is applied first.



SOLUTION

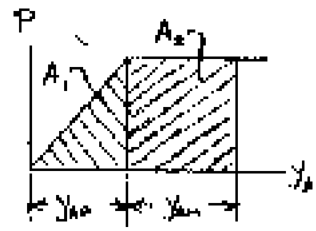
From Appendix D, Case 1

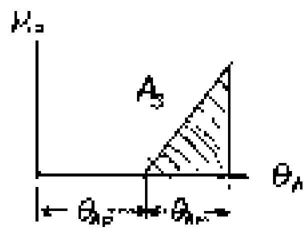
$$y_{AP} = \frac{PL^3}{3EI} \qquad \theta_{AP} = \frac{PL^2}{2EI}$$

From Appendix D, Case 3

$$y_{AM} = \frac{M_0L^2}{2EI} \qquad \theta_{AM} = \frac{M_0L}{EI}$$

(a) First P, then M_0.



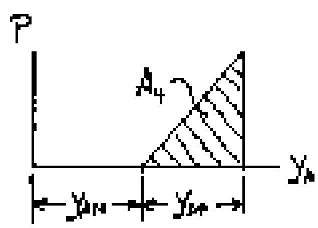


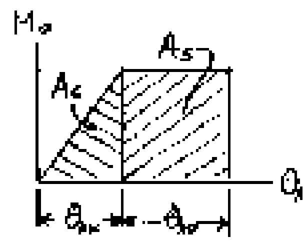
$$U = A_1 + A_2 + A_3$$

$$= \frac{1}{2}Py_{AP} + Py_{AM} + \frac{1}{2}M_0\theta_{AM}$$

$$= \frac{P^2L^3}{6EI} + \frac{PM_0L^2}{2EI} + \frac{M_0^2L}{2EI}$$

(b) First M_0, then P





$$U = A_4 + A_5 + A_6$$

$$= \frac{1}{2}Py_{AP} + M_0\theta_{AP} + \frac{1}{2}M_0\theta_{AM}$$

$$= \frac{P^2L^3}{6EI} + \frac{M_0PL^2}{2EI} + \frac{M_0^2L}{2EI}$$

PROBLEM 11.91

11.91 Using the information provided in Appendix D, compute the work of the loads as they are applied to the beam (*a*) if the load **P** is applied first, (*b*) if the couple $\mathbf{M}_0$ is applied first.

SOLUTION

Appendix D Cases 1 and 3

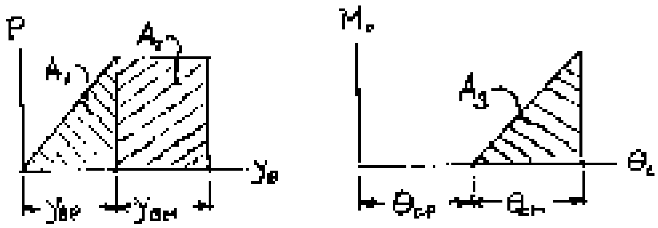
$$y_{BP} = \frac{P(L/2)^3}{3EI} = \frac{PL^3}{24EI}$$

$$\theta_{CP} = \frac{P(L/2)^2}{2EI} = \frac{PL^2}{8EI}$$

$$y_{BM} = \frac{M_0(L/2)^2}{2EI} = \frac{M_0L^2}{8EI}$$

$$\theta_{CM} = \frac{M_0L}{EI}$$

(a) First P, then M_0

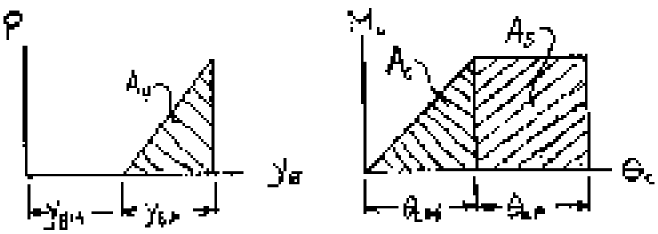


$$U = A_1 + A_2 + A_3$$

$$= \tfrac{1}{2}Py_{BP} + Py_{BM} + \tfrac{1}{2}M_0\theta_{CM}$$

$$= \frac{P^2L^3}{48EI} + \frac{PM_0L^2}{8EI} + \frac{M_0^2L}{2EI}$$

(b) First M_0, then P



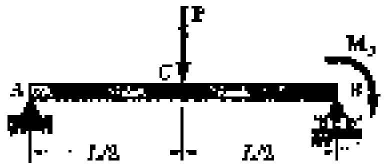
$$U = A_4 + A_5 + A_6$$

$$= \tfrac{1}{2}Py_{BP} + M_0\theta_{CP} + \tfrac{1}{2}M_0\theta_{CM}$$

$$= \frac{P^2L^3}{48EI} + \frac{M_0PL^2}{8EI} + \frac{M_0^2L}{2EI}$$

PROBLEM 11.92

11.92 Using the information provided in Appendix D, compute the work of the loads as they are applied to the beam (*a*) if the load **P** is applied first, (*b*) if the couple $\mathbf{M}_0$ is applied first.



SOLUTION

From Appendix D, Case 4

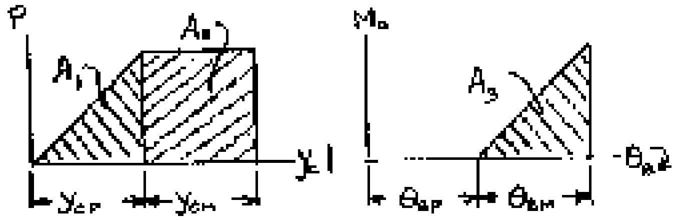
$$\downarrow y_C = \frac{PL^3}{48EI} \qquad \circlearrowleft \theta_B = -\frac{PL^2}{16EI}$$

From Appendix D, Case 7

$$\downarrow y_C = \frac{M_0}{6EIL}\left((L/2)^3 - L^2(L/2)\right) = -\frac{M_0 L^2}{16EI}$$

$$\circlearrowright \theta_B = \frac{M_0 L}{3EI}$$

(a) First P, then M_0

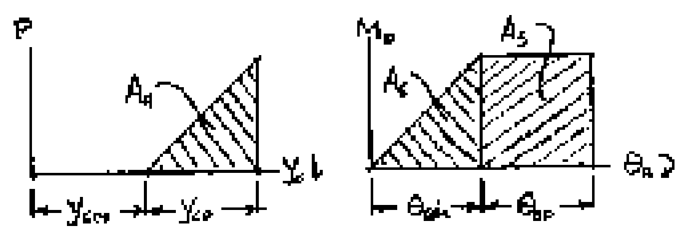


$$U = A_1 + A_2 + A_3$$

$$= \tfrac{1}{2} P y_{CP} + P y_{CM} + \tfrac{1}{2} M_0 \theta_{BM}$$

$$= \frac{P^2 L^3}{96EI} - \frac{PM_0 L^2}{16EI} + \frac{M_0^2 L}{6EI}$$

(b) First M_0, then P



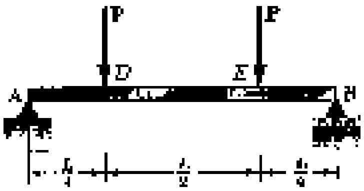
$$U = A_4 + A_5 + A_6$$

$$= \tfrac{1}{2} P y_{CP} + M_0 \theta_{BP} + \tfrac{1}{2} M_0 \theta_{BM}$$

$$= \frac{P^2 L^3}{96EI} - \frac{M_0 P L^2}{16EI} + \frac{M_0^2 L}{6EI}$$

PROBLEM 11.93

11.93 For the beam and loading shown, (*a*) compute the work of the loads as they are applied successively to the beam, using the information provided in Appendix D, (*b*) compute the strain energy of the beam by the method of Sec. 11.4 and show that it is equal to the work obtained in part *a*.



SOLUTION

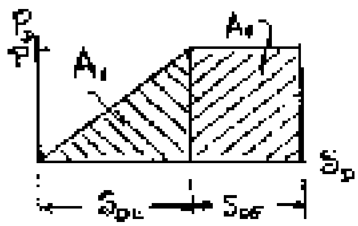
(a) Label the forces P_D and P_E.

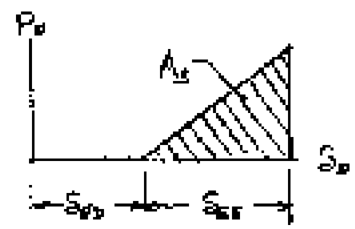
Using Appendix D Case 5

$$\delta_{EE} = \frac{P_E a^2 b^2}{3EIL} = \frac{P_E\left(\frac{3L}{4}\right)^2\left(\frac{L}{4}\right)^2}{3EIL} = \frac{3}{256}\frac{P_E L^3}{EI}$$

$$\delta_{DE} = \frac{P_E b}{6EIL}\left[(L^2 - b^2)x - x^3\right] = \frac{P_E\left(\frac{L}{4}\right)}{6EIL}\left[\left(L^2 - \left(\frac{L}{4}\right)^2\right)\left(\frac{L}{4}\right) - \left(\frac{L}{4}\right)^3\right] = \frac{7}{768}\frac{P_E L^3}{EI}$$

Likewise $\delta_{DD} = \frac{3}{256}\frac{P_D L^3}{EI}$ and $\delta_{ED} = \frac{7}{768}\frac{P_D L^3}{EI}$





Let P_D be applied first.

$$U = A_1 + A_2 + A_3$$

$$U = \frac{1}{2}P_D\delta_{DD} + P_D\delta_{DE} + \frac{1}{2}P_E\delta_{EE} = \frac{3}{512}\frac{P_D^2L^3}{EI} + \frac{7}{768}\frac{P_D P_E L^3}{EI} + \frac{3}{512}\frac{P_E^2 L^3}{EI}$$

With $P_D = P_E = P$ $\qquad U = \frac{1}{48}\frac{P^2L^3}{EI}$ ◄

(b) Reactions $R_A = R_B = P$

Over portion AD $\quad 0 < x < \frac{L}{4} \qquad M = Px$

$$U_{AD} = \int_0^{L/4}\frac{M^2}{2EI}dx = \frac{P^2}{2EI}\int_0^{L/4}x^2\,dx = \frac{P^2}{2EI}\cdot\frac{1}{3}\left(\frac{L}{4}\right)^3 = \frac{1}{384}\frac{P^2L^3}{EI}$$

Over portion DE $\quad M = \frac{PL}{4} \qquad U_{DE} = \frac{M^2\left(\frac{L}{2}\right)}{2EI} = \frac{P^2L^2}{2EI}\cdot\frac{1}{16}\cdot\frac{L}{2} = \frac{P^2L^3}{64EI}$

Over portion EB: $\quad$ By symmetry $\quad U_{EB} = U_{AD} = \frac{1}{384}\frac{P^2L^3}{EI}$

Total $\quad U = U_{AD} + U_{DE} + U_{EB} = \left(\frac{1}{384} + \frac{1}{64} + \frac{1}{384}\right)\frac{P^2L^3}{EI} = \frac{1}{48}\frac{P^2L^3}{EI}$ ◄

PROBLEM 11.94

11.94 For the beam and loading shown, (*a*) compute the work of the loads as they are applied successively to the beam, using the information provided in Appendix D, (*b*) compute the strain energy of the beam by the method of Sec. 11.4 and show that it is equal to the work obtained in part *a*.

SOLUTION

(a) Label the forces P_B and P_C

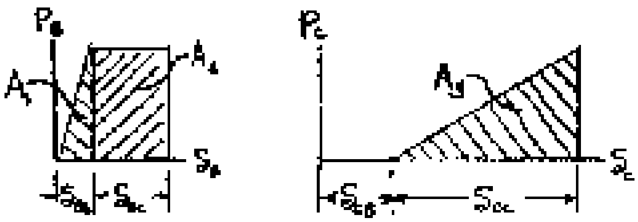
Using Appendix D Case 1

$$\delta_{BB} = \frac{P_B(L/2)^3}{3EI} = \frac{1}{24}\frac{P_B L^3}{EI}$$

$$\delta_{CB} = \delta_{BB} + \frac{L}{2}\theta_B = \frac{1}{24}\frac{P_B L^3}{EI} + \frac{L}{2}\frac{P_B(L/2)^2}{2EI} = \left(\frac{1}{24}+\frac{1}{16}\right)\frac{P_B L^3}{EI} = \frac{5}{48}\frac{P_B L^3}{EI}$$

$$\delta_{CC} = \frac{1}{3}\frac{P_C L^3}{EI}$$

$$\delta_{BC} = \frac{P_C}{6EI}(3Lx^2 - x^3) = \frac{P_C}{6EI}\left(3L\left(\frac{L}{2}\right)^2 - \left(\frac{L}{2}\right)^3\right) = \frac{5}{48}\frac{P_C L^3}{EI}$$



Apply P_B first, then P_C

$U = A_1 + A_2 + A_3$

$$U = \frac{1}{2}P_B\delta_{BB} + P_B\delta_{BC} + \frac{1}{2}P_C\delta_{CC} = \frac{1}{48}\frac{P_B^2L^3}{EI} + \frac{5}{48}\frac{P_BP_CL^3}{EI} + \frac{1}{6}\frac{P_C^2L^3}{EI}$$

With $P_B = P_C = P$ $\qquad U = \left(\frac{1}{48}+\frac{5}{48}+\frac{1}{6}\right)\frac{P^2L^3}{EI} = \frac{7}{24}\frac{P^2L^3}{EI}$

Over AB $\qquad M = Pv + P\left(v - \frac{L}{2}\right) = P\left(2v - \frac{L}{2}\right)$

$$U_{AB} = \int_{\frac{L}{2}}^{L}\frac{M^2}{2EI}dv = \frac{P^2}{2EI}\int_{\frac{L}{2}}^{L}\left(4v^2 - 2Lv + \frac{1}{4}L^2\right)dv$$

$$= \frac{P^2}{2EI}\left\{\frac{4}{3}\left[L^3 - \left(\frac{L}{2}\right)^3\right] - 2L\cdot\frac{1}{2}\left[L^2 - \left(\frac{L}{2}\right)^2\right] + \frac{1}{4}L^2\left[L - \frac{L}{2}\right]\right\}$$

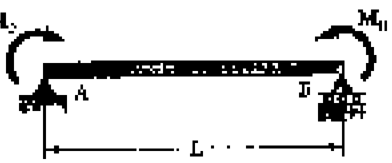
$$= \frac{P^2}{2EI}\left\{\frac{7}{6}L^3 - \frac{3}{4}L^3 + \frac{1}{8}L^3\right\} = \frac{13}{48}\frac{P^2L^3}{EI}$$

Over BC $\qquad M = Pv \qquad U_{BC} = \int_0^{\frac{L}{2}}\frac{M^2}{2EI}dv = \frac{P^2}{2EI}\int_0^{\frac{L}{2}}v^2dv = \frac{P^2}{2EI}\cdot\frac{1}{3}\left(\frac{L}{2}\right)^3$

$$= \frac{P^2L^3}{48EI}$$

Total $\qquad U = U_{AB} + U_{BC} = \left(\frac{13}{48}+\frac{1}{48}\right)\frac{P^2L^3}{EI} = \frac{7}{24}\frac{P^2L^3}{EI}$

PROBLEM 11.95



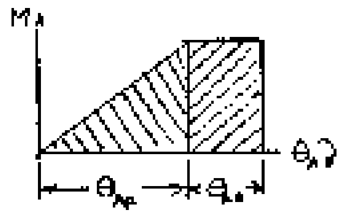
11.95 For the beam and loading shown, (*a*) compute the work of the loads as they are applied successively to the beam, using the information provided in Appendix D, (*b*) compute the strain energy of the beam by the method of Sec. 11.4 and show that it is equal to the work obtained in part *a*.

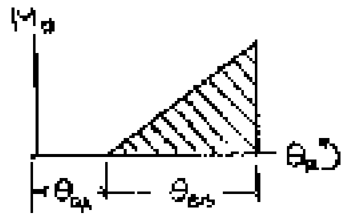
SOLUTION

(a) Label the couples M_A and M_B

Using Appendix D, Case 7

$$\circlearrowleft \theta_{AA} = \frac{M_A L}{3EI} \qquad \circlearrowleft \theta_{BA} = \frac{M_A L}{6EI} \qquad \circlearrowleft \theta_{BB} = \frac{M_B L}{3EI} \qquad \circlearrowleft \theta_{AB} = \frac{M_B L}{6EI}$$





Apply M_A first, then M_B.

$$U = A_1 + A_2 + A_3$$

$$U = \tfrac{1}{2} M_A \theta_{AA} + M_A \theta_{AB} + \tfrac{1}{2} M_B \theta_{BB} = \frac{1}{6}\frac{M_A^2 L}{EI} + \frac{1}{6}\frac{M_A M_B L}{EI} + \frac{1}{6}\frac{M_B^2 L}{EI}$$

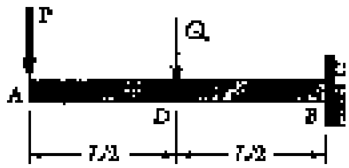
With $M_A = M_B = M_0$ $\qquad U = \frac{1}{2}\frac{M_0^2 L}{EI}$

(b) Bending moment $\qquad M = M_0$

$$U = \int_0^L \frac{M^2}{2EI}\,dx = \frac{M_0^2 L}{2EI}$$

PROBLEM 11.96

11.96 For the prismatic beam shown, determine the deflection at point *D*.



SOLUTION

Add force Q at point D.

$$U = \int_0^L \frac{M^2}{2EI}\,dx$$

$$\delta_D = \frac{\partial U}{\partial Q} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial Q}\,dx = \frac{1}{EI}\int_0^L M\frac{\partial M}{\partial Q}\,dx$$

Over portion AD $\quad 0 < x < \frac{L}{2} \qquad M = -Px, \qquad \frac{\partial M}{\partial Q} = 0$

Over portion DB $\quad \frac{L}{2} < x < L \qquad M = -Px - Q\left(x - \frac{L}{2}\right), \qquad \frac{\partial M}{\partial Q} = -\left(x - \frac{L}{2}\right)$

Set $Q = 0$

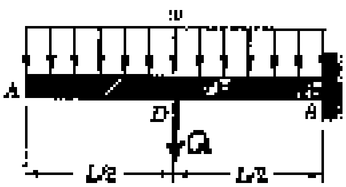
$$\delta_D = \frac{1}{EI}\int_0^{L/2}(-Px)(0)\,dx + \frac{1}{EI}\int_{L/2}^{L}(-Px)\left[-\left(x - \frac{L}{2}\right)\right]dx$$

$$= \frac{P}{EI}\int_{L/2}^{L}\left(x^2 - \frac{L}{2}x\right)dx = \frac{P}{EI}\left\{\frac{1}{3}L^3 - \frac{1}{3}\left(\frac{L}{2}\right)^3 - \left(\frac{L}{2}\right)\frac{1}{2}L^2 + \frac{L}{2}\cdot\frac{1}{2}\left(\frac{L}{2}\right)^2\right\}$$

$$= \left(\frac{1}{3} - \frac{1}{24} - \frac{1}{4} + \frac{1}{16}\right)\frac{PL^3}{EI} = \frac{5}{48}\frac{PL^3}{EI} \quad \blacktriangleleft$$

PROBLEM 11.97

11.97 For the prismatic beam shown, determine the deflection at point *D*.



SOLUTION

Add force Q at point D.

$$U = \int_0^L \frac{M^2}{2EI}\,dx$$

$$\delta_D = \frac{\partial U}{\partial Q} = \frac{1}{EI}\int_0^L M\frac{\partial M}{\partial Q}\,dx$$

Over portion AD $\quad 0 < x < \frac{L}{2} \qquad M = -\frac{1}{2}wx^2 \qquad \frac{\partial M}{\partial Q} = 0$

Over portion DB $\quad \frac{L}{2} < x < L \qquad M = -\frac{1}{2}wx^2 - Q\left(x - \frac{L}{2}\right) \qquad \frac{\partial M}{\partial Q} = -\left(x - \frac{L}{2}\right)$

Set $Q = 0$

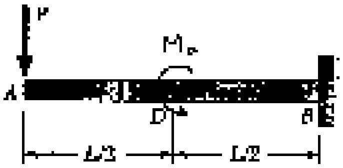
$$\delta_D = \frac{1}{EI}\int_0^{L/2}\left[-\frac{1}{2}wx^2\right](0)\,dx + \frac{1}{EI}\int_{L/2}^{L}\left(-\frac{1}{2}wx^2\right)\left[-\left(x - \frac{L}{2}\right)\right]dx$$

$$= \frac{w}{2EI}\int_{L/2}^{L}\left(x^3 - \frac{L}{2}x^2\right)dx = \frac{w}{2EI}\left\{\frac{1}{4}L^4 - \frac{1}{4}\left(\frac{L}{2}\right)^4 - \left(\frac{L}{2}\right)\frac{1}{3}L^3 + \left(\frac{L}{2}\right)\frac{1}{3}\left(\frac{L}{2}\right)^3\right\}$$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{64} - \frac{1}{6} + \frac{1}{48}\right)\frac{wL^4}{EI} = \frac{17}{384}\frac{wL^4}{EI} = 0.04427\frac{wL^4}{EI} \quad \blacktriangleleft$$

PROBLEM 11.98

11.98 For the prismatic beam shown, determine the slope at point D.



SOLUTION

Add couple M_0 at point D.

$$U = \int_0^L \frac{M^2}{2EI}\,dx$$

$$\theta_D = \frac{\partial U}{\partial M_0} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_0}\,dx = \frac{1}{EI}\int_0^L M\frac{\partial M}{\partial M_0}\,dx$$

Over portion AD $\quad 0 < x < \frac{L}{2} \qquad M = -Px \qquad \frac{\partial M}{\partial M_0} = 0$

Over portion DB $\quad \frac{L}{2} < x < L \qquad M = -Px - M_0 \qquad \frac{\partial M}{\partial M_0} = -1$

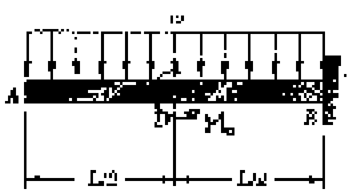
Set $M_0 = 0$. $\quad \theta_D = \frac{1}{EI}\int_0^{L/2}(-Px)(0)\,dx + \frac{1}{EI}\int_{L/2}^{L}(-Px)(-1)\,dx$

$$= \frac{P}{EI}\int_{L/2}^{L} x\,dx = \frac{P}{EI}\left[\frac{1}{2}L^2 - \frac{1}{2}\left(\frac{L}{2}\right)^2\right]$$

$$= \left(\frac{1}{2} - \frac{1}{8}\right)\frac{PL^2}{EI} = \frac{3}{8}\frac{PL^2}{EI}$$

PROBLEM 11.99

11.99 For the prismatic beam shown, determine the slope at point D.



SOLUTION

Add couple M_0 at point D.

$$U = \int_0^L \frac{M^2}{2EI}\,dx$$

$$\theta_D = \frac{\partial U}{\partial M_0} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_0}\,dx = \frac{1}{EI}\int_0^L M\frac{\partial M}{\partial M_0}\,dx$$

Over portion AD $\quad 0 < x < \frac{L}{2} \qquad M = -\frac{1}{2}wx^2 \qquad \frac{\partial M}{\partial M_0} = 0$

Over portion DB $\quad \frac{L}{2} < x < L \qquad M = -\frac{1}{2}wx^2 - M_0 \qquad \frac{\partial M}{\partial M_0} = -1$

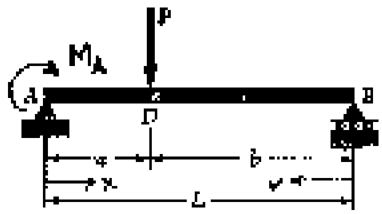
Set $M_0 = 0$. $\quad \theta_D = \frac{1}{EI}\int_0^{L/2}\left(-\frac{1}{2}wx^2\right)(0)\,dx + \frac{1}{EI}\int_{L/2}^{L}\left(-\frac{1}{2}wx^2\right)(-1)\,dx$

$$= \frac{w}{2EI}\int_{L/2}^{L} x^2\,dx = \frac{w}{2EI}\left[\frac{1}{3}L^3 - \frac{1}{3}\left(\frac{L}{2}\right)^3\right]$$

$$\frac{1}{6}\left(1 - \frac{1}{8}\right)\frac{wL^3}{EI} = \frac{7}{48}\frac{wL^3}{EI}$$

PROBLEM 11.100

11.100 and *11.101* For the prismatic beam shown, determine the slope at point A.



SOLUTION

Add couple M_A at point A.

Reactions: $R_A = \frac{Pb}{L} - \frac{M_A}{L}$, $R_B = \frac{Pa}{L} + \frac{M_A}{L}$

$$U = \int_0^L \frac{M^2}{2EI}dx = \frac{1}{2EI}\int_0^a M^2 dx + \frac{1}{2EI}\int_0^b M^2 dv$$

$$\theta_A = \frac{\partial U}{\partial M_A} = \frac{1}{EI}\int_0^a M\frac{\partial M}{\partial M_A}dx + \frac{1}{EI}\int_0^b M\frac{\partial M}{\partial M_A}dv$$

Over portion AD $(0<x<a)$ $\quad M = M_A + R_A x = M_A\left(1-\frac{x}{L}\right) + \frac{Pbx}{L}$, $\quad \frac{\partial M}{\partial M_A} = 1 - \frac{x}{L}$

Over portion DB $(0<v<b)$ $\quad M = R_B v = \frac{Pav}{L} + \frac{M_A v}{L}$, $\quad \frac{\partial M}{\partial M_A} = \frac{v}{L}$

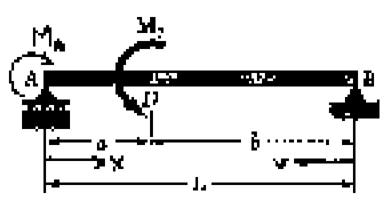
Set $M_A = 0$ $\quad \theta_A = \frac{1}{EI}\int_0^a \left(\frac{Pbx}{L}\right)\left(1-\frac{x}{L}\right)dx + \frac{1}{EI}\int_0^b \frac{Pav}{L}\frac{v}{L}dv$

$$= \frac{P}{EIL^2}\left(\tfrac{1}{2}bLa^2 - \tfrac{1}{3}ba^3 + \tfrac{1}{3}ab^3\right)$$

$$= \frac{Pab}{6EIL^2}\left(3La - 2a^2 + 2b^2\right)$$

PROBLEM 11.101

11.100 and *11.101* For the prismatic beam shown, determine the slope at point A.



SOLUTION

Add couple M_A at point A.

Reactions: Positive if upward

$R_A = \frac{M_0 - M_A}{L}$, $\quad R_B = \frac{M_A - M_0}{L}$

$$U = \int_0^L \frac{M^2}{2EI}dx = \frac{1}{2EI}\int_0^a M^2 dx + \frac{1}{2EI}\int_0^b M^2 dv$$

$$\theta_A = \frac{\partial U}{\partial M_A} = \frac{1}{EI}\int_0^a M\frac{\partial M}{\partial M_A}dx + \frac{1}{EI}\int_0^b M\frac{\partial M}{\partial M_A}dv$$

Over portion AD $(0<x<a)$ $\quad M = M_A + R_A x = M_A\left(1-\frac{x}{L}\right) + \frac{M_0 x}{L}$, $\quad \frac{\partial M}{\partial M_A} = 1 - \frac{x}{L}$

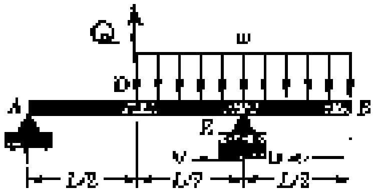
Over portion DB $(0<v<b)$ $\quad M = R_B v = \frac{(M_A - M_0)v}{L}$ $\quad \frac{\partial M}{\partial M_A} = \frac{v}{L}$

Set $M_A = 0$ $\quad \theta_A = \frac{1}{EI}\int_0^a \left(\frac{M_0 x}{L}\right)\left(1-\frac{x}{L}\right)dx + \frac{1}{EI}\int_0^b \left(\frac{M_0 v}{L}\right)\left(-\frac{v}{L}\right)dv$

$$= \frac{M_0}{EIL^2}\left(\tfrac{1}{2}La^2 - \tfrac{1}{3}a^3 - \tfrac{1}{3}b^3\right)$$

$$= \frac{M_0}{6EIL^2}\left(3La^2 - 2a^3 - 2b^3\right)$$

PROBLEM 11.102



11.102 For the prismatic beam shown, determine the deflection at point D.

SOLUTION

Add force Q at point D.

Reactions: $R_A = -\frac{1}{2}Q$, $\quad R_B = wL - \frac{1}{2}Q$

$U = U_{AD} + U_{DE} + U_{EB}$; $\qquad \delta_D = \frac{\partial U}{\partial Q}$

Over portion AD: $\quad$ with $Q = 0 \qquad M = 0 \qquad \frac{\partial U_{AD}}{\partial Q} = 0$

Over portion DE: $\quad M = R_B v - \frac{1}{2}w\left(v + \frac{L}{2}\right)^2 = wLv - \frac{1}{2}w\left(v+\frac{L}{2}\right)^2 - \frac{1}{2}Qv$

$$\frac{\partial M}{\partial Q} = -\frac{1}{2}v \qquad U_{DE} = \frac{1}{2EI}\int_0^{L/2} M^2\,dv \qquad \text{Set } Q = 0$$

$$\frac{\partial U_{DE}}{\partial Q} = \frac{1}{EI}\int_0^{L/2} M\frac{\partial M}{\partial Q}\,dv = \frac{1}{EI}\int_0^{L/2}\left[wLv - \frac{1}{2}w\left(v+\frac{L}{2}\right)^2\right]\left(-\frac{1}{2}v\right)dv$$

$$= \frac{w}{2EI}\int_0^{L/2}\left[-Lv^2 + \frac{1}{2}\left(v^3 + Lv^2 + \frac{1}{4}L^2 v\right)\right]dv$$

$$= \frac{w}{2EI}\left[-L\cdot\frac{1}{3}\left(\frac{L}{2}\right)^3 + \frac{1}{2}\left(\frac{1}{4}\left(\frac{L}{2}\right)^4 + L\cdot\frac{1}{3}\left(\frac{L}{2}\right)^3 + \frac{1}{4}L^2\frac{1}{2}\left(\frac{L}{2}\right)^2\right)\right]$$

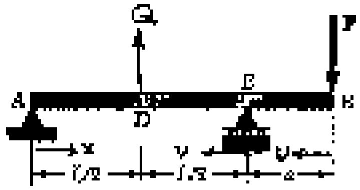
$$= \frac{1}{2}\left(-\frac{1}{24} + \frac{1}{128} + \frac{1}{48} + \frac{1}{64}\right)\frac{wL^4}{EI} = \frac{1}{768}\frac{wL^4}{EI}$$

Over portion EB: $\quad M = -\frac{1}{2}wu^2 \qquad \frac{\partial M}{\partial Q} = 0 \qquad \frac{\partial U_{EB}}{\partial Q} = 0$

$$\delta_D = \frac{\partial U_{AD}}{\partial Q} + \frac{\partial U_{DE}}{\partial Q} + \frac{\partial U_{EB}}{\partial Q} = 0 + \frac{1}{768}\frac{wL^4}{EI} + 0 = \frac{1}{768}\frac{wL^4}{EI}$$

PROBLEM 11.103

11.103 For the prismatic beam shown, determine the deflection at point D.



SOLUTION

Add force Q at point D.

Reactions $R_A = -\frac{Pa}{L} - \frac{1}{2}Q$, $\quad R_E = \frac{P(a+L)}{L} + \frac{1}{2}Q$

$U = U_{AD} + U_{DE} + U_{EB}$; $\quad \delta_D = \frac{\partial U}{\partial Q}$

Over portion AD: $U_{AD} = \int_0^{L/2} \frac{M^2}{2EI}dx$, $\quad M = R_A x = -\frac{Pa}{L}x - \frac{1}{2}Qx$, $\quad \frac{\partial M}{\partial Q} = -\frac{1}{2}x$

Set $Q = 0$. $\quad \frac{\partial U_{AD}}{\partial Q} = \frac{1}{EI}\int_0^{L/2}\left(-\frac{Pa}{L}x\right)\left(-\frac{1}{2}x\right)dx = \frac{Pa}{2EIL}\int_0^{L/2} x^2 dx$

$$= \frac{Pa}{2EIL}\,\frac{1}{3}\left(\frac{L}{2}\right)^3 = \frac{PaL^2}{48EI}$$

Over portion DE: $U_{DE} = \int_0^{L/2}\frac{M^2}{2EI}dv$ $\qquad \frac{\partial M}{\partial Q} = -\frac{1}{2}v$

$M = R_E v - P(a+v) = \frac{P(a+L)}{L}v - \frac{1}{2}Qv - P(a+v) = \frac{Pa}{L}v - Pa - \frac{1}{2}Qv$

Set $Q = 0$. $\quad \frac{\partial U_{DE}}{\partial Q} = \frac{1}{EI}\int_0^{L/2}\left(\frac{Pa}{L}v - Pa\right)\left(-\frac{1}{2}v\right) = \frac{Pa}{2EIL}\int_0^{L/2}(-v^2 + Lv)\,dv$

$$= \frac{Pa}{2EIL}\left[-\frac{1}{3}\left(\frac{L}{2}\right)^3 + (L)\frac{1}{2}\left(\frac{L}{2}\right)^2\right] = \frac{Pa}{2EIL}\left[-\frac{L^3}{24} + \frac{L^3}{8}\right]$$

$$= \frac{1}{24}\frac{PaL^2}{EI}$$

Over portion EB $\qquad M = -Pu \qquad \frac{\partial M}{\partial Q} = 0 \qquad \frac{\partial U_{EB}}{\partial Q} = 0$

$$\delta_D = \frac{\partial U_{AD}}{\partial Q} + \frac{\partial U_{DE}}{\partial Q} + \frac{\partial U_{EB}}{\partial Q} = \frac{PaL^2}{48EI} + \frac{PaL^2}{24EI} + 0 = \frac{1}{16}\frac{PaL^2}{EI}\uparrow$$

PROBLEM 11.104

11.104 For the prismatic beam shown, determine the slope at point *D*.

SOLUTION

Add couple M_0 at point D.

Reactions: $R_A = \frac{M_0}{L}$, $\quad R_B = wL - \frac{M_0}{L}$

$U = U_{AD} + U_{DE} + U_{EB}$, $\qquad \theta_D = \frac{\partial U}{\partial M_0}$

Over portion AD: $\quad M = \frac{M_0}{L} x = 0$ with $M_0 = 0$ $\qquad \frac{\partial U_{AD}}{\partial M_0} = 0$

Over portion DE: $\quad M = R_B v - \frac{1}{2} w \left(v + \frac{L}{2}\right)^2 = wLv - \frac{1}{2} w \left(v + \frac{L}{2}\right)^2 - \frac{M_0}{L} v$

$\frac{\partial M}{\partial M_0} = -\frac{1}{L} v$, $\qquad U_{DE} = \frac{1}{2EI} \int_0^{L/2} M^2 \, dv \qquad$ Set $M_0 = 0$

$$\frac{\partial U_{DE}}{\partial M_0} = \frac{1}{EI} \int_0^{L/2} M \frac{\partial M}{\partial M_0} \, dv = \frac{1}{EI} \int_0^{L/2} \left[wLv - \frac{1}{2} w \left(v + \frac{L}{2}\right)^2\right]\left(-\frac{1}{L} v\right) dv$$

$$= \frac{w}{EIL} \int_0^{L/2} \left[-Lv^2 + \frac{1}{2}\left(v^3 + Lv^2 + \frac{1}{4}L^2 v\right)\right] dv$$

$$= \frac{w}{EIL}\left[-L \cdot \frac{1}{3}\left(\frac{L}{2}\right)^3 + \frac{1}{2}\left(\frac{1}{4}\left(\frac{L}{2}\right)^4 + L \cdot \frac{1}{3}\left(\frac{L}{2}\right)^3 + \frac{1}{4} L^2 \frac{1}{2}\left(\frac{L}{2}\right)^2\right)\right]$$

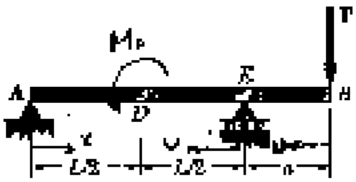
$$= \left(-\frac{1}{24} + \frac{1}{128} + \frac{1}{48} + \frac{1}{64}\right)\frac{wL^3}{EI} = \frac{1}{384} \frac{wL^3}{EI}$$

Over portion EB: $\quad M = -\frac{1}{2} w u^2 \qquad \frac{\partial M}{\partial M_0} = 0 \qquad \frac{\partial U_{EB}}{\partial M_0} = 0$

$$\theta_D = \frac{\partial U_{AD}}{\partial M_0} + \frac{\partial U_{DE}}{\partial M_0} + \frac{\partial U_{EB}}{\partial M_0} = 0 + \frac{1}{384} \frac{wL^3}{EI} + 0 = \frac{1}{384} \frac{wL^3}{EI}$$

PROBLEM 11.105

11.105 For the prismatic beam shown, determine the slope at point D.



SOLUTION

Add couple M_0 at point D.

Reactions: $R_A = -\frac{Pa}{L} + \frac{M_0}{L}$, $\quad R_E = \frac{P(a+L)}{L} - \frac{M_0}{L}$

$U = U_{AD} + U_{DE} + U_{EB} \qquad \theta_D = \frac{\partial U}{\partial M_0}$

Over portion AD: $\quad U_{AD} = \int_0^{\frac{L}{2}} \frac{M^2}{2EI}\,dx, \quad M = R_A x = -\frac{Pa}{L}x + \frac{M_0}{L}x, \quad \frac{\partial M}{\partial M_0} = \frac{1}{L}x$

Set $M_0 = 0$, $\quad \frac{\partial U_{AD}}{\partial M_0} = \frac{1}{EI}\int_0^{\frac{L}{2}}\left(-\frac{Pa}{L}x\right)\left(\frac{1}{L}x\right) = -\frac{Pa}{EIL^2}\int_0^{\frac{L}{2}} x^2\,dx$

$= -\frac{Pa}{EIL^2}\,\frac{1}{3}\left(\frac{L}{2}\right)^3 = -\frac{PaL}{24EI}$

Over portion DE: $\quad U_{DE} = \int_0^{\frac{L}{2}} \frac{M^2}{2EI}\,dv \qquad\qquad \frac{\partial M}{\partial M_0} = -\frac{1}{L}v$

$M = R_E v - P(a+v) = \frac{P(a+L)}{L}v - \frac{M_0}{L}v - P(a+v) = \frac{Pa}{L}v - Pa - \frac{M_0}{L}v$

Set $M_0 = 0$, $\quad \frac{\partial U_{DE}}{\partial M_0} = \frac{1}{EI}\int_0^{\frac{L}{2}}\left(\frac{Pa}{L}v - Pa\right)\left(-\frac{1}{L}v\right)dv = -\frac{Pa}{EIL^2}\int_0^{\frac{L}{2}}(v^2 - Lv)\,dv$

$= -\frac{Pa}{EIL^2}\left[\frac{1}{3}\left(\frac{L}{2}\right)^3 - L\cdot\frac{1}{2}\left(\frac{L}{2}\right)^2\right] = -\frac{Pa}{EIL^2}\left[\frac{1}{24}L^3 - \frac{1}{8}L^3\right]$

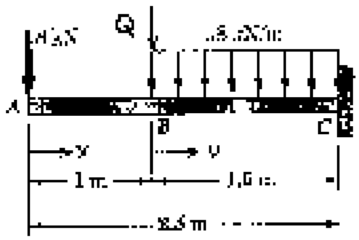
$= \frac{1}{12}\frac{PaL}{EI}$

Over portion EB: $\quad M = -Pu \qquad \frac{\partial M}{\partial M_0} = 0 \qquad \frac{\partial U_{EB}}{\partial M_0} = 0$

Total $\quad \theta_D = \frac{\partial U_{AD}}{\partial M_0} + \frac{\partial U_{DE}}{\partial M_0} + \frac{\partial U_{EB}}{\partial M_0} = -\frac{1}{24}\frac{PaL}{EI} + \frac{1}{12}\frac{PaL}{EI} + 0 = \frac{1}{24}\frac{PaL}{EI}$

PROBLEM 11.106

11.106 For the beam and loading shown, determine the deflection at point B. Use $E = 200$ GPa.



W250 × 22.3

SOLUTION

Add force Q at point B.

Units: Forces in kN, lengths in m.

Over AB $\quad M = -8x \qquad \frac{\partial M}{\partial Q} = 0$

Over BC $\quad M = -8(v+1) - \frac{1}{2}(18)v^2 - Qv \qquad \frac{\partial M}{\partial Q} = -v$

$E = 200\times10^9$ Pa, $\quad I = 28.9\times10^6\ \text{mm}^4 = 28.9\times10^{-6}\ \text{m}^4$

$EI = (200\times10^9)(28.9\times10^{-6}) = 5.78\times10^6\ \text{N}\cdot\text{m}^2 = 5780\ \text{kN}\cdot\text{m}^2$

$$U = \int_0^1 \frac{M^2}{2EI}dx + \int_0^{1.5}\frac{M^2}{2EI}dv$$

$$\delta_B = \frac{\partial U}{\partial Q} = \frac{1}{EI}\left\{\int_0^1 M\frac{\partial M}{\partial Q}dx + \int_0^{1.5} M\frac{\partial M}{\partial Q}dv\right\}$$

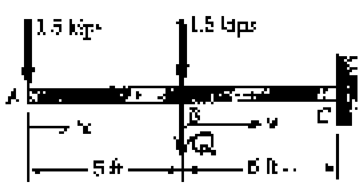
$$= \frac{1}{EI}\left\{0 + \int_0^{1.5}\left[-8(v+1) - \tfrac{1}{2}(18)v^2\right](-v)\,dv\right\} = \frac{1}{EI}\int_0^{1.5}(9v^3 + 8v^2 + 8v)\,dv$$

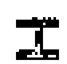
$$= \frac{1}{EI}\left\{\tfrac{9}{4}(1.5)^4 + \tfrac{8}{3}(1.5)^3 + \tfrac{8}{2}(1.5)^2\right\} = \frac{29.391}{EI} = \frac{29.391}{5780}$$

$$= 5.08\times10^{-3}\ \text{m} = 5.08\ \text{mm} \downarrow$$

PROBLEM 11.107

11.107 For the beam and loading shown, determine the deflection at point B. Use $E = 29\times10^6$ psi.





W8 × 13

SOLUTION

Add force Q at point B

Units: forces in kips, lengths in ft.

$E = 29\times10^3$ ksi $\qquad I = 39.6\ \text{in}^4$

$EI = (29\times10^3)(39.6) = 1.148\times10^6\ \text{kip}\cdot\text{in}^2 = 7975\ \text{kip}\cdot\text{ft}^2$

$$U = \int_0^5 \frac{M^2}{2EI}dx + \int_0^5 \frac{M^2}{2EI}dv \qquad \delta_B = \frac{\partial U}{\partial Q} = \frac{1}{EI}\left\{\int_0^5 M\frac{\partial M}{\partial Q}dx + \int_0^5 M\frac{\partial M}{\partial Q}dv\right\}$$

Over AB: $\quad M = -1.5x, \quad \frac{\partial M}{\partial Q} = 0 \qquad \int_0^5 M\frac{\partial M}{\partial Q}dx = 0$

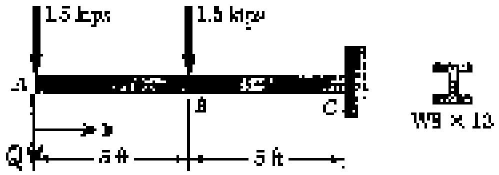
Over BC: $\quad M = -1.5(v+5) - 1.5v - Qv = -3v - 7.5 - Qv, \qquad \frac{\partial M}{\partial Q} = -v$

$$\int_0^5 M\frac{\partial M}{\partial Q}dv = \int_0^5 (3v^2 + 7.5v)\,dv = (3)(\tfrac{1}{3})(5)^3 + (7.5)(\tfrac{1}{2})(5)^2 = 218.75$$

$$\delta_B = \frac{1}{EI}\{0 + 143.75\} = \frac{218.75}{7975} = 27.43\times10^{-3}\ \text{ft} = 0.329\ \text{in.} \downarrow$$

PROBLEM 11.108

11.108 For the beam and loading shown, determine the deflection at point A. Use $E = 29 \times 10^3$ ksi.



SOLUTION

Add force Q at point A.

Units: Forces in kips, lengths in ft.

$E = 29 \times 10^3$ ksi, $\quad I = 39.6\ \text{in}^4$

$$EI = (29\times10^3)(39.6) = 1.148\times10^6\ \text{kip}\cdot\text{in}^2 = 7975\ \text{kip}\cdot\text{ft}^2$$

$$U = \int_0^{10} \frac{M^2}{2EI}\,dx \qquad \delta_A = \frac{\partial U}{\partial Q} = \frac{1}{EI}\int_0^{10} M\frac{\partial M}{\partial Q}\,dx$$

Over portion AB $\quad 0<x<5, \quad M = -1.5x - Qx \qquad \frac{\partial M}{\partial Q} = -x$

$$\int_0^5 M\frac{\partial M}{\partial Q}\,dx = \int_0^5 (1.5x)(x)\,dx = 1.5\int_0^5 x^2\,dx = (1.5)(\tfrac{1}{3})(5)^3 = 62.5$$

Over portion BC $\quad 5<x<10 \quad M = -1.5x - 1.5(x-5) - Qx$

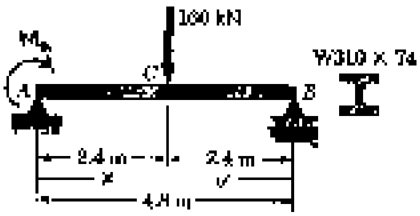
$$M = -3x + 7.5 - Qx \qquad \frac{\partial M}{\partial Q} = -x$$

$$\int_5^{10} M\frac{\partial M}{\partial Q}\,dx = \int_5^{10}(3x^2 - 7.5x)\,dx = (3)(\tfrac{1}{3})(10^3 - 5^3) - (7.5)(\tfrac{1}{2})(10^2 - 5^2)$$
$$= 593.75$$

$$\delta_A = \frac{1}{EI}\{62.5 + 593.75\} = \frac{656.25}{7975} = 82.29\times10^{-3}\ \text{ft} = 0.987\ \text{in.}\downarrow$$

PROBLEM 11.109

11.109 For the beam and loading shown, determine the slope at end A. Use $E = 200$ GPa.



SOLUTION

Add couple M_A at point A.

Units: forces in kN, lengths in m.

$E = 200 \times 10^9$ Pa, $I = 165 \times 10^6$ mm^4 $= 165 \times 10^{-6}$ m^4

$EI = (200 \times 10^9)(165 \times 10^{-6}) = 33 \times 10^6$ N·m^2 $= 33000$ kN·m^2

Reactions: $R_A = 80 - \dfrac{M_A}{4.8}$ $\qquad R_B = 80 + \dfrac{M_A}{L}$

$$U = U_{AB} + U_{BC} = \int_0^{2.4} \frac{M^2}{2EI}dx + \int_0^{2.4} \frac{M^2}{2EI}dv \qquad \theta_A = \frac{\partial U}{\partial M_A} = \frac{\partial U_{AB}}{\partial M_A} + \frac{\partial U_{BC}}{\partial M_A}$$

Over AB: $\quad M = M_A + R_A x = M_A + 80x - \dfrac{M_A}{4.8}x \qquad \dfrac{\partial M}{\partial M_A} = \left(1 - \dfrac{x}{4.8}\right)$

Set $M_A = 0$ $\quad \dfrac{\partial U_{AB}}{\partial M_A} = \dfrac{1}{EI}\displaystyle\int_0^{2.4}(80x)\left(1 - \frac{x}{4.8}\right)dx = \frac{1}{EI}\int_0^{2.4}(80x - 16.6667x^2)dx$

$$= \frac{1}{EI}\left\{(80)\left(\tfrac{1}{2}\right)(2.4)^2 - (16.6667)\left(\tfrac{1}{3}\right)(2.4)^3\right\} = \frac{153.6}{EI}$$

Over BC: $\quad M = R_B v = 80v + \dfrac{M_A}{4.8}v, \qquad \dfrac{\partial M}{\partial M_A} = \dfrac{1}{4.8}v$

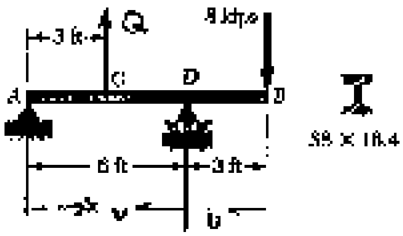
Set $M_A = 0$ $\quad \dfrac{\partial U_{BC}}{\partial M_A} = \dfrac{1}{EI}\displaystyle\int_0^{2.4}(80v)\left(\frac{1}{4.8}v\right)dv = \frac{16.6667}{EI}\int_0^{2.4} v^2\,dv$

$$= \frac{(16.6667)(2.4)^3}{3\,EI} = \frac{76.8}{EI}$$

$$\theta_A = \frac{1}{EI}\{153.6 + 76.8\} = \frac{230.4}{33000} = 6.98 \times 10^{-3} \text{ rad.}$$

PROBLEM 11.110

11.110 For the beam and loading shown, determine the deflection at point C. Use $E = 29 \times 10^3$ ksi.



SOLUTION

Units: Forces in kip, lengths in ft.

$E = 29 \times 10^3$ ksi $\quad I = 57.6$ in^4

$EI = (29 \times 10^3)(57.6) = 1.6704 \times 10^6$ kip·in^2 $= 11600$ kip·ft^2

Add dummy force Q at point C. Reactions $R_A = 4 + \frac{1}{2}Q \downarrow$, $R_D = 12 - \frac{1}{2}Q \uparrow$

$$U = U_{AC} + U_{CD} + U_{DB} \qquad \delta_C = \frac{\partial U}{\partial Q} = \frac{\partial U_{AC}}{\partial Q} + \frac{\partial U_{CD}}{\partial Q} + \frac{\partial U_{DB}}{\partial Q}$$

Over AC $\quad 0 < x < 3 \quad M = -(4 + \frac{1}{2}Q)x \quad \frac{\partial M}{\partial Q} = -\frac{1}{2}x \quad$ Set $Q = 0$.

$$\frac{\partial U_{AC}}{\partial Q} = \frac{1}{EI}\int_0^3 (4x)(\tfrac{1}{2}x)\,dx = \frac{2}{EI}\int_0^3 x^2\,dx = \frac{(2)(3)^3}{3EI} = \frac{18}{EI}$$

Over CD $\quad 0 < v < 3 \quad M = R_D v - 8(v+3) = 12v - \frac{1}{2}Q - 8v - 24 = 4v - 24 - \frac{1}{2}Qv$

$$\frac{\partial M}{\partial Q} = -\frac{1}{2}v \qquad \text{Set } Q = 0$$

$$\frac{\partial U_{CD}}{\partial Q} = \frac{1}{EI}\int_0^3 (24 - 4v)(\tfrac{1}{2}v)\,dv = \frac{1}{EI}\int_0^3 (12v - 2v^2)\,dv = \frac{1}{EI}\left\{(12)\frac{(3)^2}{2} - (2)\frac{(3)^3}{3}\right\}$$

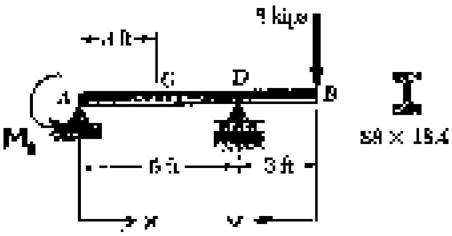
$$= \frac{36}{EI}$$

Over DB $\quad 0 < u < 3 \quad M = -8u \quad \frac{\partial M}{\partial Q} = 0 \quad \frac{\partial U_{DB}}{\partial Q} = 0$

$$\delta_C = \frac{18}{EI} + \frac{36}{EI} + 0 = \frac{54}{11600} = 4.655 \times 10^{-3}\ \text{ft} = 0.0559\ \text{in.} \uparrow$$

PROBLEM 11.111

11.111 For the beam and loading shown, determine the slope at end *A*. Use $E = 29 \times 10^3$ ksi.



SOLUTION

Units: Forces in kips, lengths in ft.

$E = 29 \times 10^3$ ksi, $\quad I = 57.6\ \text{in}^4$

$EI = (29 \times 10^3)(57.6) = 1.6704 \times 10^6\ \text{kip}\cdot\text{in}^2 = 11600\ \text{kip}\cdot\text{ft}^2$

Add dummy couple M_A at end A. Reactions: $R_A = -4 + \frac{M_A}{6}$, $\quad R_B = 12 - \frac{M_A}{6}$

$$U = U_{AD} + U_{DB} \qquad \theta_A = \frac{\partial U}{\partial M_A} = \frac{\partial U_{AD}}{\partial M_A} + \frac{\partial U_{DB}}{\partial M_A}$$

Over AD $\quad 0 < x < 6 \qquad M = -M_A + R_A x = -M_A - 4x + \frac{M_A}{6}x$

$$\frac{\partial M}{\partial M_A} = -\left(1 - \frac{x}{6}\right) \qquad \text{Set } M_A = 0.$$

$$\frac{\partial U_{AD}}{\partial M_A} = \frac{1}{EI}\int_0^6 (4x)\left(1 - \frac{x}{6}\right)dx = \frac{1}{EI}\int_0^6 \left(4x - \frac{2}{3}x^2\right)dx = \frac{1}{EI}\left\{(4)\frac{6^2}{2} - \frac{2}{3}\,\frac{6^3}{3}\right\}$$

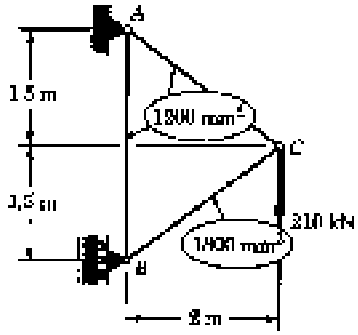
$$= \frac{24}{EI}$$

Over DB $\quad 0 < v < 3 \qquad M = -9v \qquad \frac{\partial M}{\partial M_A} = 0 \qquad \frac{\partial U_{DB}}{\partial M_A} = 0$

$$\theta_A = \frac{24}{EI} + 0 = \frac{24}{11600} = 2.07 \times 10^{-3}\ \text{rad}$$

PROBLEM 11.112

11.112 Each member of the truss shown is made of steel and has the cross-sectional area shown. Using $E = 200$ GPa, determine the vertical deflection of joint C.

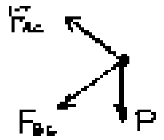


SOLUTION

Call the vertical load P. The vertical deflection of joint C is δ_P

$$\delta_P = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\sum\frac{F^2L}{2EA} = \frac{1}{E}\sum\frac{FL}{A}\frac{\partial F}{\partial P}$$

Joint C: $\xrightarrow{+}\Sigma F_x = 0 \qquad -\frac{4}{5}F_{AC} - \frac{4}{5}F_{BC} = 0$

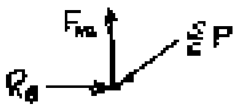


$+\uparrow\Sigma F_y \qquad \frac{3}{5}F_{AC} + \frac{3}{5}F_{BC} - P = 0$

Solving simultaneously

$F_{AC} = \frac{5}{6}P \qquad F_{BC} = -\frac{5}{6}P$

Joint B $\quad +\uparrow\Sigma F_y = 0$



$F_{AB} - \frac{3}{5}\cdot\frac{5}{6}P = 0$

$F_{AB} = \frac{1}{2}P$

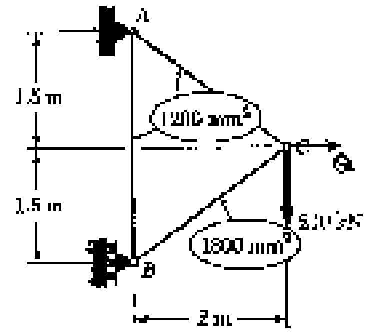
Member	F	$\partial F/\partial P$	L (m)	A (10^{-6} m²)	$F(\partial F/\partial P)L/A$
AB	$\frac{1}{2}P$	$\frac{1}{2}$	3	1200	625 P
AC	$\frac{5}{6}P$	$\frac{5}{6}$	2.5	1200	1446.76 P
BC	$-\frac{5}{6}P$	$-\frac{5}{6}$	2.5	1800	964.51 P
Σ					3036.27 P

$$\delta_P = \frac{1}{E}(3036.27\,P) = \frac{(3036.27)(210\times10^3)}{200\times10^9} = 3.19\times10^{-3}\text{ m}$$

$= 3.19$ mm $\downarrow$

PROBLEM 11.113

11.113 Each member of the truss shown is made of steel and has the cross-sectional area shown. Using $E = 200$ GPa, determine the horizontal deflection of joint C.



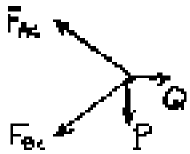
SOLUTION

Call the vertical force P. Add a dummy horizontal force Q (positive →) at joint C.

The horizontal deflection of joint C is

$$\delta_Q = \frac{\partial U}{\partial Q} = \frac{\partial}{\partial Q}\Sigma\frac{F^2L}{2EA} = \frac{1}{E}\Sigma\frac{FL}{A}\frac{\partial F}{\partial Q}$$

Joint C $\quad \xrightarrow{+} \Sigma F_x = 0$



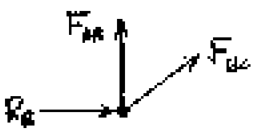
$$-\tfrac{4}{5}F_{AC} - \tfrac{4}{5}F_{BC} + Q = 0$$

$+\uparrow \Sigma F_y = 0$

$$\tfrac{3}{5}F_{AC} - \tfrac{3}{5}F_{BC} - P = 0$$

Solving simultaneously $\quad F_{AC} = \frac{5}{6}P + \frac{5}{8}Q \qquad F_{BC} = -\frac{5}{6}P + \frac{5}{8}Q$

Joint B $\quad +\uparrow\Sigma F_y = 0$



$$F_{AB} + \tfrac{3}{5}F_{BC} = 0$$

$$F_{AB} = -\tfrac{3}{5}F_{BC} = \tfrac{1}{2}P - \tfrac{3}{8}Q$$

Member	F	$\partial F/\partial Q$	L (m)	A (10^{-6} m²)	with Q = 0 $F(\partial F/\partial Q)L/A$
AB	$\frac{1}{2}P - \frac{3}{8}Q$	$-\frac{3}{8}$	3	1200	$-468.75\ P$
AC	$\frac{5}{6}P + \frac{5}{8}Q$	$+\frac{5}{8}$	2.5	1200	$1085.07\ P$
BC	$-\frac{5}{6}P + \frac{5}{8}Q$	$+\frac{5}{8}$	2.5	1800	$-723.38\ P$
					$-107.06\ P$

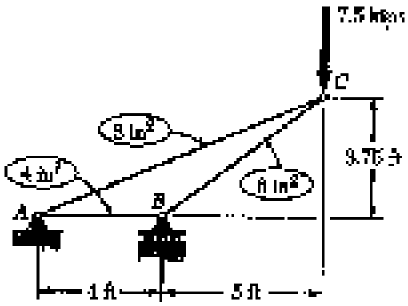
$$\delta_Q = \frac{1}{E}(-107.06\,P) = -\frac{(107.06)(210\times10^3)}{200\times10^9} = -0.1124\times10^{-3}\text{ m}$$

$$= 0.1124\text{ mm} \leftarrow$$

PROBLEM 11.114

11.114 and 11.115 Each member of the truss shown is made of steel and has the cross-sectional area shown. Using $E = 29 \times 10^6$ psi, determine the deflection indicated.

11.114 Vertical deflection of joint C.



SOLUTION

Call the vertical load P. The vertical deflection of joint C is δ_P

$$\delta_P = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\sum \frac{F^2 L}{2EA} = \frac{1}{E}\sum \frac{FL}{A}\frac{\partial F}{\partial P}$$

Geometry $\overline{AC} = \sqrt{9^2 + 3.75^2} = 9.75 \text{ ft} = 117 \text{ in}$

$\overline{BC} = \sqrt{5^2 + 3.75^2} = 6.25 \text{ ft} = 75 \text{ in}$

$4 \text{ ft} = 48 \text{ in.}, \quad 5 \text{ ft} = 60 \text{ in}, \quad 3.75 \text{ ft} = 45 \text{ in}$

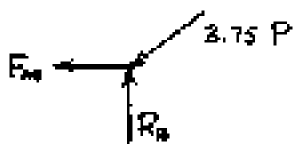
Joint C

$\xrightarrow{+} \Sigma F_x = 0 \qquad -\frac{108}{117}F_{AC} - \frac{60}{75}F_{BC} = 0$

$+\uparrow \Sigma F_y = 0 \qquad -\frac{45}{117}F_{AC} - \frac{45}{75}F_{BC} - P = 0$

Solving simultaneously $\quad F_{AC} = 3.25\,P, \quad F_{BC} = -3.75\,P$

Joint B



$\xrightarrow{+} \Sigma F_x = 0 \qquad -F_{AB} - \frac{60}{75}F_{BC} = 0$

$F_{AB} = -3.00\,P$

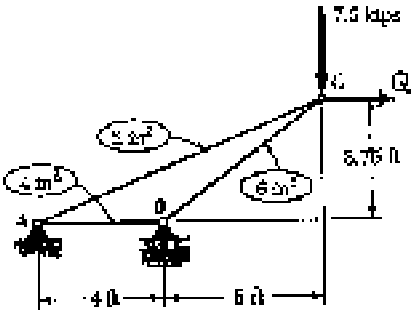
Member	F	∂F/∂P	L (in)	A (in²)	F(∂F/∂P)L /A
AB	-3.00P	-3.00	48	4	108.00 P
AC	3.25P	3.25	117	2	617.91 P
BC	-3.75P	-3.75	75	6	175.78 P
Σ					901.69 P

$$\delta_P = \frac{901.69\,P}{E} = \frac{(901.69)(7.5\times10^3)}{29\times10^6} = 0.233 \text{ in.} \downarrow$$

PROBLEM 11.115

11.114 and 11.115 Each member of the truss shown is made of steel and has the cross-sectional area shown. Using $E = 29 \times 10^6$ psi, determine the deflection indicated.

11.115 Horizontal deflection of joint C.



SOLUTION

Call the vertical load P. Add horizontal dummy load Q at joint C. The horizontal deflection of joint C is

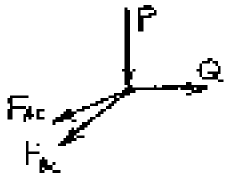
$$\delta_Q = \frac{\partial U}{\partial Q} = \frac{\partial}{\partial Q}\sum\frac{F^2L}{2EA} = \frac{1}{E}\sum\frac{FL}{A}\frac{\partial F}{\partial Q}$$

Geometry $\overline{AC} = \sqrt{9^2 + 3.75^2} = 9.75 \text{ ft} = 117 \text{ in}$

$\overline{BC} = \sqrt{5^2 + 3.75^2} = 6.25 \text{ ft} = 75 \text{ in}$

$4 \text{ ft} = 48 \text{ in}, \quad 5 \text{ ft} = 60 \text{ in}, \quad 3.75 \text{ ft} = 45 \text{ in}$

Joint C



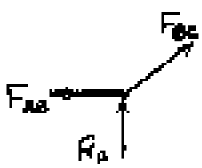
$\xrightarrow{+} \Sigma F_x = 0 \qquad -\frac{108}{117}F_{AC} - \frac{60}{75}F_{BC} + Q = 0$

$+\uparrow \Sigma F_y = 0 \qquad -\frac{45}{117}F_{AC} - \frac{45}{75}F_{BC} - P = 0$

Solving simultaneously $\qquad F_{AC} = 3.25 P + 2.4375 Q$

$F_{BC} = -3.75 P - 1.5625 Q$

Joint B



$\xrightarrow{+} \Sigma F_x = 0 \qquad \frac{4}{5}F_{BC} - F_{AB} = 0$

$F_{AB} = \frac{4}{5}F_{BC} = -3.00 P - 1.25 Q$

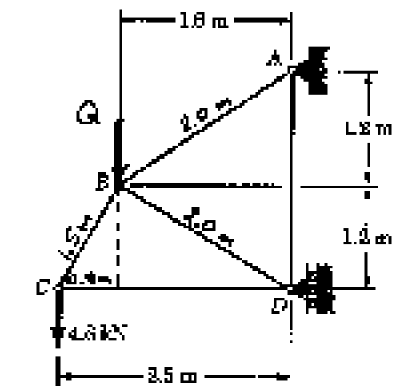
Member	F	$\partial F/\partial Q$	L (in.)	A (in²)	$F(\partial F/\partial Q)L/A$ (Q = 0)
AB	$-3.00\,P - 1.25\,Q$	-1.25	48	4	45.00 P
AC	$3.25\,P + 2.4375\,Q$	2.4375	117	2	463.43 P
BC	$-3.75\,P - 1.5625\,Q$	-1.5625	75	6	73.24 P
Σ					581.67 P

$$\delta_Q = \frac{581.67\,P}{E} = \frac{(581.67)(7.5\times10^3)}{29\times10^6} = 0.1504 \text{ in.} \longrightarrow$$

PROBLEM 11.116

11.116 and 11.117 Each member of the truss shown is made of steel and has a cross-sectional area of 500 mm². Using E = 200 GPa, determine the deflection indicated.
11.116 Vertical deflection of joint B.

SOLUTION

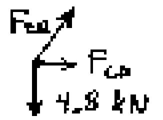


Find the length of each member as shown.

Add dummy vertical force Q at joint B.

$$\delta_B = \frac{\partial U}{\partial Q} = \frac{\partial}{\partial Q}\sum \frac{F^2 L}{2EA} = \frac{1}{EA}\sum F\frac{\partial F}{\partial Q}L$$

Joint C

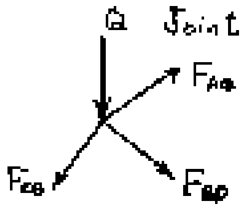


$+\uparrow \Sigma F_y = 0 \qquad \frac{4}{5}F_{CB} - 4.8 = 0$

$F_{CB} = 6.0$ kN

$\xrightarrow{+} \Sigma F_x = 0 \qquad \frac{3}{5}F_{CB} + F_{CD} = 0$

$F_{CD} = -3.6$ kN



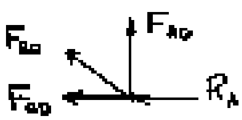
$\xrightarrow{+} \Sigma F_x = 0 \qquad \frac{4}{5}F_{AB} + \frac{4}{5}F_{BD} - 3.6 = 0$

$+\uparrow \Sigma F_y = 0 \qquad \frac{3}{5}F_{AB} - \frac{3}{5}F_{BD} - 4.8 - Q = 0$

Solving simultaneously $\qquad F_{AB} = 6.25 + 0.8333\,Q$ kN

$F_{BD} = -1.75 - 0.8333\,Q$ kN

Joint D



$+\uparrow \Sigma F_y = 0 \qquad \frac{3}{5}F_{BD} + F_{AD} = 0$

$F_{AD} = -\frac{3}{5}F_{BD} = 1.05 + 0.5\,Q$

Member	F $(10^3$ N$)$	$\partial F/\partial Q$	L (m)	with $Q = 0$: $F(\partial F/\partial Q)L$ $(10^3$ N·m$)$
AB	$6.25 + 0.8333\,Q$	0.8333	2.0	10.4167
AD	$1.05 + 0.5Q$	0.5	2.4	1.26
BD	$-1.75 - 0.8333\,Q$	−0.8333	2.0	2.9167
BC	6.0	0	1.5	0
CD	−3.6	0	2.5	0
Σ				14.593

$$\delta_B = \frac{1}{EA}\sum F(\partial F/\partial Q)L = \frac{14.593\times10^3}{(200\times10^9)(500\times10^{-6})} = 145.9\times10^{-6}\text{ m}$$

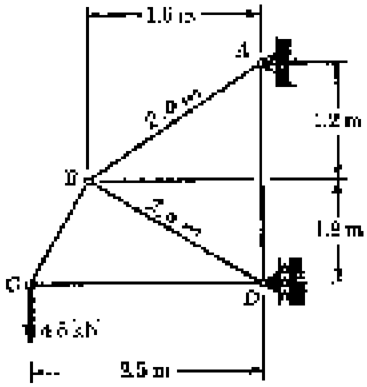
$$= 0.1459 \text{ mm} \downarrow$$

PROBLEM 11.117

11.116 and 11.117 Each member of the truss shown is made of steel and has a cross-sectional area of 500 mm². Using $E = 200$ GPa, determine the deflection indicated.

11.117 Horizontal deflection of joint B.

SOLUTION



Find the length of each member as shown.

Add dummy horizontal force Q at joint B.

$$\delta_B = \frac{\partial U}{\partial Q} = \frac{\partial}{\partial Q}\sum \frac{F^2 L}{2EA} = \frac{1}{EA}\sum F\frac{\partial F}{\partial Q}L$$

Joint C $\quad +\uparrow \Sigma F_y = 0 \qquad \frac{4}{5}F_{CB} - 4.8 = 0$

$F_{CB} = 6.0$ kN

$+\rightarrow \Sigma F_x = 0 \qquad \frac{3}{5}F_{CB} + F_{CD} = 0$

$F_{CD} = -3.6$ kN

Joint B

$+\rightarrow \quad \Sigma F_x = 0 \qquad \frac{4}{5}F_{AB} + \frac{4}{5}F_{BD} - 3.6 - Q = 0$

$+\uparrow \quad \Sigma F_y = 0 \qquad \frac{3}{5}F_{AB} - \frac{3}{5}F_{BD} - 4.8 = 0$

Solving simultaneously $\qquad F_{AB} = 6.25 + 0.625Q$ kN

$F_{BD} = -1.75 + 0.625Q$ kN

Joint D

$+\uparrow \Sigma F_y = 0 \qquad \frac{3}{5}F_{BD} + F_{AD} = 0$

$F_{AD} = -\frac{3}{5}F_{BD} = 1.05 - 0.375Q$

Member	F (10^3 N)	$\partial F/\partial Q$	L (m)	$F(\partial F/\partial Q)L$ (10^3 N·m)
AB	$6.25 + 0.625Q$	0.625	2.0	7.8125
AD	$1.05 + 0.375Q$	-0.375	2.4	-0.9450
BD	$-1.75 + 0.625Q$	0.625	2.0	-2.1875
BC	6.0	0	1.5	0
CD	-3.6	0	2.5	0
Σ				4.680

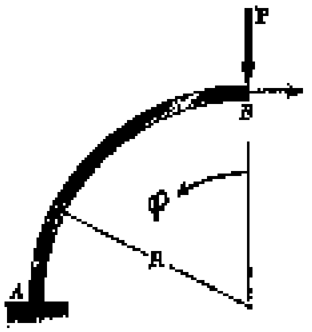
$$\delta_B = \frac{1}{EA}\Sigma F(\partial F/\partial Q)L = \frac{4.680\times10^3}{(200\times10^9)(500\times10^{-6})} = 46.8\times10^{-6}\text{ m}$$

$$= 0.0468\text{ mm} \rightarrow$$

PROBLEM 11.118

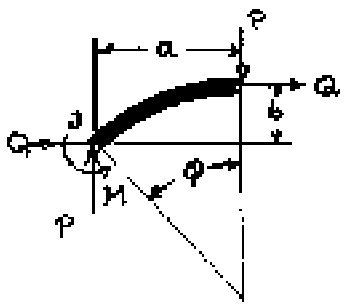
*11.118 For the uniform rod and loading shown and using Castigliano's theorem, determine (a) the horizontal deflection of point B, (b) the vertical deflection of point B.

SOLUTION



Add dummy load Q at point B.

Use polar coordinate φ



$$U = \int_0^{\frac{\pi}{2}} \frac{M^2}{2EI} R\,d\varphi$$

Bending moment

$$+\circlearrowleft \Sigma M_J = 0 \qquad M - Pa - Qb = 0$$

$$M = Pa + Qb$$
$$= PR\sin\varphi + QR(1-\cos\varphi)$$

$$\frac{\partial M}{\partial P} = R\sin\varphi \qquad \frac{\partial M}{\partial Q} = R(1-\cos\varphi) \qquad \text{Set } Q = 0$$

(a) $$\delta_a = \frac{\partial U}{\partial Q} = \frac{1}{EI}\int_0^{\frac{\pi}{2}} M\frac{\partial M}{\partial Q} R\,d\varphi = \frac{1}{EI}\int_0^{\frac{\pi}{2}} PR\sin\varphi\, R(1-\cos\varphi)R\,d\varphi$$

$$= \frac{PR^3}{EI}\int_0^{\frac{\pi}{2}}(\sin\varphi - \sin\varphi\cos\varphi)\,d\varphi = \frac{PR^3}{EI}\left(-\cos\varphi - \frac{1}{2}\sin^2\varphi\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{PR^3}{EI}\left(-\cos\frac{\pi}{2} + \cos 0 - \frac{1}{2}\sin^2\frac{\pi}{2} + \frac{1}{2}\sin^2 0\right)$$

$$= \frac{PR^3}{EI}\left(0 + 1 - \frac{1}{2} + 0\right) = \frac{1}{2}\frac{PR^3}{EI} \longleftarrow$$

(b) $$\delta_P = \frac{\partial U}{\partial P} = \frac{1}{EI}\int_0^{\frac{\pi}{2}} M\frac{\partial M}{\partial P} R\,d\varphi = \frac{1}{EI}\int_0^{\frac{\pi}{2}} PR\sin\varphi\, R\sin\varphi\, R\,d\varphi$$

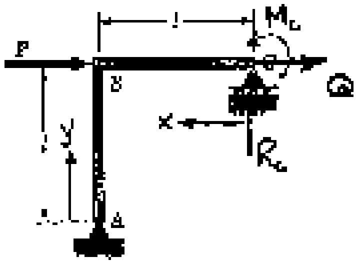
$$= \frac{PR^3}{EI}\int_0^{\frac{\pi}{2}} \sin^2\varphi\,d\varphi = \frac{PR^3}{EI}\int_0^{\frac{\pi}{2}}\frac{1}{2}(1-\cos 2\varphi)\,d\varphi$$

$$= \frac{PR^3}{EI}\left(\frac{1}{2}\varphi - \frac{1}{4}\sin 2\varphi\right)\Big|_0^{\frac{\pi}{2}} = \frac{PR^3}{EI}\left(\frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot 0 - \frac{1}{4}\sin\pi + \frac{1}{4}\cdot\sin 0\right)$$

$$= \frac{PR^3}{EI}\left(\frac{\pi}{4} - 0 - 0 + 0\right) = \frac{\pi}{4}\frac{PR^3}{EI} \longleftarrow$$

PROBLEM 11.119

11.119 Two rods AB and BC of the same flexural rigidity EI are welded together at B. For the loading shown, determine (a) the deflection of point C, (b) the slope of member BC at point C.



SOLUTION

Add dummy force Q and dummy couple M_c at C.

$\circlearrowleft \Sigma M_A = 0 \qquad R_c l + M_c + (P+Q)l = 0$

$$R_c = P + Q + \frac{M_c}{l}$$

$\xleftarrow{+} \Sigma F_x = 0 \qquad P + Q + R_{Ax} = 0 \qquad R_{Ax} = P + Q \leftarrow$

Member AB: $\quad M = R_{Ax} y = (P+Q)y, \quad \dfrac{\partial M}{\partial Q} = y, \quad \dfrac{\partial M}{\partial M_c} = 0$

$U_{AB} = \displaystyle\int_0^l \frac{M^2}{2EI}\,dy \qquad\qquad$ Set $Q = 0$ and $M_c = 0$

$$\frac{\partial U_{AB}}{\partial Q} = \frac{1}{EI}\int_0^l M\frac{\partial M}{\partial Q}\,dy = \frac{1}{EI}\int_0^l (Py)(y)\,dy = \frac{1}{3}\frac{Pl^3}{EI}$$

$$\frac{\partial U_{AB}}{\partial M_c} = \frac{1}{EI}\int_0^l M\frac{\partial M}{\partial M_c}\,dx = 0$$

Member BC: $\quad M = M_c + R_c x = M_c + \left(P + Q + \dfrac{M_c}{l}\right)x$

$$\frac{\partial M}{\partial Q} = x, \qquad \frac{\partial M}{\partial M_c} = 1 - \frac{x}{l}$$

$U_{BC} = \displaystyle\int_0^l \frac{M^2}{2EI}\,dx \qquad\qquad$ Set $Q = 0$ and $M_c = 0$

$$\frac{\partial U_{BC}}{\partial Q} = \frac{1}{EI}\int_0^l M\frac{\partial M}{\partial Q}\,dx = \frac{1}{EI}\int_0^l (Px)x\,dx = \frac{1}{3}\frac{Pl^3}{EI}$$

$$\frac{\partial U}{\partial M_c} = \frac{1}{EI}\int_0^l M\frac{\partial M}{\partial M_c}\,dx = \frac{1}{EI}\int_0^l (Px)\left(1 - \frac{x}{l}\right)dx = \frac{P}{EI}\int_0^l \left(x - \frac{x^2}{l}\right)dx$$

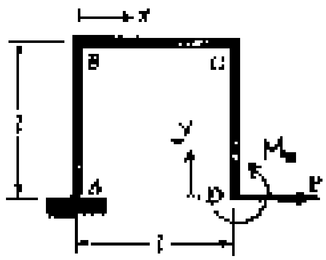
$$= \frac{P}{EI}\left(\frac{1}{2}l^2 - \frac{1}{3}l^2\right) = \frac{1}{6}\frac{Pl^2}{EI}$$

(a) Deflection at C $\qquad \delta_c = \dfrac{\partial U_{AB}}{\partial Q} + \dfrac{\partial U_{BC}}{\partial Q} = \dfrac{2}{3}\dfrac{Pl^3}{EI} \rightarrow$ ◄

(b) Slope at C $\qquad \circlearrowleft \theta_c = \dfrac{\partial U_{AB}}{\partial M_c} + \dfrac{\partial U_{BC}}{\partial M_c} = \dfrac{1}{6}\dfrac{Pl^2}{EI} \circlearrowleft$ ◄

PROBLEM 11.120

11.120 A uniform rod of flexural rigidity EI is bent and loaded as shown. Determine (a) the horizontal deflection of point D, (b) the slope at point D.



SOLUTION

Add dummy couple M_D at point D.

Reactions at A: $R_{Ay} = 0$, $R_{Ax} = P \leftarrow$, $M_A = M_D$

Member AB: $M = M_A + R_x y = M_D + Py$ $\qquad \frac{\partial M}{\partial P} = y$, $\frac{\partial M}{\partial M_D} = 1$

$U_{AB} = \int_0^l \frac{M^2}{2EI} dy$ $\qquad$ Set $M_D = 0$

$$\frac{\partial U_{AB}}{\partial P} = \frac{1}{EI}\int_0^l M \frac{\partial M}{\partial P} dy = \frac{1}{EI}\int_0^l (Py) y \, dy = \frac{Pl^3}{3EI}$$

$$\frac{\partial U_{AB}}{\partial M_D} = \frac{1}{EI}\int_0^l M \frac{\partial M}{\partial M_D} dy = \frac{1}{EI}\int_0^l (Py)(1) dy = \frac{Pl^2}{2EI}$$

Member BC $\quad M = M_A + R_x l = M_D + Pl$ $\qquad \frac{\partial M}{\partial P} = l$, $\frac{\partial M}{\partial M_D} = 1$

$U_{BC} = \int_0^l \frac{M^2}{2EI} dx$ $\qquad$ Set $M_D = 0$

$$\frac{\partial U_{BC}}{\partial P} = \frac{1}{EI}\int_0^l M \frac{\partial M}{\partial P} dx = \frac{1}{EI}\int_0^l (Pl)(l) dx = \frac{Pl^3}{EI}$$

$$\frac{\partial U_{BC}}{\partial M_D} = \frac{1}{EI}\int_0^l M \frac{\partial M}{\partial M_D} dx = \frac{1}{EI}\int_0^l (Pl)(1) dx = \frac{Pl^2}{EI}$$

Member CD $\quad M = M_D + Py$ $\qquad \frac{\partial M}{\partial P} = y$ $\qquad \frac{\partial M}{\partial M_D} = 1$

$U_{CD} = \int_0^l \frac{M^2}{2EI} dy$ $\qquad$ Set $M_D = 0$

$$\frac{\partial U_{CD}}{\partial P} = \frac{1}{EI}\int_0^l M \frac{\partial M}{\partial P} dy = \frac{1}{EI}\int_0^l (Py)(y) dy = \frac{Pl^3}{3EI}$$

$$\frac{\partial U}{\partial M_D} = \frac{1}{EI}\int_0^l M \frac{\partial M}{\partial M_D} dy = \frac{1}{EI}\int_0^l (Py)(1) dy = \frac{Pl^2}{2EI}$$

(a) horizontal deflection of point D.

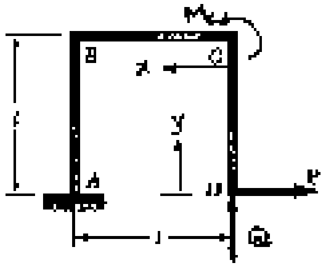
$$\delta_P = \frac{\partial U_{AB}}{\partial P} + \frac{\partial U_{BC}}{\partial P} + \frac{\partial U_{CD}}{\partial P} = \left(\frac{1}{3} + 1 + \frac{1}{3}\right)\frac{Pl^3}{EI} = \frac{5}{3}\frac{Pl^3}{EI} \rightarrow$$

(b) slope at point D

$$\theta_D = \frac{\partial U_{AB}}{\partial M_D} + \frac{\partial U_{BC}}{\partial M_D} + \frac{\partial U_{CD}}{\partial M_D} = \left(\frac{1}{2} + 1 + \frac{1}{2}\right)\frac{Pl^2}{EI} = 2\frac{Pl^2}{EI}$$

PROBLEM 11.121

11.121 A uniform rod of flexural rigidity EI is bent and loaded as shown. Determine (a) the vertical deflection of point D, (b) the slope of BC at point C.



SOLUTION

Add dummy force Q at point D and dummy couple M_c at point C.

Reactions at A: $R_{Ax} = P \leftarrow$, $R_{Ay} = Q \uparrow$

$M_A = Q\ell + M_c$ ↺

Member AB: $M = M_A + R_{Ax}y = Q\ell + M_c + Py$, $\frac{\partial M}{\partial Q} = \ell$, $\frac{\partial M}{\partial M_c} = 1$

$U_{AB} = \int_0^\ell \frac{M^2}{2EI}dy$ Set $Q = 0$ and $M_c = 0$

$$\frac{\partial U_{AB}}{\partial Q} = \frac{1}{EI}\int_0^\ell M\frac{\partial M}{\partial Q}dy = \frac{1}{EI}\int_0^\ell (Py)(\ell)\,dy = \frac{P\ell^3}{2EI}$$

$$\frac{\partial U_{AB}}{\partial M_c} = \frac{1}{EI}\int_0^\ell M\frac{\partial M}{\partial M_c}dy = \frac{1}{EI}\int_0^\ell (Py)(1)\,dy = \frac{P\ell^2}{2EI}$$

Member BC $M = M_c + P\ell + Qx$ $\frac{\partial M}{\partial Q} = x$ $\frac{\partial M}{\partial M_c} = 1$

$U_{BC} = \frac{1}{EI}\int_0^\ell \frac{M^2}{2EI}dx$ Set $Q = 0$ and $M_c = 0$

$$\frac{\partial U_{BC}}{\partial Q} = \frac{1}{EI}\int_0^\ell M\frac{\partial M}{\partial Q}dx = \frac{1}{EI}\int_0^\ell (P\ell)(x)\,dx = \frac{P\ell^3}{2EI}$$

$$\frac{\partial U_{BC}}{\partial M_c} = \frac{1}{EI}\int_0^\ell M\frac{\partial M}{\partial M_c}dx = \frac{1}{EI}\int_0^\ell (P\ell)(1)\,dx = \frac{P\ell^2}{EI}$$

Member CD $M = Py$ $\frac{\partial M}{\partial Q} = 0$ $\frac{\partial M}{\partial M_c} = 0$

$$\frac{\partial U_{CD}}{\partial Q} = 0 \qquad \frac{\partial U_{CD}}{\partial M_c} = 0$$

(a) vertical deflection of point D

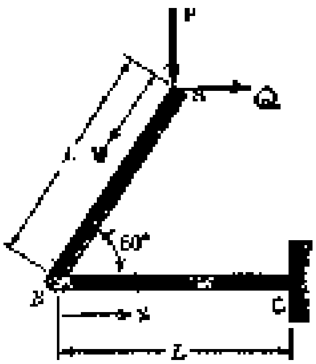
$$\delta_D = \frac{\partial U_{AB}}{\partial Q} + \frac{\partial U_{BC}}{\partial Q} + \frac{\partial U_{CD}}{\partial Q} = \left(\frac{1}{2} + \frac{1}{2} + 0\right)\frac{P\ell^3}{EI} = \frac{P\ell^3}{EI} \uparrow$$

(b) slope of BC at C

$$\theta_c = \frac{\partial U_{AB}}{\partial M_c} + \frac{\partial U_{BC}}{\partial M_c} + \frac{\partial U_{CD}}{\partial M_c} = \left(\frac{1}{2} + 1 + 0\right)\frac{P\ell^2}{EI} = \frac{3}{2}\frac{P\ell^2}{EI} \measuredangle$$

PROBLEM 11.112

11.112 A uniform rod of flexural rigidity EI is bent and loaded as shown. Determine (a) the vertical deflection of point A, (b) the horizontal deflection of point A.



SOLUTION

Add dummy horizontal force Q at point A.

Over AB $\quad M = \frac{1}{2}Pv + \frac{\sqrt{3}}{2}Qv$

$$\frac{\partial M}{\partial P} = \frac{1}{2}v \qquad \frac{\partial M}{\partial Q} = \frac{\sqrt{3}}{2}v$$

$$U_{AB} = \int_0^L \frac{M^2}{2EI}\,dv \qquad \text{Set } Q = 0$$

$$\frac{\partial U_{AB}}{\partial P} = \frac{1}{EI}\int_0^L M\frac{\partial M}{\partial P}\,dv = \frac{1}{EI}\int_0^L \left(\tfrac{1}{2}Pv\right)\left(\tfrac{1}{2}v\right)dv$$

$$= \frac{1}{12}\frac{PL^3}{EI}$$

$$\frac{\partial U_{AB}}{\partial Q} = \frac{1}{EI}\int_0^L M\frac{\partial M}{\partial Q}\,dv = \frac{1}{EI}\int_0^L \left(\tfrac{1}{2}Pv\right)\frac{\sqrt{3}}{2}\,dv$$

$$= \frac{\sqrt{3}}{12}\frac{PL^3}{EI}$$

Over BC $\quad M = -P\left(x - \frac{L}{2}\right) + \frac{\sqrt{3}}{2}QL, \qquad \frac{\partial M}{\partial P} = -\left(x - \frac{L}{2}\right), \quad \frac{\partial M}{\partial Q} = \frac{\sqrt{3}}{2}L$

$$U_{BC} = \int_0^L \frac{M^2}{2EI}\,dx \qquad \text{Set } Q = 0$$

$$\frac{\partial U_{BC}}{\partial P} = \frac{1}{EI}\int_0^L M\frac{\partial M}{\partial P}\,dx = \frac{1}{EI}\int_0^L P\left(x - \tfrac{L}{2}\right)^2 dx = \frac{P}{3EI}\left(x - \tfrac{L}{2}\right)^3\Big|_0^L = \frac{1}{12}\frac{PL^3}{EI}$$

$$\frac{\partial U_{BC}}{\partial Q} = \frac{1}{EI}\int_0^L M\frac{\partial M}{\partial Q}\,dx = \frac{-1}{EI}\int_0^L P\left(x - \tfrac{L}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right)L\,dx = -\frac{\sqrt{3}PL}{4EI}\left(x - \tfrac{L}{2}\right)^2\Big|_0^L = 0$$

(a) vertical deflection of point A

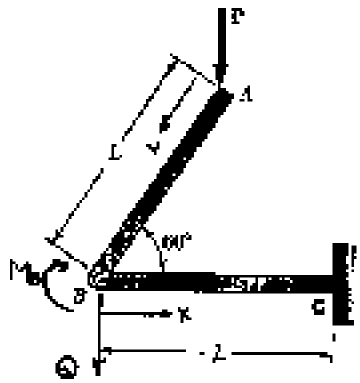
$$\delta_P = \frac{\partial U_{AB}}{\partial P} + \frac{\partial U_{BC}}{\partial P} = \frac{1}{6}\frac{PL^3}{EI} \downarrow$$

(b) horizontal deflection of point A

$$\delta_Q = \frac{\partial U_{AB}}{\partial Q} + \frac{\partial U_{BC}}{\partial Q} = \frac{\sqrt{3}}{12}\frac{PL^3}{EI} = 0.1443\frac{PL^3}{EI} \rightarrow$$

PROBLEM 11.123

11.123 A uniform rod of flexural rigidity EI is bent and loaded as shown. Determine (*a*) the vertical deflection of point B, (*b*) the slope of BC at point B.



SOLUTION

Add dummy vertical Q and dummy couple M_o at B.

Over AB $\quad M = \frac{1}{2}Pv, \quad \frac{\partial M}{\partial Q} = 0, \quad \frac{\partial M}{\partial M_o} = 0$

$$U_{AB} = \int_0^L \frac{M^2}{2EI}dv$$

$$\frac{\partial U_{AB}}{\partial Q} = 0 \qquad \frac{\partial U_{AB}}{\partial M_o} = 0$$

Over BC $\quad M = -P(x - \frac{L}{2}) - Qx + M_o \qquad \frac{\partial M}{\partial Q} = -x, \quad \frac{\partial M}{\partial M_o} = 1$

$$U_{BC} = \int_0^L \frac{M^2}{2EI}dx \qquad \text{Set } Q = 0 \text{ and } M_o = 0$$

$$\frac{\partial U_{BC}}{\partial Q} = \frac{1}{EI}\int_0^L M\frac{\partial M}{\partial Q}dx = \frac{1}{EI}\int_0^L P(x - \frac{L}{2})x\,dx = \frac{P}{EI}\left[\frac{L^3}{3} - \left(\frac{L}{2}\right)\frac{L^2}{2}\right]$$

$$= \frac{1}{12}\frac{PL^3}{EI}$$

$$\frac{\partial U_{BC}}{\partial M_o} = \frac{1}{EI}\int_0^L M\frac{\partial M}{\partial M_o}dx = \frac{-1}{EI}\int_0^L P(x - \frac{L}{2})dx = 0$$

(a) vertical deflection of point B

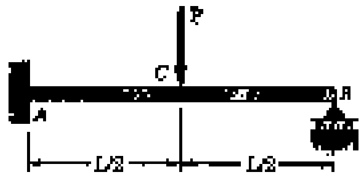
$$\delta_B = \frac{\partial U_{AB}}{\partial Q} + \frac{\partial U_{BC}}{\partial Q} = \frac{1}{12}\frac{PL^3}{EI} \downarrow$$

(b) slope of BC at point B

$$\theta_B = \frac{\partial U_{AB}}{\partial M_o} + \frac{\partial U_{BC}}{\partial M_o} = 0$$

PROBLEM 11.124

11.124 Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.

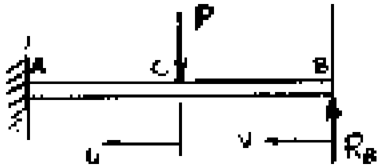


SOLUTION

Remove support B and add reaction R_B as a load.

$$U = U_{AC} + U_{CB} = \int_0^{L/2} \frac{M^2}{2EI}\,du + \int_0^{L/2} \frac{M^2}{2EI}\,dv$$

$$y_B = \frac{\partial U}{\partial R_B} = \frac{\partial U_{AC}}{\partial R_B} + \frac{\partial U_{CB}}{\partial R_B} = 0$$



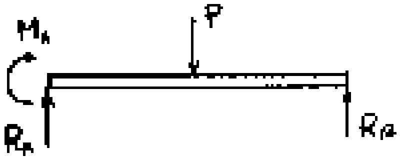
Over AC: $M = R_B\left(u + \frac{L}{2}\right) - Pu, \quad \frac{\partial M}{\partial R_B} = \left(u + \frac{L}{2}\right)$

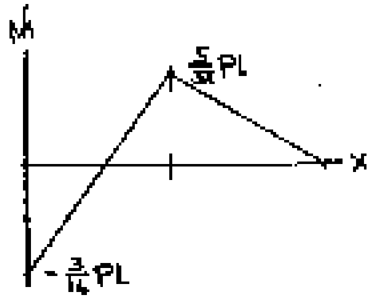
$$\frac{\partial U_{AC}}{\partial R_B} = \frac{1}{EI}\int_0^{L/2} \left[R_B\left(u + \frac{L}{2}\right) - Pu\right]\left(u + \frac{L}{2}\right) du$$

$$= \frac{R_B}{EI}\int_0^{L/2}\left(u + \frac{L}{2}\right)^2 du - \frac{P}{EI}\int_0^{L/2} u\left(u + \frac{L}{2}\right) du$$

$$= \frac{R_B}{3EI}\left[L^3 - \left(\frac{L}{2}\right)^3\right] - \frac{P}{EI}\left[\frac{1}{3}\left(\frac{L}{2}\right)^3 + \frac{L}{2}\cdot\frac{1}{2}\left(\frac{L}{2}\right)^2\right]$$

$$\frac{7}{24}\frac{R_B L^3}{EI} - \frac{5}{48}\frac{PL^3}{EI}$$





Over CB: $M = R_B v \qquad \frac{\partial M}{\partial R_B} = v$

$$\frac{\partial U_{CB}}{\partial R_B} = \frac{1}{EI}\int_0^{L/2}(R_B v)v\,dv = \frac{R_B}{3EI}\left(\frac{L}{2}\right)^3 = \frac{1}{24}\frac{R_B L^3}{EI}$$

$$y_B = \left(\frac{7}{24} + \frac{1}{24}\right)\frac{R_B L^3}{EI} - \frac{5}{48}\frac{PL^3}{EI} = 0$$

$$R_B = \frac{5}{16}P \uparrow$$

$$M_C = R_B\frac{L}{2} = \frac{5}{32}PL$$

$$M_A = R_B L - P\frac{L}{2} = \left(\frac{5}{16} - \frac{1}{2}\right)PL = -\frac{3}{16}PL$$

PROBLEM 11.125

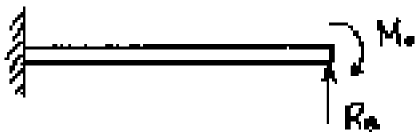
11.125 Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.

SOLUTION

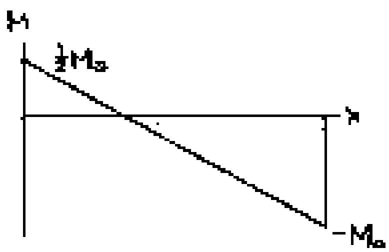
Remove support B and add reaction R_B as a load.

$$U = \int_0^L \frac{M^2}{2EI}\,dv$$

$$y_B = \frac{\partial U}{\partial R_B} = \frac{1}{EI}\int_0^L M \frac{\partial M}{\partial R_B}\,dv = 0$$



$$M = R_B v - M_0 \qquad \frac{\partial M}{\partial R_B} = v$$



$$y_B = \frac{1}{EI}\int_0^L (R_B v - M_0)\,v\,dv$$

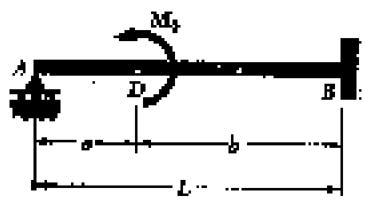
$$= \frac{R_B}{EI}\int_0^L v^2 dv - \frac{M_0}{EI}\int_0^L v\,dv$$

$$= \frac{R_B L^3}{3EI} - \frac{M_0 L^2}{2EI} = 0 \qquad R_B = \frac{3}{2}\frac{M_0}{L}\uparrow$$

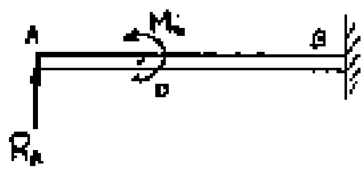
$$M_A = R_B - M_0 = \frac{3}{2}M_0 - M_0 = \frac{1}{2}M_0$$

PROBLEM 11.126

11.126 Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.



SOLUTION



Remove support A and add reaction R_A as a load.

$$U = \int_0^L \frac{M^2}{2EI}\,dx$$

$$\delta_A = \frac{\partial U}{\partial R_A} = \frac{1}{EI}\int_0^L M \frac{\partial M}{\partial R_A}\,dx = 0$$

Portion AD $\quad 0 < x < a \qquad M = R_A x \qquad \frac{\partial M}{\partial R_A} = x$

$$\frac{\partial U_{AD}}{\partial R_A} = \frac{1}{EI}\int_0^a (R_A x)(x)\,dx = \frac{R_A a^3}{3EI}$$

Portion DB $\quad (a < x < L) \qquad M = R_A x - M_0 \qquad \frac{\partial M}{\partial R_A} = x$

$$\frac{\partial U_{DB}}{\partial R_A} = \frac{1}{EI}\int_a^L (R_A x - M_0)(x)\,dx = \frac{1}{EI}\left\{ \frac{1}{3} R_A(L^3 - a^3) - \frac{1}{2} M_0 (L^2 - a^2)\right\}$$

$$\delta_A = \frac{\partial U_{AD}}{\partial R_A} + \frac{\partial U_{DB}}{\partial R_A} = \frac{1}{EI}\left\{ R_A\left(\tfrac{1}{3}a^3 + \tfrac{1}{3}L^3 - \tfrac{1}{3}a^3\right) - \tfrac{1}{2}M_0\left(L^2 - a^2\right)\right\} = 0$$

$$R_A = \frac{3}{2}\,\frac{M_0(L^2 - a^2)}{L^3} = \frac{3}{2}\,\frac{M_0 b (L+a)}{L^3} \uparrow$$

$$M_A = 0$$

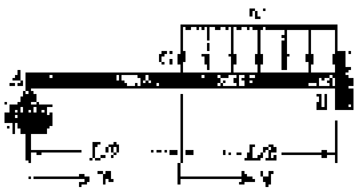
$$M_{D-} = R_A a = \frac{3}{2}\,\frac{M_0 a b (L+a)}{L^3}$$

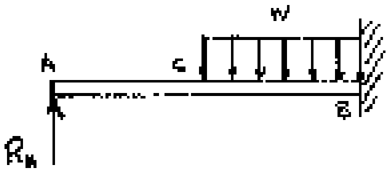
$$M_{D+} = M_{D-} + M_0 = \frac{3}{2}\,\frac{M_0 a b (L+a)}{L^3} - M_0$$

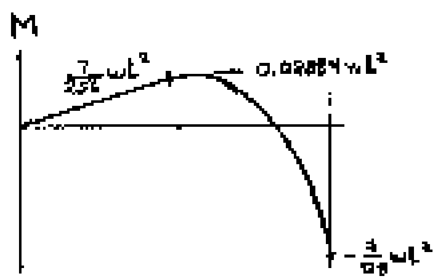
$$M_B = R_A L - M_0 = \frac{3}{2}\,\frac{M_0 b (L+a)}{L^2} - M_0$$

PROBLEM 11.127

11.127 Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.







SOLUTION

Remove support A and add reaction R_A as a load.

$$U = \int_0^{L/2} \frac{M^2}{2EI}\,dx + \int_0^{L/2} \frac{M^2}{2EI}\,dv$$

$$\delta_A = \frac{\partial U}{\partial R_A} = \frac{1}{EI}\int_0^{L/2} M \frac{\partial M}{\partial R_A}\,dx + \frac{1}{EI}\int_0^{L/2} M \frac{\partial M}{\partial R_A}\,dv = 0$$

Portion AC: $0 < x < \frac{L}{2}$ $\quad M = R_A x \quad \frac{\partial M}{\partial R_A} = x$

$$\frac{\partial U_{AC}}{\partial R_A} = \frac{1}{EI}\int_0^{L/2} (R_A x)(x)\,dx = \frac{R_A L^3}{24EI}$$

Portion CB $\quad 0 < v < \frac{L}{2}$

$$M = R_A\left(v + \frac{L}{2}\right) - \frac{1}{2}wv^2 \qquad \frac{\partial M}{\partial R_A} = \left(v + \frac{L}{2}\right)$$

$$\frac{\partial U_{CB}}{\partial R_A} = \frac{1}{EI}\int_0^{L/2}\left[R_A\left(v+\frac{L}{2}\right) - \frac{1}{2}wv^2\right]\left(v+\frac{L}{2}\right)dv$$

$$= \frac{1}{EI}\left\{R_A\int_0^{L/2}\left(v+\frac{L}{2}\right)^2 dv - \frac{1}{2}w\int_0^{L/2}\left(v^3 + \frac{L}{2}v^2\right)dv\right\}$$

$$= \frac{R_A}{EI}\left[\frac{1}{3}L^3 - \frac{1}{3}\left(\frac{L}{2}\right)^3\right] - \frac{w}{2EI}\left[\frac{1}{4}\left(\frac{L}{2}\right)^4 + \frac{L}{2}\frac{1}{3}\left(\frac{L}{2}\right)^3\right]$$

$$= \left(\frac{1}{3} - \frac{1}{24}\right)\frac{R_A L^3}{EI} - \frac{7}{384}\frac{wL^4}{EI}$$

$$\delta_A = \frac{\partial U_{AC}}{\partial R_A} + \frac{\partial U_{CB}}{\partial R_A} = \frac{1}{3}\frac{R_A L^3}{EI} - \frac{7}{384}\frac{wL^4}{EI} = 0$$

$$R_A = \frac{7}{128}wL \uparrow$$

Over AC $\quad M = \frac{7}{128}wLx$

$$M_C = \frac{7}{256}wL^2 = 0.02734\,wL^2$$

Over CB: $\quad M = \frac{7}{128}wL\left(v + \frac{L}{2}\right) - \frac{1}{2}wv^2$

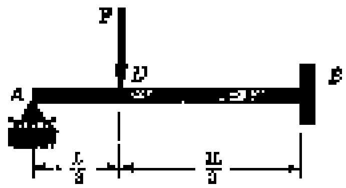
$$M_B = \frac{7}{128}wL^2 - \frac{1}{2}w\left(\frac{L}{2}\right)^2 = -\frac{9}{128}wL^2$$
$$= -0.07031\,wL^2$$

$$\frac{dM}{dv} = \frac{7}{128}wL - wv_m = 0 \qquad v_m = \frac{7}{128}L$$

$$M_m = \frac{7}{128}wL\left(\frac{7}{128}L + \frac{L}{2}\right) - \frac{1}{2}w\left(\frac{7}{128}L\right)^2$$
$$= \frac{945}{32768}wL^2 = 0.02884\,wL^2$$

PROBLEM 11.128

11.128 Determine the reaction at the roller support and draw the bending moment diagram for the beam and loading shown.

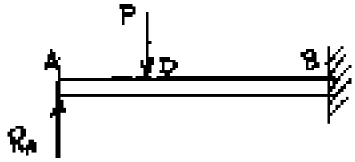


SOLUTION

Remove support A and add reaction R_A as a load.

$$U = \int_0^L \frac{M^2}{2EI}\,dx$$

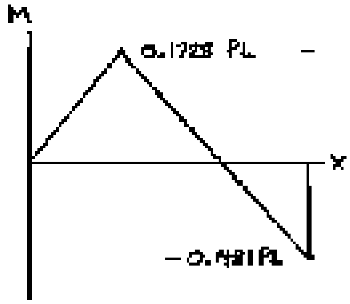
$$\delta_A = \frac{\partial U}{\partial R_A} = \frac{1}{EI}\int_0^L M\frac{\partial M}{\partial R_A}\,dx$$



Portion AD $\quad 0 < x < \frac{L}{3} \qquad M = R_A x \qquad \frac{\partial M}{\partial R_A} = x$

$$\frac{\partial U_{AD}}{\partial R_A} = \frac{1}{EI}\int_0^{\frac{L}{3}} M\frac{\partial M}{\partial R_A}\,dx = \frac{1}{EI}\int_0^{\frac{L}{3}}(R_A x)(x)\,dx$$

$$= \frac{R_A}{3EI}\left(\frac{L}{3}\right)^3 = \frac{1}{81}\frac{R_A L^3}{EI}$$



Portion DB $\quad \frac{L}{3} < x < L \qquad M = R_A x - P\left(x - \frac{L}{3}\right)$

$$\frac{\partial M}{\partial R_A} = x$$

$$\frac{\partial U_{DB}}{\partial R_A} = \frac{1}{EI}\int_{\frac{L}{3}}^{L} M\frac{\partial M}{\partial R_A}\,dx = \frac{1}{EI}\int_{\frac{L}{3}}^{L}\left[R_A - P\left(x - \frac{L}{3}\right)\right]x\,dx$$

$$= \frac{R_A}{EI}\int_{\frac{L}{3}}^{L} x^2\,dx - \frac{P}{EI}\int_{\frac{L}{3}}^{L}\left(x^2 - \frac{L}{3}x\right)dx$$

$$= \frac{R_A}{3EI}\left[L^3 - \left(\frac{L}{3}\right)^3\right] - \frac{P}{EI}\left[\frac{1}{3}\left(L^3 - \left(\frac{L}{3}\right)^3\right) - \frac{L}{6}\left(L^2 - \left(\frac{L}{3}\right)^2\right)\right]$$

$$= \left(\frac{1}{3} - \frac{1}{81}\right)\frac{R_A L^3}{EI} - \left(\frac{1}{3} - \frac{1}{81} - \frac{1}{6} + \frac{1}{54}\right)\frac{PL^3}{EI}$$

$$\delta_A = \frac{\partial U_{AD}}{\partial R_A} + \frac{\partial U_{DB}}{\partial R_A} = \left(\frac{1}{81} + \frac{1}{3} - \frac{1}{81}\right)\frac{R_A L^3}{EI} - \frac{14}{81}\frac{PL^3}{EI} = \frac{1}{3}\frac{R_A L^3}{EI} - \frac{14}{81}\frac{PL^3}{EI} = 0$$

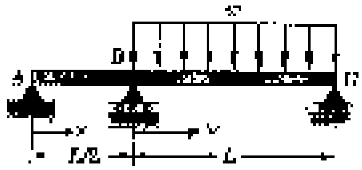
$$R_A = \frac{14}{27}P \quad \blacktriangleleft$$

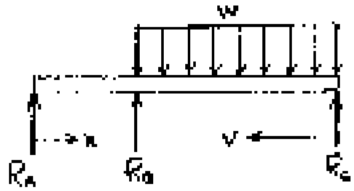
Bending moments $\quad M_D = R_A\left(\frac{L}{3}\right) = \frac{14}{81}PL = 0.1728\,PL \quad \blacktriangleleft$

$$M_B = R_A L - P\left(\frac{2L}{3}\right) = -\frac{4}{27}PL = -0.1481\,PL \quad \blacktriangleleft$$

PROBLEM 11.129

11.129 For the uniform beam and loading shown, determine the reaction at each support.





SOLUTION

Remove support A and add reaction R_A as a load.

$$\Sigma M_B = 0 \qquad -R_A \frac{L}{2} - \frac{1}{2}wL^2 + R_C L = 0$$

$$R_C = \frac{1}{2}R_A + \frac{1}{2}wL$$

$$U = U_{AB} + U_{BC} = \int_0^{L/2} \frac{M^2}{2EI}dx + \int_0^L \frac{M^2}{2EI}dv$$

$$\delta_A = \frac{\partial U}{\partial R_A} = \frac{\partial U_{AB}}{\partial R_A} + \frac{\partial U_{BC}}{\partial R_A} = 0$$

Portion AB: $\quad M = R_A x, \quad \frac{\partial M}{\partial R_A} = x$

$$\frac{\partial U_{AB}}{\partial R_A} = \frac{1}{EI}\int_0^{L/2} M\frac{\partial M}{\partial R_A}dx = \frac{1}{EI}\int_0^{L/2}(R_A x)(x)dx = \frac{R_A}{3EI}\left(\frac{L}{2}\right)^3 = \frac{1}{24}\frac{R_A L^3}{EI}$$

Portion BC: $\quad M = R_C v - \frac{1}{2}wv^2 = \frac{1}{2}R_A v + \frac{1}{2}wLv - \frac{1}{2}wLv^2$

$$\frac{\partial M}{\partial R_A} = \frac{1}{2}v$$

$$\frac{\partial U_{BC}}{\partial R_A} = \frac{1}{EI}\int_0^L \left[\frac{1}{2}R_A v + \frac{1}{2}w(Lv - v^2)\right]\left(\frac{1}{2}v\right)dv = \frac{1}{4EI}\int_0^L [R_A v^2 + w(Lv^2 - v^3)]dv$$

$$= \frac{1}{4EI}\left[R_A \frac{L^3}{3} + w\left(\frac{L^4}{3} - \frac{L^4}{4}\right)\right] = \frac{R_A L^3}{12EI} + \frac{wL^4}{48EI}$$

$$\delta_A = \frac{\partial U_{AB}}{\partial R_A} + \frac{\partial U_{BC}}{\partial R_A} = \left(\frac{1}{24} + \frac{1}{12}\right)\frac{R_A L^3}{EI} + \frac{wL^4}{48EI} = 0$$

$$R_A = -\frac{1}{6}wL = \frac{1}{6}wL \downarrow$$

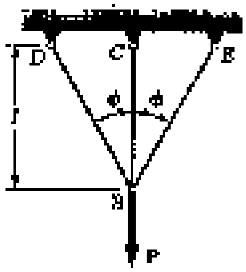
$$R_C = \frac{1}{2}\left(-\frac{1}{6}wL\right) + \frac{1}{2}wL = \frac{5}{12}wL \uparrow$$

$$+\uparrow \Sigma F_y = 0 \qquad R_A + R_B + R_C - wL = 0$$

$$-\frac{1}{6}wL + R_B + \frac{5}{12}wL - wL = 0 \qquad R_B = \frac{3}{4}wL$$

PROBLEM 11.130

11.130 Three members of the same material and same cross-sectional area are used to support the load P. Determine the force in member BC.



SOLUTION

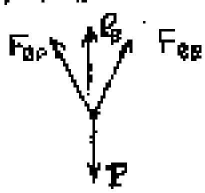
Detach member BC at support C.

Add reaction R_C as a load

$$U = \sum \frac{F^2 L}{2EA} \qquad y_C = \frac{\partial U}{\partial R_C} = \sum \frac{FL}{EA}\frac{\partial F}{\partial R_C} = 0$$

Joint C, $F_{BC} = R_C$

Joint B



$\xrightarrow{+} \sum F_x = 0 \qquad F_{BE} \sin\varphi - F_{BD} \sin\varphi = 0 \qquad F_{BE} = F_{BD}$

$+\uparrow \sum F_y = 0 \qquad F_{BD}\cos\varphi + F_{BE}\cos\varphi + R_B - P$

$$F_{BD} = F_{BE} = \frac{P - R_B}{2\cos\varphi}$$

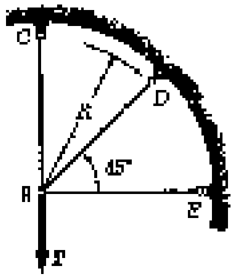
Member	F	$\partial F/\partial R_B$	L	$(FL/EA)\,\partial F/\partial R_B$
BD	$(P-R_B)/2\cos\varphi$	$-1/2\cos\varphi$	$l/\cos\varphi$	$(R_B-P)l/4EA\cos^2\varphi$
BE	$(P-R_B)/2\cos\varphi$	$-1/2\cos\varphi$	$l/\cos\varphi$	$(R_B-P)l/4EA\cos^2\varphi$
BC	R_B	1	l	$R_B l/EA$

$$y_B = -Pl/2EA\cos^2\varphi + R_B l/2EA\cos^2\varphi + R_B l/EA = 0$$

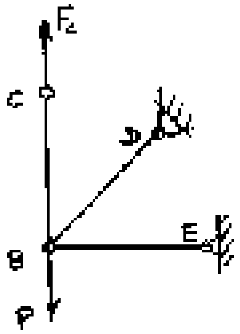
$$R_B = \frac{P}{1 + 2\cos^2\varphi} \qquad F_{BC} = R_B = \frac{P}{1+2\cos^2\varphi}$$

PROBLEM 11.131

11.131 Three members of the same material and same cross-sectional area are used to support the load **P**. Determine the force in member *BC*.

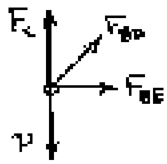


SOLUTION



Detach member BC from its support at point C. Add reaction F_c as a load.

Joint B.



$+\uparrow \Sigma F_y = 0$

$$\frac{\sqrt{2}}{2}F_{BD} + F_c - P = 0$$

$$F_{BD} = \sqrt{2}P - \sqrt{2}F_c$$

$\xrightarrow{+} \Sigma F_x = 0$

$$\frac{\sqrt{2}}{2}F_{BD} + F_{BE} = 0$$

$$F_{BE} = -P + F_c$$

$$U = \Sigma \frac{F^2R}{2EA} = \frac{R}{2EA}\Sigma F^2$$

$$\delta_c = \frac{\partial U}{\partial F_c} = \frac{R}{EA}\Sigma F\frac{\partial F}{\partial F_c} = 0$$

Member	F	$\partial F/\partial F_c$	$F(\partial F/\partial F_c)$
BC	F_c	1	F_c
BD	$\sqrt{2}P - \sqrt{2}F_c$	$-\sqrt{2}$	$-2P + 2F_c$
BE	$-P + F_c$	1	$-P + F_c$
Σ			$-3P + 4F_c$

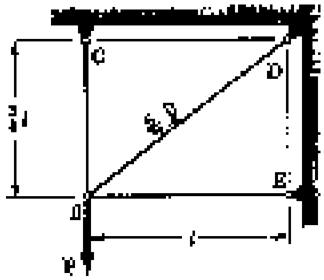
$$\delta_c = \frac{R}{EA}(-3P + 4F_c) = 0$$

$$F_c = \frac{3}{4}P$$

$$F_{BC} = F_c = \frac{3}{4}P$$

PROBLEM 11.132

11.132 Three members of the same material and same cross-sectional area are used to support the load **P**. Determine the force in member *BC*.



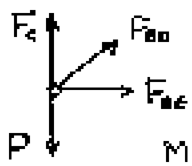
SOLUTION

Detach member BC from support C. Add reaction F_c as a load.

$$U = \Sigma \frac{F^2 L}{2EA} = \frac{1}{2EA}\Sigma F^2 L$$

$$\delta_c = \frac{\partial U}{\partial F_c} = \frac{1}{EA}\Sigma F \frac{\partial F}{\partial F_c} L$$

Joint B



$+\uparrow \Sigma F_y = 0 \qquad F_c - P + \frac{3}{5}F_{BD} = 0 \qquad F_{BD} = \frac{5}{3}P - \frac{5}{3}F_c$

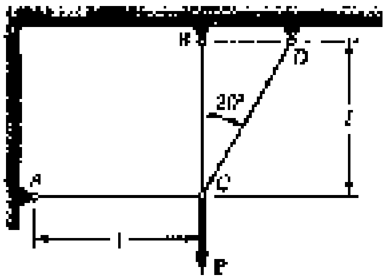
$\xrightarrow{+} \Sigma F_x = 0 \qquad F_{BE} + \frac{4}{5}F_{BD} = 0 \qquad F_{BE} = -\frac{4}{3}P + \frac{4}{3}F_c$

Member	F	$\partial F/\partial F_c$	L	$F(\partial F/\partial F_c)L$
BC	F_c	1	$\frac{3}{4}\ell$	$\frac{3}{4}F_c\ell$
BD	$\frac{5}{3}P - \frac{5}{3}F_c$	$-\frac{5}{3}$	$\frac{5}{4}\ell$	$-\frac{125}{36}P\ell + \frac{125}{36}F_c\ell$
BE	$-\frac{4}{3}P + \frac{4}{3}F_c$	$\frac{4}{3}$	ℓ	$-\frac{16}{9}P\ell + \frac{16}{9}F_c\ell$
Σ				$-\frac{21}{4}P\ell + 6F_c\ell$

$\delta_c = \frac{1}{EA}\left(-\frac{21}{4}P\ell + 6F_c\ell\right) = 0 \qquad F_c = \frac{7}{8}P \qquad F_{BC} = F_c = \frac{7}{8}P$

PROBLEM 11.133

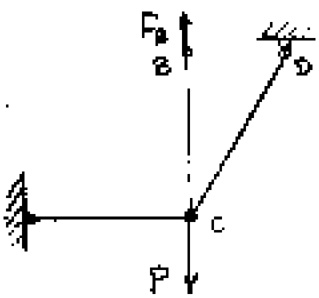
11.133 Three members of the same material and same cross-sectional area are used to support the load **P**. Determine the force in member *BC*.



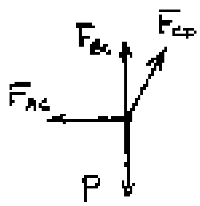
SOLUTION

Cut member BC at end B and replace member force F_{BC} by load F_B acting on member BC at B.

$$S_B = \frac{\partial U}{\partial F_B} = \frac{\partial}{\partial F_B}\sum \frac{F^2 L}{2EA} = \frac{1}{EA}\sum F\frac{\partial F}{\partial F_B}L$$



Joint C



$+\uparrow \sum F_y = 0 \qquad \frac{\sqrt{3}}{2}F_{CD} + F_{BC} - P = 0$

$$F_{CD} = \frac{2}{\sqrt{3}}P - \frac{2}{\sqrt{3}}F_B$$

$\xrightarrow{+} \sum F_x = 0 \qquad F_{AC} - \frac{1}{2}F_{CD} = 0$

$$F_{AC} = \frac{1}{\sqrt{3}}P - \frac{1}{\sqrt{3}}F_B$$

Member	F	$\partial F/\partial F_B$	L	$F(\partial F/\partial F_B)L$
AC	F_B	1	ℓ	$F_B\ell$
BC	$\frac{1}{\sqrt{3}}P - \frac{1}{\sqrt{3}}F_B$	$-\frac{1}{\sqrt{3}}$	ℓ	$-\frac{1}{3}P\ell + \frac{1}{3}F_B\ell$
CD	$\frac{2}{\sqrt{3}}P - \frac{2}{\sqrt{3}}F_B$	$-\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}\ell$	$-\frac{8}{3\sqrt{3}}P\ell + \frac{8}{3\sqrt{3}}F_B\ell$
Σ				$-\left(\frac{1}{3}+\frac{8}{3\sqrt{3}}\right)P\ell + \left(\frac{4}{3}+\frac{8}{3\sqrt{3}}\right)F_B\ell$

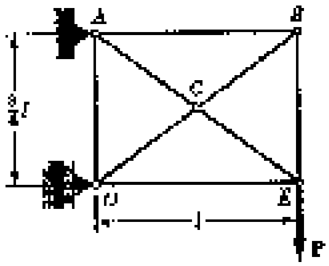
$$S_B = -\left(\frac{1}{3}+\frac{8}{3\sqrt{3}}\right)\frac{P\ell}{EA} + \left(\frac{4}{3}+\frac{8}{3\sqrt{3}}\right)\frac{F_B\ell}{EA}$$

$$F_B = \frac{\frac{1}{3}+\frac{8}{3\sqrt{3}}}{\frac{4}{3}+\frac{8}{3\sqrt{3}}}P = \frac{8+\sqrt{3}}{8+4\sqrt{3}}P = 0.652\,P$$

$$F_{BC} = F_B = 0.652\,P$$

PROBLEM 11.134

11.134 Knowing that the eight members of the indeterminate truss shown have the same uniform cross-sectional area, determine the force in member *AB*.

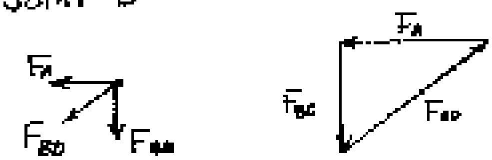


SOLUTION

Cut member AB at end A and replace member force F_{AB} by load $F_A \leftarrow$ acting on member AB at end A.

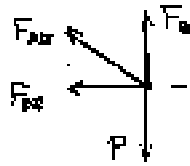
$$\delta_A = \frac{\partial U}{\partial F_A} = \frac{\partial}{\partial F_A}\sum \frac{F^2 L}{2EA} = \frac{1}{EA}\sum F\frac{\partial F}{\partial F_A}L = 0$$

Joint B



$F_{BD} = -\frac{5}{4}F_A \qquad F_{BE} = \frac{3}{4}F_A$

Joint E



$+\uparrow \Sigma F_y = 0$

$F_{BE} - P + \frac{3}{5}F_{AE}$

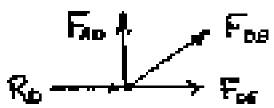
$F_{AE} = \frac{5}{3}P - \frac{5}{3}F_{BE}$

$= \frac{5}{3}P - \frac{5}{4}F_A$

$+\rightarrow \Sigma F_x = 0 \qquad -\frac{4}{5}F_{AE} - F_{DE} = 0$

$F_{DE} = -\frac{4}{5}F_{AE} = -\frac{4}{3}P + F_A$

Joint D



$+\uparrow \Sigma F_y = 0 \qquad F_{AD} + \frac{3}{5}F_{DB} = 0$

$F_{AD} = -\frac{3}{5}F_{DB} = -\frac{3}{4}F_A$

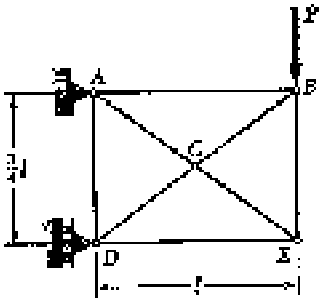
Member	F	$\partial F/\partial F_A$	L	$F(\partial F/\partial F_A)L$
AB	F_A	1	l	$F_A l$
AD	$-\frac{3}{4}F_A$	$-\frac{3}{4}$	$\frac{3}{4}l$	$\frac{27}{64}F_A l$
AE	$\frac{5}{3}P - \frac{5}{4}F_A$	$-\frac{5}{4}$	$\frac{5}{4}l$	$-\frac{125}{48}Pl + \frac{125}{64}F_A l$
BD	$-\frac{5}{4}F_A$	$-\frac{5}{4}$	$\frac{5}{4}l$	$\frac{125}{64}F_A l$
BE	$\frac{3}{4}F_A$	$\frac{3}{4}$	$\frac{3}{4}l$	$\frac{27}{64}F_A l$
DE	$-\frac{4}{3}P + F_A$	1	l	$-\frac{4}{3}Pl + F_A l$
Σ				$-\frac{63}{16}Pl + \frac{27}{4}F_A l$

$$\delta_A = \frac{1}{EA}\left(-\frac{63}{16}Pl + \frac{27}{4}F_A l\right) = 0 \qquad F_A = \frac{7}{12}P$$

$$F_{AB} = F_A = \frac{7}{12}P = 0.583\,P$$

PROBLEM 11.135

11.135 Knowing that the eight members of the indeterminate truss shown have the same uniform cross-sectional area, determine the force in member *AB*.



SOLUTION

Cut member AB at end A and replace member force F_{AB} by load $F_A \leftarrow$ acting on member AB at end A.

$$\delta_A = \frac{\partial U}{\partial F_A} = \frac{\partial}{\partial F_A}\sum \frac{F^2 L}{2EA} = \frac{1}{EA}\sum F \frac{\partial F}{\partial F_A} L = 0$$

Joint B $\quad \overset{+}{\leftarrow}\Sigma F_x = 0 \qquad -F_A - \frac{4}{5}F_{BD} = 0 \qquad F_{BD} = -\frac{5}{4}F_A$

$+\uparrow \Sigma F_y = 0 \qquad -P - F_{BE} - \frac{3}{5}F_{BD} = 0 \qquad F_{BE} = -P + \frac{3}{4}F_A$

Joint E $\quad +\uparrow \Sigma F_y = 0 \qquad F_{BE} + \frac{3}{5}F_{AE} = 0 \qquad F_{AE} = \frac{5}{3}P - \frac{5}{4}F_A$

$\overset{+}{\leftarrow}\Sigma F_x = 0 \qquad -\frac{4}{5}F_{AE} - F_{DE} = 0 \qquad F_{DE} = -\frac{4}{3}P + F_A$

Joint D $\quad +\uparrow \Sigma F_y = 0 \qquad F_{AD} + \frac{3}{5}F_{BD} = 0$

$F_{AD} = \frac{3}{4}F_A$

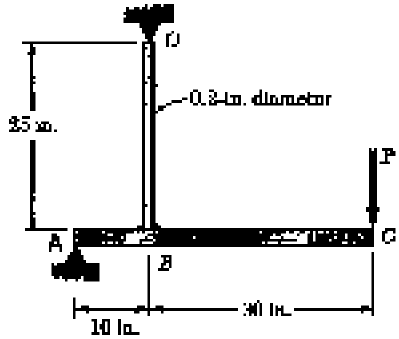
Member	F	$\partial F/\partial F_A$	L	$F(\partial F/\partial F_A)L$
AB	F_A	1	l	$F_A l$
AD	$\frac{3}{4}F_A$	$\frac{3}{4}$	$\frac{3}{4}l$	$\frac{27}{64}F_A l$
AE	$\frac{5}{3}P - \frac{5}{4}F_A$	$-\frac{5}{4}$	$\frac{5}{4}l$	$-\frac{125}{48}Pl + \frac{125}{64}F_A l$
BD	$-\frac{5}{4}F_A$	$-\frac{5}{4}$	$\frac{5}{4}l$	$\frac{125}{64}F_A l$
BE	$-P + \frac{3}{4}F_A$	$\frac{3}{4}$	$\frac{3}{4}l$	$-\frac{9}{16}Pl + \frac{27}{64}F_A l$
DE	$-\frac{4}{3}P + F_A$	1	l	$-\frac{4}{3}Pl + F_A l$
Σ				$-\frac{9}{2}Pl + \frac{27}{4}F_A l$

$$\delta_A = \frac{1}{EA}\left(-\frac{9}{2}Pl + \frac{27}{4}F_A l\right) = 0 \qquad F_A = \frac{2}{3}P$$

$$F_{AB} = F_A = \frac{2}{3}P = 0.667P$$

PROBLEM 11.136

11.136 The steel bar ABC has a square cross section of side 0.75 in. and is subjected to a 50-lb load **P**. Using $E = 29 \times 10^6$, determine the deflection of point C.



SOLUTION

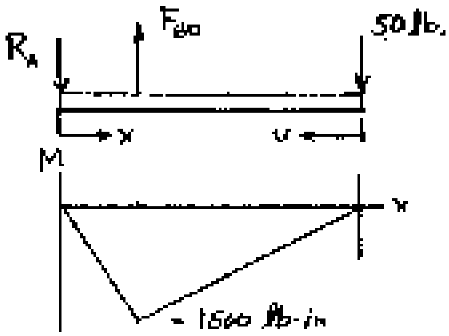
Assume member BD is a two-force member.

$\Sigma M_A = 0 \qquad 10 F_{BD} - (40)(50) = 0 \qquad F_{BD} = 200 \text{ lb.}$

$A_{BD} = \frac{\pi}{4}(0.2)^2 = 31.416 \times 10^{-3} \text{ in}^2$

$U_{BD} = \frac{F_{BD}^2 L_{BD}}{2EA} = \frac{(200)^2(25)}{(2)(29\times10^6)(31.416\times10^{-3})}$

$= 0.5488 \text{ in-lb.}$



Member ABC

$I = \frac{1}{12}(0.75)(0.75)^3 = 26.367 \times 10^{-3} \text{ in}^4$

Portion AB $\qquad M = -1500\frac{x}{10} = -150x$

$U_{AB} = \int_0^{10} \frac{M^2}{2EI}dx = \frac{150^2}{2EI}\int_0^{10} x^2 dx$

$= \frac{(150)^2(10^3)}{(2)(29\times10^6)(26.367\times10^{-3})(3)} = 4.904 \text{ in-lb}$

Portion BC: $\quad M = -50v \qquad U_{BC} = \int_0^{30}\frac{M^2}{2EI}dv = \frac{50^2}{2EI}\int_0^{30} v^2 dv$

$= \frac{(50)^2(30)^3}{(2)(29\times10^6)(26.367\times10^{-3})(3)} = 14.713 \text{ in-lb.}$

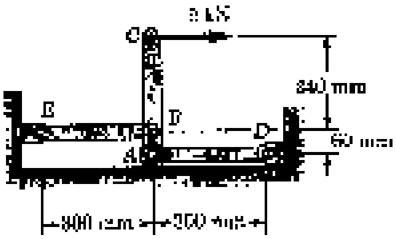
Total $\quad U = U_{BD} + U_{AB} + U_{BC} = 20.166 \text{ in-lb.}$

$\frac{1}{2}P\delta_C = U$

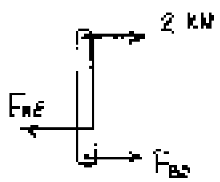
$\delta_C = \frac{2U}{P} = \frac{(2)(20.166)}{50} = 0.807 \text{ in.} \downarrow$

PROBLEM 11.137

11.137 The steel bars *BE* and *AD* have each a 5 × 15-mm cross section. Assuming that lever *ABC* is rigid and using $E = 200$ GPa, determine the deflection of point *C*.



SOLUTION



$\circlearrowleft \Sigma M_A = 0 \qquad 60 F_{BE} - (300)(2) = 0$

$F_{BE} = 10$ kN

$\circlearrowleft \Sigma M_B = 0 \qquad 60 F_{AD} - (240)(2) = 0$

$F_{AD} = 8$ kN

For bars BE and AD $\qquad A = 5 \times 15 = 75\ mm^2 = 75 \times 10^{-6}\ m^2$

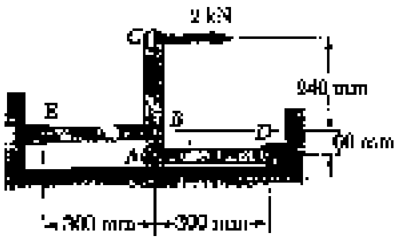
$$U = \frac{F_{BE}^2 L_{BE}}{2EA} + \frac{F_{AD}^2 L_{AD}}{2EA} = \frac{(10 \times 10^3)^2 (300 \times 10^{-3})}{(2)(200 \times 10^9)(75 \times 10^{-6})} + \frac{(8 \times 10^3)^2 (300 \times 10^{-3})}{(2)(200 \times 10^9)(75 \times 10^{-6})}$$

$$= 1.0000 + 0.6400 = 1.6400\ \text{N-m}$$

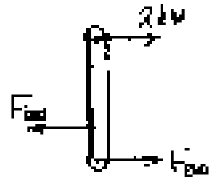
$$\frac{1}{2} P \delta_C = U \qquad \delta_C = \frac{2U}{P} = \frac{(2)(1.6400)}{2 \times 10^3} = 1.64 \times 10^{-3}\ m = 1.64\ mm \rightarrow$$

PROBLEM 11.138

11.138 The steel bars *BE* and *AD* have each a 5 × 15-mm cross section and the steel lever *ABC* has a square cross section of side 25 mm. Using $E = 200$ GPa, determine the deflection of point C.



SOLUTION



$+\circlearrowleft \Sigma M_A = 0 \qquad 60F_{BE} - (300)(2) = 0$

$F_{BE} = 10$ kN

$+\circlearrowleft \Sigma M_B = 0 \qquad 60F_{AD} - (240)(2) = 0$

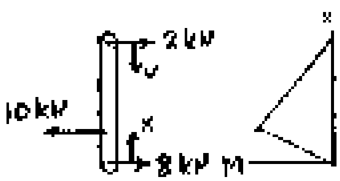
$F_{AD} = 8$ kN

For bars BE and AD $\qquad A = 5 \times 15 = 75\ \text{mm}^2 = 75 \times 10^{-6}\ \text{m}^2$

$$U_{BE} = \frac{F_{BE}^2 L_{BE}}{2EA} = \frac{(10)^2(300 \times 10^{-6})}{(2)(200 \times 10^9)(75 \times 10^{-6})} = 1.0000\ \text{J}$$

$$U_{AD} = \frac{F_{AD}^2 L_{AD}}{2EA} = \frac{(8)^2(300 \times 10^{-6})}{(2)(200 \times 10^9)(75 \times 10^{-6})} = 0.6400\ \text{J}$$

Beam ABC: $\quad I = \frac{1}{12}(25)(25)^3 = 32.552 \times 10^3\ \text{mm}^4 = 32.552 \times 10^{-9}\ \text{m}^4$



Bending moment at B.

$$M_B = (2 \times 10^3)(240 \times 10^{-3}) = 480\ \text{N·m}$$

Portion AB: $\quad M = \frac{480}{L_{AB}} x$

$$U_{AB} = \int_0^{L_{AB}} \frac{M^2}{2EI} = \frac{(480)^2}{2EI L_{AB}^2} \int_0^{L_{AB}} x^2\,dx$$

$$= \frac{(480^2) L_{AB}^3}{6EI L_{AB}^2} = \frac{(480)^2 L_{AB}}{6EI}$$

$$= \frac{(480)^2 (60 \times 10^{-3})}{(6)(200 \times 10^9)(32.552 \times 10^{-9})} = 0.3539\ \text{J}$$

Portion BC: $\quad M = \frac{480}{L_{BC}} v$

$$U_{BC} = \int_0^{L_{BC}} \frac{M^2}{2EI}\,dv = \frac{(480)^2}{2EI L_{BC}^2} \int_0^{L_{BC}} v^2\,dv = \frac{(480)^2 L_{BC}^3}{6EI L_{BC}^2} = \frac{(480)^2 L_{BC}}{6EI}$$

$$= \frac{(480)^2 (240 \times 10^{-3})}{(6)(200 \times 10^9)(32.552 \times 10^{-9})} = 1.4156\ \text{J}$$

Total $\quad U = U_{BE} + U_{AD} + U_{AB} + U_{BC}$

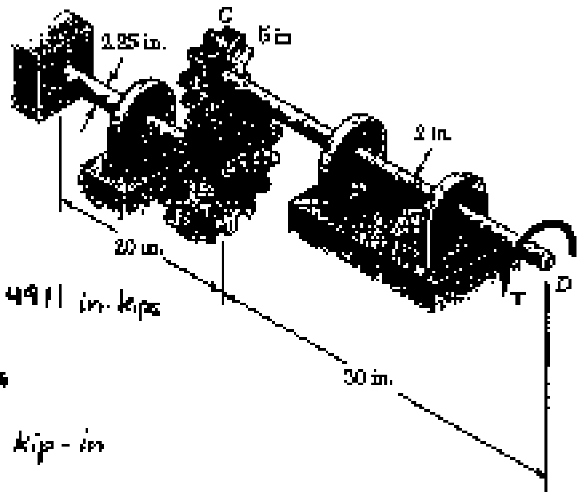
$$= 1.0000 + 0.6400 + 0.3539 + 1.4156 = 3.4095\ \text{J}$$

$\frac{1}{2} P \delta_C = U \qquad \delta_C = \frac{2U}{P} = \frac{(2)(3.4095)}{2 \times 10^3} = 3.41 \times 10^{-3}$ m

$= 3.41$ mm $\rightarrow$

PROBLEM 11.139

11.139 Two solid steel shafts are connected by the gears shown. Using $G = 11.2 \times 10^6$ psi, determine the strain energy in each shaft when a 24 kip·in. torque is applied at *D*. (Ignore the strain energy due to bending of the shafts.)



SOLUTION

Shaft CD: $T_{CD} = T_C = 24$ kip·in

$$J_{CD} = \frac{\pi}{2}\left(\frac{d}{2}\right)^4 = \frac{\pi}{2}\left(\frac{2.0}{2}\right)^4 = 1.5708 \text{ in}^4$$

$L_{CD} = 30$ in., $G = 11.2 \times 10^6$ psi $= 11.2 \times 10^3$ ksi

$$U_{CD} = \frac{T_{CD}^2 L_{CD}}{2GJ_{CD}} = \frac{(24)^2(30)}{(2)(11.2\times 10^3)(1.5708)} = 0.4911 \text{ in·kips}$$

Gear C $\qquad F_{CB} = \dfrac{T_C}{r_C} = \dfrac{T_{CD}}{r_C} = \dfrac{24}{5} = 4.8$ kips

Gear B $\qquad T_B = r_B F_{CB} = (8)(4.8) = 38.4$ kip·in

Shaft AB $\qquad T_{AB} = T_B = 38.4$ kip·in $\qquad L_{AB} = 20$ in

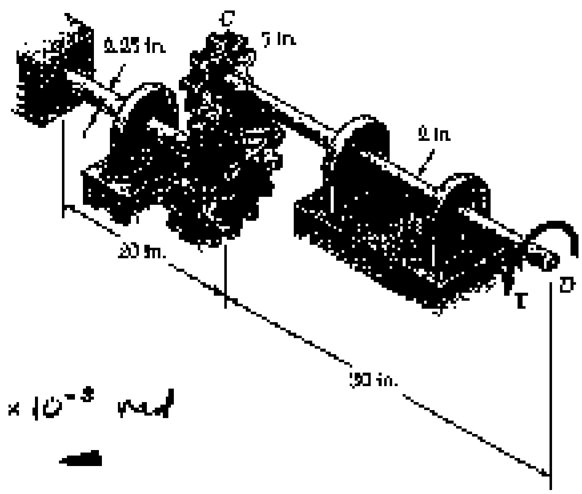
$$J_{AB} = \frac{\pi}{2}\left(\frac{d}{2}\right)^4 = \frac{\pi}{2}\left(\frac{2.25}{2}\right)^4 = 2.5161 \text{ in}^4$$

$$U_{AB} = \frac{T_{AB}^2 L_{AB}}{2GJ_{AB}} = \frac{(38.4)^2(20)}{(2)(11.2\times 10^3)(2.5161)} = 0.5233 \text{ in·kips}$$

Total $\qquad U = U_{AB} + U_{CD} = 0.5233 + 0.4911 = 1.0144$ in·kips. ◄

PROBLEM 11.140

11.140 Two solid steel shafts are connected by the gears shown. Using $G = 11.2 \times 10^6$ psi, determine the angle through which end *D* rotates when $T = 24$ kip·in.



(Ignore the strain energy of bending of the shafts.)

SOLUTION

From Prob. 11.139

$U = 1.0144$ in·lb

$$\tfrac{1}{2} T_D \varphi_D = U$$

$$\varphi_D = \frac{2U}{T_D} = \frac{(2)(1.0144)}{24} = 84.5 \times 10^{-3} \text{ rad}$$

$$= 4.84^\circ \quad ◄$$

PROBLEM 11.141

11.141 (*a*) Determine the modulus of resilience of a grade of structural steel for which $\sigma_Y = 300$ MPa and $E = 200$ GPa. (*b*) Determine the required yield strength of an aluminum alloy for which $E = 72$ GPa if the modulus of resilience of the alloy is to be the same as that of the structural steel.

SOLUTION

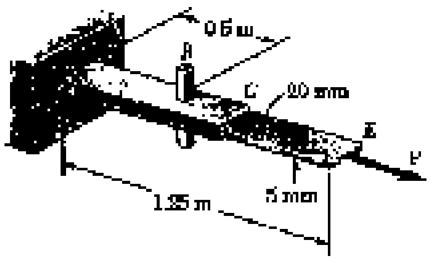
(a) $E = 200\times10^9$ Pa, $\quad \sigma_Y = 300\times10^6$ Pa

$$u_Y = \frac{\sigma_Y^2}{2E_s} = \frac{(300\times10^6)^2}{(2)(200\times10^9)} = 225\times10^3\ \text{N}\cdot\text{m/m}^3 = 225\ \text{kJ/m}^3$$

(b) $$\sigma_{Ya} = \sqrt{2E_a u_Y} = \sqrt{(2)(72\times10^9)(225\times10^3)} = 180\times10^6\ \text{Pa} = 180\ \text{MPa}$$

PROBLEM 11.142

11.142 A single 6-mm-diameter steel pin B is used to connect the steel strip DE to two aluminum strips, each of 20-mm width and 5-mm thickness. The modulus of elasticity is 200 GPa for the steel and 70 GPa for the aluminum. Knowing that for the pin at B the allowable shearing stress is $\tau_{all} = 85$ MPa, determine, for the loading shown, the maximum strain energy that can be acquired by the assembled strips.



SOLUTION

$$A_{pin} = \frac{\pi}{4}d^2 = \frac{\pi}{4}(6)^2 = 28.274\ \text{mm}^2 = 28.274\times10^{-6}\ \text{m}^2$$

$$\tau_{all} = 85\times10^6\ \text{Pa}$$

Double shear $\quad P = 2A\tau = (2)(28.274\times10^{-6})(85\times10^6) = 4.8066\times10^3\ \text{N}$

For strips AB, DB, BE $\quad A = (20)(5) = 100\ \text{mm}^2 = 100\times10^{-6}\ \text{m}^2$

$$F_{AB} = F_{DB} = \tfrac{1}{2}P = 2.4033\times10^3\ \text{N}$$

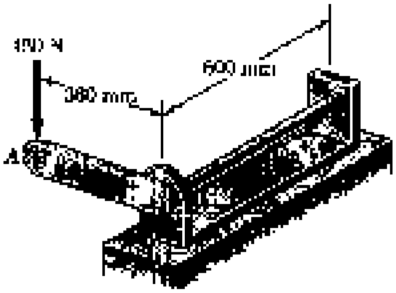
$$U_{AB} = U_{DB} = \frac{F_{AB}^2 L_{AB}}{2E_a A_{AB}} = \frac{(2.4033)^2(0.5)}{(2)(70\times10^9)(100\times10^{-6})} = 206.3\times10^{-3}\ \text{J}$$

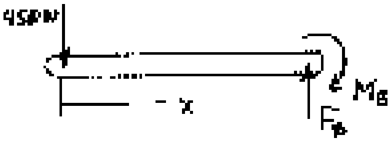
$$U_{BE} = \frac{F_{BE}^2 L_{BE}}{2E_s A_{BE}} = \frac{(4.8066\times10^3)^2(1.25-0.5)}{(2)(200\times10^9)(100\times10^{-6})} = 433.2\times10^{-3}\ \text{J}$$

Total: $\quad U = U_{AB} + U_{DB} + U_{BE} = 846\times10^{-3}\ \text{J} = 0.846\ \text{J}$

PROBLEM 11.143

11.143 The 18-mm-diameter steel rod *BC* is attached to the lever *AB* and to the fixed support *C*. The uniform steel lever *AB* is 9 mm wide and 24 mm deep. Using $E = 200$ GPa, $G = 77$ GPa, and the method of work and energy, determine the deflection of point *A*.





SOLUTION

Member AB

$$I = \frac{1}{12}(9)(24)^3 = 10.368 \times 10^3\ \text{mm}^4 = 10.368 \times 10^{-9}\ \text{m}^4$$

$$E = 200 \times 10^9$$

$$M = 450\,x \qquad M_B = 162\ \text{N}\cdot\text{m}$$

$$U_{AB} = \int_0^{L_{AB}} \frac{M^2}{2EI}\,dx = \frac{(450)^2}{2EI}\int_0^{L_{AB}} x^2\,dx$$

$$= \frac{(450)^2 L_{AB}^3}{6EI} = \frac{(450)^2(360 \times 10^{-3})^3}{(6)(200 \times 10^9)(10.368 \times 10^{-9})}$$

$$= 0.75938\ \text{J}$$

Member BC $\qquad T = M_B = 162\ \text{N}\cdot\text{m} \qquad L = 600 \times 10^{-3}\ \text{m}$

$$J = \frac{\pi}{2}\left(\frac{d}{2}\right)^4 = \frac{\pi}{2}\left(\frac{18}{2}\right)^4 = 10.306 \times 10^3\ \text{mm}^4 = 10.306 \times 10^{-9}\ \text{m}^4$$

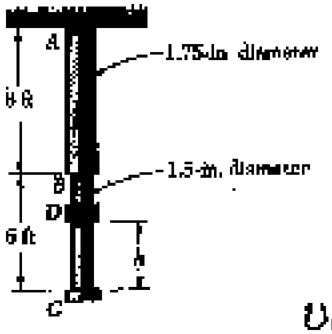
$$U_{BC} = \frac{T^2 L}{2GJ} = \frac{(162)^2(600 \times 10^{-3})}{(2)(77 \times 10^9)(10.306 \times 10^{-9})} = 9.9213\ \text{J}$$

Total $\qquad U = U_{AB} + U_{BC} = 10.681\ \text{J}$

$$\frac{1}{2}P\delta_A = U \qquad \delta_A = \frac{2U}{P} = \frac{(2)(10.681)}{450} = 47.5 \times 10^{-3}\ \text{m} = 47.5\ \text{mm} \downarrow$$

PROBLEM 11.144

11.144 The 75-lb collar D is released from rest in the position shown and is stopped by a plate attached at end C of the vertical rod ABC. Knowing that $E = 29 \times 10^6$ psi for both portions of the rod, determine the distance h for which the maximum stress in the rod is 36 ksi.



SOLUTION

Portion BC: $\quad A_{BC} = \frac{\pi}{4} d_{BC}^2 = \frac{\pi}{4}(1.5)^2 = 1.76715 \text{ in}^2$

$\sigma_{BC} = 30000$ psi $\qquad L_{BC} = 6 \text{ ft} = 72 \text{ in.}$

Force at C $\quad P = \sigma_{BC} A_{BC} = 53014$ lb.

$$U_{BC} = \frac{P^2 L_{BC}}{2EA_{BC}} = \frac{(53014)^2(72)}{(2)(29 \times 10^6)(1.76715)} = 1974.3 \text{ in}\cdot\text{lb}$$

Portion AB: $\quad A_{AB} = \frac{\pi}{4} d_{AB}^2 = \frac{\pi}{4}(1.75)^2 = 2.40528 \text{ in}^2. \qquad L_{AB} = 8 \text{ ft} = 96 \text{ in.}$

$$U_{AB} = \frac{P^2 L_{AB}}{2EA_{AB}} = \frac{(53014)^2(96)}{(2)(29 \times 10^6)(2.40528)} = 1934.0 \text{ in}\cdot\text{lb}$$

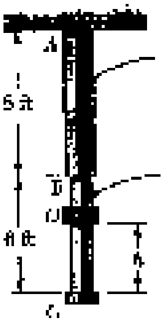
Total $\quad U = U_{AB} + U_{BC} = 3908.3 \text{ in}\cdot\text{lb}$

$$\tfrac{1}{2} P \delta_c = U \qquad \delta_c = \frac{2U}{P} = \frac{(2)(3908.3)}{53014} = 0.14744 \text{ in.}$$

$$W(h + \delta_c) = U \qquad h = \frac{U}{W} - \delta_c = \frac{3908.3}{75} - 0.14744 = 52.0 \text{ in.}$$

PROBLEM 11.145

11.145 The 75-lb collar D is released from rest when $h = 20$ in. and is stopped by a plate attached at end C of the vertical rod ABC. Knowing that $E = 29 \times 10^6$ psi for both portions of the rod, determine (*a*) the maximum deflection of end C, (*b*) the equivalent static load, (*c*) the maximum stress that occurs in the rod.



SOLUTION

Let P_m be the equivalent static load in lb.

Portion AB: $A_{AB} = \frac{\pi}{4} d_{AB}^2 = \frac{\pi}{4}(1.75)^2 = 2.40528 \text{ in}^2$

$L_{AB} = 8 \text{ ft} = 96 \text{ in}$

$$U_{AB} = \frac{P_m^2 L_{AB}}{2EA} = \frac{P_m^2 (96)}{(2)(29\times10^6)(2.40528)} = 688.14\times10^{-9} P_m^2$$

Portion BC: $A_{BC} = \frac{\pi}{4} d_{BC}^2 = \frac{\pi}{4}(1.5)^2 = 1.76715 \text{ in}^2$ $\quad L_{BC} = 6 \text{ ft} = 72 \text{ in}$

$$U_{BC} = \frac{P_m^2 L_{BC}}{2EA_{BC}} = \frac{P_m^2 (72)}{(2)(29\times10^6)(1.76715)} = 702.48\times10^{-9} P_m^2$$

Total: $U = U_{AB} + U_{BC} = 1.39062\times10^{-6} P_m^2$

$\frac{1}{2} P_m \delta_m = U, \quad \delta_m = \frac{2U}{P_m} = 2.78124 \times 10^{-6} P_m \quad \therefore \quad P_m = 359.552\times10^3 \delta_m$

$U = \frac{1}{2} P_m \delta_m = 179.776\times10^3 \delta_m^2$

Work of falling weight $\quad W(h+\delta_m) = 75(20+\delta_m) = 1500 + 75\delta_m$

Equating work and energy $\quad 1500 + 75\delta_m = 179.776\times10^3 \delta_m^2$

$$\delta_m^2 - 417.185\times10^{-6}\delta_m - 8.3437\times10^{-3} = 0$$

(a) $\delta_m = \frac{1}{2}\left\{417.185\times10^{-6} + \sqrt{(417.185\times10^{-6})^2 + (4)(8.3437\times10^{-3})}\right\}$

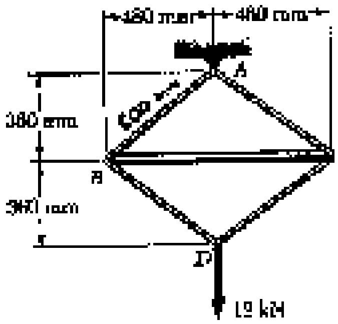
$= 0.091553$ in. $\qquad \delta_m = 0.0916$ in. ◄

(b) $P_m = (359.552\times10^3)(0.091553) = 32917$ lb $\qquad P_m = 32900$ lb ◄

$\sigma_m = \frac{P_m}{A_{min}} = \frac{32917}{1.76715} = 18630$ psi $= 18.63$ ksi ◄

PROBLEM 11.146

11.146 The steel rod BC has a 24-mm diameter and the steel cable $ABDCA$ has a 12-mm diameter. Using $E = 200$ GPa, determine the deflection of point D caused by the 12-kN load.



SOLUTION

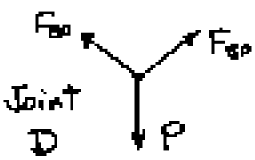
Owing to symmetry $\quad F_{AB} = F_{BD} = F_{DC} = F_{CA}$

$\qquad U_{AB} = U_{BD} = U_{DC} = U_{CA}$

$$U = 4U_{BD} + U_{BC} = 4\frac{F_{BD}^2 L_{BD}}{2EA_{BD}} + \frac{F_{BC}^2 L_{BC}}{2EA_{BC}}$$

Let P be the load at D

$$\delta_D = \frac{\partial U}{\partial P} = 4\frac{F_{BD}L_{BD}}{EA_{BD}}\frac{\partial F_{BD}}{\partial P} + \frac{F_{BC}L_{BC}}{EA_{BC}}\frac{\partial F_{BC}}{\partial P}$$

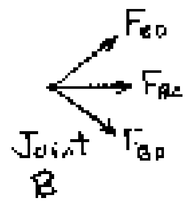


$+\uparrow \Sigma F_y = 0$

$2\frac{3}{5}F_{BD} - P = 0$

$F_{BD} = \frac{5}{6}P$

$\frac{\partial F_{BD}}{\partial P} = \frac{5}{6}$



$\overset{+}{\rightarrow} \Sigma F_x = 0$

$F_{BC} + (2)\frac{4}{5}F_{BD} = 0$

$F_{BC} = -\frac{8}{5}F_{BD} = -\frac{4}{3}P$

$\frac{\partial F_{BC}}{\partial P} = -\frac{4}{3}$

$$\delta_D = 4\left(\frac{5}{6}\right)^2\frac{PL_{BD}}{EA_{BD}} + \left(\frac{4}{3}\right)^2\frac{PL_{BC}}{EA_{BC}} = \frac{P}{E}\left\{\frac{25}{9}\frac{L_{BD}}{A_{BD}} + \frac{16}{9}\frac{L_{BC}}{A_{BC}}\right\}$$

Data: $\quad P = 12 \times 10^3$ N $\qquad E = 200 \times 10^9$ Pa

$L_{BD} = 600 \times 10^{-3}$ m $\qquad A_{BD} = \frac{\pi}{4}(12)^2 = 113.097 \text{ mm}^2 = 113.097 \times 10^{-6} \text{ m}^2$

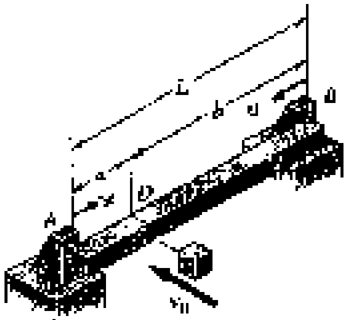
$L_{BC} = 960 \times 10^{-3}$ m $\qquad A_{BC} = \frac{\pi}{4}(24)^2 = 452.39 \text{ mm}^2 = 452.39 \times 10^{-6} \text{ m}^2$

$$\delta_D = \frac{12 \times 10^3}{200 \times 10^9}\left\{\frac{25}{9}\cdot\frac{600 \times 10^{-3}}{113.097 \times 10^{-6}} + \frac{16}{9}\cdot\frac{960 \times 10^{-3}}{452.39 \times 10^{-6}}\right\} = 1.111 \times 10^{-3} \text{ m}$$

$$= 1.111 \text{ mm} \downarrow$$

PROBLEM 11.147

11.147 The simply supported beam AB is struck squarely at D by a block of mass m moving horizontally with a velocity v_0. Show that the resulting maximum normal stress σ_m in the beam due to bending is independent of the location of point D.



SOLUTION

Let P_m be the equivalent static load at point D

Reactions: $R_A = \frac{P_m b}{L}$, $R_B = \frac{P_m a}{L}$

Maximum bending moment $M_m = R_A a = \frac{P_m a b}{L}$

Portion AD $\quad U_{AD} = \int_0^a \frac{M^2}{2EI}dx = \int_0^a \frac{(R_A x)^2}{2EI}dx = \frac{P_m^2 b^2}{2EIL^2}\int_0^a x^2 dx = \frac{P_m^2 b^2 a^3}{6EIL^2}$

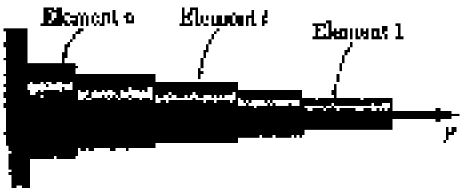
Portion DB $\quad U_{DB} = \int_0^b \frac{M^2}{2EI}dv = \int_0^b \frac{(R_B v)^2}{2EI}dv = \frac{P_m^2 a^2}{2EIL^2}\int_0^b v^2 dv = \frac{P_m^2 a^2 b^3}{6EIL^2}$

Total $\quad U = \frac{P_m^2 a^2 b^2 (a+b)}{6EIL^2} = \frac{P_m^2 a^2 b^2}{6EIL} = \frac{M_m^2 L}{6EI}$

$\frac{1}{2} m v_0^2 = U = \frac{M_m^2 L}{6EI} \qquad M_m = \sqrt{\frac{3EImv_0^2}{L}}$

Stress $\quad \sigma_m = \frac{M_m c}{I} = \sqrt{\frac{3Emv_0^2 c^2}{IL}}$, which is independent of a or b.

PROBLEM 11.C1



11.C1 A rod consisting of n elements, each of which is homogeneous and of uniform cross section, is subjected to a load **P** applied at its free end. The length of element i is denoted by L_i and its diameter by d_i. (a) Denoting by E the modulus of elasticity of the material used in the rod, write a computer program that can be used to determine the strain energy acquired by the rod and the deformation measured at the free end. (b) Use this program to determine the strain energy and deformation of the rods of Probs. 11.9 and 11.12.

SOLUTION

ENTER: P AND E

FOR EACH ELEMENT

ENTER A_i AND L_i

COMPUTE: NORMAL STRESS: $\sigma_i = \dfrac{P}{A_i}$

STRAIN ENERGY: $U_i = \dfrac{P^2 L_i}{2 A_i E}$

STRAIN ENERGY DENSITY:

$u_i = \dfrac{\sigma_i^2}{2E}$

TOTAL STRAIN ENERGY

UPDATE THROUGH n ELEMENTS

$U = U + U_i$

TOTAL DEFORMATION

$\frac{1}{2} P\Delta = U \quad ; \quad \Delta = \dfrac{2U}{P}$

PROGRAM OUTPUT

```
Problem 11.9

Axial load =   9.000 kips    Modulus of elasticity =  29 x 10^6 psi

Element    Length     delta L     Stress   Strain Energy   Strain Energy Density
            in.        in.         ksi        in·lb          lb·in./in.^3

  1       24.000      0.022       26.08      86.32            11.72
  2       36.000      0.022       18.11      89.92             5.65

Total Strain Energy  =   176.24 in-lb
Total Deformation    =    0.0441 in.


Problem 11.12

Axial load =  25.000 kN     Modulus of elasticity = 200 GPa

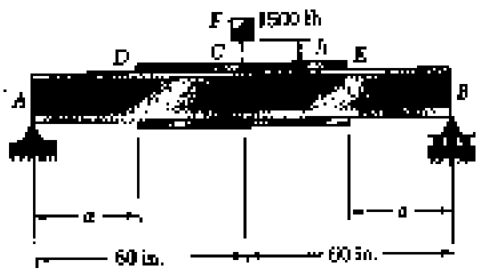
Element    Length    delta L    Stress   Strain Energy  Strain Energy Density
             m         mm        MPa          J               kJ/m^3

  1         0.80     0.497     124.34       6.22            38.65
  2         1.20     0.477      79.58       5.97            15.83

Total Strain Energy  =   12.1853 J
Total Deformation    =    0.9748 mm
```

PROBLEM 11.C2

11.C2 Two 0.75 × 6-in. cover plates are welded to a W8 × 18 rolled-steel beam as shown. The 1500-lb block is to be dropped from a height $h = 2$ in. onto the beam. (*a*) Write a computer program to calculate the maximum normal stress on transverse sections just to the left of *D* and at the center of the beam for values of *a* from 0 to 60 in., using 5-in. increments. (*b*) From the values considered in part *a*, select the distance *a* for which the maximum normal stress is as small as possible. Use $E = 29 \times 10^6$ psi.



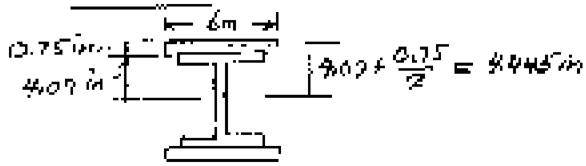
SOLUTION

COMPUTE AND ENTER MOMENTS OF INERTIA AND SECTION MODULI

FOR AD AND EB: W8x18

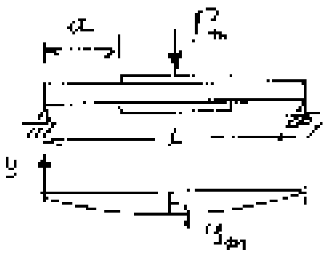
$I_1 = 61.9 \text{ in}^4 \qquad S_1 = 15.2 \text{ in}^3$ ◁

FOR DCE: W8x18 PLUS COVER PLATES



$$I_2 = 61.9 + 2(6\times0.75)(4.445)^2 = 239.72 \text{ in}^4 \quad ◁$$

$$S_2 = \frac{I_2}{(4.07+0.75)} = \frac{239.72}{4.82} = 49.7 \text{ in}^3 \quad ◁$$



$y_m = P_m \alpha$ WHERE α = INFLUENCE COEFFICIENT
SEE NEXT PAGE FOR DETERMINATION OF α

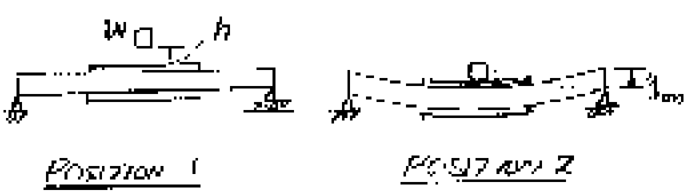
P_m = EQUIVALENT STATIC LOAD

$$U_2 = \frac{1}{2} P_m y_m = \frac{1}{2} \frac{y_m^2}{\alpha}$$

WORK DONE BY W IS $W(h + y_m)$

$$\frac{1}{2}\frac{y_m^2}{\alpha} = Wh + W y_m$$

OR: $y_m^2 - 2W\alpha\, y_m - 2Wh\alpha$ (A)



PROGRAM SOLUTION OF (A) FOR y_m

ENTER $L = 120$ in., $h = 2$ in., $W = 1500$ lb, $E = 29\times10^6$ psi

FOR $a = 0$ TO 60 in, STEP 5 in.:

SOLVE (A) FOR y_m, $P_m = y_m/\alpha$, $y_{ST} = W\alpha$

$$\sigma_D = \sigma_1 = \tfrac{1}{2} P_m a / S_1 \quad ; \quad \sigma_C = \sigma_2 = \tfrac{1}{4} P_m L / S_2$$

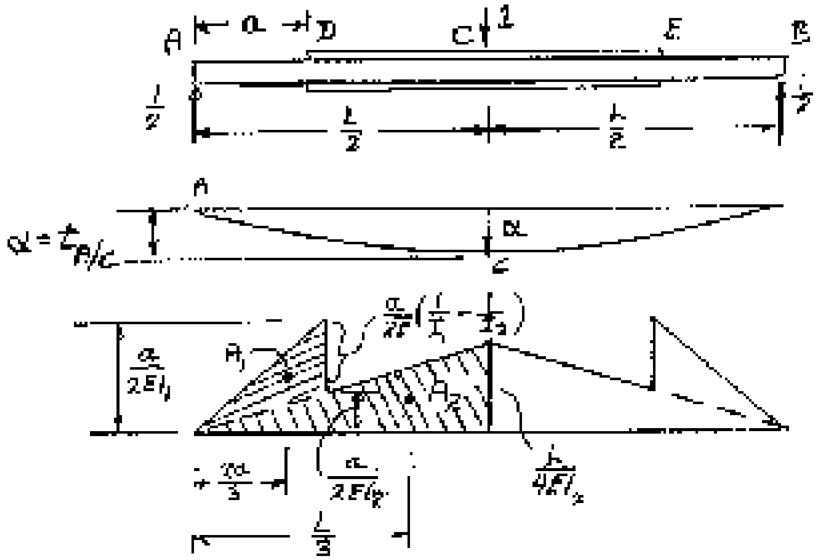
PRINT: a, y_{ST}, y_m, P_m, σ_1, σ_2, AND $(\sigma_1 - \sigma_2)$

REPEAT WITH SMALLER INTERVALS TO FIND a FOR $(\sigma_1 - \sigma_2) = 0$
THIS IS THE DISTANCE a FOR σ_{max} AS SMALL AS POSSIBLE

CONTINUED

PROBLEM 11.C2 - CONTINUED

DETERMINATION OF α: α IS DEFLECTION AT C FOR A UNIT LOAD AT C.



$$\alpha = t_{A/C} = A_1\left(\frac{2a}{3}\right) + A_2\left(\frac{L}{3}\right) = \left[\frac{1}{2}\frac{a}{EI_1}\left(\frac{1}{I_1}-\frac{1}{I_2}\right)\frac{a}{2}\right]\frac{2a}{3} + \left[\frac{1}{2}\cdot\frac{L}{4EI_2}\cdot\frac{L}{2}\right]\frac{L}{3}$$

$$\alpha = \left[\left(\frac{1}{I_1}-\frac{1}{I_2}\right)a^3 + \frac{1}{8I_2}L^3\right]\Big/6E$$

PROGRAM OUTPUT

Beam = W 8x18 with two 6 by 0.75-in. cover plates
h = 2 in. W = 1500 lb L = 120 in.

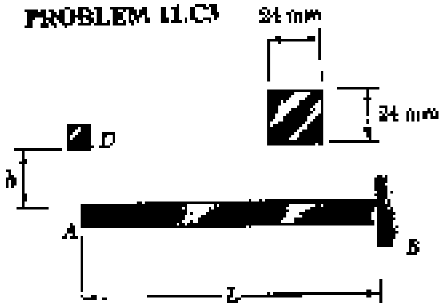
a in.	ystat in.	ymax in.	Pmax lb	σ1 ksi	σ2 ksi	σ1 - σ2 ksi
0.00	0.00777	0.1842	35572	0.00	21.46	-21.46
5.00	0.00778	0.1844	35544	5.85	21.44	-15.59
10.00	0.00787	0.1855	35348	11.63	21.32	-9.69
15.00	0.00812	0.1885	34834	17.19	21.01	-3.82
20.00	0.00859	0.1942	33896	22.30	20.45	1.85
25.00	0.00938	0.2033	32509	26.73	19.61	7.13
30.00	0.01056	0.2163	30736	30.33	18.54	11.79
35.00	0.01220	0.2334	28706	33.05	17.32	15.73
40.00	0.01438	0.2546	26563	34.95	16.02	18.93
45.00	0.01718	0.2799	24436	36.17	14.74	21.43
50.00	0.02068	0.3090	22415	36.87	13.52	23.35
55.00	0.02496	0.3419	20550	37.18	12.40	24.78
60.00	0.03008	0.3783	18862	37.23	11.38	25.85

Use smaller increments to seek the smallest maximum normal stress

18.33	0.00840	0.1919	34259	20.657	20.665	-0.01
18.34	0.00840	0.1920	34257	20.667	20.664	0.00
18.35	0.00841	0.1920	34258	20.677	20.663	0.01

Max stress small as possible for a = 18.34in.
Smallest max stress = 20.67 ksi

PROBLEM 11.C3



11.C3 The 16-kg block D is dropped from a height h onto the free end of the steel bar AB. For the steel used $\sigma_{all} = 120$ MPa and $E = 200$ GPa. (*a*) Write a computer program to calculate the maximum allowable height h for values of the length L from 100 mm to 1.2 m, using 100-mm increments. (*b*) From the values considered in part *a*, select the length corresponding to the largest allowable height.

SOLUTION

ENTER $\sigma_{all} = 120$ MPa, $E = 200$ GPa, $d = 0.024$ m

$m = 16$ kg, $g = 9.81$ m/s^2

$I = d^4/12 \qquad S = \frac{I}{c} = \frac{I}{d/2} = d^3/6$

FOR L = 100 mm TO 1200 mm STEP 100 mm

$L = L/1000$

$y_{st} = mg\,L^3/3EI$

$M_{max} = \sigma_{all}\,S$

$P_{max} = M_{max}/L$

$y_{max} = P_{max}\,L^3/3EI$

From Prob. 11.69, page 705

$$y_m = y_{st}\left[1+\sqrt{1+\frac{2h}{y_{st}}}\right] \quad \xrightarrow{\text{Solve For}} \quad h = \left[\left(\frac{y_{max}}{y_{st}}-1\right)^2-1\right]\frac{y_{st}}{2}$$

PRINT: L, y_{st}, y_{max}, P_{max}, M_{max}, h

RETURN

PROGRAM OUTPUT

Problem 11.C3

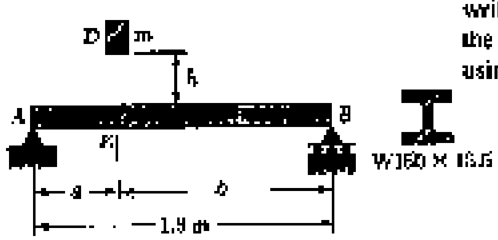
m = 16.0 kg   d = 24 mm   σ = 120 MPa   E = 200 GPa

L mm	ystat mm	ymax mm	Pmax N	Mmax N·m	h mm
100	0.00946	0.167	2764.8	276.48	1.301
200	0.07569	0.667	1382.4	276.48	2.269
300	0.25547	1.500	921.6	276.48	2.904
400	0.60556	2.667	691.2	276.48	3.205
500	1.18273	4.167	553.0	276.48	3.173
600	2.04375	6.000	460.8	276.48	2.807
700	3.24540	8.167	395.0	276.48	2.109
800	4.84445	10.667	345.6	276.48	1.076
900	6.89766	13.500	307.2	276.48	-0.289
1000	9.46181	16.667	276.5	276.48	-1.988
1100	12.59367	20.167	251.3	276.48	-4.020
1200	16.35000	24.000	230.4	276.48	-6.385

Use smaller increments to seek the largest height h

435	0.77883	3.154	635.6	276.48	3.2316
440	0.80599	3.227	628.4	276.48	3.2320
445	0.83378	3.300	621.3	276.48	3.2317

PROBLEM 11.C4



11.C4 The block D of mass $m = 8$ kg is dropped from a height $h = 750$ mm onto the rolled-steel beam AB. Knowing that $E = 200$ GPa, write a computer program to calculate the maximum deflection of point E and the maximum normal stress in the beam for values of a from 100 to 900 mm, using 100-mm increments.

SOLUTION

ENTER: $L = 1.8$ m, $E = 200$ GPa, $h = 0.75$ m, $m = 8$ kg, $g = 9.81$ m/s^2

$I = 6.87 \times 10^{-6}$ m^4

$S = 91.6 \times 10^{-6}$ m^3

FOR a = 100 mm TO 900 mm STEP 100 mm

$a = a/1000$

$b = L - a$

SEE PROB. 11.71, page 705 → $y_{st} = mg\,a^2 b^2/3EIL$

INFLUENCE COEFFICIENT FOR Δ_E FOR UNIT LOAD AT E → $\alpha = a^2 b^2/3EIL$

SEE PROB. 11.64, page 705 → $y_m = y_{st}\left[1 + \sqrt{1 + \dfrac{2h}{y_{st}}}\right]$

$P_{max} = y_m/\alpha$

$M_{max} = P_{max}\, a\, b/L$

$\sigma_{max} = M_{max}/S$

PRINT: a, y_{st}, y_m, P_{max}, σ_{max}

RETURN

```
Problem 11.C4

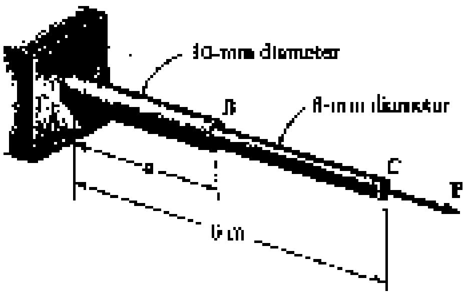
Beam:  W 150 x 13.5
  I = 6.87 x10^-6 m^4    S = 91.6x10^-6 m^3

  L = 1.8 m    h = 750 mm    m = 8 kg    g = 9.81 m/s^2
```

a mm	ystat mm	ymax mm	Pmax N	σ_{max} MPa
100	0.0003	0.6775	173.93	179.33
200	0.0011	1.2757	92.43	179.40
300	0.0021	1.7946	65.75	179.46
400	0.0033	2.2339	52.85	179.51
500	0.0045	2.5936	45.55	179.55
600	0.0055	2.8734	41.13	179.59
700	0.0063	3.0734	38.46	179.61
800	0.0068	3.1934	37.02	179.63
900	0.0069	3.2334	36.56	179.63

NOTE: THE SMALL VARIATION IN σ_{max}. THIS IS DUE TO THE ENERGY REQUIRED BY THE MASS AS IT FALLS THROUGH y_{max}. SEE PROB. 11.147, page 731, FOR A CASE WHERE ENERGY DELIVERED IS CONSTANT AND σ_{max} IS ALSO CONSTANT.

PROBLEM 11.C5



11.C5 The steel rods AB and BC are made of a steel for which $\sigma_Y = 300$ MPa and $E = 200$ GPa. (*a*) Write a computer program to calculate, for values of a from 0 to 6 m, using 1-m increments, the maximum strain energy that can be acquired by the assembly without causing any permanent deformation. (*b*) For each value of a considered, calculate the diameter of a uniform rod of length 6 m and of the same mass as the original assembly, and the maximum strain energy that could be acquired by this uniform rod without causing permanent deformation.

SOLUTION

ENTER: $\sigma_Y = 300\text{ MPa}$, $E = 200\text{ GPa}$, $L = 6\text{ m}$

$AREA_{AB} = \frac{\pi}{4}(0.010\text{ m})^2$, $AREA_{BC} = \frac{\pi}{4}(0.006\text{ m})^2$

$P_m = \sigma_Y \, AREA_{BC}$

FOR $a = 0$ TO 6 m, STEP 1 m

$$U = \frac{P_m^2}{2E}\left(\frac{a}{AREA_{AB}} + \frac{L-a}{AREA_{BC}}\right)$$

FOR UNIFORM ROD OF SAME VOLUME

$$Vol = a\,(AREA_{AB}) + (L-a)(AREA_{BC})$$

$$d = \sqrt{\frac{4\,Vol}{\pi L}}$$

$$AREA_{NEW} = \frac{\pi}{4} d^2$$

$$P_{NEW} = \sigma_Y\,(AREA_{NEW})$$

$$U_{NEW} = \frac{P_{NEW}^2 L}{2E(AREA_{NEW})}$$

PRINT a, U, Vol, d, P_{NEW}, U_{NEW}

RETURN

PROGRAM OUTPUT

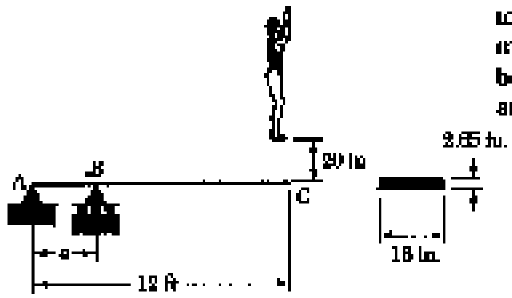
Problem 11C5

sigmaY = 300 MPa, Pm = 8482 N, L = 6 m, E = 200 GPA

a m	U J	Vol m^3	d mm	New P N	newU J
0.00	38.17	169.65	6.00	8482.30	38.17
1.00	34.10	219.91	6.83	10995.58	49.48
2.00	30.03	270.18	7.57	13508.85	60.79
3.00	25.96	320.44	8.25	16022.12	72.10
4.00	21.88	370.71	8.87	18535.40	83.41
5.00	17.81	420.97	9.45	21048.67	94.72
6.00	13.74	471.24	10.00	23561.95	106.03

PROBLEM 11.C6

11.C6 A 160-lb diver jumps from a height of 20 in. onto end C of a diving board having the uniform cross section shown. Write a computer program to calculate for values of a from 10 to 50 in., using 10-in. increments, (*a*) the maximum deflection of point C, (*b*) the maximum bending moment in the board, (*c*) the equivalent static load. Assume that the diver's legs remain rigid and use $E = 1.8 \times 10^6$ psi.

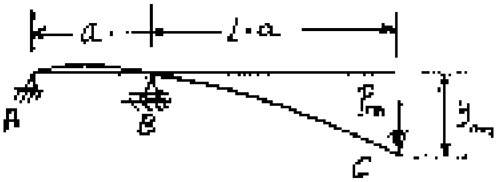


SOLUTION

ENTER: $L = 12$ ft, $h = 20$ in., $W = 160$ lb
$E = 1.8 \times 10^6$ psi

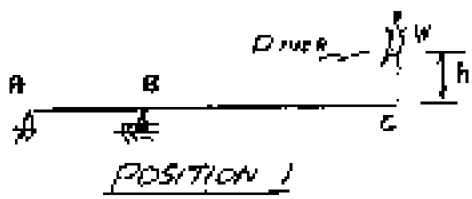
$I = (18\text{ in.})(2.65\text{ in.})^3/12$

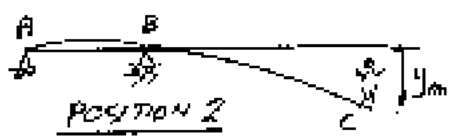
$S = (18\text{ in.})(2.65\text{ in.})^2/6$



$y_m = P_m \alpha$ WHERE α = INFLUENCE COEFFICIENT
SEE BELOW FOR DETERMINATION OF α

WHERE P_m = EQUIVALENT STATIC LOAD





$U_m = \frac{1}{2} P_m y_m = \frac{1}{2} \frac{y_m^2}{\alpha}$

WORK $= W(h + y_m)$

WORK $= U_m$

$W(h + y_m) = \frac{1}{2} \frac{y_m^2}{\alpha}$ A

PROGRAM SOLUTION OF α FOR y_m. ENTER α

FOR $a = 10$ in. TO 50 in. STEP 10 in.

SOLVE A FOR y_m; $P_m = y_m/\alpha$

$M_{max} = M_B = P_m(L - a)$

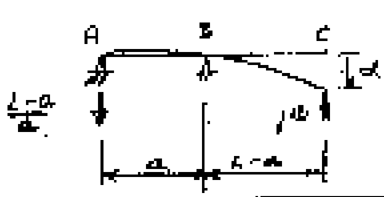
$\sigma = M_{max}/S$

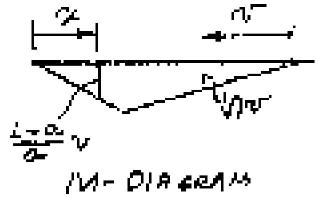
PRINT a; y_m; P_m; M_{max}; σ

PROGRAM OUTPUT

a in.	ym in.	Pm lb	Max M kip·in.	sigma psi
10	14.622	757.7	101.532	5422
20	13.262	802.6	99.519	5314
30	11.950	855.6	97.536	5208
40	10.683	919.1	95.583	5104
50	9.462	996.4	93.661	5001

DETERMINATION OF INFLUENCE COEFFICIENT α





$U = \frac{1}{2}(1^{lb})\alpha = \sum \int \frac{M^2}{2EI} dx$

$\frac{\alpha}{2} = \frac{1}{2EI}\left[\int_0^a \left(\frac{L-a}{a}\right)^2 x^2 dx + \int_0^{L-a} v^2 dv\right]$

$\alpha = \frac{1}{EI}\left[\frac{(L-a)^2}{a^2}\frac{a^3}{3} + \frac{(L-a)^3}{3}\right]$

$\alpha = \frac{1}{3EI}\left[(L-a)^2 a + (L-a)^3\right]$ ◁