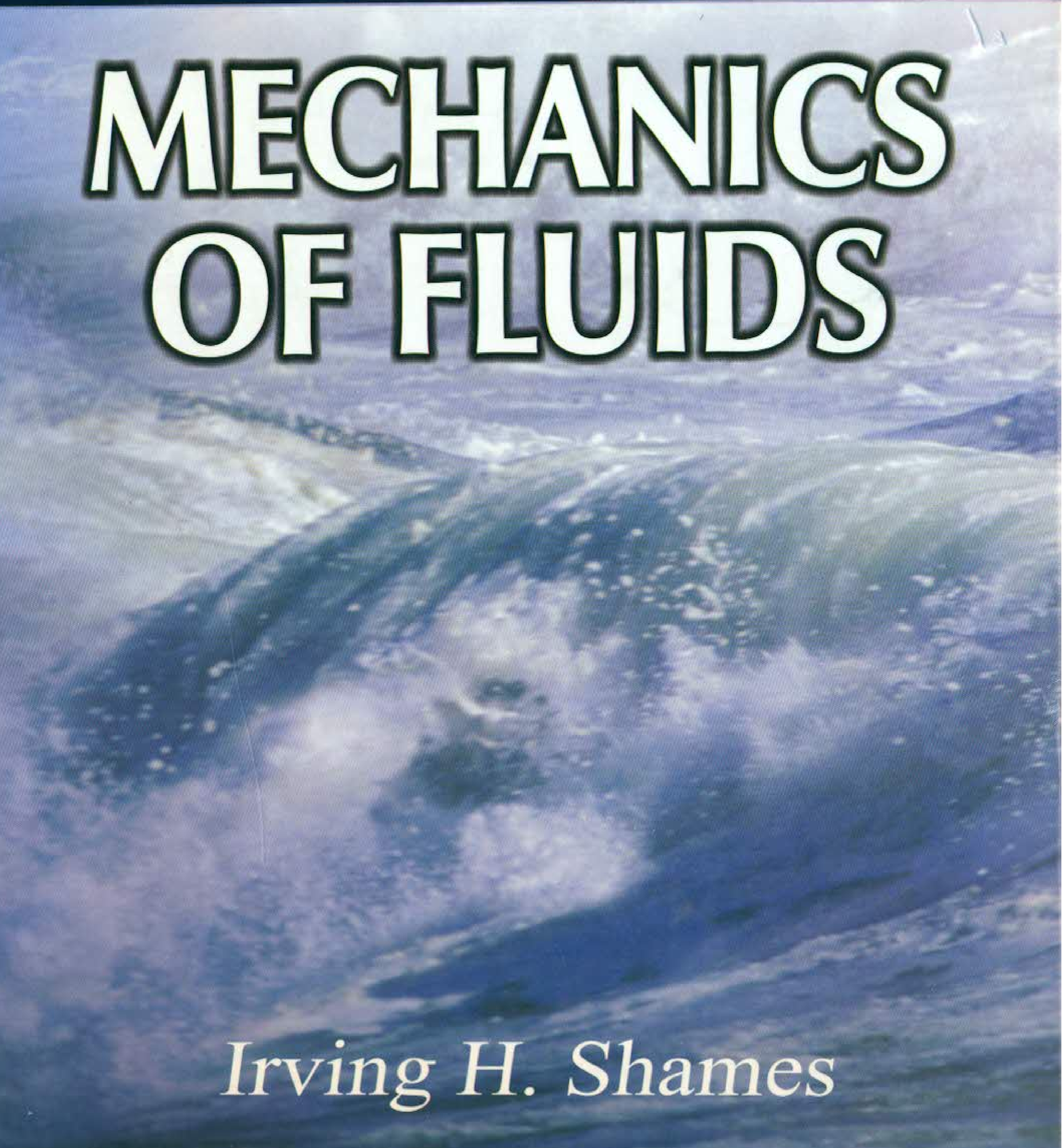
Fourth Edition

MECHANICS OF FLUIDS



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Mechanics of Fluids

Fourth Edition

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McGraw Hill

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PREFACE

This book was first published in 1962. Subsequent editions followed with a continual buildup of the contents and the level of treatment. Eventually, long-time users indicated to me that parts of the book had become too long and too advanced for some of their students. The advanced material was not being used and had the tendency to intimidate and discourage these students. With the advice of seven excellent reviewers who were familiar with previous editions, I am, in this edition, attempting to rectify these problems and, at the same time, attempting to make the treatment more student friendly without knowingly "dumbing down" the book. I am inserting changes developed during the past three years that I believe will improve the pedagogy of the text. For starters, this preface will be short and crisp.

First of all, I have deleted about 150 pages of the third edition, and they are available on the website. These pages include such topics as:

1. The tensor development of the differential forms of the basic laws.
2. A careful development of the Stokes viscosity law.
3. Blasius' development of boundary-layer thickness.
4. Axisymmetric potential flow.
5. Noninertial control volume analysis.
6. Heating and friction in constant area ducts.

I have also deleted detailed chapters on turbomachines and measurement techniques. In their place I have inserted, at appropriate places, instructive examples and useful homework problems to partially compensate for the deletion of these chapters.

In addition, I have overhauled all the past examples and have added new examples, all of which have the following clearly delineated format:

Problem Statement for describing the problem.

Strategy for attacking the problem.

Execution for carrying out the computation.

Debriefing for discussing the accuracy of the solutions and the trade-off made in using the assumptions, as well as other pertinent remarks.

At the end of each chapter, in addition to the usual closure, I have included a *Highlights* section. Here we verbally go over the essence of the chapter with minimum mathematical and development details, the goal being to engender a physical feel for the material.

I have also added about 40 computer problem examples that are solved using MATLAB. These examples provide detailed discussion for the programming. The chapter on computational fluid mechanics from the previous editions by my distinguished Buffalo colleague, Dr. Dale Taulbee, has been retained, including a list of projects.

I have attempted to open-end the treatment in this book with the likely course work in particle and solid mechanics, not to mention thermodynamics.

To facilitate use, the homework problems are delineated in the following way:

- Problems that are particularly challenging or longer than usual are identified with a single asterisk.
- Computer problems are identified with a computer icon and are placed at the end of the problem list.
- On the website, homework problems are separated into three groups, according to difficulty.

Students are encouraged to use whatever code they are comfortable with. At the end of Chapter 14, we have provided a projects section for which the student can use FORTRAN or C language.

There is more than enough material for most users to cover in a one-semester, three-credit course. The instructor will not find it hard to choose which material to cover. What is left may be inserted as needed in other follow-on courses since much effort was made for flexibility of use.

Finally, I want to thank the excellent reviewers who gave me much valuable advice, all of which I followed.

Giles J. Brereton, *Michigan State University*

Marilyn Lightstone, *McMaster University*

Ephraim Gutmark, *University of Cincinnati*

John R. Biddle, *California Polytechnic Institute, Pomona*

Jay Khodada, *Auburn University*

Chiang Shih, *Florida State University*

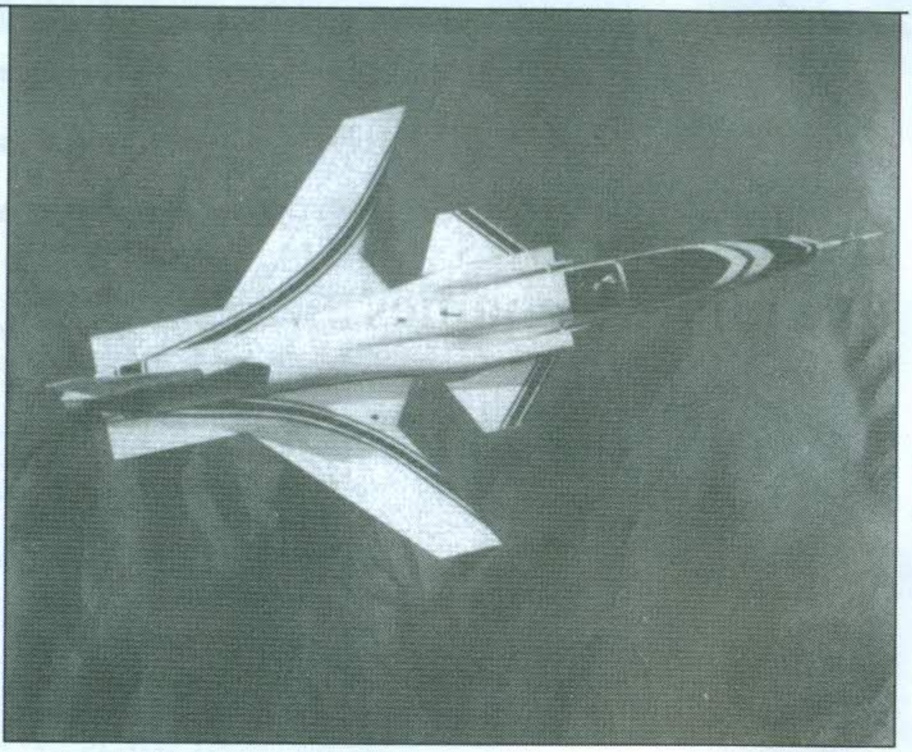
Russell Daines, *Brigham Young University*

I was very fortunate in having engineering student Aerik Lafave available to program my computer examples. His academic and U.S. Navy experience proved to be of great value to me, and I thank him profusely. I also thank my esteemed colleagues in the civil and environmental department and in the mechanical and aerospace department at GW for their continued support and friendship. They have honored me by using my books for their mechanics sequence, covering statics, dynamics, solids, and fluids in four courses. I can think of nothing more gratifying than that show of support and confidence. I wish next to thank Mrs. Anne Sheffield and Miss Sparkle Todman, who administer my department, for their help and kindness in keeping this absent-minded professor on the right track. And of course, most of all, I thank my dear wife Sheila for her sound advice and unfailing affection.

PART 1

strong belief /

Basic Principles of Fluid Mechanics



X-29 advanced technology demonstrator. (*Courtesy Grumman Corporation, Bethpage, NY.*)

German engineers began experimenting with forward swept wings during the Second World War. Grumman Aviation began experimenting with forward swept wings in 1981. The research program has shown that a forward swept wing will produce approximately 20 percent better performance in the transonic regime than an equivalent aft swept wing. The advantage of a lower drag across its entire operational envelope, particularly at speeds around Mach 1, permits the use of a smaller engine.

Compared to an aft swept wing, a forward swept wing offers higher maneuverability, improved slow-speed handling, and lower stall speeds with good post-stall characteristics. And since forward swept wings are placed further back on the fuselage, more flexibility in fuselage design is possible.

However, unfavorable aeroelastic effects are prevalent for forward swept metal wings requiring stiffer and hence heavier wings, thus negating their potential gains. The advent of advanced composite materials provides a solution. Aeroelastic tailoring of graphite epoxy composites allows the forward swept wing to twist its leading edge down to counteract the upward bending motion that the wing experiences due to flight loads.

Finally, we point out there are control problems for subsonic flight conditions for this kind of aircraft. The instability is controlled by an advanced digital flight control system, which adjusts the control surfaces up to 40 times each second. The system is driven by three computers.

CHAPTER 1

Fundamental Notions

PART A FLUID CONCEPTS

1.1 HISTORICAL NOTE

Prior to the 20th century the study of fluids was undertaken essentially by two groups—hydraulicians and mathematicians. Hydraulicians worked along empirical lines, while mathematicians concentrated on analytical lines. The vast and often ingenious experimentation of the former group yielded much information of indispensable value to the practicing engineer of the day. However, lacking the generalizing benefits of workable theory, these results were of restricted and limited value in novel situations. Mathematicians, meanwhile, by not availing themselves of experimental information, were forced to make assumptions so simplified as to render their results very often completely at odds with reality.

It became clear to such eminent investigators as Reynolds, Froude, Prandtl, and von Kármán that the study of fluids must be a blend of theory and experimentation. Such was the beginning of the science of fluid mechanics as it is known today. Our modern research and test facilities employ mathematicians, physicists, engineers, and skilled technicians, who, working in teams, bring both viewpoints in varying degrees to their work.

محیط پیوسته

1.2 FLUIDS AND THE CONTINUUM

We define a fluid as a substance which must continue to change shape as long as there is a shear stress, however small, present. By contrast a solid undergoes a definite displacement (or breaks completely) when subjected to a shear stress. For instance, the solid block shown on the left in Fig. 1.1 changes shape in a manner conveniently characterized by the angle $\Delta\alpha$ when subjected to a shear stress τ. If this were an element of fluid (as shown on the right in Fig. 1.1), there would be no fixed

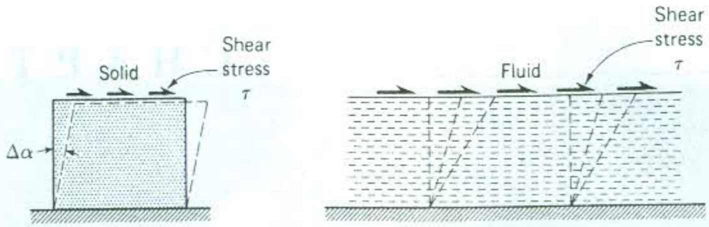


Figure 1.1
Shear stress on a solid and on a fluid.

$\Delta\alpha$ even for an infinitesimal shear stress. Instead, a continual deformation persists as long as a shear stress τ is applied. In materials which we sometimes call *plastic,* such as paraffin, either type of shear deformation may be found depending on the shear-stress magnitude. Shear stress below a certain magnitude will induce definite displacements akin to those of a solid body, whereas shear stress above this value causes continuous deformation similar to that of the fluid. This dividing shear-stress magnitude is dependent on the type and state of the material. Certain of these materials are called *Bingham materials.*

In considering various types of fluids under *static* conditions, we find that certain fluids undergo very little change in density despite the existence of large pressures. These fluids are invariably in the liquid state for such behavior. Under such circumstances, the fluid is termed *incompressible,* and it is assumed during computations that the density is constant. The study of incompressible fluids under static conditions is called *hydrostatics.* Where the density cannot be considered constant under static conditions, as in a gas, the fluid is termed *compressible,* and we sometimes use the name *aerostatics* to identify this class of problems.

The classifications of compressibility given above are reserved for statics. In fluid *dynamics,* the question of when the density may be treated as constant involves more than just the nature of the fluid. Actually, it depends mainly on a certain dimensionless flow ratio (the Mach number). We then speak of incompressible and compressible *flows,* rather than incompressible or compressible *fluids.* Whenever density variations in a problem are inconsequential, gases and liquids submit to the same manner of analysis. For instance, for flow about fully submerged bodies the basic formulations for low-speed aerodynamics (under about 300 mi/h) are the same as for hydrodynamics. In fact, it is entirely possible to examine certain performance characteristics of low-speed airfoils in a water tunnel.

Fluids are composed of molecules in constant motion and collision. To be exact in an analysis, one would have to account for the action of each molecule or group of molecules in a flow. Such procedures are adopted in the kinetic theory of gases and statistical mechanics but are, in general, too cumbersome for use in engineering applications. In most engineering computations, we are interested in average, measurable manifestations of the many molecules—as, for example, density, pressure, and temperature. These manifestations can be conveniently assumed to arise from a *hypothetical continuous distribution of matter,* called the *continuum,* instead of the

Figure 1.2
Noncontinuum effect on area element.

actual complex conglomeration of discrete molecules. The continuum concept affords great simplification in analysis and already has been used as an idealization in earlier mechanics courses in the form of a rigid body or a perfectly elastic body.

The continuum approach must be used only where it may yield reasonably correct results. For instance, the continuum approach breaks down when the mean free path[1] of the molecules is of the same order of magnitude as the smallest significant length in the problem. Under such circumstances we may no longer detect meaningful, gross manifestations of molecules. The action of each molecule or group of molecules is then of significance and must be treated accordingly.

To illustrate this, examine the action of a gas on an inside circular element of area of a closed container. With even relatively small amounts of enclosed fluid the innumerable collisions of molecules on the surface result in the gross, non-time-dependent manifestation of force. A truly continuous substance would simulate such action quite well. If only a very tiny amount of gas is now permitted in the container so that the mean free path is of the same order of magnitude as the diameter of the area element, an erratic activity is experienced, as individual molecules or groups of molecules bombard the surface. We can no longer speak of a constant force but must cope with an erratic force variation, as indicated graphically in Fig. 1.2. This action is not what is expected of a continuous distribution of mass. Thus, it is seen that in the first situation the continuum approach would be applicable but in the second case the continuum approach, ignoring as it does individual molecular effects, would be of questionable value.

We may reach the same situation for any amount of enclosed gas by decreasing the size of the area element until irregular molecular effects become significant. Since the continuum approach takes no cognizance of action "in the small," it can yield no accurate information "in the small."

1.3 DIMENSIONS AND UNITS

To study mechanics, we must establish abstractions to describe those manifestations of the body that interest us. These abstractions are called *dimensions.* The dimensions that we pick, which are independent of all other dimensions, are termed *primary,* or *basic, dimensions;* the ones that are then developed in terms of the basic dimensions are called *secondary dimensions.* Of the many possible sets of basic

[1]Mean free path is the average distance traversed by the molecules between collisions.

dimensions that we could use, we will confine ourselves at present to the set that includes the dimensions of length, time, mass, and temperature. Also we can use force in place of mass in the list of basic dimensions. For quantitative purposes, units have been established for these basic dimensions by various groups and countries. The U.S. Customary System (USCS) employs the pound-force, foot, second, and degree Rankine as the units for the basic dimensions. The International System of Units (SI) uses the newton, meter, second, and degree Kelvin. Table 1.1 lists several common systems of units.

It is convenient to identify these dimensions in the following manner:

Length	L
Time	T
Force	F
Temperature	θ

These formal expressions of identification for basic dimensions and the more complicated groupings to be presented for secondary dimensions are called *dimensional representations.*

Secondary dimensions are related by law or by definition to the basic dimensions. Accordingly, the dimensional representation of such quantities will be in terms of the basic dimensions. For instance, the dimensional representation of velocity V is

$$V \equiv \frac{L}{T}$$

By this scheme, pressure then has the dimensions F/L^2 and acceleration is expressed dimensionally as L/T^2.

Table 1.1 Common systems of units

Metric			
Centimeter-gram-second (cgs)		**SI**	
Mass	gram (g)	Mass	kilogram (kg)
Length	centimeter (cm)	Length	meter (m)
Time	second (s)	Time	second (s)
Force	dyne (dyn)	Force	newton (N)
Temperature	degree Kelvin (K)	Temperature	degree Kelvin (K)
U.S. Customary System			
Type I		**Type II**	
Mass	pound-mass (lbm)	Mass	slug (slug)
Length	foot (ft)	Length	foot (ft)
Time	second (s)	Time	second (s)
Force	pound-force (lbf)	Force	pound-force (lbf)
Temperature	degree Rankine (°R)	Temperature	degree Rankine (°R)

A change to a new system of units generally entails a change in the scale of measure for the secondary dimensions. The use of the dimensional representation given above permits a simple evaluation of the change of scale. For example, the handbook tells us that one scale unit of pressure in USCS, 1 lb of force per 1 ft^2, is equivalent to 47.9 scale units of pressure in the SI system, or 47.9 N/m^2 (= 47.9 Pa). The unit N/m^2 is called a pascal (Pa) in the SI system. We may arrive at this conclusion by writing the pressure dimensionally, substituting basic units of USCS, and then changing these units to equivalent SI units, as follows:

$$p \equiv \frac{F}{L^2} \equiv \frac{\text{lbf}}{\text{ft}^2} \equiv \frac{4.45 \text{ N}}{0.0929 \text{ m}^2} = 47.9 \text{ N/m}^2$$

Hence,

$$1 \text{ lbf/ft}^2 \equiv 47.9 \text{ N/m}^2 = 47.9 \text{ Pa}$$

On the inside covers of this book we list the physical equivalences of some of the common units of fluid mechanics.

Another technique is to form the ratio of a unit of a basic or secondary dimension and the proper number or fraction of another unit for the basic or secondary dimension such that there is physical equivalence between the quantities. The ratio is then considered as unity because of the one-to-one relation between numerator and denominator from this viewpoint. Thus,

$$\left(\frac{1 \text{ ft}}{12 \text{ in}}\right) \equiv 1$$

Or for another unit, we could say

$$\left(\frac{12 \text{ in}}{305 \text{ mm}}\right) \equiv 1$$

These are to be taken as statements of equivalence, not as algebraic relations in the ordinary sense. Multiplying an expression by such a ratio does not change the measure of the physical quantity represented by the expression. Hence, to change a unit in an expression, we then multiply this unit by a ratio physically equivalent to unity in such a way that the old unit is canceled out, leaving the desired unit. We can then perform a change of units on the previous case in a more convenient way, using the formalism given above on the expressions in the numerator and denominator. Thus,

$$p \equiv \frac{\text{lbf}}{\text{ft}^2} \equiv \frac{\cancel{\text{lbf}}\left(\dfrac{4.45 \text{ N}}{1 \cancel{\text{lbf}}}\right)}{\left[\cancel{\text{ft}}\left(\dfrac{0.305 \text{ m}}{1 \cancel{\text{ft}}}\right)\right]^2} \equiv 47.9 \frac{\text{N}}{\text{m}^2} = 47.9 \text{ Pa}$$

You are urged to employ the latter technique in your work, for the use of less-formal intuitive methods is an invitation to error.

In fluid mechanics, as noted earlier, we deal with secondary dimensions which stem from gross, measurable, molecular manifestations such as pressure and density. Manifestations which are primarily characteristic of a particular fluid and not the manner of flow are called *fluid properties.* Viscosity and surface tension are examples of fluid properties, whereas pressure and density of gases are primarily flow-dependent and hence are not considered fluid properties.

1.4 LAW OF DIMENSIONAL HOMOGENEITY

In order to determine the dimensions of properties established by laws, we must first discuss the law of dimensional homogeneity. This states that *an analytically derived equation representing a physical phenomenon must be valid for all systems of units.* Thus, the equation for the frequency of a pendulum, $f = (1/2\pi)\sqrt{g/L}$, is properly stated for any system of units. A plausible explanation for the law of dimensional homogeneity is that natural phenomena proceed completely oblivious to man-made units, and hence fundamental equations representing such phenomena should have a validity for *any* system of units. For this reason, the fundamental equations of physics are dimensionally homogeneous, so all relations derived from these equations must also be dimensionally homogeneous.

What restriction does this independence of units place on the equation? To answer this, examine the following arbitrary equation:

$$x = y\eta\zeta^3 + \alpha^{3/2}$$

For this equation to be dimensionally homogeneous, the numerical equality between both sides of the equation must be maintained for all systems of units. To accomplish this, the change of scale for each expression must be the same during changes of units. That is, if one expression such as $y\eta\zeta^3$ is doubled in numerical measure for a new systems of units, so must be the expressions x and $\alpha^{3/2}$. *For this to occur under all systems of units, it is necessary that each grouping in the equation have the same dimensional representation.*

As a further illustration, consider the following dimensional representation of an equation which is not dimensionally homogeneous:

$$L = T^2 + T$$

Changing the units from feet to meters will change the value of the left side while not affecting the right side, thus invalidating the equation in the new system of units. We are concerned almost entirely with dimensionally homogeneous equations in this text.

With this in mind, examine a common form of Newton's law, which states that the force on a body is proportional to the resulting acceleration. Thus

$$\mathbf{F} \propto \mathbf{a}$$

We may call the proportionality factor the mass (M). From the law of dimensional homogeneity the dimensions of mass must be

$$M \equiv \frac{FT^2}{L}$$

Mass may be considered as that property of matter which resists acceleration. Hence, it is entirely possible to choose mass as a basic dimension. Force would then be a dependent entity given dimensionally from Newton's law as

$$F \equiv \frac{ML}{T^2}$$

and our basic system of dimensions would then be mass (M), length (L), time (T), and temperature (θ).

1.5 A NOTE ON FORCE AND MASS

In USCS units, we define the amount of mass which accelerates at the rate of 1 ft/s^2 under the action of 1 lbf in accordance with Newton's law as the *slug*. The pound-force could be defined in terms of the deformation of an elastic body such as a spring at prescribed conditions of temperature. Unfortunately a unit of mass stipulated independently of Newton's law is also in common usage. This stems from the law of gravitational attraction, wherein it is posited that the force of attraction between two bodies is proportional to the masses of the bodies—the very same property of a material that enters into Newton's law. Hence, the *pound-mass* (lbm) has been defined as the amount of matter which at the earth's surface is drawn by gravity toward the earth by 1 lbf.

We have thus formulated two units of mass by two different actions, and to relate these units we must subject them to the same action. Thus we can take the pound-mass and see what fraction or multiple of it will accelerate at 1 ft/s^2 under the action of 1 lb of force. This fraction, or multiple, will then represent the number of units of pound-mass that are physically equivalent to 1 slug. It turns out that this coefficient is g_0, where g_0 has the value corresponding to the acceleration of gravity at a position on the earth's surface where the pound-mass was standardized.[2] The value of g_0 is 32.2, to three significant figures. We may then make the statement of equivalence that

$$1 \text{ slug} \equiv 32.2 \text{ lbm} \qquad [1.1]$$

How does weight fit into this picture? *Weight is defined as the force of gravity on a body.* Its value will depend on the position of the body relative to the earth's surface. At a location on the earth's surface where the pound-mass is standardized, a mass of 1 lbm has the weight of 1 lbf; but with increasing altitude, the weight will become smaller than 1 lbf. The mass remains at all times a pound-mass, however. If the altitude is not exceedingly high, the measure of weight, in pound-force, will practically equal the measure of mass, in pound-mass. Therefore, it is an unfortunate practice in engineering to think erroneously of weight at positions other than on the earth's surface as the measure of mass and consequently to use the same symbol W to represent pound-mass and pound-force. In this age of rockets and missiles, it behooves us to be careful about the proper usage of units of mass and weight throughout the entire text.

[2]The notation g_c is also extensively used for this constant.

If we know the weight of a body at some point, we can determine its mass very easily, provided that we know the acceleration of gravity g at that point. Thus, according to Newton's law,

$$W(\text{lbf}) = M(\text{slugs})\, g(\text{ft/s}^2)$$

Therefore,

$$M(\text{slugs}) = \frac{W(\text{lbf})}{g(\text{ft/s}^2)} \qquad [1.2]$$

In USCS there are two units of *mass,* namely, the slug and the lbm. In contrast, SI units, as used by many people, involve two units of *force,* as we shall soon see. The basic unit for mass in SI is the *kilogram,* which is the amount of mass that will accelerate 1 m/s^2 under the action of 1 N force. Unfortunately, the kilogram is also used as a measure of force. That is, one often comes across such statements as "body C weighs 5 kg." A kilogram of force is the weight measured at the earth's surface of a body A having a mass of 1 kg. Note that at positions appreciably above the earth's surface, the weight of the body A will decrease; but the mass remains at all times 1 kg. Therefore, the weight in kilograms equals numerically the mass in kilograms *only* at the earth's surface where the acceleration of gravity is 9.806 m/s^2. Care must be taken accordingly in using the kilogram as a measure of weight. In this text we use only the newton, the kilonewton, and so on, as the unit for force.

What is the relation between the kilogram force and the newton force? This is easily established when one makes the following observation:

1 N accelerates 1 kg mass 1 m/s^2

1 kg force accelerates 1 kg mass 9.806 m/s^2

Clearly 1 kg force is equivalent to 9.806 N. Furthermore, a newton is about one-fifth of a pound.

What is the mass M of a body weighing W newtons at a location where the acceleration of gravity is g meters per second squared? For this we need only use Newton's law. Thus,

$$W = Mg$$

$$\therefore\ M(\text{kg}) = \frac{W(\text{N})}{g(\text{m/s}^2)} \qquad [1.3]$$

In this text we use both systems of units, but with greater emphasis on SI units.

1.6 NEWTON'S VISCOSITY LAW: THE COEFFICIENT OF VISCOSITY

A very important property will now be introduced as a consequence of Newton's viscosity law. For a well-ordered flow[3] whereby fluid particles move in *straight,*

[3]Such a flow, called *laminar,* is free of macroscopic velocity fluctuations. This will be discussed in detail in Chap. 8.

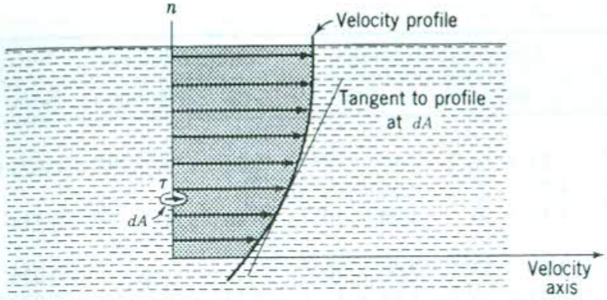


Figure 1.3
Well-ordered parallel flow.

parallel lines (parallel flow), the law states that for certain fluids, called *newtonian fluids,* the shear stress on an interface tangent to the direction of flow is proportional to the distance rate of change of velocity, wherein the differentiation is taken in a direction normal to the interface. Mathematically this is stated as

$$\tau \propto \frac{\partial V}{\partial n}$$

Figure 1.3 may further explain this relationship. An infinitesimal area in the flow is chosen parallel to the horizontal velocity axis, as shown. The normal n to this area is drawn. The fluid velocities at points along this normal are plotted, thus forming a velocity profile. The slope of the profile toward the n axis at the position corresponding to the area element is the value $\partial V/\partial n$, which is related, as stated above, to the shear stress τ shown on the interface.

Inserting the coefficient of proportionality into Newton's viscosity law leads to the result

$$\tau = \mu \frac{\partial V}{\partial n} \qquad [1.4]$$

where μ is called the *coefficient of viscosity,* having the dimensions $(F/L^2)T$, or M/LT. In the cgs system of units, the unit for viscosity is the *poise,* corresponding to 1 g/cm · s. The *centipoise* is $\frac{1}{100}$ of a poise. The SI unit for viscosity is 1 kg/m · s. It has no particular name. It is 10 times the size of the poise, as is clear from the basic units. In USCS, the coefficient of viscosity has the unit 1 slug/ft · s and like the SI system has no name. Viscosity coefficients for common liquids at 1 atm and 20°C temperature are given in Table 1.2.

The characteristics of viscosity are also given in Fig. B.1 in the appendix for a number of significant fluids. We point out first that viscosity does *not* depend appreciably on *pressure.* However, note from Fig. B.1 that the viscosity of a liquid *decreases* with an *increase* in temperature, whereas a gas, curiously, does quite the opposite. The explanation for these tendencies is as follows. In a *liquid,* the molecules have limited mobility with large cohesive forces present between the molecules. This manifests itself in the property of the fluid which we have called viscosity. An increase

Table 1.2 Properties of common liquids at 1 atm and 20°C

	Viscosity μ		Kinematic viscosity ν		Bulk modulus κ		Surface tension σ	
Liquid	kg/(m · s)	slug/(ft · s)	m²/s	ft²/s	GPa	lb/in²	N/m	lb/ft
Alcohol (ethyl)	1.2×10^{-3}	2.51×10^{-5}	1.51×10^{-6}	1.62×10^{-5}	1.21	1.76×10^{5}	0.0223	1.53×10^{-3}
Gasoline	2.9×10^{-4}	6.06×10^{-6}	4.27×10^{-7}	4.59×10^{-6}				
Mercury	1.5×10^{-3}	3.14×10^{-5}	1.16×10^{-7}	1.25×10^{-6}	26.20	3.80×10^{6}	0.514	3.52×10^{-2}
Oil (lubricant)	0.26	5.43×10^{-3}	2.79×10^{-4}	3.00×10^{-3}	...	...	0.036	2.47×10^{-3}
Water	1.005×10^{-3}	1.67×10^{-5}	0.804×10^{-6}	8.65×10^{-6}	2.23	3.23×10^{5}	0.0730	4.92×10^{-3}

in temperature decreases this cohesion between molecules (they are on the average farther apart) and there is a decrease of "stickiness" of the fluid—i.e., a decrease in the viscosity. In a *gas,* the molecules have great mobility and are in general far apart. In contrast to a liquid, there is little cohesion between the molecules. However, the molecules do interact by *colliding* with each other during their rapid movements. The property of viscosity results from these collisions. To illustrate this, consider two small but finite adjacent chunks of fluid A and B at a time t in a simple, parallel flow of a gas of the kind discussed at the outset of this section. This is shown in Fig. 1.4. As seen from the diagram, chunk A is moving faster than chunk B. This means that, *on the average,* molecules in chunk A move faster to the right than do molecules in chunk B. But in addition to the aforestated average movement of the molecules, there is also a random migration of molecules from chunk A into chunk B across their interface, and vice versa. Let us first consider the migration from A to B. When the A molecules move to B, there will be some collisions between A molecules and B molecules. Because A molecules are on the average faster in the x direction than are B molecules, there will be a tendency to speed B molecules up in the x direction. This means that there will be a tendency for chunk B macroscopically to speed up. From

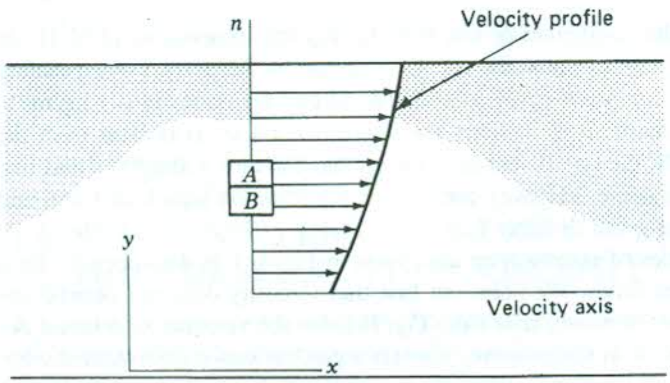


Figure 1.4
Parallel flow of a gas at time t.

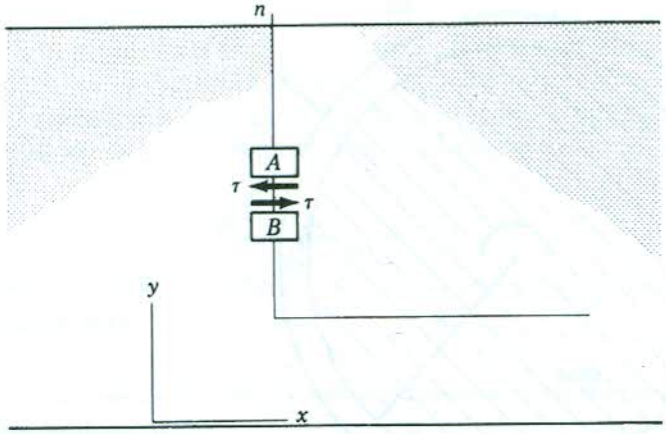


Figure 1.5
Shear stress on chunks A and B.

a continuum point of view, it would appear that there is a shear stress τ at the upper face of B acting to speed B up. This is shown in Fig. 1.5. By a similar action, slow molecules migrating from B into A tend to slow chunk A. Macroscopically this can be considered as resulting from a shear stress τ on the bottom interface of A. Such stresses on other chunks of fluid where there is a macroscopic velocity variation with position gives rise to a stickiness of the gas and this in turn gives rise to the macroscopic property of viscosity. Now the higher the temperature, the greater will be the migration tendency of the molecules and the greater will be τ for our simple case, because more collisions will be expected from molecules of A going to B, and vice versa. This will result in greater stickiness and thus greater viscosity.

In summation, viscosity in a liquid results from cohesion between molecules. This cohesion, and hence viscosity, *decreases* when the temperature increases. On the other hand, viscosity in a gas results from random movement of molecules. This random movement *increases* with temperature, so the viscosity increases with temperature. We note again that pressure has only a small effect on viscosity—an effect that is usually neglected.

The variation of viscosity of gases with temperature can be approximated by either of two laws called, respectively, the *Sutherland law* and the *power law* and given as follows:

$$\mu = \frac{\mu_0 (T/T_0)^{3/2}(T_0 + S)}{T + S} \qquad \textit{Sutherland law} \qquad [1.5]$$

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^n \qquad \textit{power law} \qquad [1.6]$$

where μ_0 is some known viscosity at absolute temperature T_0 and where S and n are constants determined by curve fitting. Note that T is the absolute temperature at which μ is being evaluated.

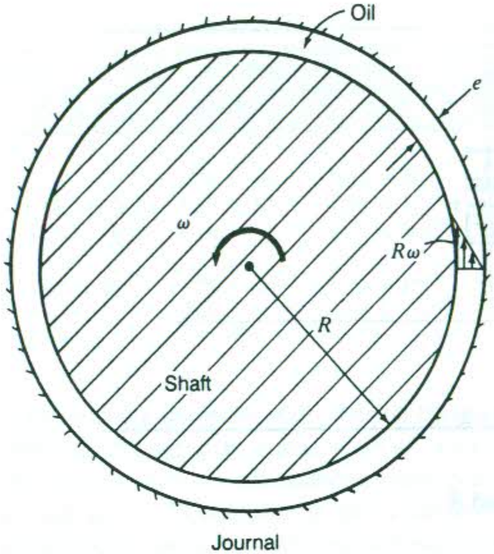


Figure 1.6
Shaft rotating in a lubricated journal.

As for liquids, the following simple formula is used to determine viscosity:

$$\mu = Ae^{-BT} \tag{1.7}$$

where A and B are constants found again by curve fitting data for a particular liquid.

Returning to our general discussion of viscosity, we can point out that most gases and most simple liquids are newtonian fluids and hence behave according to Newton's viscosity law for the conditions outlined. Pastes, slurries, greases, and high density polymers are examples of fluids that cannot be considered newtonian fluids.

There is a more general viscosity law, called *Stokes' viscosity law,* that is applicable to considerably more general flows of newtonian fluids than is undertaken in this section. This will also be examined in Chap. 9. However, in such applications as bearing-lubrication problems it is permissible to disregard the curvature of the flow and to use the relatively simple Newton viscosity law. This is permissible since the lubrication film thickness is very small compared to the bearing radius. Therefore domains of such flows having dimensions comparable to the film thickness involve very little change of direction of flow and may be thought of in such domains as *parallel* flow[4] with the attending permissible use of Newton's viscosity law (for Newtonian fluids). Furthermore, in flows of *real* fluids (which always have some measure of viscosity), in contrast to *hypothetical frictionless,* or as we say *inviscid* flows, the fluid touching a solid boundary must "stick" to such boundaries and thus must have the same velocity as the boundary.[5]

[4]An intuitive explanation for this can be achieved by noting that when looking at a small region around you, where the dimension of this region is much smaller than the radius of the earth, you are not aware of the overall curvature of the earth where you stand.

[5]In very high speed flows, five or more times the speed of sound, there can take place slip of real fluids relative to solid boundaries. We call such flows *slip flows.*

For instance, we consider a shaft A shown in Fig. 1.6 as a cross section rotating with speed ω rad/s inside of a bearing journal with a thin oil film of thickness e separating the bodies. We can approximate the velocity profile here, because e is small compared to the radius, as a *linear profile,* as has been shown in the diagram. The shear stress on all interfaces of oil normal to radial lines can then be given as follows:

$$\tau = \mu \frac{\partial V}{\partial n} = \mu \left[\frac{(R\omega - 0)}{e} \right]$$

We will examine such problems as homework. Now we consider an oil film problem with actual parallel flow.

EXAMPLE 1.1

■ Problem Statement

A solid cylinder A of mass 2.5 kg is sliding downward inside a pipe, as shown in Fig. 1.7. The cylinder is perfectly concentric with the centerline of the pipe having a film of oil between the cylinder and inside pipe surface. The coefficient of viscosity of the oil is 7×10^{-3} N · s/m². What is the *terminal* speed V_T of the cylinder—i.e., the final constant speed of the cylinder? Neglect the effects of air pressure.

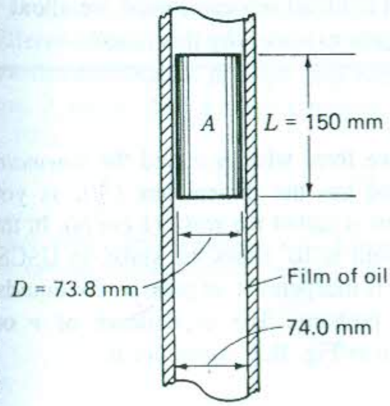


Figure 1.7
Cylinder A slides in a lubricated pipe.

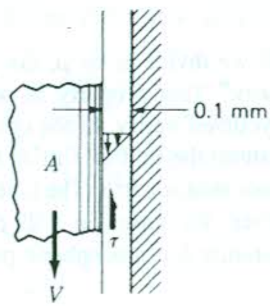


Figure 1.8
Linear velocity profile in film.

■ Strategy

We will assume a linear velocity profile for the film of oil as shown in Fig. 1.8. The oil is taken as a Newtonian fluid for which we can apply Newton's viscosity law. The resulting shear stress on the surface of the cylinder will be in equilibrium with the force of gravity on the cylinder at the condition of terminal speed.

■ Execution

The value of $\partial V/\partial n$ that we shall need for Newton's viscosity law then becomes

$$\frac{\partial V}{\partial n} = \frac{V - 0}{0.0001} = 10{,}000V \quad \mathrm{s}^{-1} \qquad \text{[a]}$$

The shear stress τ on the cylinder wall is then

$$\tau = \mu \frac{\partial V}{\partial n} = (7 \times 10^{-3})(10{,}000V) = 70V \quad \mathrm{Pa} \qquad \text{[b]}$$

We may now equate the weight of the cylinder with the viscous force for the condition of equilibrium which occurs when the cylinder has reached its terminal velocity V_T. Thus,

$$W = (\tau)(\pi D)(L)$$

$$\therefore (2.5)(9.81) = (70V_T)(\pi)(0.0738)(0.150) \qquad \text{[c]}$$

We get for V_T:

$$V_T = 10.07 \text{ m/s} \qquad \text{[d]}$$

■ Debriefing

The downward moving cylinder acted on only by viscous friction from the oil film and by-gravity never reaches so-called terminal velocity unless we allow, theoretically, an infinite span of time. Can you explain why this is so?

If we divide μ by ρ, the mass density, we form what is called the *kinematic iscosity.*[6] This property is denoted as ν and has the dimensions L^2/t, as you may yourself verify. In the cgs system, the unit is called the *stoke* (1 cm^2/s). In the SI system, the unit is 1 m^2/s. Clearly the SI unit is 10^4 times the stoke. In USCS, the basic unit is 1 ft^2/s. The kinematic viscosity is independent of pressure for liquids. However, for gases, ν will depend on the pressure. The dependence of ν on temperature for atmospheric pressure is shown in Fig. B.2, Appendix B.

1.7 THE PERFECT GAS: EQUATION OF STATE

If molecules of a fluid are presumed to have a mutual effect arising solely from perfectly elastic collisions, then the kinetic theory of gases indicates that for such a fluid, called a *perfect gas,* there exists a simple formulation relating absolute pressure, specific volume, and absolute temperature. This relation, called the

[6]The viscosity itself is often called the *absolute,* or *dynamic,* viscosity to distinguish it more clearly from the kinematic viscosity.

equation of state, has the following form for a perfect gas at equilibrium:

$$pv = RT \tag{1.8}$$

where R, the *gas constant,* depends only on the molecular weight of the fluid, v is the specific volume (volume per unit mass), and T is the absolute temperature. Values of R for various gases at low pressure are given in Appendix Table B.3.

In actuality, the behavior of many gases such as air, oxygen, and helium very closely approximates the perfect gas under most conditions and may, with good accuracy, be represented by the above equation of state.[7] Since the essence of the perfect gas is the complete lack of intermolecular attraction, gases near condensation conditions depart greatly from perfect-gas behavior. For this reason, steam, ammonia, and Freon at atmospheric pressure and room temperature and, in addition, oxygen and helium at very high pressures cannot properly be considered as perfect gases in many computations.

There are equations of state for other than perfect gases, but these lack the simplicity and range of the relation above. It should be emphasized that all such relations are developed for fluids which are under macroscopic mechanical and thermal equilibrium. This essentially means that the fluid bulk is undergoing no accelerative motion relative to an inertial reference and is free of heat transfer. The term p in the equation of state above and other such equations is usually referred to as the pressure. However, because of the equilibrium nature of this property as used in equations of state, we use the designation *thermodynamic pressure* to differentiate it from quantities involved in dynamic situations. The relations between thermodynamic pressure and nonequilibrium concepts will be discussed in Chaps. 2 and 8. And a more complete discussion of the perfect gas is given in Sec. 10.2.

---

[7]You have already used the perfect-gas law for special cases in your chemistry course. Thus, *Charles' law for constant pressure* becomes, from Eq. 1.8,

$$\frac{p}{R} = \text{const} = \frac{T}{v}$$

Hence,

$$v = \frac{1}{(\text{const})} T$$

$$\therefore v \propto T$$

This means that the specific volume v is directly proportional to T. Also *Boyles' law* applies for constant temperature. Thus from Eq. 1.8,

$$pv = RT = \text{const}$$

$$\therefore v \propto \frac{1}{p}$$

The specific volume of a gas varies inversely with the pressure. For a given mass of gas, we can replace the specific volume in the preceding statements by the volume V of the gas.

EXAMPLE 1.2

■ Problem Statement

Air is kept at a pressure of 200 kPa absolute and a temperature of 30°C in a 500-L container. What is the mass of the air?

■ Strategy

We may use the equation of state with the gas constant R as 287 N · m/(kg · K) and solve for the specific volume v. Then it is a simple matter to get the mass M.

■ Execution

Thus the equation of state is first presented.

$$pv = RT$$
$$[(200)(1000)]v = (287)(273 + 30)$$
$$\therefore v = 0.435 \text{ m}^3/\text{kg}$$

We may now compute the mass M of the air in the following manner:

$$M = \frac{V}{v} = \frac{[500/1000]}{0.435} = 1.149 \text{ kg}$$

■ Debriefing

We remind you to use absolute temperatures in the equation of state and that the equation of state used here loses validity if the pressure is excessive. Also the relation, strictly speaking, is for static conditions. However, we can generally use it for most dynamic conditions excluding explosions or shock waves where there is violent change with time.

1.8 SURFACE TENSION

A property that we will discuss is *surface tension* at the interface of a liquid and a gas. This phenomenon which is a tensile force distributed along the surface is due primarily to molecular attraction between *like* molecules (*cohesion*) and molecular attraction between *unlike* molecules (*adhesion*). In the interior of a liquid (see Fig. 1.9), the cohesive forces cancel, but at the free surface the liquid cohesive

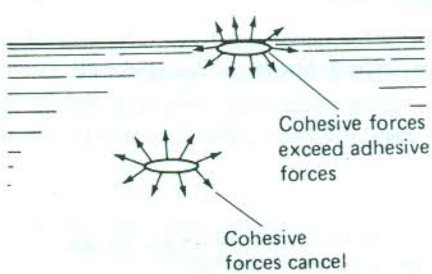


Figure 1.9
Cohesive and adhesive forces.

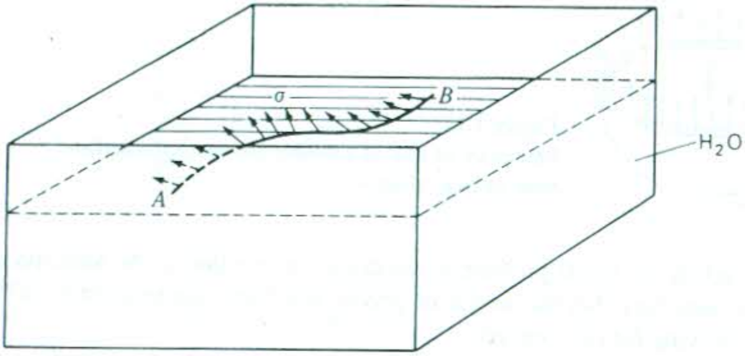


Figure 1.10
Surface tension σ.

forces from below exceed the adhesive forces from the gas above resulting in surface tension. It is for this reason that a droplet of water will assume a spherical shape. And it is for this reason likewise that small insects can alight on the free surface of a pond and not sink. The surface tension is measured as a *line-loading* intensity σ *tangential* to the surface and given per unit length of a line drawn on the free surface.[8]

Furthermore, the loading is normal to the line, as is shown in Fig. 1.10, where the AB is on the free surface. σ is called the *surface tension coefficient* and is the force per unit length transmitted from the fluid surface just to the left of AB to the fluid surface just to the right of AB with a direction normal to line AB. Thus, the vertical force distribution on the edge of the free body of a half water droplet (see Fig. 1.11) is the surface tension σ on the droplet surface. On the interior cross section, we have shown the distribution of force coming from the pressure p_i inside the droplet. For a droplet of liquid in equilibrium, we can then say that

$$-(p_i)_g(\pi R^2) + (\sigma)(2\pi R) = 0$$

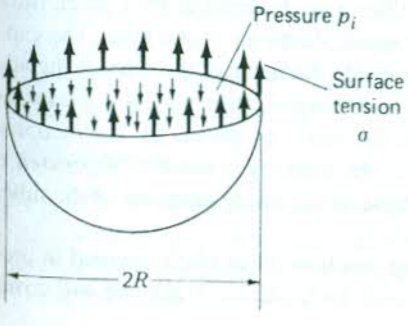


Figure 1.11
Surface tension on half of a water droplet.

[8]This loading is similar to the line loading $w(x)$ on beams used in strength of materials.

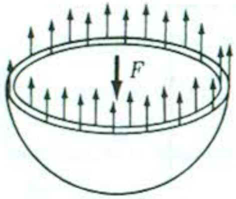


Figure 1.12
Free body of half of a bubble showing two surfaces with surface tensions.

where $(p_i)_g$ is the inside pressure in the droplet above that of the atmosphere. (We are assuming here that the weight of gravity has been counteracted by an outside agent.) Solving for $(p_i)_g$ we get

$$(p_i)_g = \frac{2\sigma}{R} \qquad [1.9]$$

At room temperature, σ for water exposed to air is 0.0730 N/m.[9] Hence, for a droplet of radius 0.5 mm we have for $(p_i)_g$

$$(p_i)g = \frac{(2)(0.0730)}{0.0005} = 292 \text{ Pa}$$

Since 1 atm is 1.013×10^5 Pa, we see that the inside pressure is 0.00288 atm.

Let us next consider the case of a *bubble.* If we cut the bubble in half to form a free body (see Fig. 1.12), we see that surface tension exists on two surfaces—the inner and the outer. Taking the radius to be approximately the same for inner and outer surfaces, we can say from equilibrium for the inside gage pressure $(p_i)_g$ that

$$-(p_i)_g \pi R^2 + 2[\sigma(2\pi R)] = 0$$
$$\therefore (p_i)_g = \frac{4\sigma}{R} \qquad [1.10]$$

Consider now the situation where a liquid is in contact with a solid such as liquid in a glass tube. If the adhesion of the liquid to the solid exceeds the cohesion in the liquid, then the liquid will *rise* in the tube and form a *meniscus* curving upward toward the solid, as shown in Fig. 1.13a for water and glass. This curvature toward the solid is measured by the angle θ. The capillary rise, h, depends for a given fluid and solid on θ which in turn depends on the inside diameter of the tube. The capillary rise will increase with a decrease in the inside diameter of the tube. If the adhesion to the glass is less than the cohesion in the liquid, then we get a meniscus curving downward as measured by θ toward the solid, as shown for the mercury and glass in Fig. 1.13b. Note that in this case the mercury column is *depressed* a distance h. Again, h will increase with a decrease in the inside diameter of the tube. These effects are called *capillary* effects.

We will now state certain simple facts that you have most likely covered in previous physics and mechanics courses dealing with hydrostatics. If not, we will cover

[9]The coefficient of surface tension for a *water-air* interface is σ = 0.0730 N/m ≡ 0.0050 lb/ft. For a *mercury-air* interface, the coefficient of surface tension is σ = 0.514 N/m ≡ 0.0352 lb/ft.

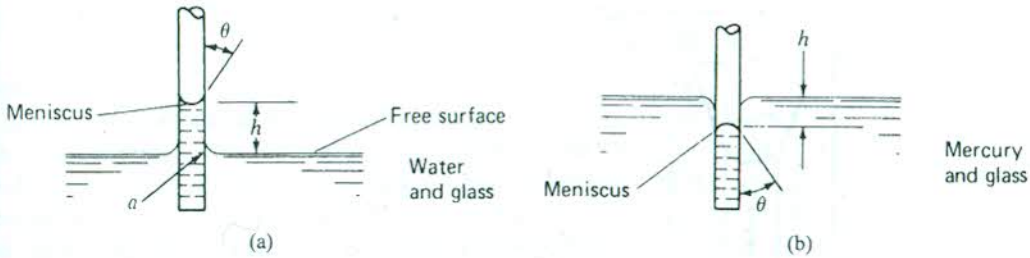


Figure 1.13
Capillary effects of cohesion and adhesion.

them in great detail in Chap. 2. First, note that the gage pressure in a liquid is computed by the product γd, where d is the depth below the free surface and γ is the specific weight of the liquid. Note however in Fig. 1.13a that the free surface corresponds to the liquid surface away from capillary effects. Thus, the free surface is *not* at the meniscus in the capillary tube. Accordingly, the pressure at a is p_{atm} since it is at the *same elevation* as the free surface where the pressure is p_{atm}. Also note that if you move up a distance l in a liquid, the pressure will decrease by the amount γl. Second, note that if a uniform pressure p acts on a curved surface, the resultant force in a given direction from this pressure is found by simply multiplying p times the area projected in the direction of the desired force. Thus, in Fig. 1.13 the vertical resultant force from the atmospheric pressure on the meniscus is simply $p_{\text{atm}}(\pi D^2/4)$, where $\pi D^2/4$ is a circular area—the projection of the meniscus seen as one looks down on it.

EXAMPLE 1.3

■ Problem Statement

Consider in Fig. 1.13a that the inside diameter of the capillary is 2 mm. If θ is 20° and σ for the water in the presence of air is 0.0730 N/m, determine the height h of the water rise in the capillary.[10]

■ Strategy

We shall use a system of fluid in the capillary going from the level of the outside free surface to the meniscus of the fluid (see Fig. 1.14). Then, we will use hydrostatics and equilibrium of the capillary column to establish its height h above the outside free surface.

■ Execution

In Fig. 1.14, we have shown an enlarged free-body diagram of the water standing above the level of the outside free surface. Note that we have atmospheric pressure at the bottom of this column.

[10] For a very clean glass wall, the angle θ for the water is close to 0°. For mercury and clean glass, the angle θ is 40°.

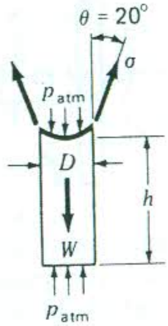


Figure 1.14
Free-body diagram of water raised by capillary action.

Summing forces in the vertical direction, we have, on neglecting the weight of water directly above elevation h and under the meniscus,

$$\sigma(\pi D)\cos\theta - W = 0$$

Hence,

$$(0.0730)(\pi)(0.002)\cos 20° - (9806)\frac{(\pi)(0.002)^2}{4}h = 0$$

$$\therefore h = 13.99 \text{ mm}$$

■ Debriefing

We see here that knowing θ and σ, we can compute h, the rise or depression of a liquid in a capillary tube.

PART B MECHANICS CONSIDERATIONS

1.9 SCALAR, VECTOR, AND TENSOR QUANTITIES: FIELDS

Before continuing with our discussion, we should classify certain types of quantities which appear in mechanics. A *scalar* quantity requires only the specification of magnitude for a complete description. Temperature, for example, is a scalar quantity. A *vector* quantity, on the other hand, requires a complete directional specification in addition to magnitude, and must add according to the parallelogram law. Usually three values associated with convenient orthogonal directions are employed to specify a vector quantity. These values are called *scalar components* of the vector. There are more complicated quantities which require the specification of nine or more scalar components for a complete designation. Among these are stress, strain, and mass moment of inertia. These particular quantities, called *tensors*, transform (i.e., change values) in a certain manner under a rotation of the reference axes at a point.[11]

A field is a continuous distribution of a scalar, vector, or tensor quantity described by continuous functions of space coordinates and time. For example, we may describe the temperature at all points in a body at any time by the scalar field expressed as $T(x, y, z, t)$. A vector field, such as the velocity field, may be designated mathematically as $\mathbf{V}(x, y, z, t)$. Usually, however, three scalar fields are employed, each field yielding the value of the velocity component in one of three orthogonal directions. Thus,

$$V_x = f(x, y, z, t)$$
$$V_y = g(x, y, z, t)$$
$$V_z = h(x, y, z, t)$$

This technique of employing a number of scalar fields may be extended to the tensor field, where there may be nine scalar fields.

Since the science of fluid mechanics deals with distributed quantities, there will be considerable opportunity to use the field approach advantageously. Scalar, vector, and tensor fields will all appear in the study of various aspects of fluid mechanics as well as in your courses in solid mechanics.

1.10 SURFACE AND BODY FORCES: STRESS

In the study of continua, we distinguish between two types of force distributions. Those force distributions which act on matter without the requirement of direct contact are called *body-force distributions*. Gravitational force on a body is the most common body-force distribution. Also, the magnetic-force distribution on a magnetized material in a magnetic field is a body-force distribution. We denote body forces as $\mathbf{B}(x, y, z, t)$ and give them on the basis of per unit mass of the material acted on. The second kind of force distribution on a body arises from direct contact of this body with other surrounding media; it is called a *surface-force distribution* or a *surface-traction distribution*. We denote surface tractions as $\mathbf{T}(x, y, z, t)$ and give them on the basis of per unit area of the material acted on. Surface tractions exist on the physical boundaries of a body or they occur when a "mathematical cut" is made of a body in order to "expose" a surface.

[11]You may have also learned in sophomore mechanics that a vector may be defined as having three scalar components associated with a reference xyz. These components must change in a certain prescribed manner to become components of a reference $x'y'z'$ rotated relative to xyz. The transformation equation yielding the new components is very similar but simpler than that which defines tensors. A scalar, on the other hand, is invariant with respect to a rotation of axes. From these definitions we can consider all such quantities as tensors of different rank with a vector as a first-order tensor and a scalar as a zero-order tensor.

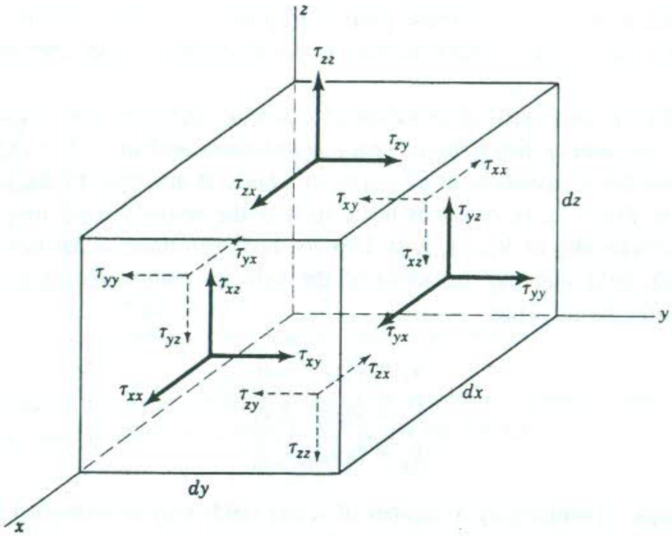


Figure 1.15
Stresses on faces of an infinitesimal rectangular parallelepiped taken from inside a body.

In your course in strength of materials or solid mechanics[12] you learned that traction forces on mathematically isolated solid or fluid internal elements (free bodies) gave rise to shear and normal stresses. In Fig. 1.15 we show an infinitesimal rectangular parallelepiped taken from a body with nine stresses acting on the outer faces. A double-index scheme has been utilized to identify the stresses. The first subscript indicates the direction of the normal to the plane associated with the stress, while the second subscript denotes the coordinate direction of the stress itself. The normal stresses have a repeated index, since the stress direction and the normal to the plane on which the stress acts are collinear. The shear stresses will then have mixed indices. For example τ_{yx} is the value of the shear stress acting on a plane whose normal is parallel to the y direction, while the stress itself is parallel to the x direction. The concept of stresses applied to solids holds also for fluids—indeed, it holds for *any* continuum.

1.11 STRESS AT A POINT FOR A STATIONARY FLUID AND FOR NONVISCOUS FLOWS

We will now investigate the relation between the stress on any interface at a point with the stresses on a set of orthogonal interfaces at the point. To do this most simply, we consider conveniently shaped infinitesimal free bodies of elements of the

[12] See the author's text with J. Pitarresi, *Introduction to Solid Mechanics,* 3rd ed., Prentice-Hall, Englewood Cliffs, NJ.

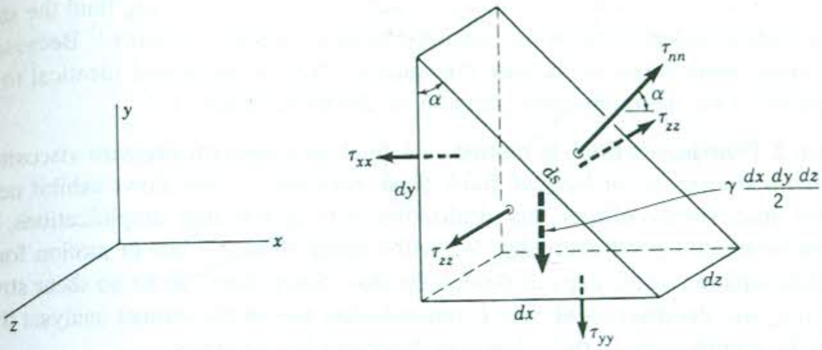


Figure 1.16
Element in a stationary or uniformly moving fluid. Note that τ_{nn} is parallel to the yx plane.

medium. By using Newton's law and taking the limit as the size of the element goes to zero, we can arrive at the relations that must exist at a point in the medium. We consider other special cases of fluids as set forth in case 1 and case 2.

Case 1. The Stationary or Uniformly Moving Fluid. Since a fluid cannot withstand a shear stress without moving, a stationary fluid must necessarily be completely free of shear stress. A uniformly moving fluid, i.e., a flow where all elements have the same velocity, is also devoid of shear stress since the variation of velocity in all directions for uniform flow must be zero ($\partial V/\partial n = 0$), and hence, by virtue of Newton's viscosity law, all the shear stresses are zero.

Assuming that the only body force is that of gravity, we consider an infinitesimal prismatic element of fluid under these conditions as shown in Fig. 1.16. Newton's law in the x direction is

$$-\tau_{xx}\,dy\,dz + \tau_{nn}\,ds\,dz\cos\alpha = 0$$

Since $\cos\alpha = dy/ds$, the equation becomes

$$\tau_{xx} = \tau_{nn}$$

In the y direction Newton's law yields

$$-\tau_{yy}\,dx\,dz + \tau_{nn}\,dz\,ds\sin\alpha - \gamma\frac{dx\,dy\,dz}{2} = 0$$

Again, by recognizing $\sin\alpha$ to be dx/ds and dividing through by $dx\,dz$, we get

$$-\tau_{yy} + \tau_{nn} - \gamma\frac{dy}{2} = 0$$

Now, letting the size of the element shrink to zero, we see that the body force of gravity drops out, so

$$\tau_{yy} = \tau_{nn}$$

Thus we can conclude that in a stationary or uniformly moving fluid the stress at a point is independent of direction and is hence a scalar quantity.[13] Because of the equilibrium nature of the case this quantity may be considered identical to the negative of the thermodynamic pressure, as discussed in Sec. 1.7.

Case 2. Nonviscous fluid in motion. A fluid with theoretically zero viscosity is called a *nonviscous,* or *inviscid,* fluid. Since portions of many flows exhibit negligibly small viscous effects, this idealization, with its resulting simplifications, can often be used to good advantage. We will employ Newton's law of motion for an infinitesimal prismatic mass of fluid in the flow. Since there can be no shear stress, we may use the diagram of case 1, remembering that in the present analysis there may be acceleration. In the y direction, Newton's law becomes

$$-\tau_{yy}\,dx\,dz + \tau_{nn}\,ds\,dz\sin\alpha - \gamma\frac{dx\,dy\,dz}{2} = \rho\frac{dx\,dy\,dz}{2}a_y$$

where a_y is the acceleration component. Note that the gravity force and the inertia term vanish in the limiting process, since both these terms are composed of the product of three infinitesimals, as compared with two for the other two terms. Upon replacing $\sin\alpha$ by dx/ds the equation becomes, on dividing through by $dx\,dz$,

$$\tau_{yy} = \tau_{nn}$$

A similar equation in the other direction leads to the conclusion that

$$\tau_{nn} = \tau_{xx} = \tau_{yy}$$

Hence, it may be concluded that for a nonviscous fluid in motion, just as in the case of the stationary or uniformly moving viscous fluid, the stress at a point is a scalar quantity. In Sec. 1.12 it will be pointed out that this quantity is also equivalent to the negative of the thermodynamic pressure.

1.12 PROPERTIES OF STRESS

Consider a vertical plate loaded by forces in the plane of the plate in Fig. 1.17. The plate is in equilibrium. This is a case of *plane stress,* you may recall from solid mechanics, where only stresses τ_{xx}, τ_{yy}, τ_{yx}, and τ_{xy} are nonzero. We have formed a free body in Fig. 1.18 of the element of plate shown in Fig. 1.17. Now set the sum of moments about corners of the element equal to zero. The normal stresses and gravity force give second-order contribution to moments and drop out in the limit. We conclude first that

$$\tau_{xy} = \tau_{yx}$$

and second that the shear stresses on the infinitesimal element must either point *toward* a corner or *away* from a corner. This is the *complementary* property of shear

[13]This is called *Pascal's* law. Also, we point out that this kind of stress distribution is called *hydrostatic stress.*

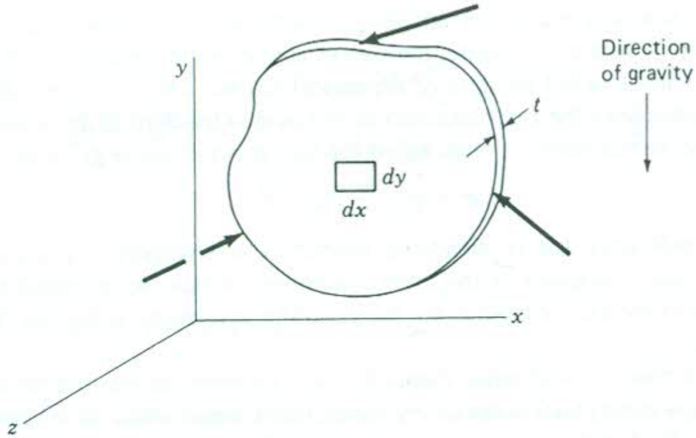


Figure 1.17
Plate in equilibrium; plane stress.

developed in your strength of materials course. If a state of equilibrium does not exist, then the inertia terms, like the body force, contribute only second-order terms which go out in the limit. The conclusions above still hold. One can extrapolate them to three dimensions wherein $\tau_{yz} = \tau_{zy}$ and $\tau_{xz} = \tau_{zx}$. Again these stresses, as shown in Fig. 1.15, must point toward or away from the edges of a vanishingly small rectangular parallelepiped. In the case of a fluid, the fact that a vanishingly small rectangular parallelepiped is in the process of deforming does not change the conclusions above about shear stress at the instant that the element is a rectangular parallelepiped. The terms involving the rate of deformation, like the body force and the inertia term, introduce only second-order contributions into the equation of motion about the corners of the element. Thus, *the complementary property of shear learned in strength of materials holds for viscous fluids.*[14]

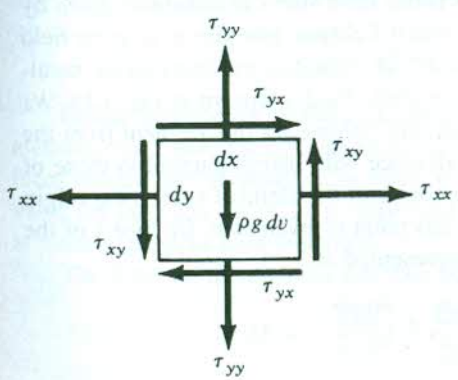


Figure 1.18
Infinitesimal rectangular parallelepiped.

[14]See footnote 12.

Let us now turn to a second important property of the stress tensor. As may have been shown in your solids course, the sum of any set of orthogonal normal stresses at a point (this is called the trace of the tensor) has but a single value independent of the orientation of the x, y, and z axes at that point. One-third of this quantity, i.e., the average normal stress, is often called the *bulk stress* $\overline{\sigma}$ and is given as

$$\overline{\sigma} = \tfrac{1}{3}(\tau_{xx} + \tau_{yy} + \tau_{zz}) \tag{1.11}$$

Since the bulk stress has no directional properties, it is properly a scalar quantity. There are other groupings of the stress components which are independent of the orientation of the axes at a point, but the use of these relations is beyond the scope of the text.

At this time, we will relate thermodynamic pressure, an equilibrium concept, with stress, a concept not restricted by conditions of equilibrium. In the case of the nonviscous fluid, it has been shown in Sec. 1.11 that all normal stresses at a point are equal and consequently must equal the bulk stress. Furthermore, it may generally be shown for such a fluid that the magnitude of the bulk stress is equal to the thermodynamic pressure. Since normally only negative normal stresses are possible in a fluid, the mathematical statement of this equivalence is given as

$$-\overline{\sigma} = p$$

Using the kinetic theory of gases, Chapman and Cowling[15] have demonstrated this relation to be good for a perfect gas. For the case of real gases, the relation above is not valid when the gas approaches the critical point. Since the vast majority of real-gas problems do not approach this extreme condition, the simple relation given above will be used in this text for all real gases. Finally, experience indicates that this relation may be used with confidence for liquids except when very close to the critical point.

1.13 THE GRADIENT

We have shown how static and frictionless fluids have stress distributions given by the scalar field p. We will now show how a scalar field can give rise to a vector field of physical significance. To do this, consider an infinitesimal rectangular parallelepiped of fluid at time t, in a frictionless or static fluid as shown in Fig. 1.19. We wish first to compute the resultant force per unit volume on this element from the pressure distribution. For this purpose, a reference with planes parallel-to those of the fluid element has also been shown. The corner of the element nearest the origin is taken as position x, y, z. The pressure at this point is given as p. On face 1 of the element, we have a pressure that may be represented as

$$p_1 = p + \frac{\partial p}{\partial x}\frac{dx}{2} + \frac{\partial p}{\partial z}\frac{dz}{2}$$

[15] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases,* Cambridge University Press, NY, 1953.

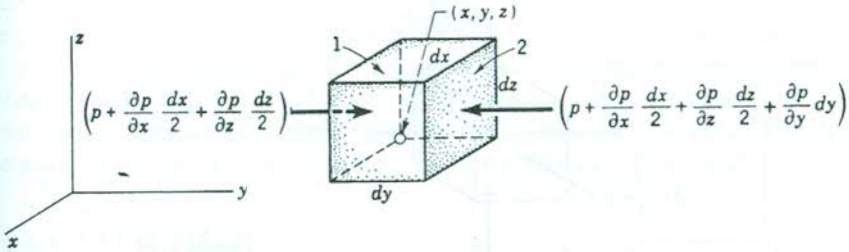


Figure 1.19
Pressure variation in the x direction.

This is reached by considering linear variations of pressure in all directions in the immediate vicinity of point x, y, z and computing the pressure in this way at the center of face 1.[16] Face 2 is positioned a distance dy from face 1 so that the pressure there can be considered equal to the pressure on face 1 plus an increment due to this shift in position. We may then say

$$p_2 = p + \frac{\partial p}{\partial x}\frac{dx}{2} + \frac{\partial p}{\partial z}\frac{dz}{2} + \frac{\partial p}{\partial y}dy$$

Note that we could express the increment of pressure from the shift more accurately, but this would bring in terms of higher order that would vanish when going to the limit. The net force in the y direction may now be computed from the pressures above. Having chosen the rectangular parallelepiped as the free body, note how we can cancel out the first-order terms of the equation and leave only the second-order terms of the equation, which give the variation "in the small" of the pressure distribution. Thus, on further cancellations, we have

$$dF_y = -\frac{\partial p}{\partial y}\,dx\,dy\,dz$$

Similarly in the x and z directions we get

$$dF_x = -\frac{\partial p}{\partial x}\,dx\,dy\,dz \qquad dF_z = -\frac{\partial p}{\partial z}\,dx\,dy\,dz$$

Before proceeding further, we should point out that the above forces on the element could have been achieved had we taken the pressures on the adjacent surfaces nearest the reference to be equal to p and added first-order variations to this value for the outer faces, as is illustrated in Fig. 1.20. It is this formulation which is usually taken in such situations.

The force on the element can then be given as

$$\mathbf{dF} = -\left(\frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j} + \frac{\partial p}{\partial z}\mathbf{k}\right)dx\,dy\,dz$$

[16]What we are doing is actually expressing pressure p as a Taylor series about point x, y, z and retaining terms involving only first-order differentials.

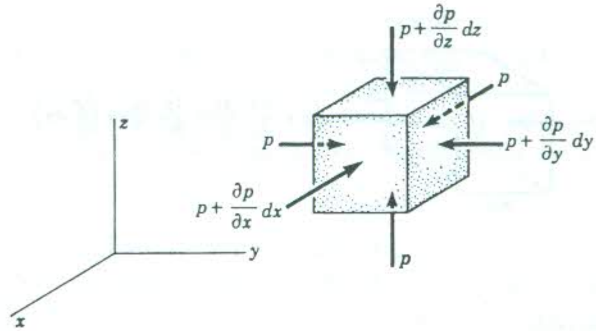


Figure 1.20
Pressure variation in *xyz* directions.

The force per unit volume is then

$$\frac{\mathbf{dF}}{dx\,dy\,dz} = \mathbf{f} = -\left(\frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j} + \frac{\partial p}{\partial z}\mathbf{k}\right) \qquad [1.12]$$

Had we used a different element of a shape suitable for computations in a different coordinate system, as, for example, cylindrical coordinates, we would have arrived at a different form of **f** from the one given above. (You will be asked to work out the case of cylindrical coordinates as an exercise.) However, all such formulations have identically the *same physical meaning,* namely force per unit volume at a point, which accordingly is quite *independent* of the coordinate systems used for evaluation purposes. For this reason, we formulate a vector operator,[17] called the *gradient,* which relates scalar and vector fields in such a way that, for the case at hand, we go from a pressure distribution p to the vector field **f**, yielding the force per unit volume at a point from surface traction. Thus we can say

$$\mathbf{f} = -\mathbf{grad}\, p \qquad [1.13]$$

where, if the gradient operator is referred to a particular coordinate system, it will take on a form dependent on the coordinate system used.[18] For Cartesian coordinates, we thus have for the gradient operator

$$\mathbf{grad} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \qquad [1.14]$$

In heat transfer, the negative of the gradient of a temperature distribution T multiplied by a constant yields a vector field **q** which is the *heat-flux* field. Thus the gradient of a scalar gives rise to a *driving action* per unit volume. In particular, $-$**grad** T gives rise to a driving action causing flow of heat. And $-$**grad** p gives rise to a driving action causing flow of fluid.

In later chapters, we set forth other vector operators such as the *divergence* and *curl* operators that are very useful in that they can describe analytically certain actions

---

[17]A vector operator per se is divorced from coordinate systems until such time as when components are desired.

[18]The gradient operator is also expressed by the symbol ∇. Therefore

$$\mathbf{f} = -\nabla p$$

occurring commonly in nature without the need for a reference. These operators are used extensively in such fields of study as electricity and magnetism, heat transfer, and theory of elasticity; and although they take on different meanings in these different disciplines, there is still a considerable carryover of meaning from one subject to the other. In the study of fluid mechanics, there is a very vivid picture to be associated with these operators, as they are ordinarily used, so we will employ them throughout the text.

1.14 CLOSURE

In Part A of this opening chapter, we have defined a fluid from a mechanics viewpoint and set forth the means of describing this substance and its actions in a quantitative manner, using dimensions and their units. There are other attributes of a physical quantity beyond its dimensional representation that are significant in analysis. For instance, you learned in mechanics that some quantities must also have directional descriptions to be meaningful. We investigated certain of these additional considerations for quantities which form the foundation for the description of fluid phenomena.

In Part B, we have made some introductory remarks concerning the stress field and its properties. One of the primary goals in fluid mechanics will be to evaluate the stress distribution and the velocity field for certain flows. From this, one can compute forces on bodies, such as airfoils, in the flow and then ascertain the probable performance of machines. We generally require use of several basic laws for such undertakings. However, in Chapter 2 we will be able, by using only Newton's law, to determine the stress field for a static fluid. In other words, static fluids are generally statically determinate, to use the language of rigid-body mechanics.

1.15 COMPUTER EXAMPLES

COMPUTER EXAMPLE 1.1

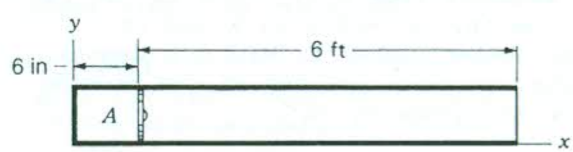


Figure C1.1
Air gun.

■ Computer Problem Statement

A pneumatic dart gun is shown in Fig. C1.1. Region A has a pressure of 2 psi gage (above atmosphere). A dart and piston assembly weighing 5 oz is held stationary by a quick-release mechanism (not shown). The tip of the dart is impregnated with poison for the purpose of killing or paralyzing game. The internal cross-sectional area of the gun barrel is 1 in^2, and there is a resistance to movement of the assembly from the wall given by the formula $f = 0.02V^{1.216}$ lb with the velocity V in ft/sec. The hunter fires the gun by activating the quick-release mechanism. If the air in region 1 expands according to the formula $pv^{1.4} = Const$, develop a plot of the speed V of the dart vs. position x. Neglect the volumes of the dart and the quick-release mechanism.

■ Strategy

The equation to be used is the work-energy equation: F*x = 0.5*m* (Vf^2 − Vo^2). The forces come from two sources. One source is the pressure which is moving the dart to the right (this force is continually decreasing as volume increases), and there is also a force resisting the dart's motion given by the equation in the problem statement.

■ Execution

```
clear all;
%Putting this at the beginning of the program
%ensures values don't overlap from previous
%programs.

area=.00694;
%This is the area of the piston in ft^2.

mass=.0097;
%This is the mass of the dart in slugs (5 oz/16=
.3125 lbm/32.2=.0097 slugs).

x=linspace (0, 6, 100);
%this assignment breaks up the barrel length
%into 100 segments from 0 ft. to 6 ft. We can
%use 0 as the initial entry in x and not
%"eps" (the lowest number in Matlab's memory
%greater than 0) since nowhere in the problem
%do we have to divide by "x".

volume=linspace (.003472, .045139, 100);
%These are increments of volume in ft^3 caused
%by 100 movements of the piston from a position
%of x=0' to x=6' (.003472 is the initial volume
%to the left of the piston corresponding to
%x=0'). Since we know how the volume will
%change we then know how the pressure will
%change due to the adiabatic equation.
```

```
velocity_i(1)=0;
%The dart starts from rest.

for j=1:100;

resist_vel(j)=velocity_i(j);
%resist_vel=the velocity put into resistance
%term.

press(j)=.86653./(volume(j)^1.4);
%Pressure due to adiabatic equation
%(pressure*Volume^1.4=constant) using
%standard atmosphere and 2 psig.

for i=1:20;

velocity_f(i)=sqrt((((press(j).*area-
.02*resist_vel(i)^1.216).*.1)./
(.5*mass))+velocity_i(j)^2);
%This is just the "work=change in K.E."
%equation solved for final velocity.

resist_vel(i+1)=velocity_f(i);
%The final velocity next yields a new
%"resist_vel" term. Plugging this
%back in the K.E. equation is how we converge
%to a proper value for velocity.

end

velocity_i(j+1)=velocity_i(1)+
sum(velocity_f(20));
%The new original velocity will be the previous
%original velocity plus all the best estimate
%velocity changes since the beginning.

vx(j)=velocity_f(20);
%We want to plot the last "velocity_f" term
%because it is the most accurate.

end

vx(1)=0;
%When the movement along the barrel=0(x=0), the
%velocity of the dart must be equal to zero.
plot(x,vx);
xlabel('Position along the barrel (ft.)');
ylabel('Velocity of dart (ft/sec)');
```

```
title('Velocity of the dart vs. position in the barrel');
grid;
%This just gives us the graph that we want.
```

■ Debriefing

Looking at the graph of Fig. C1.2 and our values, we can see that the dart will leave the end of the gun with a velocity of 31.23 ft/sec (31.23 = vx (100)). By taking the resistance term out of this problem you can see that the final velocity would be 68.8 ft/sec.

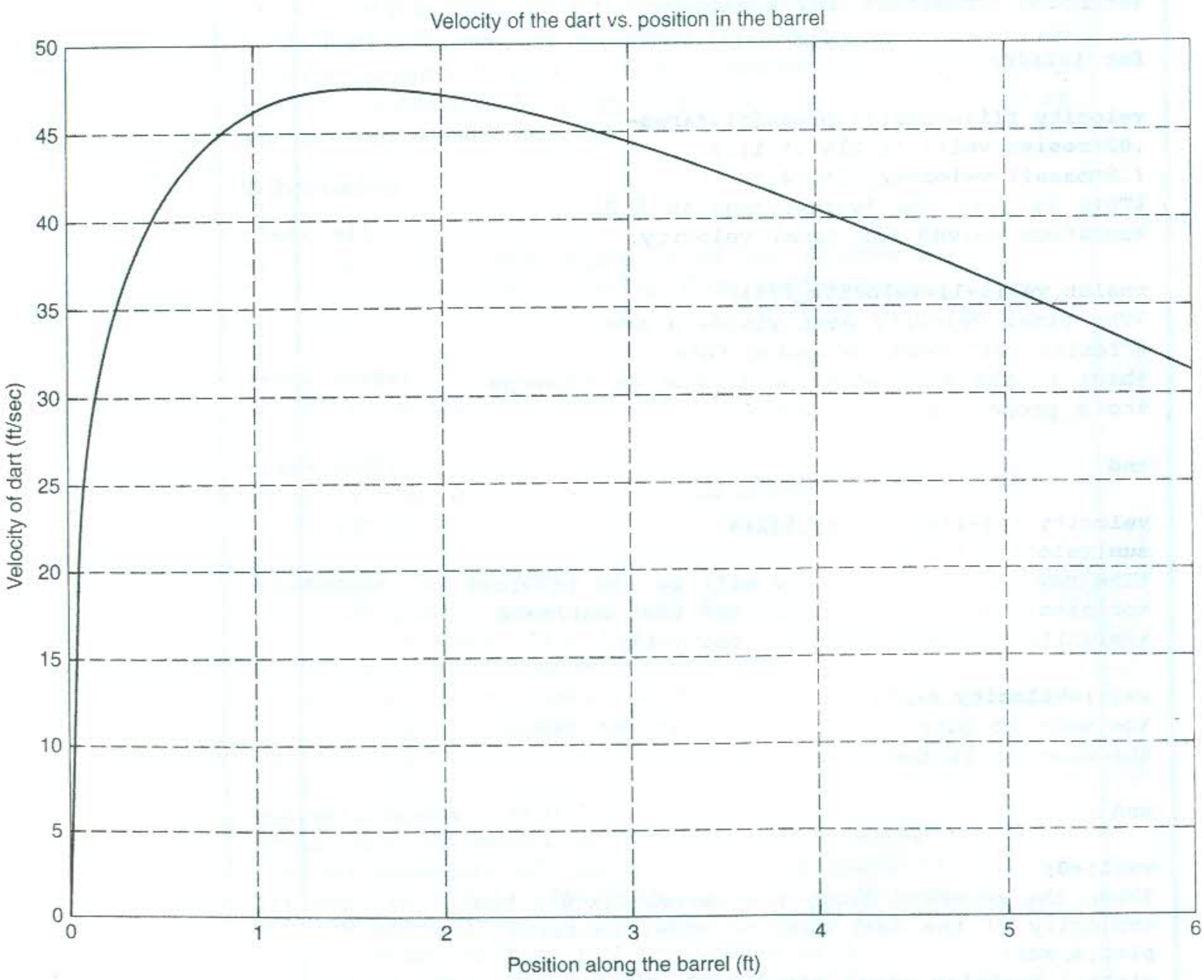


Figure C1.2
Air gun with resistance.

COMPUTER EXAMPLE 1.2

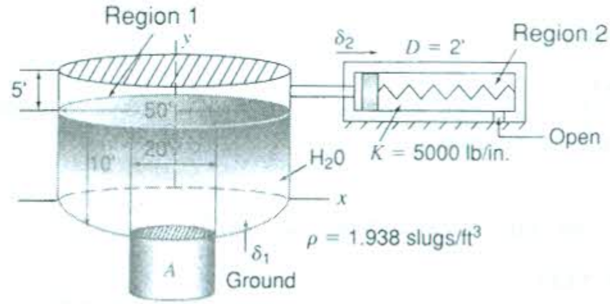


Figure C1.3

■ Computer Problem Statement

We start with a cylindrical tank of (Fig. C1.3) diameter 50 ft containing water up to a depth of 10 ft. Initially, the solid vertical piston A, having a diameter of 20 ft, is positioned flush with the bottom of the tank. A second smaller horizontal cylinder connects to the large tank and contains a piston which is attached to a spring having a spring force of $60{,}000\delta_2$ lb/ft. Move the vertical piston up slowly, and plot the movement of this piston δ_1 against the movement of the smaller piston δ_2. Plot these results against each other. Take the air above the water in the large tank as a perfect gas which undergoes adiatic compression and expansion. Region 2 is open.

■ Strategy

We will move the lower piston (piston 1) in increments of 0.1 ft. This movement will be controlled by an outer loop. For each of these movements of piston 1, we will enter an inner loop. First the pressure will increase to a new value due to the movement of piston 1, and we will let piston 2 move in response to this pressure, which we will take as constant during movement of piston 2. Next we will take the new total volume of the air at the end of the movement of piston 2, and we will compute a new pressure p_i to start the process over again with piston 1 moving 0.1 ft from its initial position while piston 2 starts from its initial position. The air is compressed from its pressure of p_i, and so on. We will let the inner loop go through 30 cycles for this first displacement of piston 1 to improve accuracy, and we will plot the distance moved by piston 2 against 0.1 ft for piston 1 using the last cycle for the best accuracy. We will then move in the outer loop to the next displacement of piston 1 going from 0.1 ft to 0.2 ft with the starting pressure from the end of the previous cycle and the starting piston 2 displacement from the final displacement from the previous cycle. Again the inner loop will be used as before. We will go through piston 1 displacements until we reach 5 ft.

■ Execution

```
Clear all;
%Putting this at the beginning of the program ensures
%values don't overlap.

k=60000;
%This is the spring constant in lb/ft.

area2=314.159;
%This is the area of piston 2 (the upper piston on the
%diagram).

area1=314.159;
%This is the area of piston 1 (the lower piston on the
%diagram).

volmain=9817.475;
%"volmain" is the original volume with no motion of
%either piston.

vmain(1)=9817.475;
%Original volume to be used in loop (there has been
%no motion yet). You can see looking at the program
%why we need the two previous terms, both with the
%same value.

delta1=linspace(0,5,30);
%This is just the movement of piston 1. 30 linear
%steps between 0 and 5 ft. This will be the
%independent variable(what we will control) in our
%problem.

volume1=delta1.*area1;
%this is how the volume will change due to piston 1
%moving. The area of piston 1 is 314.159 ft^2 and it
%is moving in 30 even steps between 0 and 5
%ft.(delta1).

for j=1:30;
%Running "j" from 1 to 30 corresponds to the 30
%movements we have arbitrarily chosen for the
%movements of piston 1.

for i=1:20;
%Will continue for 20 iterations giving better and
%better estimates.
```

```
p(i)=(8.21.*10.^8)./vmain(i)^1.4;
%Pressure given according to adiabatic equation. The
%constant was found using values for the standard
%atmosphere and p*V^1.4=constant (Used total volume).

d2(i)=sqrt(p(i).*area2./k);
%Movement of piston 2.

vm2(i)=d2(i).*area2;
%vm2=volume added due to the movement of piston 2.

vmain(i+1)=vmain(1)+vm2(i);
%Gives a new main chamber volume due to vm2.

end
%We come out of this loop with a vector d2 with 20
%values and each subsequent d2 value is a better
%estimate to what the displacement of piston 2 would
%actually be.

delta2(j)=d2(20);
%We want the d2 value which will be the closest
%approximation (the last one since we are converging
%in an iterative fashion).

volume2(j)=delta2(j).*area2;
%volume2 is the volume obtained from the last
%calculation (d2(20)) and is the best estimate for the
%volume added by the movement of piston 2.

vmain(1)=volmain+sum(volume2)-volume1(j);
%the new vmain(1) to be put into the inner loop is
%the original volume with no displacement(volmain),
%plus the sum of all the volume2 values, minus the
%volume1 value that corresponds to the outer loop
%iteration we are currently on.

end

plot(delta1,delta2);
xlabel('Movement of piston 1 (ft)');
ylabel('Movement of piston 2 (ft)');
title('Movement of piston 1 vs. piston 2'),grid;
%This just gives us the type of graph (Fig. C1.4) that
we want.
```

■ Debriefing

What we have done here is to link up two interacting systems in a self-correcting manner via loops. You will have the opportunity to use this technique in homework assignments.

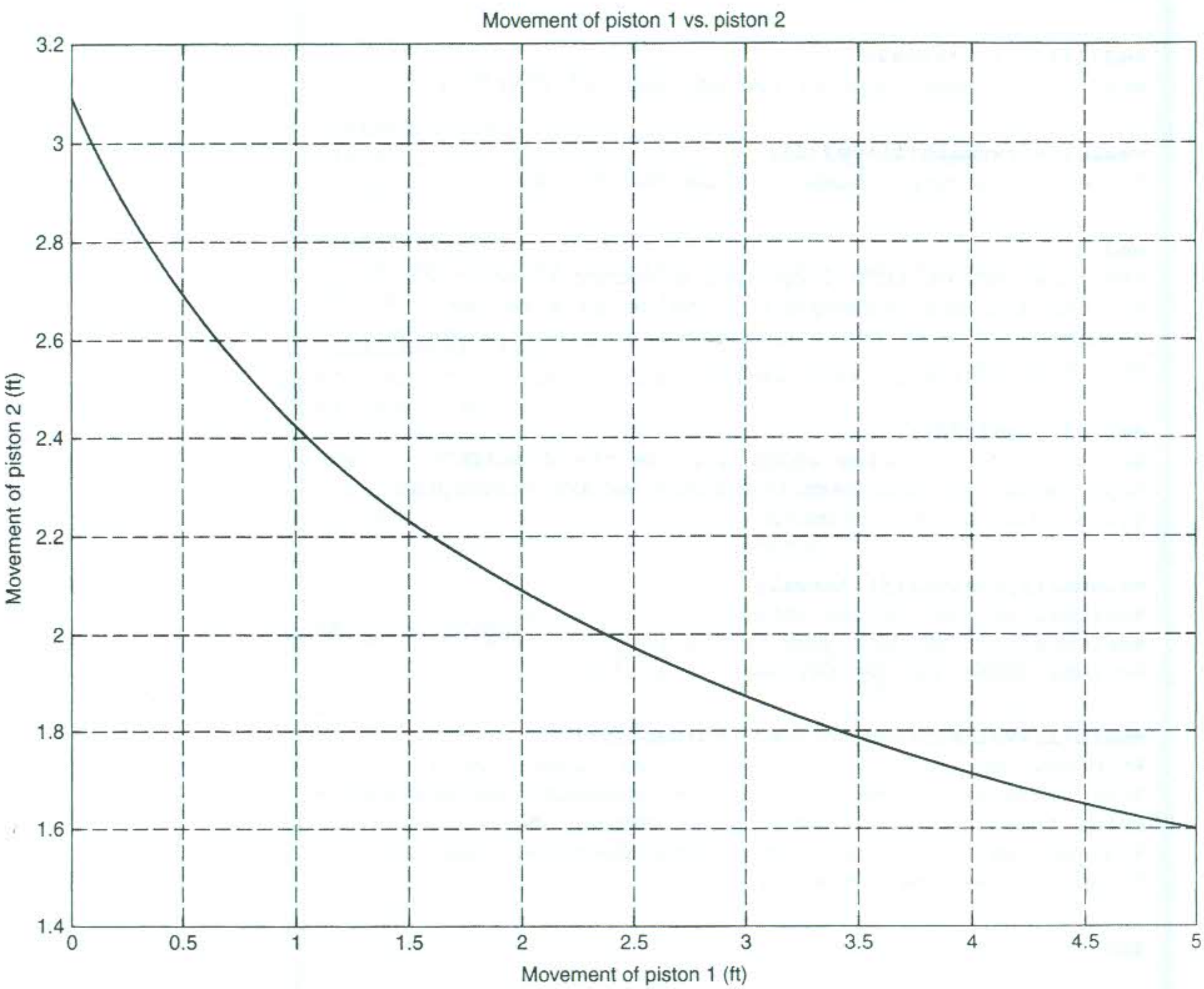


Figure C1.4

PROBLEMS

Problem Categories

Definitions 1.1–1.2

Dimensions and units 1.3–1.11

Viscosity and shear stress 1.12–1.22

Gases (definitions and equations of state) 1.23–1.29

Surface tension and capillary action 1.30–1.35

Problems involving fields 1.36–1.37

Traction forces and stresses 1.38–1.43

Gradient operator 1.44–1.46

Computer problems 1.47–1.50

1.1 Does the definition of a fluid as used in mechanics differ from what you learned in physics? In chemistry? If so, explain the various definitions and why you think they are different or the same.

1.2 In strength of materials, we used the concept of an elastic, perfectly plastic stress-strain diagram, as shown in Fig. P1.2. Does such a material satisfy the definition of a fluid? Explain.

Figure P1.2

1.3 What is the dimensional representation of:

a. Power
b. Modulus of elasticity
c. Specific weight
d. Angular velocity
e. Energy
f. Moment of a force
g. Poisson's ratio
h. Strain

1.4 What is the relation between a scale unit of acceleration in USCS (pound-mass-foot-second) and SI (kilogram-meter-second)?

1.5 How many scale units of power in SI using newtons, meters, and seconds are there to a scale unit in USCS using pounds of force, feet, and seconds?

1.6 Is the following equation a dimensionally homogeneous equation:

$$a = \frac{2d}{t^2} - \frac{2V_0}{t}$$

where
$a \equiv$ acceleration
$d \equiv$ distance
$V_0 \equiv$ velocity
$t \equiv$ time

1.7 The following equation is dimensionally homogeneous:

$$F = \frac{4Ey}{(1 - \nu^2)(Rd^2)}\left[(h - y)\left(h - \frac{y}{2}\right)A - A^3\right]$$

where
$E \equiv$ Young's modulus
$\nu \equiv$ Poisson's ratio
$d, y, h \equiv$ distances
$R \equiv$ ratio of distances
$F \equiv$ force

What are the dimensions of A?

1.8 The shape of a hanging drop of liquid is expressible by the following formulation developed from photographic studies of the drop:

$$T = \frac{(\gamma - \gamma_0)(d_e)^2}{H}$$

where
γ = specific weight of liquid drop
γ_0 = specific weight of vapor around it
d_e = diameter of drop at its equator
T = surface tension, i.e., force per unit length
H = a function determined by experiment

For the equation above to be dimensionally homogeneous, what dimensions must H possess?

1.9 In the study of elastic solids we must solve the following partial differential equation for the case of a plate where body forces are conservative:

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2\,\partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = -(1-\nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

where ϕ = stress function
ν = Poisson's ratio
V = scalar function whose gradient $[(\partial V/\partial x)\mathbf{i} + (\partial V/\partial y)\mathbf{j}]$ is the body-force distribution where the body force is given per unit volume

What would the dimensions have to be of the stress function?

1.10 Convert the coefficient of viscosity μ from units of dynes, seconds, and centimeters (i.e., *poises*) to units of pound-force, seconds, and feet.

1.11 What are the dimensions of kinematic viscosity? If the viscosity of water at 68°F is 2.11×10^{-5} lb · s/ft^2, what is the kinematic viscosity at these conditions? How many stokes of kinematic viscosity does the water have?

1.12 Water is moving through a pipe. The velocity profile at some section is shown and is given mathematically as

$$V = \frac{\beta}{4\mu}\left(\frac{D^2}{4} - r^2\right)$$

where β = a constant
r = radial distance from centerline
V = velocity at any position r

What is the shear stress at the wall of the pipe from the water? What is the shear stress at a position $r = D/4$? If the profile above persists a distance L along the pipe, what drag is induced on the pipe by the water in the direction of flow over this distance?

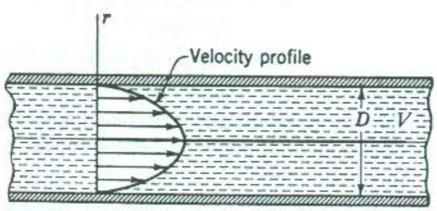


Figure P1.12

1.13 A large plate moves with speed V_0 over a stationary plate on a layer of oil. If the velocity profile is that of a parabola, with the oil at the plates having the same velocity as the plates, what is the shear stress on the moving plate from the oil? If a linear profile is assumed, what is then the shear stress on the upper plate?

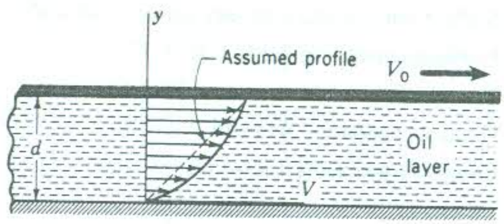


Figure P1.13

1.14 A block weighing 1 kN and having dimensions 200 mm on an edge is allowed to slide down an incline on a film of oil having a thickness of 0.0050 mm. If we use a linear velocity profile in the oil, what is the terminal speed of the block? The viscosity of the oil is 7×10^{-2} P.

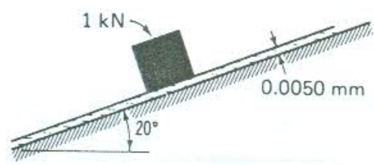


Figure P1.14

1.15 A cylinder of weight 20 lb slides in a lubricated pipe. The clearance between cylinder and pipe is 0.001 in. If the cylinder is observed to decelerate at a rate of 2 ft/s^2 when the speed is 20 ft/s, what is the viscosity of the oil? The diameter of the cylinder D is 6.00 in and the length L is 5.00 in.

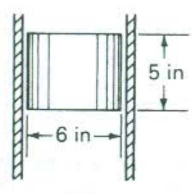


Figure P1.15

1.16 A plunger is moving through a cylinder at a speed of 20 ft/s. The film of oil separating the plunger from the cylinder has a viscosity of 0.020 lb · s/ft^2. What is the force required to maintain this motion?

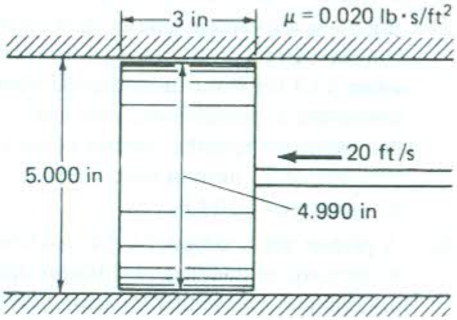


Figure P1.16

1.17 A vertical shaft rotates in a bearing. It is assumed that the shaft is concentric with the bearing journal. A film of oil of thickness e and viscosity μ separates the shaft from the bearing journal. If the shaft rotates at a speed of ω radians per second and has a diameter D, what is the frictional torque to be overcome at this speed? Neglect centrifugal effects at the bearing ends and assume a linear velocity profile. What is the power dissipated?

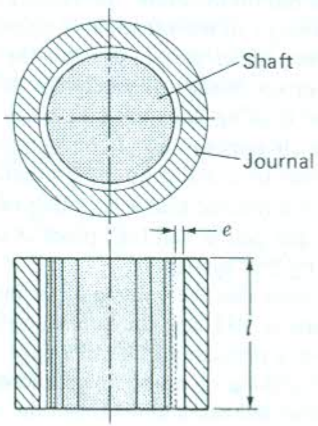


Figure P1.17

1.18 In some electric measuring devices, the motion of the pointer mechanism is dampened by having a circular disc turn (with the pointer) in a container of oil. In this way, extraneous rotations are damped out. What is the damping torque for $\omega = 0.2$ rad/s if the oil has a viscosity of 8×10^{-3} N · s/m^2? Neglect effects on the outer edge of the rotating plate.

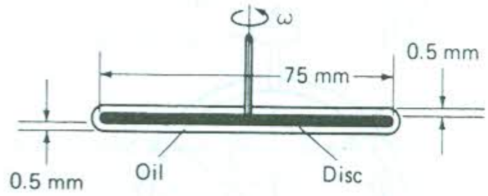


Figure P1.18

1.19 For the apparatus in Prob. 1.18, develop an expression giving the damping torque as a function of x (the distance that the midplane of the rotating plate is from its center position). Do this for an angular rotation $\omega = 0.2$ rad/s.

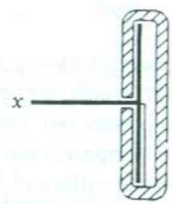


Figure P1.19

1.20 A conical body is made to rotate at a constant speed of 10 rad/s. A film of oil having a viscosity of 4.5×10^{-5} lb · s/ft^2 separates the cone from the container. The film thickness is 0.01 in. What torque is required to maintain this motion? The cone has a 2-in radius at the base and is 4 in tall. Use the straight-line-profile assumption and Newton's viscosity law.

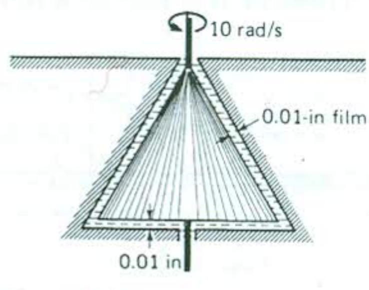


Figure P1.20

1.21 A sphere of radius R rotates at constant speed of ω rad/s. A thin film of oil separates the rotating sphere from a stationary spherical container. Develop an expression for the resisting torque in terms of R, ω, μ, and e. Spherical coordinates are shown.

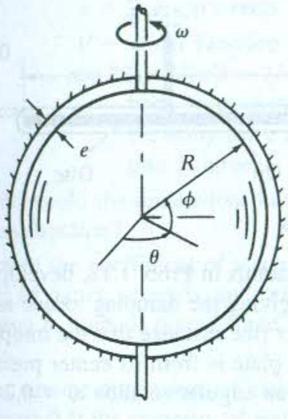


Figure P1.21

1.22 An African hunter is operating a blow gun with a poison dart. He maintains a constant pressure of 5 kPa gage behind the poison dart, which has a weight of $\frac{1}{2}$ N and a peripheral area directly adjacent to the inside surface of the blow gun of 1500 mm². The average clearance of this 1500-mm² peripheral area of the dart with the inside surface of the gun is 0.01 mm when shooting directly upward (at a bird in a tree). What is the speed of the dart on leaving the blow gun when fired directly upward? The inside surface of the gun is dry with air and vapor from the hunter's breath as the lubricating fluid between dart and gun. This mixture has a viscosity of 3×10^{-5} N · s/m². *Hint:* Express dV/dt as $V(dV/dx)$ in Newton's law.

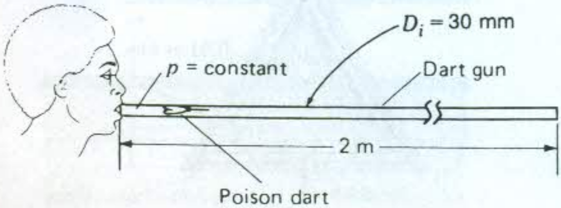


Figure P1.22

1.23 If specific volume v is given in units of volume per unit mass, and density ρ is given in terms of mass per unit volume, how are they related? Also, if specific weight γ is given in units of weight per unit volume, how is it related to the other quantities?

1.24 What are the dimensions of R, the gas constant, in Eq. 1.10? Using for air the value 53.3 for R for units degrees Rankine, pound-mass, pound-force, and feet, determine the specific volume of air at a pressure of 50 lb/in² absolute and a temperature of 100°F.

1.25 A perfect gas undergoes a process whereby its pressure is doubled and its specific volume is decreased by two-thirds. If the initial temperature is 100°F, what is the final temperature in degrees Fahrenheit?

1.26 In order to reduce gasoline consumption in city driving, the Department of Energy of the federal government is studying the so-called inertial transmission system. In this system, when drivers want to slow up, the wheels are made to drive pumps which pump oil into the compressor tank so as to increase the pressure of the trapped air in the tank. The pumps thus act as brakes. As long as the pressure in the tank stays above a certain minimum value, the tank can supply energy to the aforestated pumps, which then act as motors to drive the wheels when a driver wishes to accelerate. If sufficient braking does not take place to keep the air pressure up, a conventional gas engine cuts in to build up the pressure in the tank. It is expected that a doubling of mileage per gallon can take place in city driving by this system.

Suppose that the volume of air initially in the tank is 80 L and the temperature is 30°C with a pressure of 200 kPa gage. As a result of braking on going down a long hill, the volume decreases to 40 L and the air reaches a pressure of 500 kPa gage. What is the final temperature of the air if there is a loss of air due to a leak of 0.003 kg?

1.27 For Prob. 1.26 suppose that the initial volume of air in the tank is 80 L at a pressure of 120 kPa at $T = 20°C$. The gasoline engine cuts in to double the pressure in the tank while the volume is decreased to 50 L. What is the final temperature and density of the air?

1.28 As you may recall from chemistry, a *pound-mole* of a gas is the number of pounds-mass of the gas equal to its molecular weight M. For 2 lb-mol of air with a molecular weight of 29, a temperature of 100°F, and a pressure of 2 atm, what is the volume V? Show that $pv = RT$ can be expressed as $pV = nMRT$, where n is the number of moles.

1.29 You may recall from chemistry that the gas constant R for a particular gas can be determined from a universal gas constant R_u, having a constant value for all perfect gases, and the molecular weight M of the particular gas. That is, $R = R_u/M$.

The value of R_u in USCS is $R_u = 49{,}700$ $\text{ft}^2/(\text{s}^2)(°\text{R})$.

Show that for SI units, we get, $R_u = 8310$ $\text{m}^2/(\text{s}^2)(\text{K})$. What is the gas constant R for helium in SI units?

1.30 A thin, circular wire is being lifted from contact with water. What force F is required for this action over and above the weight of the wire? The water forms a zero degree contact angle with the outermost and innermost peripheries of the wire for certain metals such as platinum. Compute F for a platinum wire. Explain how you could use this system to measure σ.

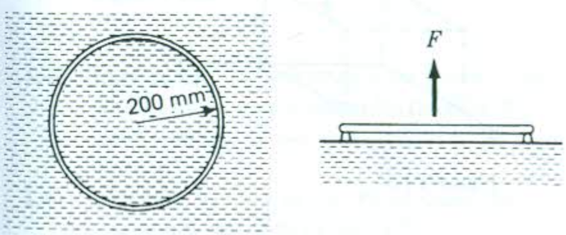


Figure P1.30

1.31 Two parallel, wide, clean, glass plates separated by a distance d of 1 mm are placed in water. How far does the water rise due to capillary action away from the ends of the plates? *Hint:* See footnote 10.

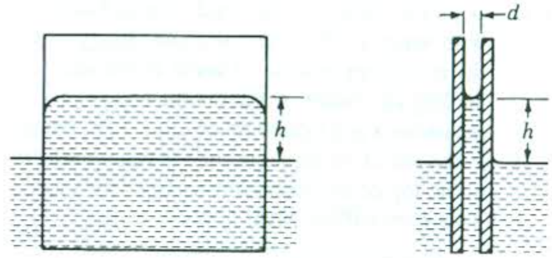


Figure P1.31

1.32 A glass tube is inserted in mercury. What is the upward force on the glass as a result of surface effects? Note that the contact angle is 50° inside and outside. Temperature is 20°C.

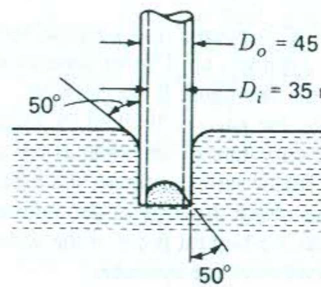


Figure P1.32

1.33 Compute an approximate distance d for mercury in a glass capillary tube. The surface tension σ for mercury and air here is 0.514 N/m, and the angle θ is 40°. The specific gravity of mercury is 13.6. *Hint:* The pressure p_{gage} below the main free surface is the specific weight times the depth below the free surface. Do your assumptions render the actual d larger or smaller than the computed d?

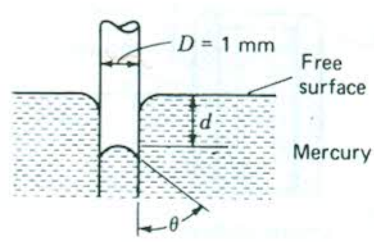


Figure P1.33

1.34 A narrow tank with one end open is filled with water at 45°C carefully and slowly to get the maximum amount of water in without spilling any water. If the pressure gage measures a gage pressure of 2943.7 Pa, what is the radius of curvature of the water surface at the top of the surface away from the ends? Take $\sigma = 0.0731$ N/m.

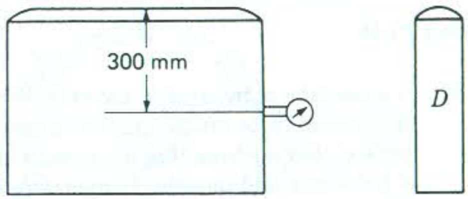


Figure P1.34

1.35 Water at 10°C is poured into a region between concentric cylinders until water appears above the top of the open end. If the pressure measured by the gage is 3970.80 Pa gage, what is the curvature of the water at the top? Using the Taylor series, estimate the height h of the water above the edge of the cylinders. Assume that the highest point of the water is at the midradius of the cylinders.

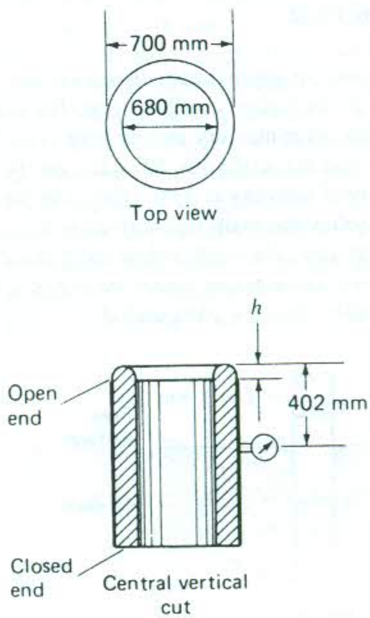


Figure P1.35

1.36 Given the velocity field

$$\mathbf{V}(x, y, z, t) = (6xy^2 + t)\mathbf{i} + (3z + 10)\mathbf{j} + 20\,\mathbf{k} \quad \text{m/s}$$

with x, y, z in meters and t in seconds, what is the velocity vector at position $x = 10$ m, $y = -1$ m, and $z = 2$ m when $t = 5$ s? What is the magnitude of this velocity?

The velocity components in a flow of fluid are known to be

$$V_x = 6xt + y^2z + 15 \quad \text{m/s}$$
$$V_y = 3xy^2 + t^2 + y \quad \text{m/s}$$
$$V_z = 2 + 3ty \quad \text{m/s}$$

where x, y, and z are given in meters and t is given in seconds. What is the velocity vector at position (3, 2, 4) m and at time $t = 3$ s? What is the magnitude of the velocity at this point and time?

1.37 A body-force distribution is given as

$$\mathbf{B} = 16x\mathbf{i} + 10\mathbf{j} \quad \text{N/kg}$$

per unit mass of the material acted on. If the density of the material is given as

$$\rho = x^2 + 2z \quad \text{kg/m}^3$$

what is the resultant body force on material in the region shown in the diagram?

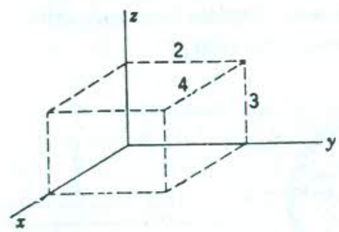


Figure P1.37

1.38 Oil is moving over a flat surface. We are observing this flow from above in the diagram. A traction force field $\mathbf{T}$ is developed on the flat surface given as

$$\mathbf{T} = (6y + 3)\mathbf{i} + (3x^2 + y)\mathbf{j} + (5 + x^2)\mathbf{k} \quad \text{lb/ft}^2$$

What is the total force on the 3×3 square of area shown in the diagram?

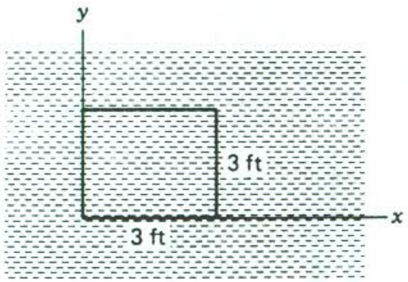


Figure P1.38

1.39 The stresses on face A of an infinitesimal rectangular parallelepiped of fluid in a flow are shown in Fig. P1.39 at time t. What is the traction vector for this face at the instant shown? What can you say about shear stresses on faces B and C at this instant?

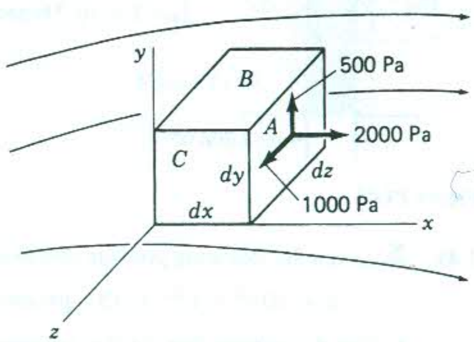


Figure P1.39

1.40 Explain why in hydrostatics the traction force exerted on an area element by the fluid is always normal to the area element of the boundary.

1.41 Label the stresses in Fig. P1.41 using the convention set forth in Sec. 1.10.

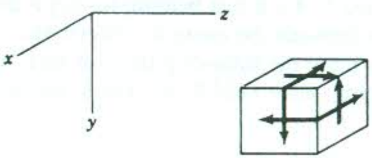


Figure P1.41

1.42 We are given the following stress field in megapascals:

$$\tau_{xx} = 16x + 10 \qquad \tau_{zz} = \tau_{xz} = \tau_{yz} = 0$$
$$\tau_{yy} = 10y^2 + 6xy$$
$$\tau_{xy} = -5x^2$$

Express the bulk stress distribution as a scalar field. What is the bulk stress at (0, 10, 2) m?

1.43 In a viscous flow, the stress tensor at a point is

$$\tau_{ij} = \begin{bmatrix} -4000 & 3000 & 1000 \\ 3000 & 2000 & -1000 \\ 1000 & -1000 & -5000 \end{bmatrix} \quad \text{lb/in}^2$$

What is the thermodynamic pressure at this point?

1.44 A vector field may be formed by taking the gradient of a scalar field. If $\phi = xy + 16t^2 + yz^3$, what is the field **grad** ϕ? What is the magnitude of the vector **grad** ϕ at position (0, 3, 2) when $t = 0$?

1.45 If we have a pressure distribution in a fluid given as

$$p = xy + (x + z^2) + 10 \quad \text{kPa}$$

what is the force per unit volume on an element of the medium in the direction

$$\mathbf{e} = 0.95\mathbf{i} + 0.32\mathbf{j} \quad \text{m}$$

at position $x = 10$ m, $y = 3$ m, $z = 4$ m?

1.46 Derive the gradient of pressure for cylindrical coordinates in the manner in which we developed the gradient of pressure for Cartesian coordinates. What is the gradient operator in cylindrical coordinates? Use the element shown in Fig. P1.46. *Hint:* Replace $\sin(d\theta/2)$ by $(d\theta/2)$ and $\cos(d\theta/2)$ by 1.

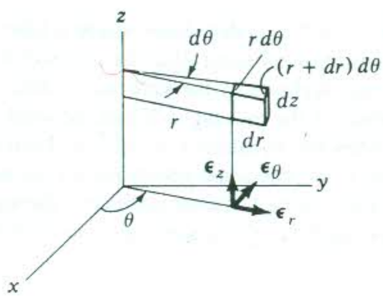


Figure P1.46

1.47 A thrust bearing surface B supports via an oil film a vertical shaft A having a diameter of 100 mm and carries a load of 1000 N. Shaft A has a clockwise angular velocity given as $\omega_A = 6t^{3/2} + 0.4$ rad/sec, while supporting surface B has a counterclockwise angular velocity given as $\omega_B = 0.006 \ln(0.2 + 0.3t^3)$ rad/sec. Plot the frictional torque needed for shaft A as a function of time starting from $t = 0$ for 10 seconds for an oil film having the thickness δ of 0.25 mm. The coefficient of viscosity μ of the oil is 7×10^{-3} N · s/m³.

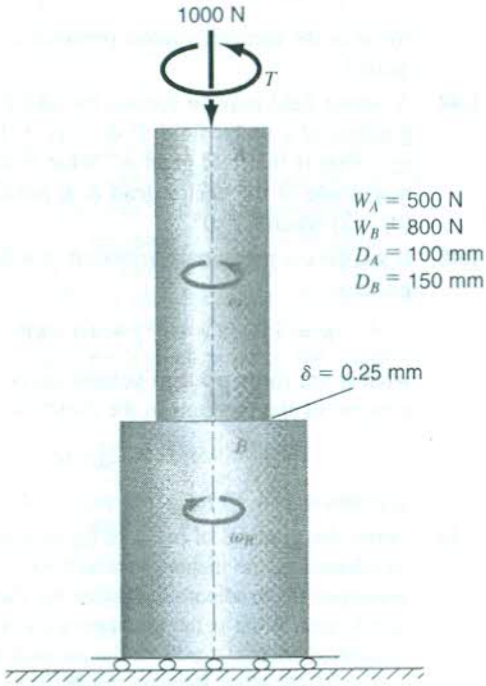


Figure P1.47

1.48 A solid cylinder slides inside a tube with a thin film of oil separating the two would-be contact surfaces. If the vertical position of the bottom of the moving cylinder, denoted as y, satisfies the equation $\dot{y} = 3t^{1/2} + 2$ mm/sec, what is the frictional resistance on the moving cylinder plotted against position y during a movement of 10 cm starting from the time $t = 0$? The diameter of the cylinder is 2.000 cm, and the inside diameter of the tube is 2.002 cm. The coefficient of viscosity is 7×10^{-3} N-sec/m³.

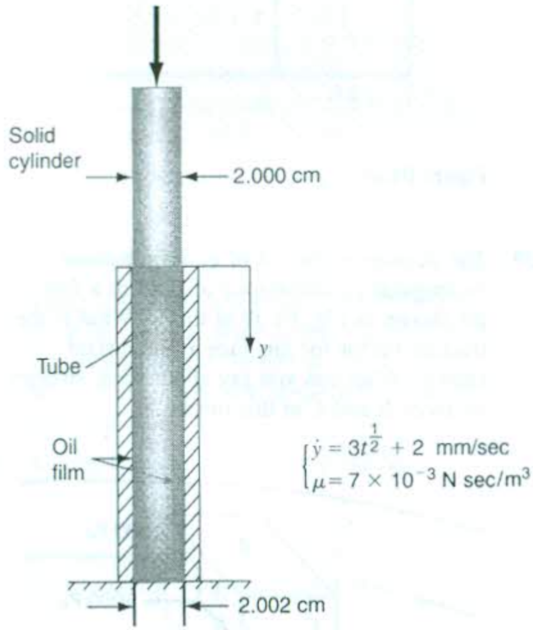


Figure P1.48

1.49 Given the following pressure distribution:

$$p = 10x^2 + y^{3/2} + 15 \quad \text{psi abs}$$

a. Plot the contour lines of this pressure distribution in the xy plane for values of pressure p = 15, 20, 25, 30 psi abs.
b. Plot the contour lines for force per unit volume generated by this pressure field. Choose df/dv having values corresponding to those values at the following positions in the xy plane: (2,5), (3,7), (5,10), and (8,15).

1.50 Develop a software package that will permit the quick evaluation of the resisting torque T of a thrust bearing having a thin oil film between the contact surfaces. Have the user input the following data for such a thrust bearing: three radii R_i in meters, the oil film thickness δ in mm, the angular velocity ω of the shaft in rad/sec, and the coefficient of viscosity μ of the oil in N-s/m^2. In your program be sure you only accept radii such that $R_4 > R_3 > R_2 > R_1$. Run your program for the data shown in Fig. P1.50. The contact surface of a thrust bearing is shown (crosshatched) having an angular velocity ω.

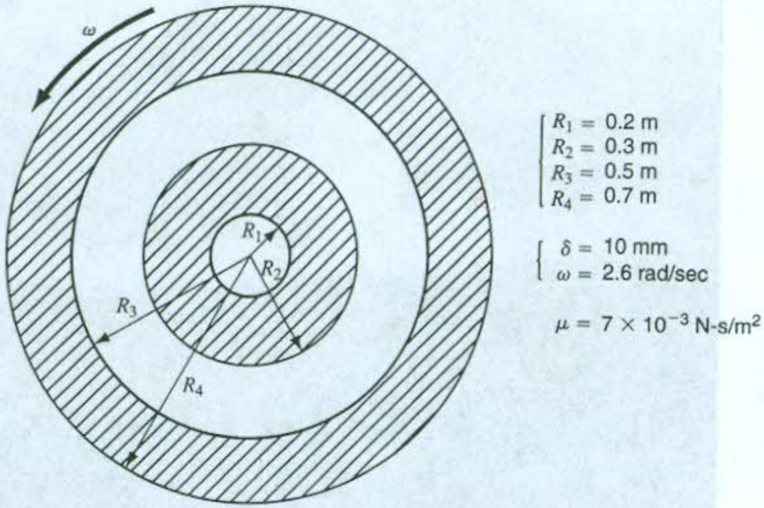
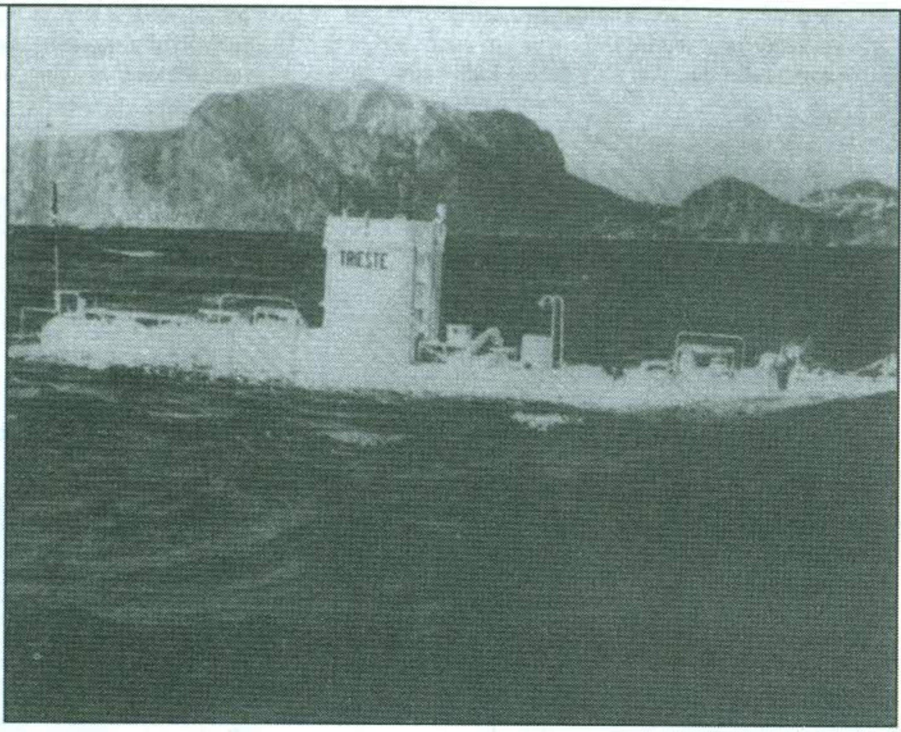
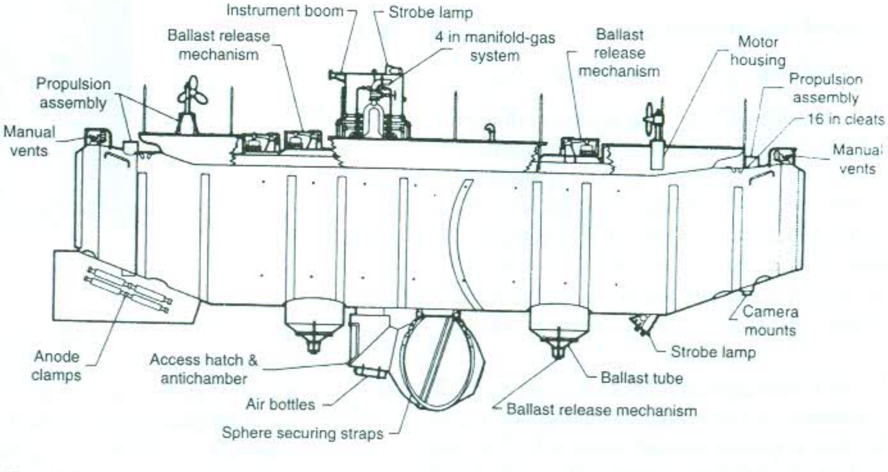


Figure P1.50

Photo of the deep sea submersible *Trieste* along with a sketch of details. (*Courtesy U.S. Navy.*)





The original bathyscaphe *Trieste* was built by the Swiss physicist Piccard to explore the ocean floor at its lowest depth of 11.3 km. The U.S. Navy purchased this bathyscaphe and has further developed it into a larger, more useful system. This is shown above in an actual scene and also a detailed sketch. An example involving the original bathyscaphe *Trieste* is presented in this chapter (Example 2.12).

CHAPTER 2

Fluid Statics

2.1 INTRODUCTION

A fluid will be considered static if all particles either are motionless or have the same constant velocity relative to an inertial reference. Hence, the conditions for case 1, Sec. 1.11 would now be properly classed as static. For such a case, it was learned, there is no shear stress, so one must deal with a scalar pressure distribution. This chapter evaluates pressure distributions in static fluids and examines some important effects attributable to such pressure distributions.

2.2 PRESSURE VARIATION IN AN INCOMPRESSIBLE STATIC FLUID

In order to ascertain pressure distribution in static fluids, we will consider the equilibrium of forces on an infinitesimal fluid element as shown in Fig. 2.1.[1] The forces acting on the element stem from pressure from the surroundings and the gravity force. For equilibrium, we have

$$-\gamma\, dx\, dy\, dz\, \mathbf{k} + (-\mathbf{grad}\, p)\, dx\, dy\, dz = \mathbf{0}$$

where γ is the specific weight. The resulting scalar equations are

$$\frac{\partial p}{\partial x} = 0 \qquad [2.1a]$$

$$\frac{\partial p}{\partial y} = 0 \qquad [2.1b]$$

$$\frac{\partial p}{\partial z} = -\gamma \qquad [2.1c]$$

[1]For simplicity we have shown a liquid with a free surface, but the resulting equation (2.2) is valid for a gas or a liquid.

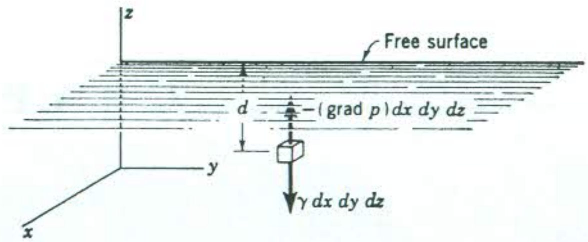


Figure 2.1
Free body of an element in a static fluid.

From this, we see that the pressure can vary only in the z direction, which has been selected as opposite the direction of gravity. (It will be left for you in Prob. 2.4 to deduce from the preceding formulations that the free surface of a liquid at rest must be at right angles to the direction of gravity.)

Since p varies only in the z direction and is not a function of x and y, we may use an ordinary derivative in Eq. 2.1c. Thus

$$\boxed{\frac{dp}{dz} = -\gamma} \qquad [2.2]$$

This differential equation applies to *any static compressible or incompressible fluid* in a gravity field. In order to evaluate the pressure distribution itself, we must integrate between conveniently chosen limits. Choosing the subscript 0 to represent conditions at the free surface, we integrate from any position z, where pressure is p, to position z_0, where the pressure is atmospheric and denoted as p_{atm}. Thus

$$\int_p^{p_{\text{atm}}} dp = \int_z^{z_0} -\gamma\, dz$$

Taking γ as constant,[2] we may readily integrate. We then get

$$p_{\text{atm}} - p = -\gamma(z_0 - z)$$

or [2.3]

$$p - p_{\text{atm}} = \gamma(z_0 - z) = \gamma d$$

where d is the distance below the free surface (see Fig. 2.1). We usually term $p - p_{\text{atm}}$, that is, the pressure difference from atmospheric pressure, as the *gage* pressure, with the symbol p_g or p gage. Hence

$$p_g = \gamma d \qquad [2.4]$$

[2]We consider here that g is constant in the range of interest, which is a step that can be taken in most hydrostatic engineering problems. In addition, we consider that the fluid is incompressible so that $\gamma = \rho g$ is constant. We are now restricted to liquids.

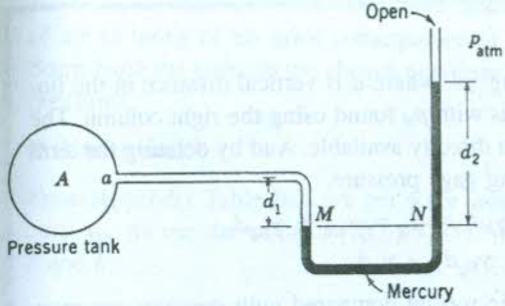


Figure 2.2
Simple manometer or U-tube.

Many pressure-measuring devices yield the pressure above or below that of the atmosphere. As a consequence, gage pressures are used quite often in engineering work.[3] In contrast to pressure p, note that p_g can be negative, with a maximum possible negative value equal to $-p_{atm}$. Note also from the preceding equations that, for any one static fluid, *pressure at a given depth below the free surface remains constant for that depth anywhere in the fluid.*

The preceding results can be used directly in a pressure-measuring technique called *manometry,* a very common technique that we shall put to use frequently in the discussions and problems to follow. The simplest type of manometer is the so-called U-tube. This is a capillary tube (see Fig. 2.2) that connects at one end to the fluid where the pressure is to be measured and is open to the atmosphere at the other end. Note in the diagram that the fluid in the tank extends into the U-tube, eventually making contact with the column of mercury. The fluids attain a configuration of equilibrium from which it is relatively easy to deduce the pressure in the tank at *a*. Because of its high specific weight, mercury is usually used as the second fluid when appreciable pressures are to be expected, since shifts of the fluids demanded by equilibrium will then be reasonably small.

EXAMPLE 2.1

■ Problem Statement

Determine the absolute pressure at a for the U-tube in Fig. 2.2 in terms of the data shown. What is the gage pressure at a?

■ Strategy

We shall look for two points on the vertical segments of the tube where we must have equal pressures. For this computation the points must be at the same elevation and must be joined by the same fluid, thereby guaranteeing equal pressures. Points M and N, you will note, satisfy these requirements. We will compute the pressure along the left leg, starting at end a, whose pressure we seek, and going to M. We will equate this with the pressure along the right leg starting with atmospheric pressure and going to N.

[3] We often term the pressure p as *absolute* pressure to distinguish it from *gage* pressure p_g.

■ Execution

We start with the left column using γd, where d is vertical distance in the liquid, to write p_M, and we equate this with p_N found using the right column. The desired absolute pressure p_a is then directly available. And by deleting the term p_{atm} we will have the corresponding gage pressure.

$$p_M = p_a + \gamma_A d_1 = p_N = p_{\text{atm}} + \gamma_{\text{Hg}} d_2$$
$$\therefore\ p_a = p_{\text{atm}} + \gamma_{\text{Hg}} d_2 - \gamma_A d_1$$

If fluid A has a very small specific weight compared with mercury, we may, under most circumstances, neglect the term $\gamma_A d_1$. Hence

$$p_a \approx p_{\text{atm}} + \gamma_{\text{Hg}} d_2$$

■ Debriefing

One need only employ a consistent measure when dealing with the meniscus of fluids at locations where two fluids touch along each leg. And in the laboratory, one must exert great care in dealing with mercury since it is a deadly poison when ingested even in the most minute amounts.

EXAMPLE 2.2

■ Problem Statement

The *differential manometer* shown in Fig. 2.3 will yield the difference in pressure between two points a and b of the tanks A and B, which contain fluids. In tank A we have water at temperature of 20°C, and in tank B we have air. The specific gravity of mercury is 13.6. The following data apply:

$$d_1 = 10 \text{ mm} \qquad d_2 = 80 \text{ mm} \qquad d_3 = 60 \text{ mm}$$

■ Strategy

We will select two points M and N along the capillary having equal pressures to work with. We will look up in our Appendix Table B.1 the specific weight of water at the specified temperature, and we will neglect the specific weight

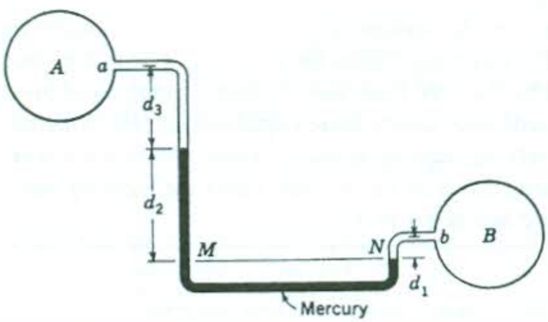


Figure 2.3
Differential manometer.

of air as being of no great consequence in this problem. We will then move down from the tanks to the chosen positions along the capillary and equate the pressures.

■ Execution

From Appendix Table B.1, we get γ for water to be 9788 N/m^3, and for mercury γ_{Hg} we use the value of (13.6)(9788). We then have, equating pressures at a and b:

$$p_a + \lambda_{H_2O} d_3 + \gamma_{Hg} d_2 = p_b$$

Hence

$$p_b - p_a = \lambda_{Hg} d_2 + \gamma_{H_2O} d_3$$

We then have the desired result.

$$\begin{aligned} p_b - p_a &= [(13.6)(9788)\ \text{N/m}^3](0.080\ \text{m}) + (9788\ \text{N/m}^3)(0.060\ \text{m}) \\ &= 11{,}240\ \text{Pa} \end{aligned}$$

■ Debriefing

Notice that we were able to neglect the weight of air compared to that of water and that of mercury. Later in this chapter we shall look at air involving large heights in our atmosphere. The specific weight of the air or that of other gases on other planets becomes a vital consideration. Note also that in problems where no temperature is specified we will use γ_{H_2O} = 62.4 lb/ft^3 and 9806 N/m^3.

EXAMPLE 2.3

■ Problem Statement

What is the absolute and gage pressure in drum A in Fig. 2.4 at position a?

■ Strategy

We will pick two points M and N in the capillary tube which are joined by the same fluid and are at the same elevation. The pressures at these points are equal; hence we equate these pressures along each leg of the capillary, thereby allowing us to determine the desired pressure p_a.

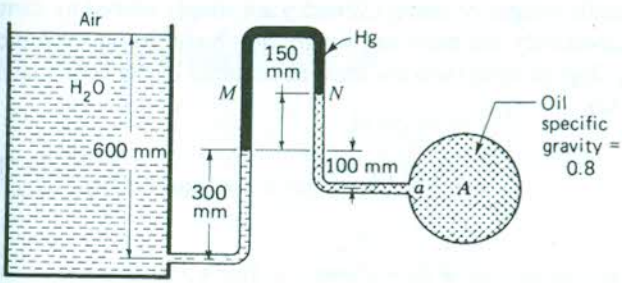


Figure 2.4
Inverted U-tube.

■ Execution

The points M and N shown in Fig. 2.4 satisfy the aforestated requirements for having equal pressures. We compute each pressure as follows:

$$p_N = p_a - (0.8)(9806)(0.100 + 0.150)$$
$$p_M = p_{\text{atm}} + (9806)(0.6 - 0.3) - (9806)(13.6)(0.15)$$

Now by equating p_M and p_N and using 101,325 Pa for the atmospheric pressure we may directly solve for p_a to get the desired information. Thus

$$p_a = 86{,}224 \text{ Pa abs}$$
$$p_a = -15{,}101 \text{ Pa gage}$$

■ Debriefing

This problem shows that a U-tube type of problem does not have to look like the letter U. However, note that the same approach has been used here as in Examples 2.1 and 2.2. Note also that the gage pressure came out negative. Any *absolute pressure* less than the value of 101,325 Pa will have a negative gage pressure. And the magnitude of a negative gage pressure cannot be smaller than 101,325. Finally, an absolute pressure cannot be less than zero.

2.3 PRESSURE VARIATION WITH ELEVATION FOR A STATIC COMPRESSIBLE FLUID

The vertical distances for the gases in the manometry problems were small and as a consequence we neglected pressure variation with height for such cases. However, in computations involving large vertical distances as may occur in considering atmospheres of planets, we must often consider pressure variation of gas with height. We shall examine two useful cases in this section.

Returning to the differential equation of Eq. 2.2, relating pressure, specific weight, and elevation for all static fluids, we now assume that γ is a *variable* and thus allow for *compressibility* effects. We restrict ourselves to the perfect gas, which is valid for air or most of its components for relatively large ranges of pressure and temperature. The *equation of state,* containing v, helps us evaluate the required functional variation of the specific weight, γ, since $1/v$ and γ are simply related by their definitions, which are, respectively, the mass and weight of a body per unit volume of the body.[4] Thus, using slugs or kilograms for mass as required by Newton's law, we have from Newton's law

$$\gamma = \frac{1}{v}g = \rho g \qquad [2.5]$$

[4]Note that the specific volume v is the reciprocal of the mass density ρ. That is $\rho = 1/v$.

If the mass unit pound-mass is used, the relation then becomes

$$\gamma = \frac{1}{v}\frac{g}{g_0} \tag{2.6}$$

and since g and g_0 can be considered to have equal values in most practical fluid applications, we often find the relation $1/v = \gamma$ employed under these circumstances. We will formulate our results in terms of slugs or kilograms and make proper conversions when necessary while solving problems.

We will now compute the pressure-elevation relation for two cases, namely, the *isothermal* (constant-temperature) fluid and the case where the temperature of the fluid varies linearly with elevation. These cases occur in certain regions of our atmosphere.

Case 1. Isothermal Perfect Gas. For this case the equation of state (Eq. 1.8) indicates that the product pv is constant. Thus, at any position in the fluid, we may say, using the subscript 1 to indicate known data,

$$pv = p_1 v_1 = C \tag{2.7}$$

where C is a constant. Solving for v in Eq. 2.5 to form $v = \frac{g}{\gamma}$ and substituting into the equation above, we get

$$p\frac{g}{\gamma} = p_1\frac{g_1}{\gamma_1} = C \tag{2.8}$$

We assume that the elevation range is not excessively large so that we can take g as constant. Thus dividing by g,

$$\frac{p}{\gamma} = \frac{p_1}{\gamma_1} = \frac{C}{g} = C' \tag{2.9}$$

Using the relation above, we may express the basic differential equation (2.2) as follows:

$$\frac{dp}{dz} = -\gamma = -\frac{p}{C'}$$

Separating variables and integrating from p_1 to p and z_1 to z, we have

$$\int_{p_1}^{p}\frac{dp}{p} = -\int_{z_1}^{z}\frac{dz}{C'}$$

Carrying out the integration, we get

$$\ln p\Big|_{p_1}^{p} = -\frac{z}{C'}\Bigg|_{z_1}^{z}$$

Putting in the limits, we have

$$\ln\frac{p}{p_1} = -\frac{1}{C'}(z - z_1)$$

Now we use $p_1/\gamma_1 = C'$ from Eq. 2.9 and solve for p:

$$\boxed{p = p_1 \exp\left[-\frac{\gamma_1}{p_1}(z - z_1)\right]} \tag{2.10}$$

This gives us the desired relation between elevation and pressure in terms of the known conditions p_1, γ_1 at elevation z_1. If the datum ($z = 0$) is placed at the position of given data, then z_1, in Eq. 2.10, can be set equal to zero. Note that the pressure decreases *exponentially* with elevation.

Case 2. Temperature Varies Linearly with Elevation. The temperature variation for this case is given by

$$T = T_1 + Kz \tag{2.11}$$

where T_1 is the temperature at the datum ($z = 0$). K is often called the *lapse rate* and is a constant. For terrestrial problems, K will be negative. In order to be able to separate the variables of Eq. 2.2, we must solve for γ from the equation of state and, in addition, determine dz from Eq. 2.11. These results are

$$\gamma = \frac{pg}{RT} \tag{2.12a}$$

$$dz = \frac{dT}{K} \tag{2.12b}$$

Substituting into the basic equation of statics (Eq. 2.2), we get, on separating the variables

$$\frac{dp}{p} = -\frac{g}{KR}\frac{dT}{T} \tag{2.13}$$

Integrating from the datum ($z = 0$) where p_1, T_1, and soon, are known, we have

$$\ln\frac{p}{p_1} = \frac{g}{KR}\ln\frac{T_1}{T} = \ln\left(\frac{T_1}{T}\right)^{g/KR}$$

Solving for p and replacing the temperature T by $T_1 + Kz$, we have for the final expression

$$\boxed{p = p_1\left(\frac{T_1}{T_1 + Kz}\right)^{g/KR}} \tag{2.14}$$

where it should be noted that T_1 must be in degrees absolute.

In concluding this section on compressible static fluids, we should point out that if we know the manner in which the specific weight varies, we can usually separate variables in the basic equation (Eq. 2.2) and integrate to an algebraic equation between pressure and elevation.

EXAMPLE 2.4

■ Problem Statement

An atmosphere on a hypothetical planet has a temperature of 15°C at sea level that drops 1°C per 500 m of elevation. The gas constant R for this atmosphere is 220 N · m/(kg)(K). At what elevation is the pressure 30 percent that of sea level? Take $g = 9.00\ \text{m/s}^2$.

■ Strategy

We will first established the lapse rate K. We can then go directly to Eq. 2.14 where, remembering to use absolute pressures, we can directly get the desired result.

■ Execution

To get the lapse rate K, we note from the problem statement that

$$\begin{aligned} T &= T_1 + Kz \\ \therefore\ T - T_1 &= Kz \end{aligned} \qquad [a]$$

For $T - T_1 = -1°\text{C}$, $z = 500$ m. We get, on applying this condition to Eq. a,

$$-1 = 500K$$

$$\therefore\ K = -\frac{1}{500}$$

Now go to Eq. 2.14 in the following form:

$$\frac{p}{p_1} = 0.30 = \left(\frac{T_1}{T_1 + Kz}\right)^{g/KR}$$

Noting that $T_1 = 15 + 273 = 288$ K, we have

$$0.30 = \left(\frac{288}{288 - z/500}\right)^{9.00/[220(-1/500)]}$$

Solving for z, we get

$$z = 8231\ \text{m}$$

This is the desired altitude.

■ Debriefing

We chose the linearly varying temperature of an atmosphere for this example because this kind of atmosphere resembles the earth's atmosphere up to a height of 36,000 ft (11,000 m) as we shall see in Sec. 2.4. Also, a portion of the earth's atmosphere is isothermal. Pressure variations with elevation are important factors in rocketry calculations. However, we shall not directly use the formulations of this section after we exit Chapter 2.

2.4 THE STANDARD ATMOSPHERE

In order to compare the performances of airplanes, missiles, and rockets, a standard atmosphere has been established which approximately resembles the actual atmosphere as it is found in many parts of the world. At sea level, the U.S. Standard Atmosphere conditions are

$$p = 29.92 \text{ in Hg} = 2116.2 \text{ lb/ft}^2 = 760 \text{ mm Hg} = 101.325 \text{ kPa}$$
$$T = 59°\text{F} = 519°\text{R} = 15°\text{C} = 288 \text{ K}$$
$$\gamma = 0.07651 \text{ lb/ft}^3 = 11.99 \text{ N/m}^3$$
$$\rho = 0.002378 \text{ slug/ft}^3 = 1.2232 \text{ kg/m}^3$$
$$\mu = 3.719 \times 10^{-7} \text{ lb}\cdot\text{s/ft}^2 = 1.777 \times 10^{-8} \text{ kN}\cdot\text{s/m}^2$$

The temperature in the U.S. Standard Atmosphere decreases *linearly with height* according to the relation

$$T = (519 - 0.00357z) \quad °\text{R}(z \text{ in ft})$$
$$T = (288 - 0.006507z) \quad \text{K}(z \text{ in m}) \qquad [2.15]$$

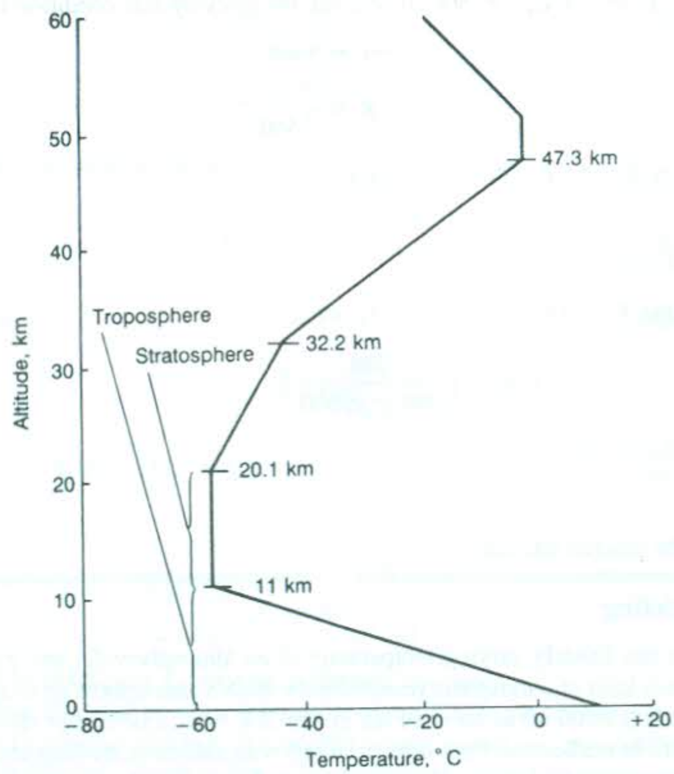


Figure 2.5
Temperature variation with height in the U.S. Standard Atmosphere.

where z is the elevation above sea level. This region is called the *troposphere*. When a height of about 36,000 ft (or about 11,000 m) is reached, the U.S. Standard Atmosphere becomes isothermal at a temperature of -69.7°F (or -56.5°C). This isothermal region is called the *stratosphere*. At about 65,000 ft (or 20,100 m), the temperature starts to increase in value. Appendix Table B.4 gives standard air properties as a function of elevation.

In Fig. 2.5 we have shown a plot of the temperature variation with height for the U.S. Standard Atmosphere.

2.5 EFFECT OF SURFACE FORCE ON A FLUID CONFINED SO AS TO REMAIN STATIC

If external pressure is exerted on a portion of the boundary of a confined fluid, compressible or incompressible, this pressure, once all fluid motion has subsided, will extend undiminished throughout the fluid. Two examples are illustrated in Fig. 2.6. The truth of this statement can be demonstrated by examining cylindrical elements of fluid projecting from the pressurized boundary, as is shown in Fig. 2.6. Equilibrium demands that the pressure increase on the interior end of the element must keep pace with the pressure applied at the boundary. Since the element may be chosen of any length and at any position, it should be clear that a pressure p developed on the boundary must extend uniformly throughout the fluid.

This principle underlies the action of the *hydraulic jack* and the *hydraulic brake*. A pressure Δp developed by piston C (Fig. 2.7) is felt throughout the fluid. Hence, the force F_C on piston C for the pressure increase Δp is $\Delta p A_C$ and the corresponding force F_B on piston B is $\Delta p A_B$. Thus,

$$\frac{F_B}{F_C} = \frac{\Delta p A_B}{\Delta p A_C} = \frac{A_B}{A_C}$$

In this way with $A_B > A_C$, a considerable *mechanical advantage* may be developed.

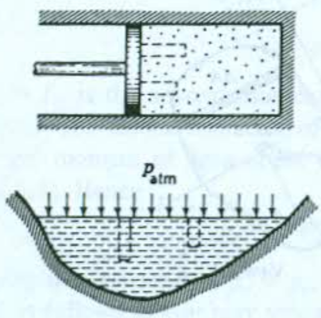


Figure 2.6
Pressure on confined fluids.

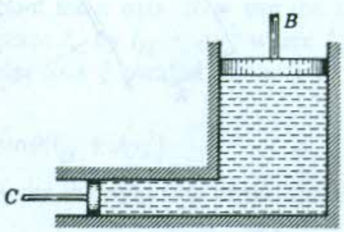


Figure 2.7
Hydraulic jack device.

2.6 HYDROSTATIC FORCE ON A PLANE SURFACE SUBMERGED IN A STATIC INCOMPRESSIBLE FLUID

Shown in Fig. 2.8 is an inclined plate on whose upper face we wish to evaluate the resultant hydrostatic force. Since there can be no shear stress, this force must be normal to the surface. For purposes of calculation, the plane of the submerged surface is extended so as to intersect with the plane of the free surface at an angle θ. The trace of the intersection shown as a dot is the x axis in Fig. 2.8. Note that the y axis is coplanar with the top surface of the plate. From previous discussion, we know that there will be superimposed on the plate a uniform pressure p_s, from atmospheric pressure on the free surface, and a uniformly increasing pressure due to the action of gravity on the liquid. It is left for the student to show that the resultant force arising from the uniform pressure p_s has the value $p_s A$ and acts at the *centroid* of the area. Let us proceed then to the analysis of the resultant force from the uniformly increasing pressure.

Note that strip dA has been selected in Fig. 2.8. at uniform depth h and consequently is subject to a constant pressure. The *magnitude* of the force on this element is then $\gamma h\, dA$. Integrating over the plate area will then give the value of the resultant force. Using the coordinate y for the position of the strip, we have:

$$F_R = \int_A \gamma h\, dA = \int_A \gamma(y \sin\theta) dA = \gamma \sin\theta \int_A y\, dA$$

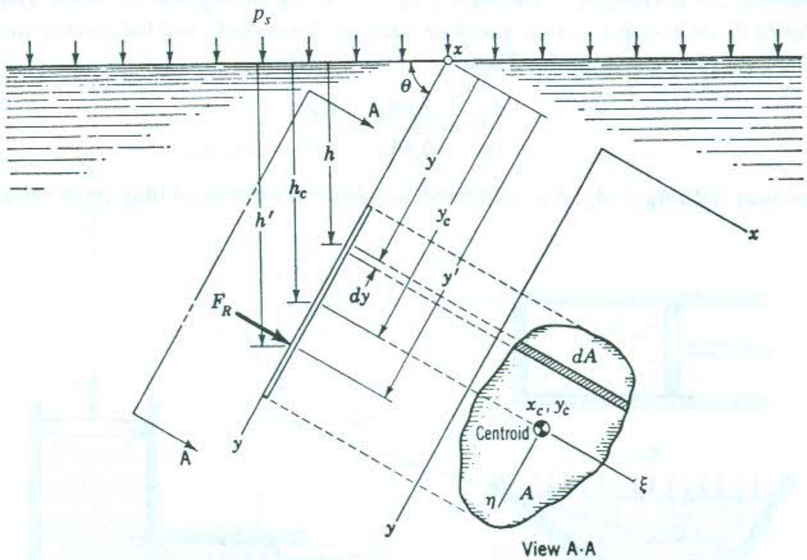


Figure 2.8
Submerged plane surface in a liquid.

Noting that $\int_A y\,dA$ is the first moment of the plate area about x axis, we may use in its stead the term Ay_c, where y_c is the y coordinate of the centroid of this surface. Thus

$$F_R = \gamma \sin\theta\, y_c A = \gamma h_c A = p_c A \tag{2.16}$$

It may be concluded from the formula above that the value of the resultant force due to a uniformly increasing pressure can be most simply evaluated by imagining the pressure at the centroid to extend uniformly over the whole area and computing accordingly.

Clearly the *total* force F_R from the uniform pressure p_s acting at the free surface *and* the uniformly increasing pressure from gravity on the liquid can then be given as

$$\boxed{F_R = (p_s + \gamma h_c)A = p_c A} \tag{2.17}$$

where now p_c is the *total* pressure at the centroid.

The *inclined position* y' of the resultant force from pressure p_s (see Fig. 2.8) on the free surface, as well as from the uniformly increasing pressure of the liquid, will now be evaluated. To do this we will equate the moment of the resultant force F_R about the x axis with the corresponding moment developed by the pressure p_s over the area plus the moment about the x axis of the uniformly increasing pressure of the liquid over the area. Thus

$$F_R y' = \int_A y(p_s + \gamma h)\,dA$$

Replacing F_R and h we have

$$p_c A y' = \int_A y[p_s + \gamma(y \sin\theta)]\,dA$$

where we remind you p_c is the total absolute pressure at y_c from both p_s and the uniformly increasing pressure of the liquid. We now rewrite the above equation:

$$\begin{aligned} p_c A y' &= p_s \int_A y\,dA + \gamma \sin\theta \int y^2\,dA \\ &= p_s A y_c + \gamma \sin\theta I_{xx} \end{aligned}$$

where I_{xx} is the second moment of area about the x axis. Now use the transfer theorem for second moments of area to replace I_{xx} by $I_{\xi\xi} + Ay_c^2$ where $I_{\xi\xi}$ is the second moment of area about the centroidal axis ξ parallel to the x axis (see Fig. 2.8). Hence

$$p_c A y' = p_s A y_c + \gamma \sin\theta (I_{\xi\xi} + Ay_c^2)$$

Noting that $\gamma y_c \sin\theta + p_s = p_c$, we may rewrite the right side of the above equation as follows as you may yourself verify:

$$p_c A y' = p_c A y_c + \gamma \sin\theta I_{\xi\xi}$$

Rearranging terms we arrive at the desired formulation, namely:

$$\boxed{y' - y_c = \frac{\gamma \sin\theta I_{\xi\xi}}{p_c A}} \tag{2.18}$$

where again p_c is the *total absolute pressure* at the centroid of the area.

The position of the point of application of the resultant force on the submerged surface is called the *center of pressure.* Since the terms of the right side of the above equation are positive, we see that the center of pressure will always be at a *lower* depth than the centroid.

Next, we investigate the *lateral position* of the resultant. For clarity, the normal view A-A of Fig. 2.8 is shown again in Fig. 2.9. The center of pressure is shown at position y', as determined previously, and at an unknown distance x' from the y axis. Equating the moment about the y axis of the resultant force with the corresponding moment from the pressure distributions, we get

$$(F_R)x' = \int_A x[p_s + \gamma h]\, dA = p_s \int_A x\, dA + \gamma \sin\theta \int_A xy\, dA$$

Replacing F_R, and noting the appearance of the first moment about the y axis and the product of area about the xy axes, we get

$$p_c A x' = p_s A x_c + \gamma \sin\theta I_{xy} \tag{2.19}$$

Next, consider the centroidal reference $\xi\eta$ parallel to the xy reference. The transfer formula for products of area between the xy and $\xi\eta$ axes is

$$I_{xy} = I_{\xi\eta} + A x_c y_c \tag{2.20}$$

and, introducing this into Eq. 2.19, we get

$$\begin{aligned} p_c A x' &= p_s A x_c + \gamma \sin\theta (I_{\xi\eta} + A x_c y_c) \\ \therefore\ p_c A x' &= p_s A x_c + \gamma \sin\theta I_{\xi\eta} + \gamma \sin\theta A x_c y_c \end{aligned} \tag{2.21}$$

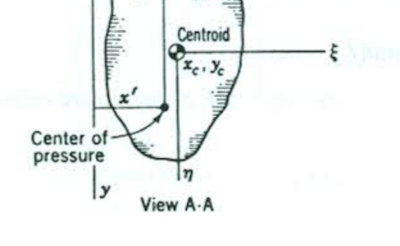


Figure 2.9
Normal view of a plane surface.

Noting once again that $(\gamma \sin\theta\, y_c + p_s) = p_c$, we can rewrite the above equation as follows:

$$p_c A x' = p_c A x_c + \gamma \sin\theta I_{\xi\eta}$$

We can now give the desired result

$$\boxed{x' - x_c = \frac{\gamma \sin\theta I_{\xi\eta}}{p_c A}} \qquad [2.22]$$

It must be remembered that $I_{\xi\eta}$ is the product of area about those centroidal axes which are parallel and perpendicular, respectively, to the trace of the plane of the area with the free surface.

2.7 PROBLEMS INVOLVING FORCES ON PLANE SURFACES

In many problems the pressure on the free surface of the liquid in contact with the submerged surface is that of the atmosphere, i.e., p_{atm} and, furthermore, the reverse side of the plate under consideration is exposed only to the atmospheric pressure as shown in Fig. 2.10. For those cases, the combined force on *both* faces of the plate will then be solely due to the action of the uniformly increasing pressure.

There is also the possibility that there may be several layers of immiscible liquids resting on top of the liquid which is in contact with the submerged surface of interest to us (see Fig. 2.11). To get the force on door AB from the liquids, we recommend that you get the pressure at the centroid of the door as follows:

$$p_c = p_o + \gamma_1 h_1 + \gamma_2 h_2 + \gamma_3 (h_3)_c$$

where $(h_3)_c$ is the height of the liquid that wets AB above the centroid of AB. We can then use Eqs. 2.17 and 2.18 to get the force and its line of action acting on the wetted part of the door. To get the *total* force inside and outside, use *gage* pressure for p_o.

As a further aid in solving problems we wish to encourage the student not to let geometric complexity obscure the basic simplicity of hydrostatics. For instance, consider the rather ominous geometry in Fig. 2.12 where a high gage pressure p_1 of air is shown acting on the free surface. Assuming the geometry is known, how do you proceed to simplify the problem so that the simple rules of hydrostatics can be used to find the force on door AB?

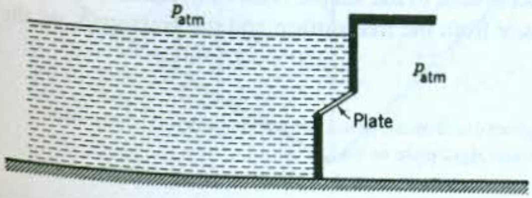


Figure 2.10
Atmospheric pressure on the dry side of the plate and at the free surface.

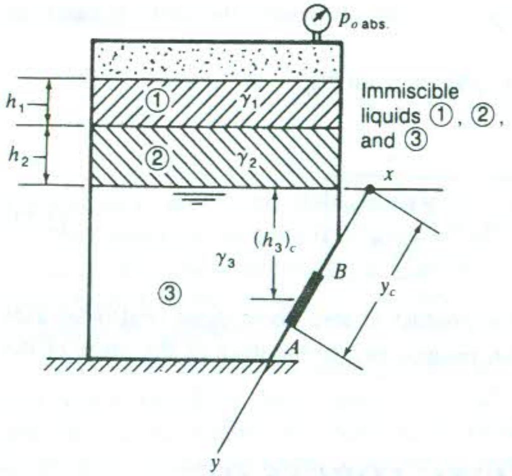


Figure 2.11
Liquids ① and ② rest on liquid ③ which wets the surface of door *AB*.

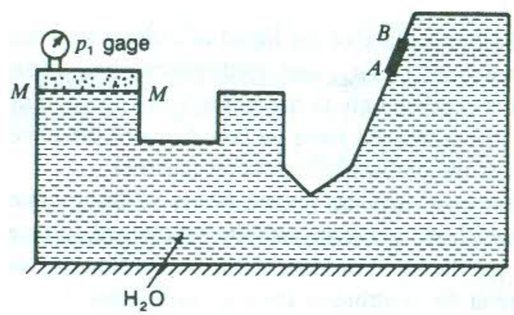


Figure 2.12
Complex geometry involving a hydrostatic problem.

In short, redraw the diagram as has been shown in Fig. 2.13. Note that it is p_1 and the vertical distance *from the free surface* that dictate pressure *anywhere* in the liquid[5] and this diagram shows the principal factors, namely p_1, the free surface, and the vertical distance h of the centroid of the door relative to this free surface. The rest of the geometry, part of which is shown dashed, is of no significance here and can be disregarded however complicated it may be. Now we have the simple case of a plane surface *AB* submerged in a liquid with free surface *MM*. All we need do is find the hydrostatic force on the right-hand face of the door from p_1 gage on the free surface and from gravity on the water measuring distance h *normal* to the free surface to the centroid of the submerged area. This is a most simple problem. You are encouraged to reduce your problems so as to make solutions accessible to the simple rules of hydrostatics, always keeping in mind that *vertical distance* from the free surface and the pressure p_1 on the

[5]The liquid must be in free contact throughout the domain where we use this rule—i.e., one part cannot be "sealed" from another part by some rigid plate or body.

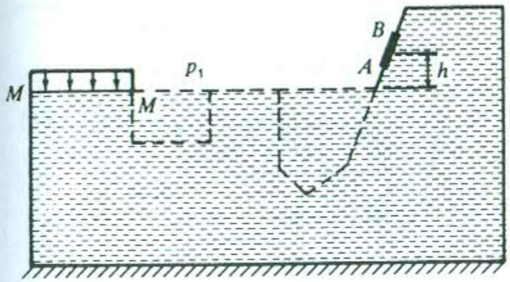


Figure 2.13
Simplified geometry noting the key factors p_1 gage and h.

free surface are the controlling factors and *not boundary geometry* which is true when the liquid is freely self-connected in the domain of interest.

As a final aid in attacking hydrostatic problems, consider the tank with two different liquids touching a door as shown in Fig. 2.14. The top of the tank is closed to the atmosphere, and we wish to find the *total* force on this door. You must observe caution in this undertaking.

First look at portion ① of the door. This is handled by our simple force formulas using p_1 gage at the free surface of the oil, as well as gravity on the oil, to find a force F_1 at the center of pressure of area ①. Now go to area ② of the door. Employing p_1 gage and the gravitational action on the oil, find the gage pressure acting on the free surface of the water. With this pressure and the uniformly increasing pressure from gravity on the water, find the force F_2 and the center of

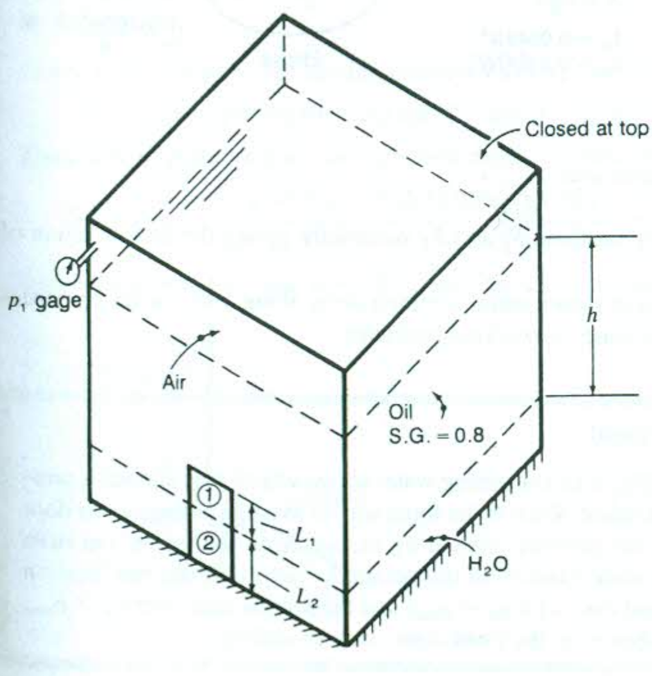


Figure 2.14
Two liquids act on door.

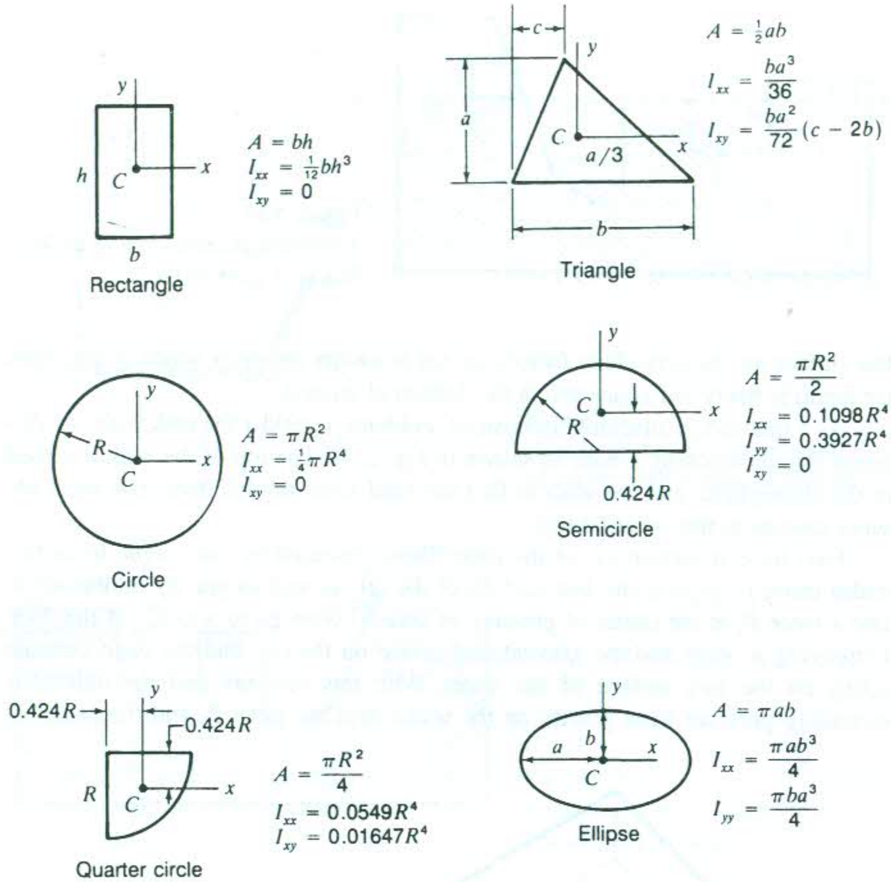


Figure 2.15
Properties of some common areas.

pressure for F_2. Finally combine F_1 and F_2 vectorially giving the line of action of the combined force.

In Fig. 2.15 we have shown some common areas along with useful properties for the students' convenience in working problems.

EXAMPLE 2.5

■ Problem Statement

A tank is shown in Fig. 2.16 containing water above whose free surface a pressure p_A may be established. What is the force and its location acting on the door AB stemming from the pressure exerted by the water on the inside and from the atmospheric pressure exerted on the outside? Find the force and location for two cases: the first case with $p_A = p_{\text{atm}}$, and the second case with $p_A > p_{\text{atm}}$. Note that we have shown, in the front view, axes xy and $\xi\eta$.

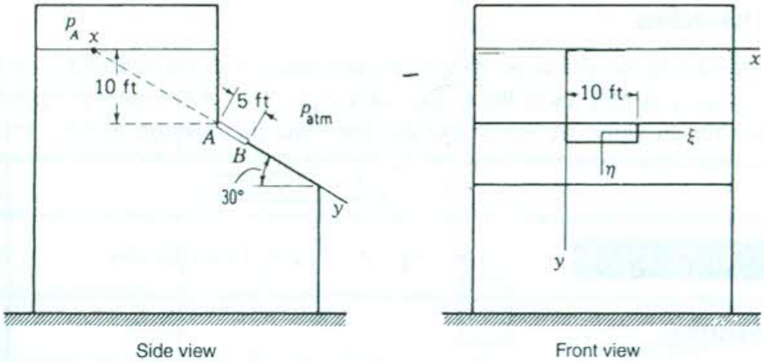


Figure 2.16
Find force on door AB.

CASE 1. $p_A = p_{atm}$

■ Strategy

In this case the effect of pressure p_A on the inside surface of the door is completely counteracted by the atmospheric pressure on the outside of the door. The resultant force from pressures on the door can then be completely determined by the uniformly increasing pressure distribution from gravitational influence on the door.

■ Execution

Going to the centroid, we use the pressure there as follows:

$$F_R = p_c A = (62.4)(10 + 2.5 \sin 30°)(50) = (702)(50) = 35{,}100 \text{ lb}$$

The center of pressure is located at a distance $y' - y_c$ beyond the centroid. Thus

$$y' - y_c = \frac{\gamma \sin\theta I_{\xi\xi}}{p_c A} = \frac{(62.4)(0.5)(\frac{1}{12})(10)(5^3)}{(702)(50)} = 0.0926 \text{ ft}$$

wherein you will note that $\theta = 150°$. See Fig. 2.17 for results.

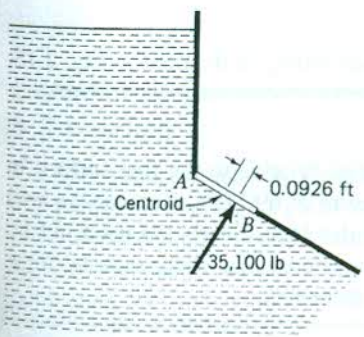


Figure 2.17
Resultant force for case $p_A = p_{atm}$.

■ Debriefing

In considering the lateral position of the resultant force, we note that $I_{\xi\eta}$ is zero owing to symmetry about the η axis, so that $x' = x_c$. In computing supporting forces for the door, we would use the value and the position of the resultant.

CASE 2. $p_A > p_{atm}$ $\quad p_A = 18.20$ psi abs

■ Strategy

In this case, we need only use the gage pressure of p_A namely $(18.20 - 14.70) = 3.50$ psi in computing the force contribution from the uniform pressures. And for the uniformly increasing pressure from gravitational influence, we go to the centroid and get this pressure there. Finally, we can add the pressures.

■ Execution

The computation is straightforward, that is,

$$\begin{aligned} F_R &= [(3.50)(144) + (10 + 2.5\sin 30^\circ)(62.4)]50 \\ &= (1206)(50) = 60{,}300 \text{ lb} \end{aligned}$$

Also

$$y' - y_c = \frac{\gamma \sin\theta I_{\xi\xi}}{p_c A} = \frac{(62.4)(0.5)(\frac{1}{12})(10)(5^3)}{(1206)(50)} = 0.0539 \text{ ft}$$

■ Debriefing

This example sets the stage for problems 2.6 and 2.7 to follow.

EXAMPLE 2.6

■ Problem Statement

Determine the force and its position from fluids acting on the door in Fig. 2.18.

■ Strategy

We will focus on the position and its inclination relative to the free surface in the left-hand tank where we have a gage pressure of 690 kPa. Because we will be using gage pressure for the inside face of the door, clearly we need not be concerned with atmospheric pressure on the outside surface of the door because of the cancellation of atmospheric forces on these faces.

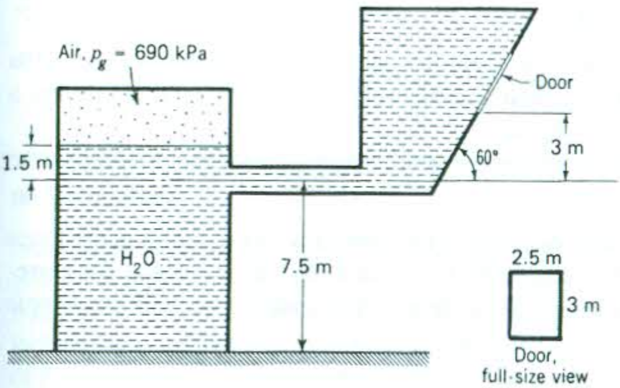


Figure 2.18
Force on a door.

■ Execution

We will compute the pressure at the centroid of the door from the water and add to this the gage pressure exerted on the free surface. Then, multiplying by the area, we will get the value of the resultant force on the door from air and water. Thus,

$$F_R = [690{,}000 + (9806)(1.5 - 3 - 1.5\sin 60°)][(2.5)(3)]$$

We get

$$F = 4.969 \times 10^6 \text{ N}$$

Clearly F is normal to the door. We next get the distance of the position of F below the centroid of the door measured along the inclined centerline of the door. This will give us the position of the center of pressure for the door. Thus, noting in Eq. 2.18 that θ is the angle between the wetted surface of the door and the free surface of the water, (60°) we have

$$y' - y_c = \frac{\gamma \sin\theta I_{\xi\xi}}{p_c A} = \frac{(9806)(0.866)\left(\frac{1}{12}\right)(2.5)(3^3)}{(6.625 \times 10^5)(2.5)(3)}$$

We get

$$y' - y_c = 0.00961 \text{ m}$$

■ Debriefing

This is a good example to illustrate the vital role played by the free surface in computing pressures on a plane surface in a complicated geometry in which the plane surface is not "sealed" from the free surface. Furthermore, note that p_c in Eq. 2.18 can be the result of uniform pressure on the surface plus a uniformly increasing pressure on the surface.

EXAMPLE 2.7

■ Problem Statement

At what height h will the static water just start to rotate the door in Fig. 2.19a clockwise if we neglect friction and the weight of the door? The door has a width of 3 m. Take $\theta = 60°$.

■ Strategy

We will draw a free body diagram of the door as we have drawn many times in statics. We will then determine the force from the water and the corresponding center of pressure, all in terms of h. Finally, we will use simple statics.

■ Execution

The free body diagram of the door is shown in Fig. 2.19b. The force F from fluids includes only that which is coming from the uniformly increasing gravitational pressure in the water, with atmospheric pressure canceling on the two sides of the door. This force is computed as

$$F = p_c A_{\text{door}} = \left[\frac{h}{2}(9806)\right]\left[(3)\left(\frac{h}{0.866}\right)\right] = 16{,}985h^2\ \text{N} = 16.985h^2\ \text{kN}$$

The center of pressure at y' measured from point O is next computed.

$$y' = y_c + \frac{\gamma \sin\theta I_{\xi\xi}}{p_c A} = \frac{1}{2}\left(\frac{h}{0.866}\right) + \frac{(9806)(0.866)\left(\frac{1}{12}\right)(3)\left(\frac{h}{0.866}\right)^3}{(4{,}903h)(3)\left(\frac{h}{0.866}\right)}$$

Hence

$$y' = 0.770h$$

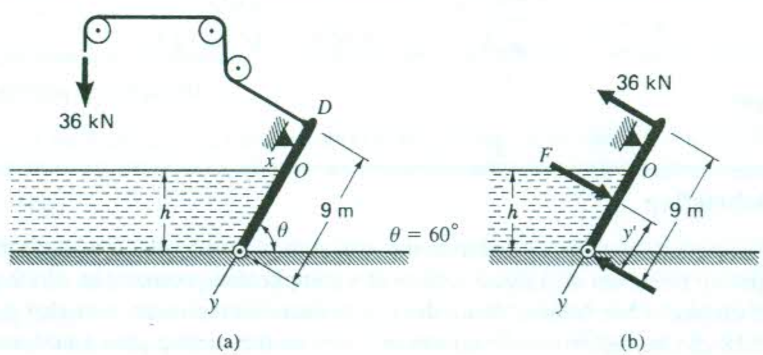


Figure 2.19
Swinging gate.

Now setting the moment about the base equal to zero, we get

$$(36)(9) - (16.985h^2)\left(\frac{h}{0.866} - 0.770h\right) = 0$$

$$h^3 = 49.58 \text{ m}^3 \qquad h = 3.67 \text{ m}$$

■ Debriefing

Notice that the stop at D in Fig. 2.19a does not appear in Fig. 2.19b. This signifies that the door is just about to move but is still touching the stop. Note that both the value and the position of F were functions of h. How would you now include the weight of the door? In Computer Example 2.2, we will plot a curve of F versus angle θ. We will use MATLAB. The student may wish to use some other software. The 36 kN force will then always be vertical at the gate.

2.8 HYDROSTATIC FORCE ON CURVED SUBMERGED SURFACES

We will now show that forces on curved surfaces submerged in any static fluid can be partially determined by methods used on plane surfaces, as presented in Sec. 2.7.

A curved surface is shown in Fig. 2.20 submerged in a static fluid. The force on any area element dA of this surface is directed along the normal to the area element and is given as

$$\mathbf{dF} = -p\,\mathbf{dA}$$

where the convention of taking $\mathbf{dA}$ pointing outward from the surface has been observed. Taking the dot product of each side of the equation above with the unit vector $\mathbf{i}$, we get the component dF_x on the left side. That is,

$$dF_x = -p\,\mathbf{dA} \cdot \mathbf{i}$$

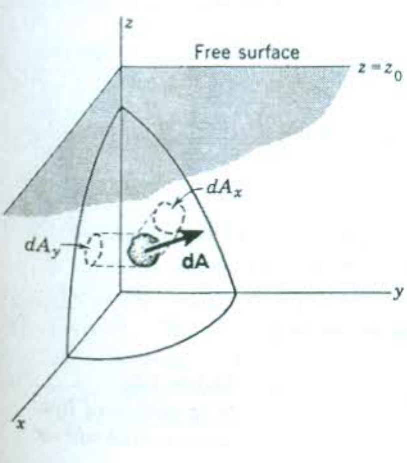


Figure 2.20
Area **dA** on curved submerged surface.

But $\mathbf{dA} \cdot \mathbf{i}$ is actually the projection of the area element onto plane yz yielding dA_x (see Fig. 2.20). To get F_x, we have

$$F_x = -\int_{S_x} p\,dA_x$$

wherein the limit of integration S_x is the projection of the curved surface onto the zy plane (or any other plane perpendicular to the x axis). The problem of finding F_x now becomes the problem of finding the force on a plane submerged surface oriented *perpendicular* to the free surface. We can accordingly use all the techniques set forth earlier for this problem. Similarly, we have for F_y

$$F_y = -\int_{S_y} p\,dA_y$$

where S_y is the projection of the curved surface onto the zx plane (or any plane perpendicular to the y axis). Therefore, two orthogonal components of the resultant force can be determined by methods of submerged plane surfaces. Note that these components are *parallel* to the free surface.

Now let us consider the component *normal* to the free surface. We note that pressure p from *gravitational action* on the fluid at a point on the curved surface is $\int \gamma\,dz$, with limits between z' on the curved surface and z_0 at the free surface (see Fig. 2.21). We can then say that

$$\begin{aligned}
\mathbf{dF} &= -p\,\mathbf{dA} \\
\therefore\ dF_z &= -p\,\mathbf{dA} \cdot \mathbf{k} = -p\,dA_z \\
&= -\left(\int_{z'}^{z_0} \gamma\,dz\right) dA_z \\
&= -\int_{z'}^{z_0} \gamma\,dz\,dA_z
\end{aligned}$$

Note next in Fig. 2.21 that $\gamma\,dz\,dA_z$ is the weight of an infinitesimal element of fluid in the prismatic column of fluid directly above dA of the curved surface.

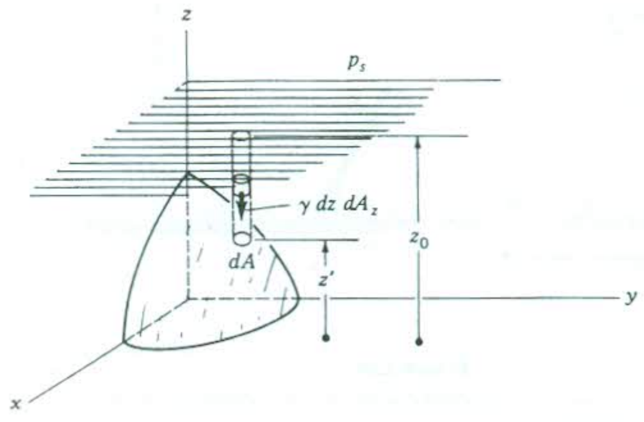


Figure 2.21
Note column of fluid above curved surface.

This column extends to the free surface above. Integrating this quantity from z' to z_0 as we have done above, we obtain from dF_z the *weight* of the column of fluid directly above dA. Clearly, when integrating dF_z over the entire curved surface we get for F_z simply the *weight of the total prismatic column of fluid directly above the curved surface*. The minus sign indicates that a curved surface with a positive dA_z projection (top side of an object) is subjected to a negative force in the z direction (down). It may be shown (see homework Prob. 2.53) that this force component has a line of action through the center of gravity of the prism of fluid "resting" on the surface.

To account for a pressure p_s on the *free surface*, we need only multiply the projected area of the curved surface, as seen from above, by p_s. Then we add this force to the weight of the column of liquid above the submerged free surface on up to the free surface. We have now formulated the means to determine orthogonal components of the resultant force on the submerged curved surface. These force components give the equivalent action in these directions of the entire surface-force distribution from the fluid on the curved surface. Their lines of action will not necessarily coincide (which means that the simplest resultant system of a curved submerged surface may not be a single force). However, in practical problems it is the components of force in directions parallel and normal to the free surface that are of greatest use.

Note that the conclusions of this section are in no way restricted to incompressible fluids. They are valid for any fluid.

2.9 EXAMPLES OF HYDROSTATIC FORCE ON CURVED SUBMERGED SURFACES

We will now consider examples that illustrate the formulations of the earlier sections. For simplicity, the fluid in the problems is incompressible.

EXAMPLE 2.8

■ Problem Statement

Find the vertical and horizontal force on the quarter circular door AB shown in Fig. 2.22 from water on one side and air on the other side. The door is 2 m in length.

■ Strategy

For the horizontal force F_x, we project the curved surface AB onto a plane parallel to the yz plane. This projected area is a 1 by 2 m rectangle shown on edge as OB in Fig. 2.22. We may now use the formulation for a plane surface to determine F_x. The atmospheric pressure on the free surface clearly develops a horizontal force on the left side of the door AB that is canceled completely by the horizontal force from the atmosphere on the right side of the door.

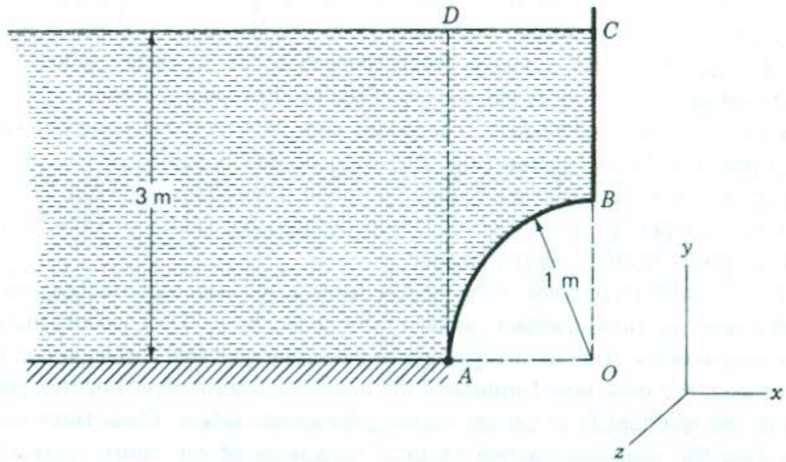


Figure 2.22
Find hydrostatic force components on door AB.

■ Execution

We need only be concerned with pressure coming from gravitational effect on the water. We then have

$$F_x = p_c A = [(9806)(2.5)](1)(2) = 49{,}030 \text{ N}$$
$$= 49.0 \text{ kN}$$

Clearly, the force component in the z direction is zero because the projected area in this direction is zero.

As for the vertical component, we need only consider the weight of the column of water directly above the door AB. We thus have (see Fig. 2.22)

$$F_y = (9806)[\underbrace{(3)(1)(2)}_{\substack{\text{Volume having} \\ \text{face } AOCD}} - \underbrace{\tfrac{1}{4}\pi(1^2)(2)}_{\substack{\text{Volume having} \\ \text{quarter-circle} \\ \text{face}}}]$$

$$\therefore F_y = 43{,}400 \text{ N} = 43.4 \text{ kN}$$

The resultant force is then

$$F_R = \sqrt{43.4^2 + 49.0^2} = 65.46 \text{ kN}$$

and is at an angle of 45° from AO (why?).

■ Debriefing

To better understand the projecting of a curved surface as we have done in this example, simply make believe you are looking at the curved surface in a direction which is normal to the plane onto which you are projecting the surface. What you would see via this procedure is the desired projected area.

EXAMPLE 2.9

■ Problem Statement

Determine the hydrostatic force components acting on the semiconical surface shown in Fig. 2.23a.

■ Strategy

For the horizontal force component F_x, we project the semiconical surface onto a plane which is perpendicular to the x axis. This projected area is an isosceles triangle of height a and base $2r$ as shown in Fig. 2.23b. Furthermore, by the same reasoning, the force component F_z is zero because there is zero projected area of the semiconical surface in the z direction.

For computing the vertical force component due to the uniformly increasing gravitational pressure, we will imagine that the semiconical surface is cut from the semicone at the free surface and is at the same time detached from the semicone. The resulting surface, do not forget, has zero thickness. It is then completely submerged in the liquid at a position shown in Fig. 2.23c. We will first concentrate on the inside part of this surface to determine the downward force from the fluid "supported" by this surface. This will then give us the negative of the vertical upward force on the outside semiconical surface.

Then, we will turn our attention to the contribution of the atmospheric pressure acting at the free surface. Using the projected area of the inside semiconical surface, this time in the vertical direction, we easily have the upward force from atmospheric pressure on the outside semiconical surface.

■ Execution

For the force component F_x, we first get the total pressure at the centroid of the triangle of Fig. 2.23b including the atmospheric pressure at the free surface which extends throughout the fluid. Thus

$$F_x = \left[p_{\text{atm}} + \gamma\left(\frac{1}{3}a\right)\right]\frac{1}{2}(2r)(a) = \gamma\left[\frac{p_{\text{atm}}}{\gamma} + \frac{a}{3}\right](ra)$$

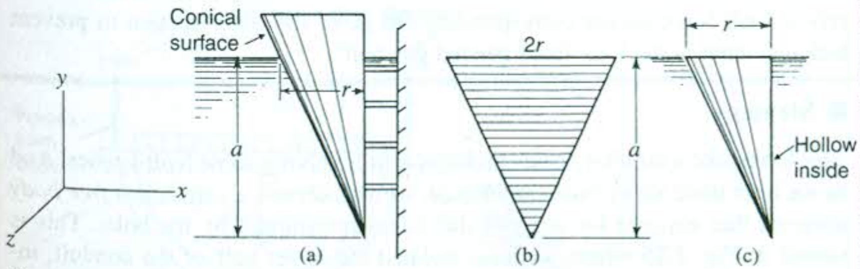


Figure 2.23
(a) Submerged semicone. (b) Projection of wetted surface. (c) Water above hypothetical conical shell.

Next, the downward force on the inside portion of the detached and submerged semiconical surface is the weight of the prismatic column of fluid above it and extending to the free surface. The upward force $(F_y)_1$ we seek is equal and opposite to this force. Thus

$$(F_y)_1 = \gamma\left(\frac{1}{2}\right)(\text{volume of cone}) = \gamma\left(\frac{1}{2}\right)\left[\frac{1}{3}\pi r^3 a\right] = \gamma\left(\frac{1}{6}\pi r^3 a\right)$$

Finally, there is a vertical force on the semiconical surface from p_{atm}. Using the vertical projected area of the semiconical surface, we have

$$(F_{y_2}) = p_{\text{atm}}(\text{semicircular area}) = p_{\text{atm}}\frac{\pi r^2}{2}$$

The total *upward* force is then

$$(F_y)_{\text{total}} = \gamma\left[\frac{1}{6}\pi r^3 a + \frac{p_{\text{atm}}}{2\gamma}\pi r^2\right]$$

■ Debriefing

We have determined hydrostatic force components on a curved, submerged surface in a liquid using some simple ingenuity. That is, we have considered one face of the surface hypothetically submerged in the liquid and "supporting" a column of liquid above it up to the free surface. We did this to get the desired vertical force component on the opposite face of the curved submerged surface. We will be able, at times, to profitably use the approach of this example on other curved surfaces.

EXAMPLE 2.10

■ Problem Statement

A large conduit is supported from above and conducts water and oil under pressure (see Fig. 2.24). It is constructed from two semicylindrical sections bolted together. Each section weighs 4 kN/m and has a length of 6 m. If 100 bolts clamp the sections together such that each of the two flanges of one section exerts a 3 kN force on the corresponding flange of the other section to prevent leakage, what is the total force needed per bolt?

■ Strategy

This looks like a familiar statics problem now involving some hydrostatics. And as we have done many times in the past, we will choose a convenient *free body diagram* that exposes for analysis the forces transmitted by the bolts. This is shown in Fig. 2.25 where we have isolated the lower half of the conduit, including the water in this portion of the conduit. A key step will be to compute the gage pressure $(p_a)_{\text{gage}}$ on surface *AA*. Then, using a simple rigid body equation of equilibrium, we can get the desired result.

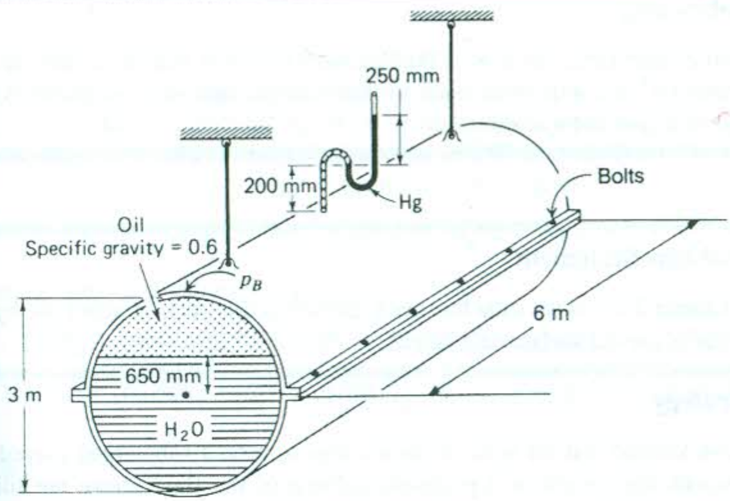


Figure 2.24
Large conduit containing water and oil.

■ Execution

For simple *hydrostatics* we have on starting from the top of the mercury column in Fig. 2.24:

$$(p_a)_{\text{gage}} = (13.6)(9806)(0.250) + (0.6)(9806)(0.2 + 1.5 - 0.650)$$
$$+ (9806)(0.650) = 45{,}892 \text{ Pa}$$

Now summing forces in the vertical direction, we have from *equilibrium*

$$(100)F_{\text{bolt}} - (45{,}892)(3)(6) - (4000)(6) - 6000 - \frac{1}{2}\frac{\pi(3^2)}{4}(6)(9806) = 0$$
$$\therefore F_{\text{bolt}} = 10.64 \text{ kN}$$

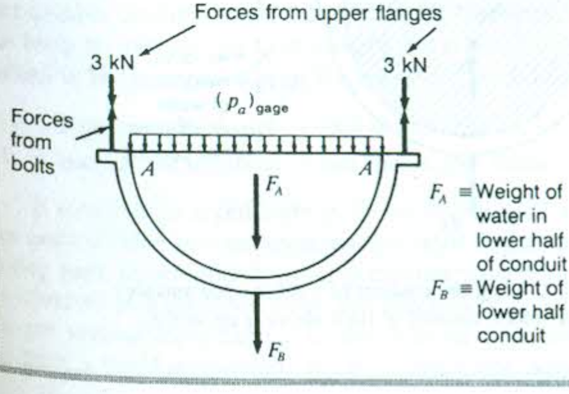


Figure 2.25
Free body diagram I.

■ Debriefing

The procedure taken here is a familiar and effective one from the past. In Example 2.11 we will come back to this example and use our knowledge of curved surfaces and hydrostatics.

EXAMPLE 2.11

■ Problem Statement

Do Example 2.10 using what has been learned in this section about the hydrostatics of a curved surface in a liquid.

■ Strategy

We have learned that the vertical force component on a submerged curved surface equals the weight of a prismatic column of the fluid above the surface where this column goes up to the free surface. The cross section of this column is identical to the projection in the vertical direction of the curved surface. The curved surface we will consider here is the inside face of the bottom semicylinder, and the free surface is the top of the mercury column in the U-tube. Finally, we will form a free body and use equilibrium to relate the weight of the column with the other forces acting on the metal semicylindrical section, including forces from the bolts. Note that the net force from atmospheric pressure is zero.

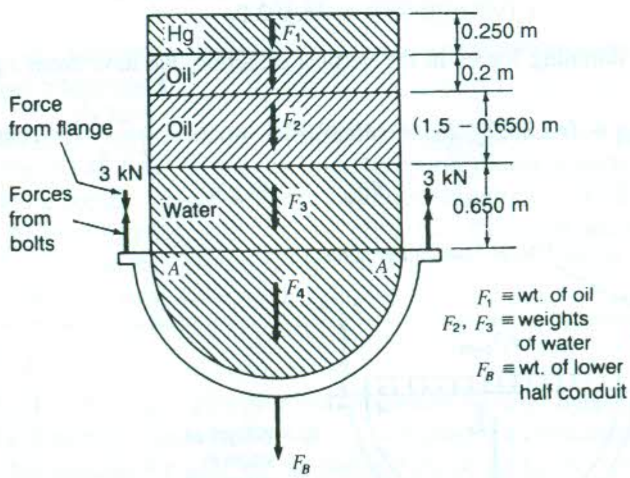


Figure 2.26
Free body diagram II using the concept of a submerged surface *AA* supporting a prismatic column of fluid above it up to the free surface.

■ Execution

In Fig. 2.26, we have shown the uniform prismatic column rising above the curved surface. This column includes regions of water, oil, and mercury. We have inserted all the forces acting on the portions of the column and also those forces acting on the metal semicylindrical section. We thereby form a free body diagram. Summing forces in the vertical direction we have

$$100F_{\text{bolt}} - F_1 - F_2 - F_3 - F_4 - F_B - (2)(3000) = 0$$

Inserting numerical values, we get

$$100F_B - (9806)[(13.6)(0.250) + (0.6)(0.2) + (0.6)(1.5 - 0.650) + (0.650)](3)(6) - (9806)\frac{\pi 3^2}{4}(6) - (4000)(6) - (2)(3000) = 0$$

Hence,

$$F_b = 10.64 \text{ kN}$$

■ Debriefing

We leave it to the student to decide which calculation was easiest, that of Example 2.10 or 2.11. In any case, we have shown here once again the idea of a submerged curved surface "supporting" a column of fluid above it to the free surface in a hydrostatic problem.

2.10 LAWS OF BUOYANCY

The buoyant force on a body is defined as the net vertical force that stems from the fluid or fluids in contact with the body. A body in *flotation* is in contact only with fluids, and the surface force from the fluids is in equilibrium with the force of gravity on the body. To ascertain the buoyant force on bodies, both in flotation and subject to other conditions, we merely compute the net vertical force on the surface of the body by methods we have already discussed. Thus no new formulations are involved in buoyancy problems. We consider the cases of:

1. A body completely submerged in one fluid
2. A body at the interface of two immiscible fluids

A totally submerged body is shown in Fig. 2.27 for case 1. The body surface has been divided into an upper portion *AUB* and a lower portion *ALB* along a dividing path, shown dotted, forming the outermost periphery of the body as seen by an observer looking down along the direction of gravity. The buoyant force is then the net vertical force exerted by the fluid on these surfaces. Assume for now that we have a liquid surrounding the body and a free surface.

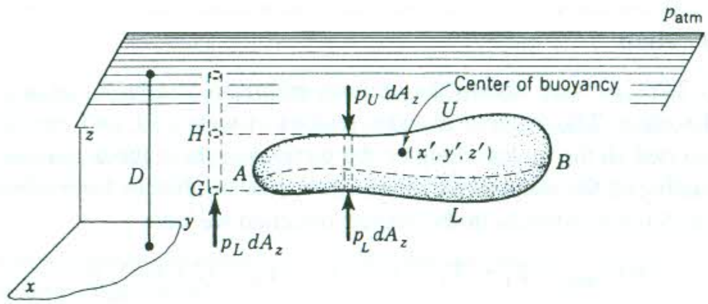


Figure 2.27
Submerged body in a fluid.

Note first that a pressure p_{atm} on the *free surface* will yield a uniform pressure p_{atm} throughout the fluid below the free surface. Clearly this will yield a zero contribution to the buoyant force, and hence we will disregard pressure p_{atm} on the free surface for the computation of buoyant forces. To determine the buoyant force, consider an infinitesimal vertical column *in the body* with cross-sectional area dA_z.

At the top of this column the vertical force, shown as $p_U\,dA_z$, equals the weight of the column of fluid above the upper boundary reaching up to the free surface and having the same cross-sectional area dA_z. At the lower end, the pressure p_L will be the same as that at the bottom of a column of fluid shown to the left where the bottom of this column is at the *same elevation* as the bottom of the column in the submerged body. If dA_z is the same for both columns, the vertical force at the bottom of the one at the left must then be the same as that on the bottom of the column in the body to the right. But the vertical force at the bottom of the left column is simply the weight of the column of fluid up to the free surface. Hence the difference between the upper force $p_U\,dA_z$ and the lower force $p_L\,dA_z$ on the body is the weight of a column of fluid GH having the same size and elevation as the column in the solid body. It is clear that considering all columns in the submerged body the *net upward force is then the weight of fluid displaced*—the familiar *Archimedes' principle.* Note that there is no restriction on compressibility involved in the development of this principle, so it is valid for liquids and gases.

We now consider the submerged body in Fig. 2.27 to be composed of infinitesimal vertical prisms, one of which is shown in the diagram. The net force on the prism is

$$dF_B = (p_L - p_U)\,dA_z$$

If we restrict ourselves to an *incompressible* fluid, we can say, using the height D of the free surface, that

$$dF_B = [(D - z_L)\gamma - (D - z_U)\gamma]\,dA_z = \gamma(z_U - z_L)\,dA_z$$

where z_L and z_U are the elevations, respectively, of the lower end of the prism and the upper end of the prism. Integrating throughout the entire body, we then get the buoyant force

$$F_B = \gamma \int (z_U - z_L)\, dA_z = \gamma V$$

where V is the volume of the submerged body. We thus verify, for the incompressible fluid, the general Archimedes principle which was presented earlier for any fluid.

We next determine the *center of buoyancy,* which is the position in space where the buoyant force may be considered to act. To find the center of buoyancy for this case, we equate the moment of the resultant force F_B about the y axis with that of the pressure distribution of the enveloping fluid. Thus

$$F_B x' = \gamma \int x(z_U - z_L)\, dA_z = \gamma \int_V x\, dv$$

where dv represents the volume of the elemental prism. Replacing F_B by γV and solving for x', we get

$$x' = \frac{\int_V x\, dv}{V} \tag{2.23}$$

We see that x' is the x component of the position vector from xyz to the *centroid* of the volume displaced by the body. We can then conclude by this argument and by similarly taking moments about the x axis that the buoyant force from an incompressible fluid goes through the centroid of the volume of liquid displaced by the body. Clearly this will not be true for a compressible fluid where γ will be some function of z.

Next, examine the case of a body in flotation at the *interface of two immiscible fluids* (Fig. 2.28). This, of course, is the case of every floating vessel, the fluids being water and air. A vertical prism of infinitesimal cross section has been designated in the floating body. The vertical force components on the upper and lower extremities of the prism are denoted dF_1 and dF_2. It is clear that the net vertical force on the prism from the fluids equals the weight of column a of fluid A plus the weight of column b of fluid B. And by integrating these forces so as to encompass the entire body, we see that the buoyant force equals the sum of the weights of the

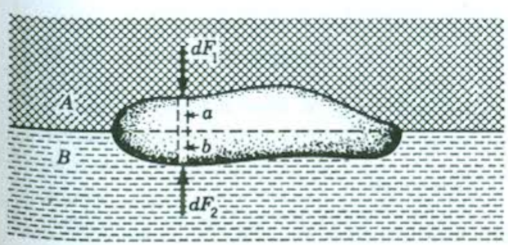


Figure 2.28
Body in flotation at interface of two immiscible fluids.

fluids displaced by the body. Note that with different values of γ present we *cannot* extend our previous argument to state that the buoyant force goes through the centroid of the total volume displaced by the body. However, in nautical work we generally neglect the specific weight of air, and we can consider then that the center of buoyancy is at the centroid of the volume of water displaced by the body.

EXAMPLE 2.12

■ Problem Statement

Shown in Fig. 2.29 is a highly idealized sketch of the bathyscaphe *Trieste,* a device developed by the Swiss physicist Piccard to explore the ocean floor at its greatest depth (11.3 km). A cylindrical tank contains gasoline, which gives the system a buoyant force. A ballast of gravel is attached to the cockpit, which is a steel sphere large enough to house an observer and instruments. The vertical motion of the ship is controlled by dropping ballast to get upward force or by releasing gasoline at A and, at the same time, admitting seawater to B to replace the lost gasoline to get a downward force.

If the cockpit and total remaining structure minus gravel and gasoline weigh 15.50 kN, how much should the gravel ballast weigh for flotation at a depth of 3 km below the free surface? The ballast tank of gravel displaces a volume of 2.85 m^3. Take the specific weight γ of seawater to be 10.150 kN/m^3 at the location of interest and the specific weight of the gasoline to be 0.65 that of the seawater.

■ Strategy

We will be using Archimedes' principle, so the immediate task will be to determine the total volume of the system. In doing so, we will neglect the volume of the supporting structural elements such as rods and wires. This permits us to get the buoyant force and thus to use equilibrium to arrive at the desired weight of the gravel ballast.

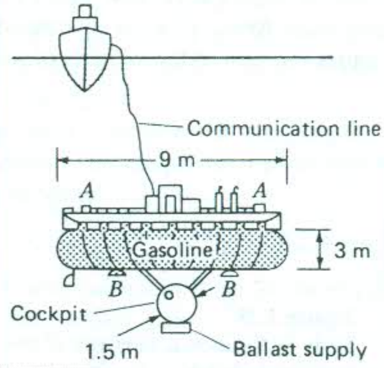


Figure 2.29
The bathyscaphe *Trieste* built by Piccard.

■ Execution

We will compute the volume of the system. The volume of the gasoline and the cockpit are, respectively (see Fig. 2.29),

$$V_{gas} = \frac{\pi(3^2)}{4}(9) = 63.6 \text{ m}^3$$

$$V_{cockpit} = \frac{4}{3}\pi\left(\frac{1.5}{2}\right)^3 = 1.767 \text{ m}^3$$

The total volume of displaced seawater then is

$$V_{total} = V_{gas} + V_{cockpit} + V_{ballast} = 63.6 + 1.767 + 2.85 = 68.2 \text{ m}^3 \quad \text{[a]}$$

where we have neglected volume displaced by supporting structural elements. The buoyant force according to the Archimedes' principle is then

$$F_{buoy} = (V_{total})(\gamma) = (68.2)(10.150) = 692 \text{ kN}$$

If W_B is the weight of the gravel ballast, W_{gas} the weight of the gasoline, and 15.50 kN the weight of the entire structure without gravel and gasoline, then *equilibrium* requires for a full tank of gasoline that

$$-W_B - W_{gas} - 15.50 + F_{buoy} = 0$$

$$\therefore W_B = 692 - 15.50 - (63.6)(0.65)(10.150)$$

$$= 257 \text{ kN}$$

The gravel ballast weight must be 257 kN to maintain neutral buoyancy.

■ Debriefing

In recent times many designs of underwater vehicles such as the *Trieste* have been used for examining sunken ships of historical interest such as the *Titanic.* And, of course, there are also treasure hunters and naturalists who use this kind of craft. You may find it interesting to read Piccard's book *Seven Miles Down: The Story of the Bathyscaphe TRIESTE,* describing the construction of the *Trieste* and the successful descent to the bottom of the Pacific Ocean at a location off Japan where the deepest ocean depth in the world is found. There, Piccard discovered the existence of marine life despite the tremendous pressure. The book describes fish that look like a cross between a jelly fish and a catfish.

EXAMPLE 2.13

■ Problem Statement

An empty bucket with its wall thickness and weight considered negligible is forced open-end first into water to a depth E (Fig. 2.30). What is the force F required to maintain this position, assuming the trapped air remains constant in temperature during the entire action?

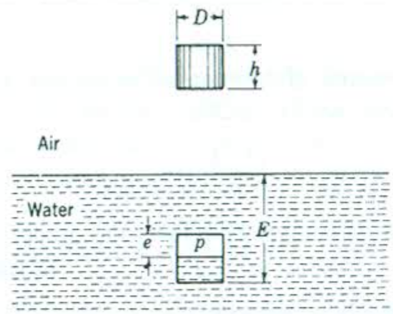


Figure 2.30
Empty bucket forced into water.

■ Strategy

We will use the buoyant force of the trapped air in the bucket. We will assume that this trapped air has been compressed according to the law of isothermal compression of a perfect gas. Finally, we shall employ *hydrostatics,* using the final position of the bucket. This will give us three independent equations for solving the desired force F_B.

■ Execution

It is clear that the buoyant force must equal the weight of the water displaced, whose volume is that of the entrapped air. Therefore,

$$F_B = \gamma e \frac{\pi D^2}{4}$$

We do not know e, so we must examine the action of the air and water in the bucket. By using the isothermal-compression formulation for a *perfect gas* to relate the initial state (atmospheric) and the final state of the entrapped air, the following equation holds:

$$p_{\text{atm}} \frac{\pi D^2}{4} h = p \frac{\pi D^2}{4} e \qquad \text{[b]}$$

This equation introduces another unknown, p. However, p must also be the pressure of the water at the free surface in the bucket, so considering the water now, we can say from hydrostatics that

$$p = p_{\text{atm}} + \gamma[E - (h - e)] \qquad \text{[c]}$$

This gives us a third independent equation by which we can solve for the unknowns F_B, e, and p, a task which we leave to the student. The desired force F is equal and opposite to F_B.

Debriefing

If the bucket is inserted quickly into the water, a better assumption concerning the compression of the trapped air is to consider the compression to be adiabatic (no heat transfer). The formula for this kind of compression is easily derived from the equation of state (Eq. 1.8) presented in Chapter 1.

EXAMPLE 2.14

Problem Statement

A tank is shown in Fig. 2.31. It is hermetically partitioned into two parts containing water and air above and oil below. A closed sphere D is welded to the thin reinforced partition plate EC and extends equally into the water above and the oil below as shown in the diagram. What is the vertical force on the sphere from the fluids?

Strategy

First, we must resist the temptation to use Archimedes' principle here. The sphere clearly is not floating at the interface between two immiscible fluids. To use Archimedes, the fluids must have a common free surface making the pressure a continuous function of height, a condition we do not have here since we can change the pressure in one of the fluids at will by some means without affecting the pressure in the other fluid. We must go back to a fundamental consideration of submerged curved surfaces.

We will use the idea of a curved surface supporting a prismatic column of fluid up to the free surface plus the force from the pressure at the free surface. For the upper half-sphere, this is a direct simple calculation. For the lower half-sphere, we recreate the same condition as above by isolating a half-spherical surface in a body of oil having the same pressure distribution but extending vertically to a hypothetical free surface. Now again we use the

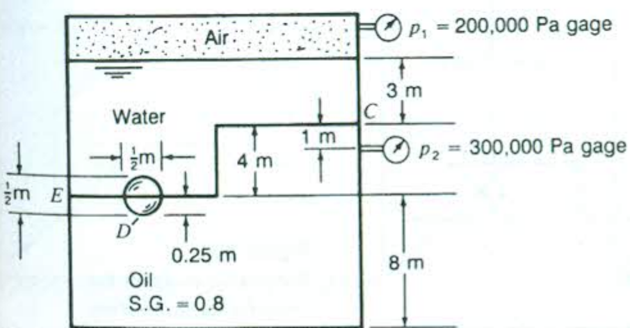


Figure 2.31
A partitioned tank.

concept of the upper face of this semispherical surface supporting a column of oil up to the free surface, and we use the negative of this value for the bottom face of this semisphere for part of the vertical force. By combining the forces from the upper and lower portions of the sphere, we will get the desired result.

■ Execution

First consider the portion of the sphere in the oil. The downward force F_1 is

$$F_1 = (9806)\left[(3+4)\frac{\pi(\frac{1}{2})^2}{4} - \left(\frac{1}{2}\right)\left(\frac{4}{3}\right)(\pi)\left(\frac{1}{4}\right)^3\right] + (200{,}000)\frac{\pi(\frac{1}{2})^2}{4}$$

$$\therefore\ F_1 = 52.43 \text{ kN}$$

Now we go to the portion of the sphere in the oil. Let us find the pressure just below the partition and directly above pressure gage 2. Calling this pressure $(p_C)_{\text{gage}}$ we get

$$(p_C)_{\text{gage}} = 300{,}000 - (0.8)(9806)(1) = 292{,}155 \text{ Pa}$$

Next we will simplify the problem to allow elementary hydrostatics to be used. Thus, imagine there is no partition EC and no water or compressed air above EC. Now raise the level of oil above EC so as to give the above pressure at C. The required height of oil above C for this purpose is

$$h = \frac{292{,}155}{(0.8)(9806)} = 37.24 \text{ m}$$

We next show in Fig. 2.32 a simplified diagram for the lower semispherical surface of interest to us here. We will imagine this body to be a semispherical "cup" with zero wall thickness completely submerged in oil having the free surface shown in the diagram. We will compute the downward force on the

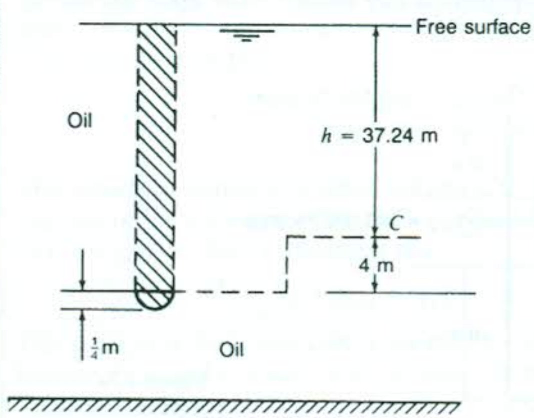


Figure 2.32
Simplified diagram for considering semisphere portion in oil.

inside surface of this cup by computing the weight of oil supported by this cup. Thus we have

$$(F_2)_{\text{down}} = (0.8)(9806)[37.24 + 4]\left(\frac{\pi}{4}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\left(\frac{4}{3}\right)(\pi)\left(\frac{1}{4}\right)^3(0.8)(9806)$$
$$= 63.78 \text{ kN}$$

Clearly then the *upward* force on the *outside* surface of the cup will equal this value. Thus the net upward force on the sphere from water *and* oil is then

$$F_{\text{net}} = 63.78 - 52.43 = 11.35 \text{ kN}$$

■ Debriefing

Note that we have used gage pressures in this problem since the atmospheric pressure yields equal and opposite forces on the upper and lower surfaces of the sphere and hence plays no role.

Before embarking on assigned problems, we wish to point out how Archimedes' principle may further be used. Looking back at Fig. 2.27 notice that the proof of the principle depended on the existence of a distinct upper surface as viewed from above and a distinct lower surface as viewed from below, both having the same outer edge. In Fig. 2.33 we have shown a case where with a moments thought you will agree the aforestated condition exists for a closed tube cantilevered to the tank wall. The tube is submerged in water. Note there is a distinct upper surface as viewed from above and a distinct, identical lower surface as viewed from below. Both

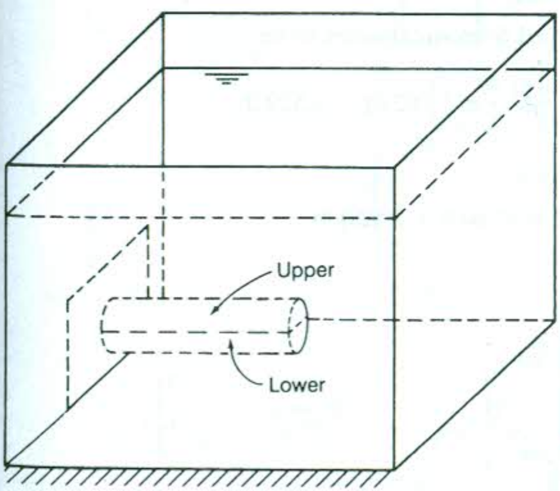


Figure 2.33
Closed tube forming a cantilever with the tank wall.

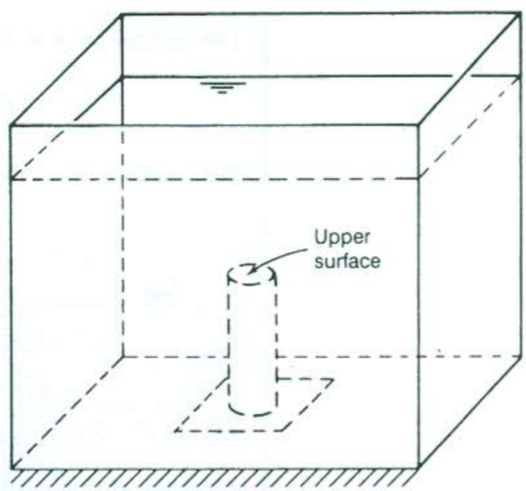


Figure 2.34
Closed tube forming a cantilever with the tank base.

surfaces are in contact only with water extending in free contact from above to below. We can find the buoyant force as the weight of water displaced. In contrast to this case consider Fig. 2.34 showing a closed tube mounted on the base of the tank. It has a distinct upper surface as viewed from above and exposed to water but no such lower surface exposed to water. Here we *cannot* use Archimedes' principle. The vertical force from the water is found by examining the upper surface. We will be able to use this way of looking at submerged bodies in this chapter and indeed other chapters. Note that these conclusions apply also to gases.

EXAMPLE 2.15

■ Problem Statement

Determine the resultant force acting on the hemispherical surface from the fluids acting on it (see Fig. 2.35).

■ Strategy

We will get the horizontal force by projecting the surface in a direction normal to the x axis and computing the force from the uniformly increasing gravitational pressure of the water (the forces from atmospheric pressure cancel). For the vertical force, we can use Archimedes' principle.

■ Execution

The horizontal force component is determined using the projected area in the x direction, which is a plane circle of radius 3 ft. We find the pressure at the centroid and multiply this by the projected area. Thus, we have

$$F_x = (\gamma_{H_2O})(10)(\pi r^2) = 17{,}643 \text{ lb}$$

The vertical force is determined from Archimedes to be

$$(F_y) = \left(\frac{1}{2}\right)\left(\frac{4}{3}\pi r^3\right)(62.4) = 3{,}529 \text{ lb}$$

The resultant force, therefore, is

$$\mathbf{F} = 17{,}643\mathbf{i} + 3{,}529\mathbf{j} \text{ lb}$$

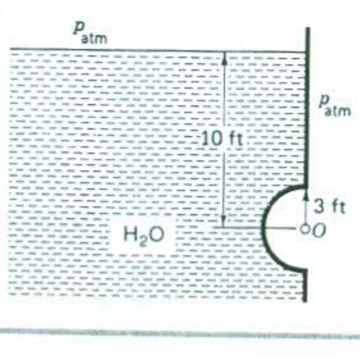


Figure 2.35
Submerged semisphere.

■ **Debriefing**

The resultant force must go through point O since the pressure on the hemispherical surface is everywhere normal to the surface and hence is directed everywhere toward the center O.

*2.11 STABILITY CONSIDERATIONS FOR BODIES IN FLOTATION

If the imposition of a small displacement on a body in equilibrium brings into action forces tending to restore the body to its original position, the system is said to be in *stable* equilibrium. For instance, in the balloon and basket shown in Fig. 2.36 note that a displacement from the normal position (1) brings into action couple Wa, tending to restore the system to the original configuration. The system is thus stable. In general, for completely submerged bodies, as in this example, *stability demands only that the center of gravity of the body be below the center of buoyancy in the normal configuration.* For bodies in flotation at the interface of fluids this requirement is *not* necessary for stability. To illustrate this, observe the vessel shown in Fig. 2.37. Here the weight of the body acts at a point above the center of buoyancy. However, on a "roll," the center of buoyancy *shifts* far enough to develop a righting couple. This explains why a wide rectangular cross section provides a highly

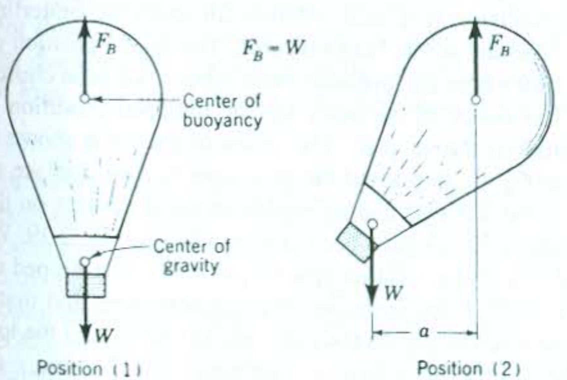


Figure 2.36
Stability of a balloon.

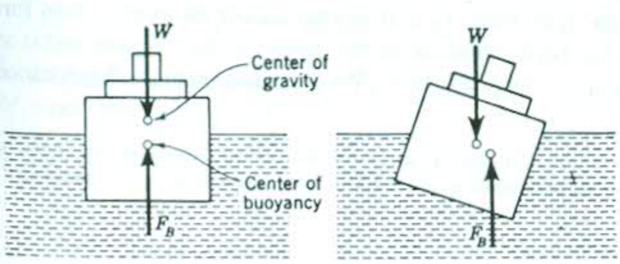


Figure 2.37
Stability of a ship.

Figure 2.38
Cross section of a ship at the water line.

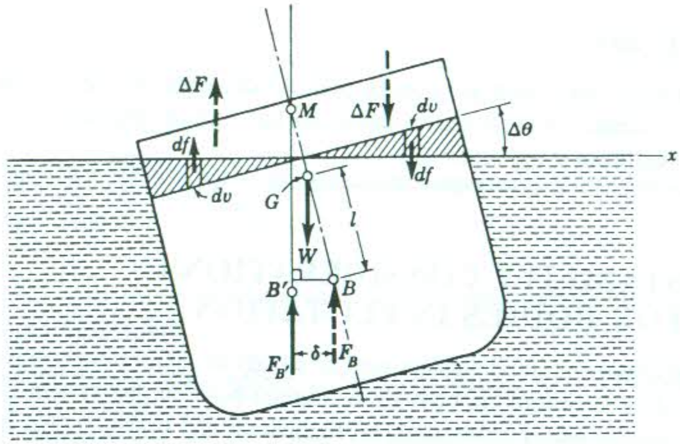


Figure 2.39
Ship rotated slightly to show metacentric height *MG*.

stable shape, since a roll causes much fluid to be displaced at one extremity at the expense of the other with the result that a large shift for the center of buoyancy takes place toward the tipped end. The weight does not shift, so there is then a comparatively large restoring torque for such shapes.

We investigate the stability of a body in flotation at a free surface with the purpose of setting forth a measure for the *degree* of stability possessed by the body. Consider a ship of arbitrary configuration whose hull section at the water line is shown in Fig. 2.38. We will give the ship a *small* rotation $\Delta\theta$ about the centerline y and study the shift in line of action of the buoyant force. The tipped position of the ship is illustrated in Fig. 2.39 where a convenient cross section has been chosen for purposes of discussion. The center of buoyancy for the untipped condition is shown at B, and the new position is shown at B'. The center of gravity is shown at the same section at G.[6] In rotating the ship about the axis, note that we displace an additional amount of water on the left side and relinquish an equal amount on the right side. The sections of these volumes have been crosshatched in Fig. 2.39. We will consider, for purposes of calculation, that an upward force ΔF is developed on the left side of the ship as a result of the increased displacement there and that a downward force of equal value is developed on the right side to take care of the loss of displaced water there. These forces then form a couple moment C in the y direction. Thus the total buoyant force system for the tipped configuration can be considered the superposition of the force F_B at B and the couple moment C from forces ΔF. This force system has been shown as dashed arrows in the diagram and is statically equivalent to the single force $F_{B'}$ at B'. We can easily express the distance δ,

[6]The precise positions of the points B, B', and G along the y direction (i.e., along the axis of the boat) will not be of significance in our computations, so we will consider them to be in the same cross section of the ship.

representing the shift in line of action of the buoyant force, by equating moments of the two systems of forces about an axis parallel to y and going through B'. Thus

$$-F_B\delta + C = 0$$

Hence

$$\delta = \frac{C}{F_B} = \frac{C}{W} \qquad [2.24]$$

and so, knowing the couple moment C and the weight of the ship, we can compute the distance δ. Noting in Fig. 2.39 that point M is the intersection of the line of action of $F_{B'}$ and the centerline of the cross section, we can next compute the distance $\overline{MB}$, using δ.

Thus

$$\frac{\delta}{\overline{MB}} = \sin \Delta\theta$$
$$\therefore \overline{MB} = \frac{\delta}{\sin \Delta\theta} \qquad [2.25]$$

If the position of point M, thus computed, is *above* G, we see that the buoyant force and the weight W form a righting couple and the boat is said to be stable. Furthermore, the greater this distance, which we denote as $\overline{MG}$, the greater this restoring couple and the more stable the vessel. Thus $\overline{MG}$ is a criterion for stability and is called the *metacentric height.* If M falls on G, we have neutral stability, and if it falls below G, we have an unstable condition.

To evaluate the metacentric height, we must determine the couple moment C. For this, we select volume elements dv for the newly displaced fluid and also for the displaced space given up by the vessel as a result of rotation. These are shown in Figs. 2.38 and 2.39, and from these diagrams we can see that

$$dv = x\,\Delta\theta\, dA$$

For each dv we can associate a force df which is the weight of the column of water from dA to the free surface and thus this force has the value $\gamma x\,\Delta\theta\, dA$. Force df points up for volume elements to the left of y and points down for volume elements to the right of y, as explained earlier. The couple moment C can be determined by taking the moment about y of this force distribution extending over the entire hull section of the ship at the water line (corresponding to the level of the free surface). Denoting the area of this section as $A_{\text{f.s.}}$, we thus have for C

$$C = \int_{A_{\text{f.s.}}} \gamma x^2\, \Delta\theta\, dA = \gamma\, \Delta\theta \int_{A_{\text{f.s.}}} x^2\, dA = \gamma\, \Delta\theta\, I_{yy} \qquad [2.26]$$

where I_{yy} is the second moment of area $A_{\text{f.s.}}$ about the y axis. Now replace C in Eq. 2.24, using the above result.

$$\delta = \frac{\gamma\, \Delta\theta\, I_{yy}}{W} \qquad [2.27]$$

The distance $\overline{MB}$ in Eq. 2.25 can then be written as

$$\overline{MB} = \frac{\gamma\,\Delta\theta\, I_{yy}}{W \sin \Delta\theta} \qquad [2.28]$$

From L'Hôpital's rule

$$\lim_{\Delta\theta \to 0} \frac{\Delta\theta}{\sin \Delta\theta} = 1$$

Hence, Eq. 2.28 becomes, in the limit as $\Delta\theta \to 0$,

$$\overline{MB} = \frac{\gamma I_{yy}}{W} \qquad [2.29]$$

Upon denoting the distance between G and B as l (see Fig. 2.39) the metacentric height $\overline{MG}$ then becomes

$$\boxed{\overline{MG} = (\overline{MB} - l) = \frac{\gamma I_{yy}}{W} - l} \qquad [2.30]$$

From this formulation we see that a negative value of $\overline{MG}$ means $\overline{MB} < l$ and hence instability, while a positive value means stability. From our assumptions, both tacit and explicit, we see that the stability criterion presented becomes less meaningful the greater the roll. (We all know that even the most stable vessels can capsize if the disturbance is great enough.) The technique of limiting oneself to small disturbances to facilitate computations is common for all engineering sciences. It is important at all times to be cognizant of the limitations associated with the results stemming from such formulations.

EXAMPLE 2.16

■ Problem Statement

A barge in Fig. 2.40 has the form of a rectangular parallelepiped having dimensions 10 m by 26.7 m by 3 m. When loaded, the barge weighs 4450 kN and has a center of gravity which is 4 m from the bottom face. Find the metacentric height for a rotation about its longest axis, and determine whether or not the barge is stable. If the barge is rotated by 10° about its longest axis, what is the restoring torque?

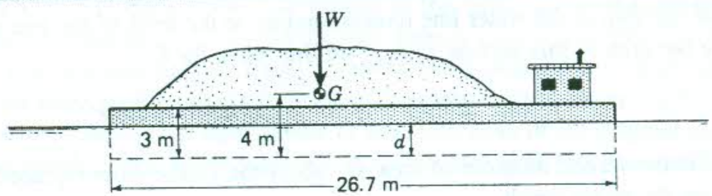


Figure 2.40
Loaded barge.

■ Strategy

To use the metacentric height formula we must make a series of simple calculations to find the following values:

1. The center of buoyancy using, in part, the Archimedes' principle.
2. The distance between the center of buoyancy and the center of gravity of the loaded barge. We denote this distance as l.
3. The second moment of area about the longest centerline of the planeview section of the barge taken at the waterline. This is a 26.7 m by 10 m rectangular area.

Finally, we can use Eq. 2.26 for determining the torque needed for the 10° roll.

■ Execution

We first determine the volume of water that the barge displaces using Archimedes' principle with d giving the depth that the water wets the barge. Thus

$$[(10)(26.7)d](9806) = W = 4450 \times 10^3$$
$$d = 1.700 \text{ m}$$

The center of buoyancy is at a distance 1.700/2 m above the bottom of the barge. The distance l, needed in Eq. 2.30, is then $l = 4 - (1.700/2) = 3.15$ m. The metacentric height $\overline{MG}$ is then

$$\overline{MG} = \frac{(9806)[(\frac{1}{12})(26.7)(10^3)]}{(4450)(1000)} - 3.15 = 1.753 \text{ m}$$

Thus the barge is stable.

The restoring couple for a rotation of 10° is given by Eq. 2.26. Thus

$$C = \gamma\, \Delta\theta\, I_{yy} = (9806)\left(\frac{10}{360}\right)(2\pi)\frac{(26.7)(10^3)}{12} = 3808 \text{ kN} \cdot \text{m}$$

■ Debriefing

This is an example of a highly stable vessel. At the other extreme, we have canoes, kayaks, and high-speed destroyers. The destroyer, in addition to undergoing rough seas while moving at speeds of over 30 kn, must also undergo rapid changes of direction, which poses an additional challenge to its stability. In the case of ocean liners, where the comfort of passengers is important, engineers have sought to control roll using underwater, retractable, stubby wings and even have used large, high-speed, vertical gyros. Can you explain how these devices could work?

HIGHLIGHTS

In this chapter we presented a very simple theory which, when coupled with direct physically oriented viewpoints, permits us to solve interesting problems with dispatch. We started with pressure in a fluid, liquid, or gas, and presented a key simple differential equation of equilibrium, $dp/dz = -\gamma$. Then with constant γ (hence, a liquid), we integrated to get a formula $p = \gamma d$, giving us the change in pressure at any position z from that of the known pressure at a location z_0 having a distance d above or below z_0. Often the position of known pressure is at the free surface, and we get the **desired pressure change as γ times the distance above or below the free surface.** Remember that going to a higher position than z_0, the pressure will drop, and going to a lower position the pressure will increase.

With this behind us, we went to the measuring technique called **manometry.** To measure a pressure, for instance, using a U-tube, we found along the capillary **two locations that had the same elevation and were joined by the same liquid.** These points will then be the same distance above or below a common elevation in the capillary and accordingly must have the same pressure. We then go along each leg of the U-tube and compute the pressure at each of the two chosen points. Equating the results, we can then get the desired information.

In manometry, we can neglect the specific weight of gases in comparison to that of liquids. However, in dealing with gases forming an atmosphere involving large elevations, we must consider the specific weight of the gas. We can use the differential equation of equilibrium as before, but when we integrate we must express the specific weight as a variable. Thus, using the perfect gas as the fluid, we formulated the pressure variation for an isothermal atmosphere and an atmosphere having a temperature varying linearly with elevation. We presented the so-called **Standard Atmosphere,** which we use in calculations to compare performance of planes and dirigibles, in an atmosphere closely resembling the earth's atmosphere.

Next, we focused our attention on the force of a liquid on a submerged plane surface. We showed that **the pressure applied to the free surface of a liquid is transmitted unchanged throughout the liquid.** To find the force on a submerged surface, you find the total pressure. This pressure stems from the external pressure on the free surface plus that which obtains from gravity action on the liquid, the latter evaluated at the centroid of the submerged surface. Now you multiply this total pressure by the area. The position of the resultant force is denoted as y', and it is called the **center of pressure.** With y locating the centroid, the value of y' is found from the formula, $y' - y = \dfrac{\gamma \sin\theta I_{\xi\xi}}{p_c A}$, where **$I_{\xi\xi}$ is the second moment of area about the centroidal axis, and p_c is the total pressure at the centroid including pressure at the free surface.** For a curved submerged surface, the horizontal force component is found by projecting the surface onto a plane normal to the direction of the desired force component. We then deal with the familiar plane submerged surface. **The vertical component equals the weight of a prismatic**

column of fluid "supported" by the surface going up to the free surface. You are referred to Example 2.9, where the curved surface is the wetted portion on the bottom of a body, to see a simple way of getting back to the idea of a curved surface supporting a column of fluid directly above it.

Next, using our knowledge of forces on submerged surfaces, we easily produced the **Archimedes' principle** familiar to every high school physics student. However, look again at Fig. 2.33 to see that we can use Archimedes at times when the liquid does not completely envelop the submerged body.

2.12 CLOSURE

In this chapter, we have been able to ascertain pressure distributions in static fluids by using Newton's law and, occasionally, an equation of state. From this we could then deduce forces on submerged surfaces and bodies. With these results we were able, in Section 2.11, to predict, to some degree, the action of bodies floating at a free surface when given slight disturbances. In the studies of dynamic fluids in Parts II and III of the text, we will follow essentially the same general procedure. That is, we will first determine the velocity field (in this chapter we knew initially by inspection that the velocity was zero relative to an inertial reference); we then get the stress field or that part of it which is of interest and compute certain practical items of interest, as, for example, the lift or drag on some object in the flow.

However, we need more sophisticated methods for quantitatively describing the motion of deformable media beyond those required in the study of particle and rigid-body dynamics. Also, additional laws other than Newton's law are required, and a new way of applying these laws will be helpful. We consider these vital requirements in Chapter 3.

2.13 COMPUTER EXAMPLES

COMPUTER EXAMPLE 2.1

■ Computer Problem Statement

On a hypothetical planet, the atmosphere has a specific weight γ given by the following formula, wherein z is the height above ground:

$$\gamma = \frac{720}{z^2 + 4}\left(\frac{1}{T^{3/2}}\right) \text{N/m}^3$$

If $T = (15 + 3z)$ °C, plot p vs. z. The pressure at the ground is 101,325 Pa.

■ Strategy

The basic equation that we will use, developed from equilibrium, is

$$\frac{dp}{dz} = -\gamma = -\frac{720}{x^2 + 4}\left(\frac{1}{T^{3/2}}\right)$$

Integrating from the ground level $z = 0$ where $p_0 = 101{,}325$ to elevation z we get

$$p = p_0 - \int_0^z \frac{720}{z^2 + 4}\left(\frac{1}{(15 + 3z)^{3/2}}\right)dz$$

■ Execution

```
clear all;
%Putting this at the beginning of the program ensures
%values don't overlap from previous programs.

p(1)=101325;
%This is the initial pressure (the pressure when the
%height off the ground=0m).

for i=1:40;

p(i+1)=p(i)-quad8('delta_p',i,i+1);
%This statement gives you a new pressure value by
%taking the previous pressure value and subtracting
%from it the value of the integral given by the
%function "delta_p." We do this 40 times.

end

z=0:40;
%The height above the surface of the planet goes
%from 0m to 40m.

plot(z,p);
grid;
xlabel('Height above planet (meters)');
ylabel('Pressure (Pa)');
title('Pressure of the atmosphere vs. height above
surface of planet');
%This just gives us the plot that we want.
```

The plot is shown in Fig. C2.1

■ Debriefing

This program integrates the function "delta_p" and subtracts the result from the previous value for pressure in an iterative fashion. The pressure is then plotted against the distance above the planet in meters. The value of the function "delta_p", which is referenced by the program in this example and has to be located in a different M-file than this program, is given as:

```
function Press_var=delta_p(z);
Press_var=(720./(z.^2+4)).*(1./(15+3.*z).^1.5);
```

The integral of this function we found, in this problem, using the "quad8" function, which uses a adaptive Newton-Cotes 8-panel rule. The value of this integral, (found using MATLAB's "sym" function) is:

```
16.552/(15.+3.*z)^(1/2)+1.498*log(31.16+3.*z+7.894
*(15.+3.*z)^(1/2))+4.67*atan(1.316*(15.+3.*z)^(1/2)
+5.193)-1.498*log(31.156+3.*z-894*(15.+3.*z)^(1/2))
+4.67*atan(1.316*(15.+3.*z)^(1/2)-5.193)
```

Thus, you can really see the advantage of using a computer for this problem. Imagine solving this integral over and over again with your calculator!

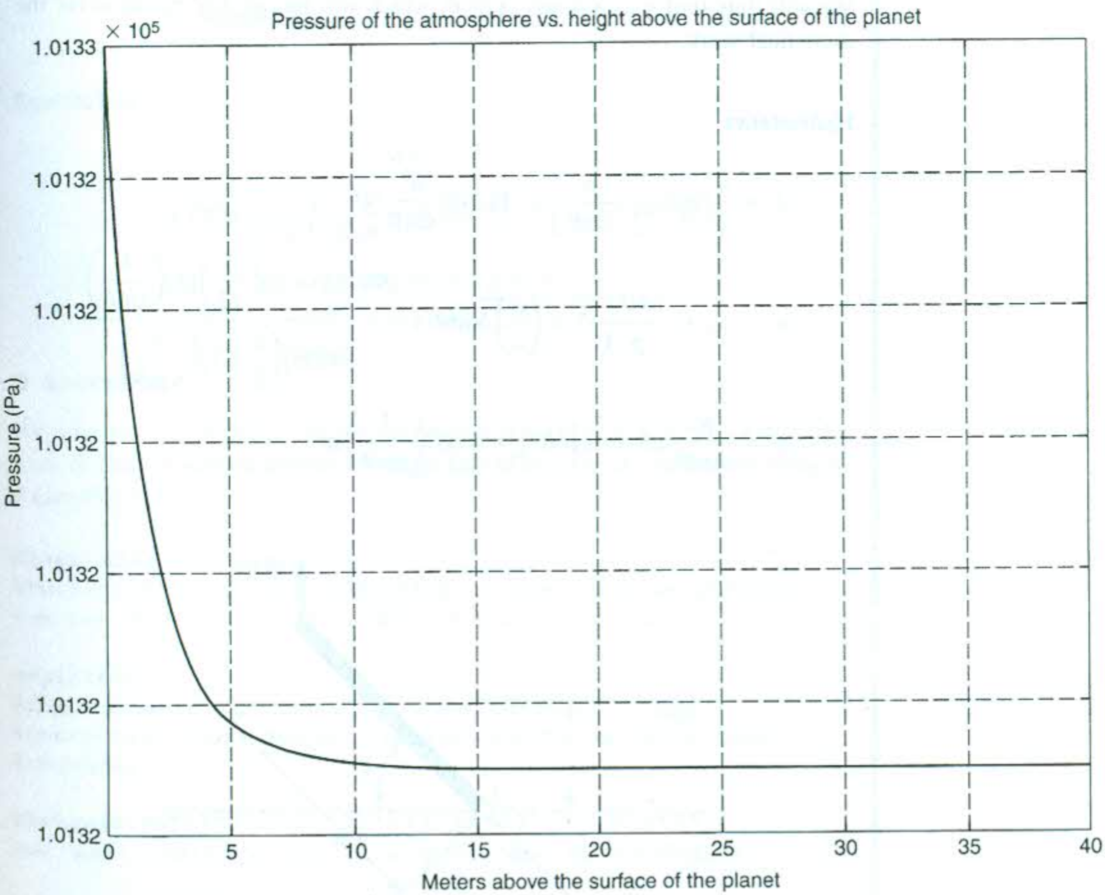


Figure C2.1

COMPUTER EXAMPLE 2.2

■ Computer Problem Statement

A vertical 36 kN force is maintained on the door, which is shown in equilibrium at an angle θ. The door maintains water whose free surface is at a height h. (See Fig. C2.2.) The door is 3 m in length and is pinned at A. Plot h vs. θ from $\theta = 0°$ to $\theta = 90°$. Neglect friction and the weight of the door.

■ Strategy

We shall first compute the hydrostatic force on the door. Then, with a free body diagram of the door, we shall use equilibrium, taking moments about the hinge. We will thus find h as a function of θ, which will be our key equation for the numerical work.

Hydrostatics

$$F = \frac{h}{2}(9806)\left[3\frac{h}{\sin\theta}\right] = 14{,}709\frac{h^2}{\sin\theta}\text{ N}$$

$$y' = y_C + \frac{\gamma \sin\theta I_{zz}}{p_c A} = \left(\frac{h}{2}\right)/\sin\theta + \frac{(9806)(\sin\theta)\left(\frac{1}{12}\right)(3)\left(\frac{h}{\sin\theta}\right)^3}{(9806)\left(\frac{h}{2}\right)(3)\left(\frac{h}{\sin\theta}\right)}$$

$$y' = \frac{h}{\sin\theta}[0.5 + 0.1667] = 0.667\left(\frac{h}{\sin\theta}\right)\text{ m}$$

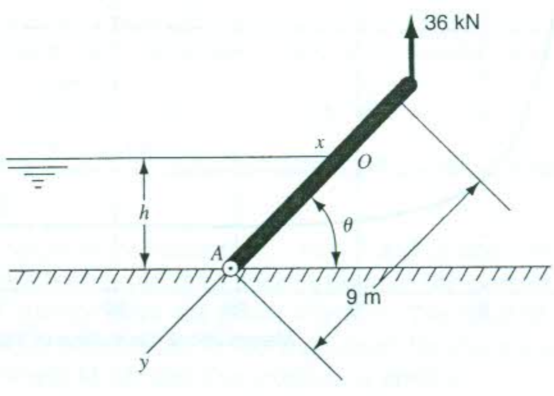


Figure C2.2

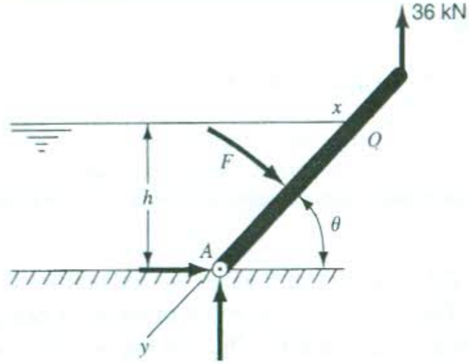


Figure C2.3

Equilibrium

$$\Sigma M_o = 0$$

$$(14.709)\left(\frac{h^2}{\sin\theta}\right)\left(\frac{h}{\sin\theta} - 0.667\frac{h}{\sin\theta}\right) = (36)(9)(\cos\theta)$$

$$h = \sqrt[3]{\frac{(36)(9)(\cos\theta)(\sin^2\theta)}{(14.709)(0.333)}}\ \text{m}$$

■ Execution

We now proceed with the solution of Eq. *a* using MATLAB. Next we will plot h vs. θ, and we will determine the angle that allows for the maximum value of θ (see Fig. C2.3).

```
clear all;
%Putting this at the beginning of the program ensures
%values don't overlap from previous programs.

c=pi/180;
%The variable "c" is a variable to plug-in for
%converting from degrees to radians(Matlab only uses
%radians).

theta=[0:90];
%We want theta to run from 0 degrees to 90 degrees.

height=((36*9.*cos(theta.*c).*sin(theta.*C).^2)/(14.7
09*.333)).^(1/3);
%This equation is just height as a function of theta.
```

```
plot(theta,height);
xlabel('theta(degrees)');
ylabel('height of liquid (m)');
grid;
title('Theta vs. height of liquid');
%This just gives us the plot that we want.
```

■ Debriefing

Examining in Fig. C2.4 the plot of h vs. θ, we see that $h = 0$ at the extremities of the angle θ. The first zero is the result of the door being horizontal, and hence it is impossible to have a static fluid of any nonzero depth resting stationary on the door. The mechanics thus verifies what common sense dictates. The case of $\theta = 90°$ obtains because the vertical 36 kN force now goes through the hinge and thus has no moment about the hinge. Clearly the door, once again, cannot support a nonzero depth of water. Mechanics and common sense again agree.

By using the "`diff`" function in MATLAB on the equation for height, we can see a critical point of this curve in Fig. C2.4 is between 55 and 56 degrees. "`diff`" in MATLAB calls up an intrinsic function that gives the difference and approximate derivative. "`diff(x)`", for a vector `x`, is `[x(2)-x(1)x(3)-x(2)... x(n)-x(n-1)]`. All you do in MATLAB's command window, in this case, is to type "`diff(height)`" after running the program, and MATLAB will give you a vector of values (if "`height`" has "`n`" values you will get a vector with "`n-1`" values) of the differences between adjacent entries in your height vector. By looking at these values, you can see that between 55 and 56 degrees is a critical point (this is where the "`diff`" values cross from positive to negative values).

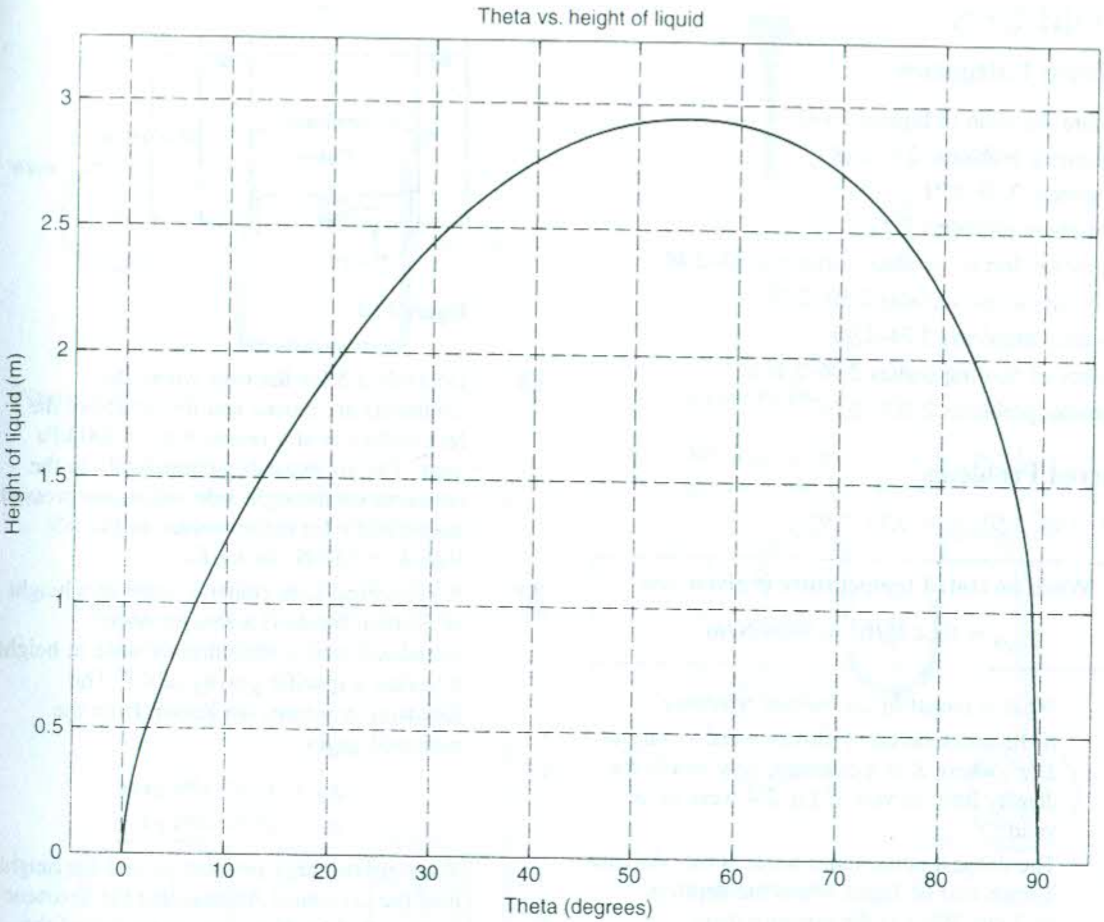

Theta vs. height of liquid
Height of liquid (m)
Theta (degrees)
0
0.5
1
1.5
2
2.5
3
0
10
20
30
40
50
60
70
80
90

Figure C2.4

PROBLEMS

Problem Categories

Pressure variation in liquids 2.1–2.7
Manometry problems 2.8–2.18
Barometers 2.19–2.21
Atmosphere problems 2.22–2.32
Hydrostatic forces on plane surfaces 2.33–2.49
Forces on curved surfaces 2.50–2.73
Buoyancy problems 2.74–2.95
Stability of floating bodies 2.96–2.101
Computer problems 2.102–2.107

Starred Problems

2.18, 2.49, 2.50, 2.59, 2.73, 2.92

When no stated temperature is given, use

$$\gamma_{H_2O} = 62.4 \text{ lb/ft}^3 = 9806 \text{ N/m}^3.$$

2.1 What is meant by an *inertial reference?*

2.2 If the acceleration of gravity were to vary as K/z^2, where K is a constant, how would the density have to vary if Eq. 2.4 were to be valid?

2.3 The deepest point under water is the Mariana Trench east of Japan where the depth is 11.3 km. What is the pressure there
a. in absolute pressure?
b. in gage pressure?

The average specific gravity of seawater there we estimate as 1.300.

2.4 Prove that the free surface of a static liquid must be normal to the direction of gravity.

2.5 Two identical containers, each open to the atmosphere, are initially filled with the same liquid ($\rho = 700$ kg/m^3) to the same level H (Fig. P2.5). The two containers are connected by a pipe in which a frictionless piston of cross section $A = 0.05$ m^2 is made to slide slowly. How much work is done by water on the piston in moving a distance of $L = 0.1$ m? The cross section area of each container is *twice* that of the pipe.

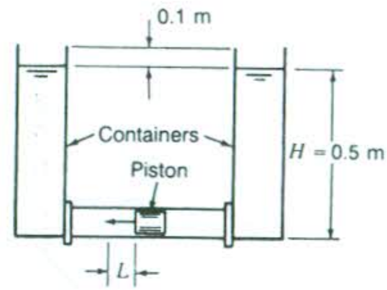


Figure P2.5

2.6 Do Prob. 2.5 for the case where the containers are closed and the air above the free surface is at a pressure p_0 of 200 kPa gage. The air expands isentropically in the container on the right side and is compressed isothermally for the container on the left side. $R = 287$ N · m/kg K.

2.7 A cylindrical tank contains water at a height of 50 mm. Inside is a smaller open cylindrical tank containing kerosene at height h having a specific gravity of 0.8. The following pressures are known from the indicated gages:

$$p_B = 13.80 \text{ kPa gage}$$
$$p_C = 13.82 \text{ kPa gage}$$

What are the gage pressure p_A and the height h of the kerosene? Assume that the kerosene is prevented from moving to the top of the tank.

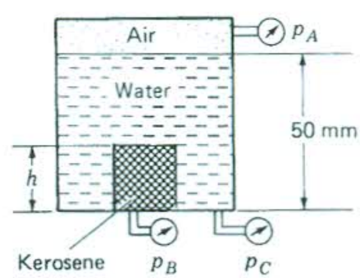


Figure P2.7

2.8 Find the difference in pressure between tanks A and B if $d_1 = 300$ mm, $d_2 = 150$ mm, $d_3 = 460$ mm, $d_4 = 200$ mm, and $S_{Hg} = 13.6$.

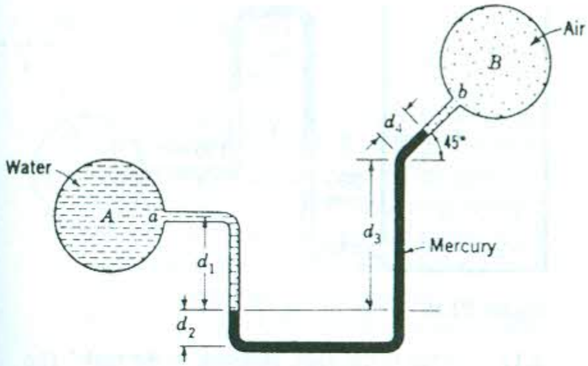


Figure P2.8

2.9 An open tube is connected to a tank. The water rises to a height of 900 mm in the tube. A tube used in this way is called a *piezometer*. What are the pressures p_A and p_B of the air above the water? Neglect capillary effects in the tube.

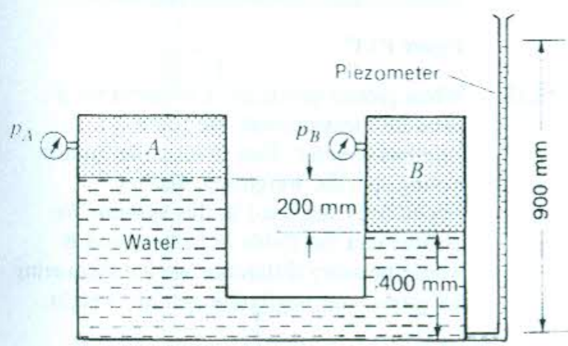


Figure P2.9

2.10 Consider the U-tube with one end closed and the other end having a funnel of height 2 in. Mercury is poured into the funnel to trap the air in the tube, which is 0.1 in in inside diameter and 3 ft in total length. Assuming that the trapped air is compressed isothermally, what is h when the funnel starts to run over? Neglect capillary effects for this problem.

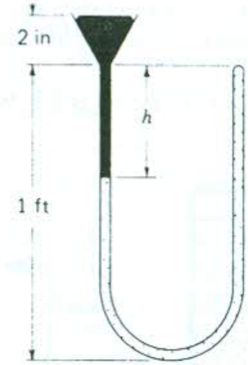


Figure P2.10

2.11 What is the pressure difference between points A and B in the tanks?

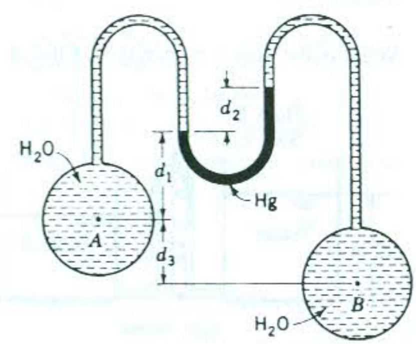


Figure P2.11

2.12 Calculate the difference in pressure between centers of tank A and tank B. If the entire system is rotated 180° about the axis MM.

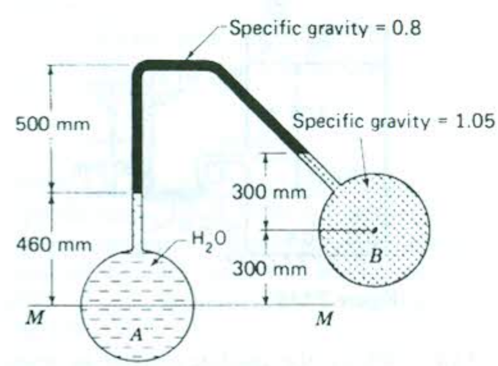


Figure P2.12

what changes in pressure between the tanks would be necessary to maintain the positions of the fluids intact?

2.13 The specific gravity of the oil is 0.8. What is the pressure p_A?

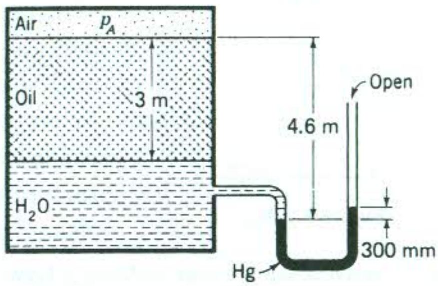


Figure P2.13

2.14 What is the specific gravity of fluid A?

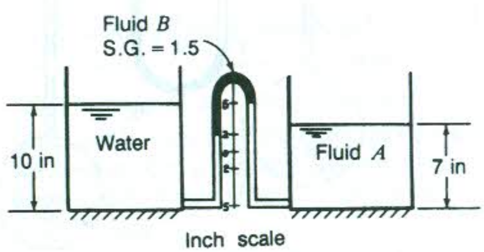


Figure P2.14

2.15 Find distance d for the U-tube.

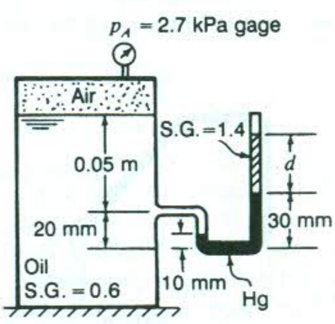


Figure P2.15

2.16 What is the absolute pressure in drum A at position a?

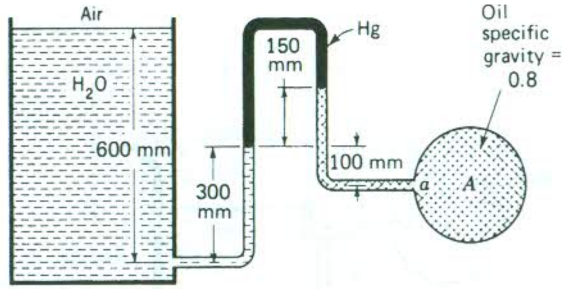


Figure P2.16

2.17 What is the gage pressure in the tank? The tank contains air.

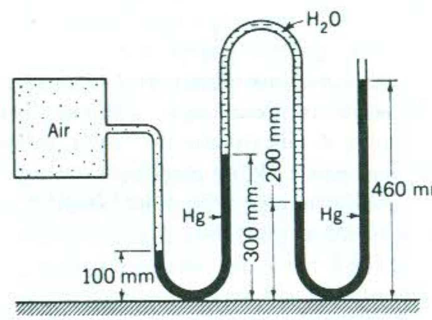


Figure P2.17

***2.18** When greater precision is required for a pressure measurement, we use a *micromanometer*. Two immiscible liquids having specific weights γ_1 and γ_2, respectively, are used in this system. We assume that the fluids in tanks E and B whose pressure difference we are measuring are gases with negligible specific weight.

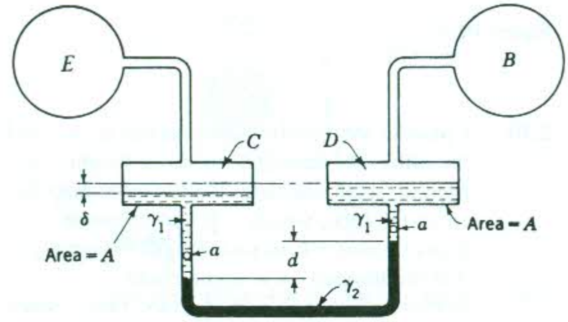


Figure P2.18

Compute the pressure difference $p_E - p_B$ in terms of δ, d, γ_1, and γ_2. If the area of the micromanometer tube is a and the cross-sectional areas of the containers C and D are A, determine δ in terms of d, by geometrical considerations. Explain how by having a/A very small and γ_1 almost equal to γ_2, a small pressure difference $p_E - p_B$ will cause a large displacement d, thus making for a sensitive instrument. When the pressures in B and E are equal, the levels of liquid in the two links are equal.

2.19 A barometer is a device for measuring atmospheric pressure. If we use a liquid having specific weight γ and invert a tube full of this material as shown, find formulas for h if the absolute vapor pressure of the liquid is p_{vap}

a. in SI units.

b. in USCS units using psi.

Show dimensions in your formulas. If we use a fluid having a specific weight of 850 lb/ft^3 and a vapor pressure of 0.2 psi abs, find h.

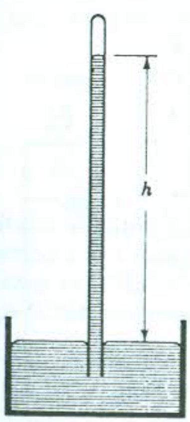


Figure P2.19

2.20 From the preceding problem the height h in a barometer is

$$h = \frac{1}{\gamma}[p_{\text{atm}} - p_{\text{vap}}](144)\text{ ft} \qquad \text{[a]}$$

with the pressure given in psi. If a barometer registers 800 mm in a pressure chamber, and a pressure gage in this chamber measures 50 psi gage, what is the absolute pressure for this gage? Take the vapor pressure of mercury to be 0.3 psi abs. The value of γ is 850 lb/ft^3.

2.21 A barometer measures 750 mm in a chamber where a pressure gage measures 10,000 Pa gage on a device in the chamber. What is the absolute pressure for this gage? The vapor pressure of the mercury is 0.5 Pa abs. See Probs. 2.19 and 2.20. What conclusion can be drawn concerning the inclusion of vapor pressure of mercury in most problems?

2.22 The Eiffel Tower in Paris is 984 ft tall with its base about 500 ft above sea level. What is the pressure and temperature at the top in a U.S. Standard Atmosphere? Do not use tables.

2.23 Outside a hot air balloon a barometer measures 690 mm mercury. What is the elevation of the balloon in a U.S. Standard Atmosphere? Do not use tables.

2.24 In the U.S. Standard Atmosphere where the temperature varies according to Eq. 2.15, the equation relating pressure and specific volume is

$$pv^n = \text{const}$$

This is called a *polytropic process.* What should the value of n be?

2.25 At what elevation in feet is the pressure in a standard atmosphere 0.92 that at sea level? Do this *without* tables. What is v at this position? Use sea level data given in Sec. 2.4.

2.26 In an *adiabatic atmosphere,* the pressure varies with the specific volume in the following manner:

$$pv^k = \text{const}$$

where k is a constant equal to the ratio of the specific heats c_p and c_v. Develop an expression for pressure as a function of elevation for this atmosphere, using the ground as a reference. When $z = 0$, take $p = p_0$ and $\gamma = \gamma_0$. Reach the following result:

$$p = \frac{1-k}{k}\gamma z + p_0\frac{\gamma}{\gamma_0}$$

2.27 An atmosphere has a temperature of 27°C at sea level and drops 0.56°C for every 152.5 m. If the gas constant is 287 N · m/(kg)(K), what is the elevation above sea level where the pressure is 70 percent that of sea level?

2.28 In Example 2.4, assume that the atmosphere is *isothermal* and compute the elevation for a pressure which is 30 percent that at sea level.

2.29 Work Example 2.4 for the case of the atmosphere being incompressible.

2.30 The wind has been considered as a possible useful source of energy. How much kinetic energy would be present in a U.S. Standard Atmosphere between the elevations of 5000 ft and 6000 ft above sea level if there is an average wind speed of 5 mi/h? Use an average density. The radius of the earth is 3960 mi. What is the kinetic energy per unit volume of air? Comment on the practical use of wind power. Area of a sphere is πD^2.

2.31 A light rubber balloon containing helium is released in a U.S. Standard Atmosphere. The stretched rubber transmits a membrane force σ proportional to the diameter and given as $5D$ lb/ft with D given in feet. What is the inside pressure in the balloon at an elevation of 5000 ft in a U.S. Standard Atmosphere? The balloon is rising slowly at a constant speed. *Hint:* The force on a curved surface from a uniform pressure equals the pressure times the projected area of the surface onto a plane normal to the direction of the force.

2.32 In a light airplane the cabin pressure is to be maintained at 80 percent that of atmospheric pressure on the ground which is 10,000 ft above sea level. If for structural reasons the outside-to-inside ambient pressure ratio is not to get smaller than 0.6, what is the maximum height h_{max} that the plane may fly in a U.S. Standard Atmosphere?

2.33 A force of 445 N is exerted on lever AB. End B is connected to a piston which fits into a cylinder having a diameter of 50 mm. What force P must be exerted on the larger piston to prevent it from moving in its cylinder which has a 250 mm diameter?

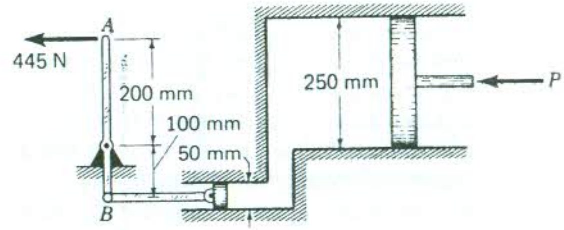


Figure P2.33

2.34 Prove that the resultant force from a uniform pressure distribution on an area acts at the centroid of the area.

2.35 Find total force on door AB and the moment of this force about the bottom of the door.

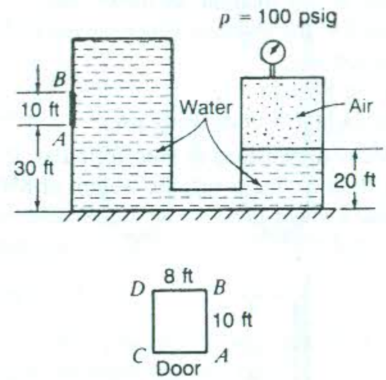


Figure P2.35

2.36 A plate is submerged vertically into the water. What is the radius r of a hole to be cut from the center of $ABCD$ to make the hydrostatic force on surface $ABCD$ equal to

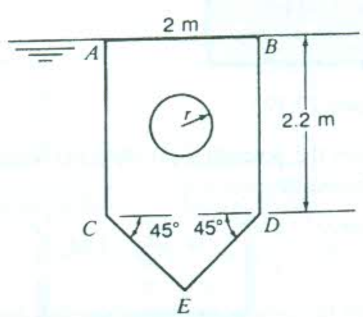


Figure P2.36

the hydrostatic force on surface *CDE*? What is the moment of the total force about *AB*? Delete p_{atm}.

2.37 A rectangular plate shown as *ABC* can rotate about hinge *B*. What length *l* should *BC* be so that there is zero torque about *B* from water and plate weight? Take the weight as 1000 N/m of length. The width is 1 m.

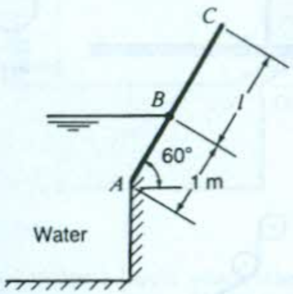


Figure P2.37

2.38 Find the total force on door *AB* from fluids. Take $S_{oil} = 0.6$. Find the position of this force from the bottom of the door.

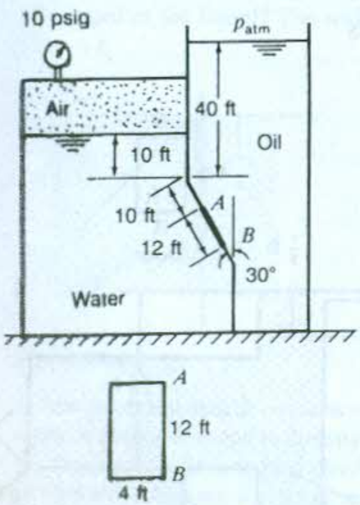


Figure P2.38

2.39 Find the resultant force on the top of the submerged surface. Give the complete position of the resultant. Disregard p_{atm}.

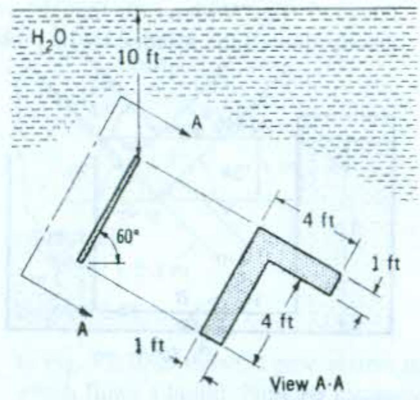


Figure P2.39

2.40 An open rectangular tank is partially filled with water. The dimensions are shown.

a. Determine the force on the bottom of the tank from the water.

b. Determine the force on the end of the tank from water. Give position also.

c. Determine the force on the door shown at the side of the tank. Be sure to state position.

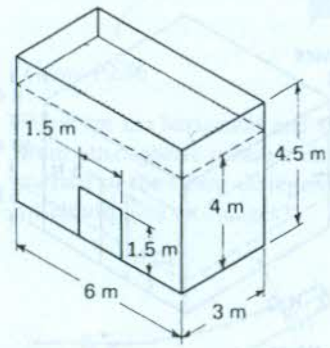


Figure P2.40

2.41 A gate *AB* is hinged at *A*. When closed, it is inclined at an angle of 60°. It is rectangular and has a length of 0.6 m and a width of 1 m. There is water on both sides of the gate. Furthermore, compressed air exerts a pressure of 20 kPa gage on the surface of the water on the left side of the gate, while the water on the right side is exposed to atmospheric pressure. What is the moment about the hinge *A* exerted by the water on the gate? *Hint:* With a little thought, you can greatly shorten the solution to the problem.

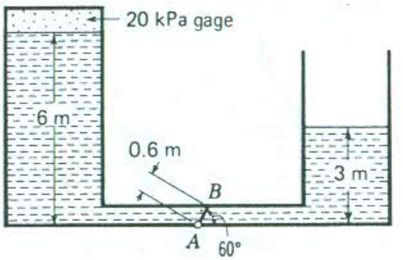


Figure P2.41

2.42 In Prob. 2.41, a 1.2-m layer of oil, having specific gravity of 0.8, is added to the top of the water on the right side of the gate. What is the total moment about A from the water on the gate? The hint of Prob. 2.41 applies here.

2.43 Find the resultant force from all fluids acting on the door. Specific gravity of the oil is 0.8.

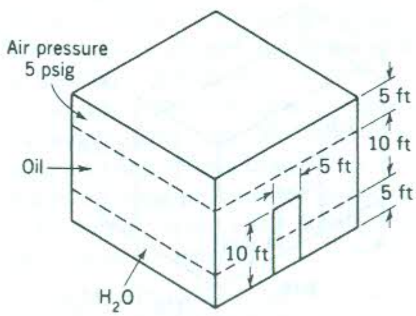


Figure P2.43

2.44 Determine the force and its position from fluids acting on the door in Fig. P2.44.

2.45 At what height h_1 will the water cause the door to rotate clockwise (Fig. P2.45)? The door is 3 m wide. Neglect friction and the weight of the door. Take $h_2 = 0.2$ m.

2.46 Find F_R on door AB from inside and outside fluids. Give distance d below B for the position of F_R. See Fig. P2.46.

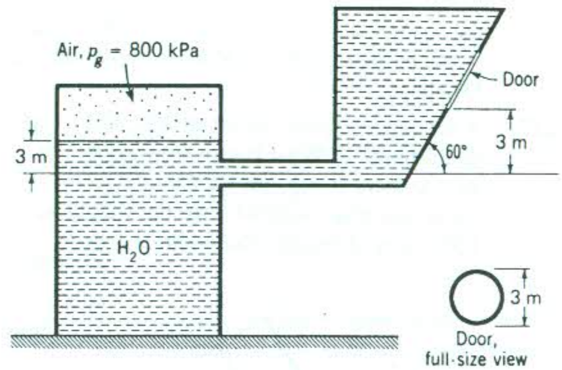


Figure P2.44

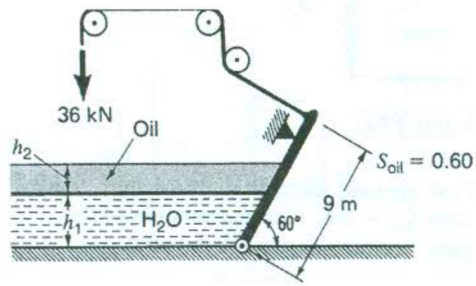


Figure P2.45

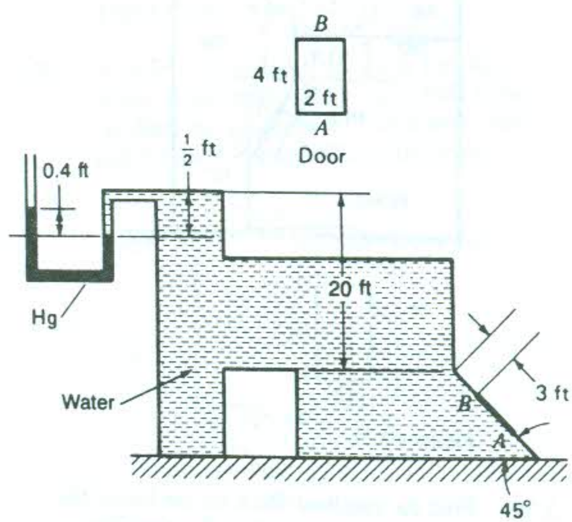


Figure P2.46

2.47 At what pressure in the air tank will the square piston be in equilibrium if one neglects friction and leakage? See Fig. P2.47.

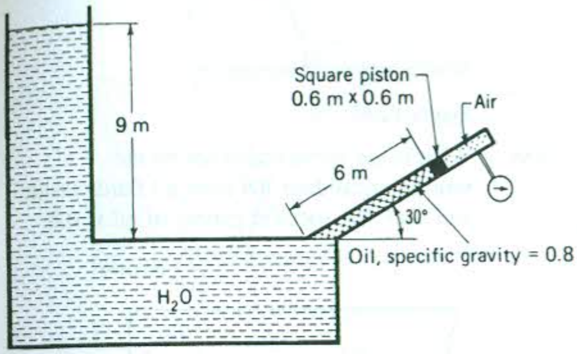


Figure P2.47

2.48 Imagine a liquid which when stationary stratifies in such a way that the specific weight is proportional to the square root of the pressure. At the free surface the specific weight is known and has the value γ_0. What is the pressure as a function of depth from the free surface? What is the resultant force on one face *AB* of a rectangular plate submerged in the liquid? The width of the plate is *b*.

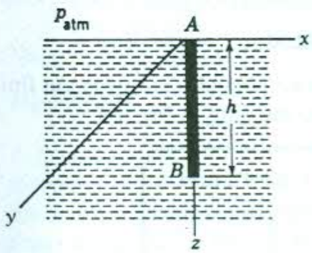


Figure P2.48

***2.49** A trough of unit length contains water. A solid of identical shape is directly touching the free surface. It is moved directly downward a distance δ relative to the ground. What is the force on door *AB* of unit width as a function of δ? What happens when $\delta \rightarrow 1$ m? Consider only the gravitational force from water. *Hint:* What is the area of a parallelogram?

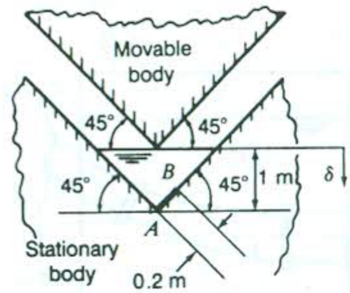


Figure P2.49

***2.50** In Fig. P2.50 is shown a pipe system in which flows a liquid. Find the force vector from the atmospheric pressure of 101,325 Pa on the *outside* surface of the pipe system.

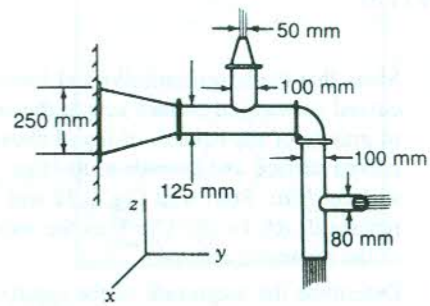


Figure P2.50

2.51 What are the horizontal and vertical forces from atmospheric pressure on the outside surface of the elbow disregarding the effects of atmosphere on flanges?

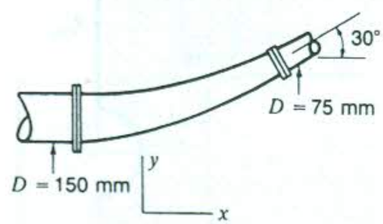


Figure P2.51

2.52 A thin-walled reducing elbow of Prob. 2.51 is shown in Fig. P2.52 inside a pressure tank. Find the horizontal force on the outside surface of the reducing elbow. Neglect pressure of flanges.

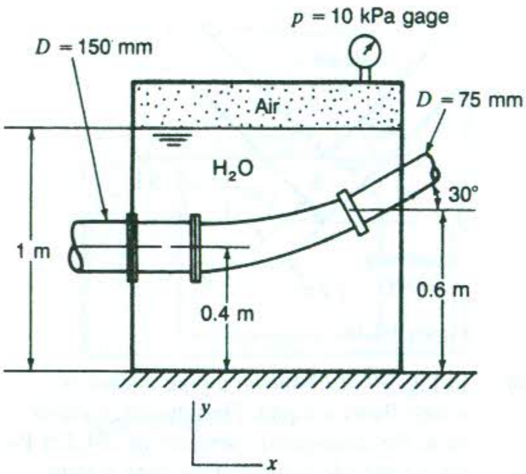


Figure P2.52

2.53 Show that the hydrostatic vertical force on a curved submerged surface acts at the center of gravity of the column of liquid above the curved surface and extends to the free surface. *Hint:* Start with Fig. 2.21 and replace $dz\, dA_z$ by dv. Use V as the volume of the prismatic column.

2.54 Determine the magnitude of the resultant force acting on the spherical surface and explain why the line of action goes through the center O.

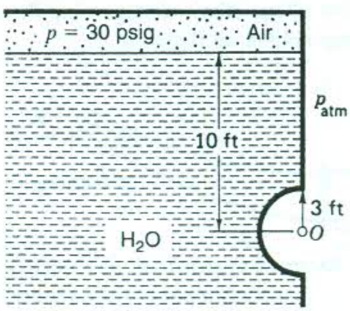


Figure P2.54

2.55 What is the resultant force from fluids acting on the door AB, which is a quarter circle? The width of the door is 1.3 m. Above the water there is air at $p = 10$ psig.

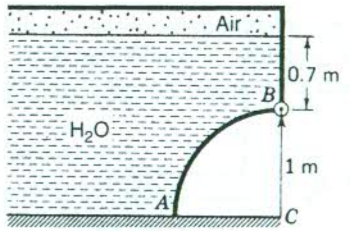


Figure P2.55

2.56 What is the horizontal force on the semispherical door AB from all fluids inside and out? The specific gravity of oil is 0.8.

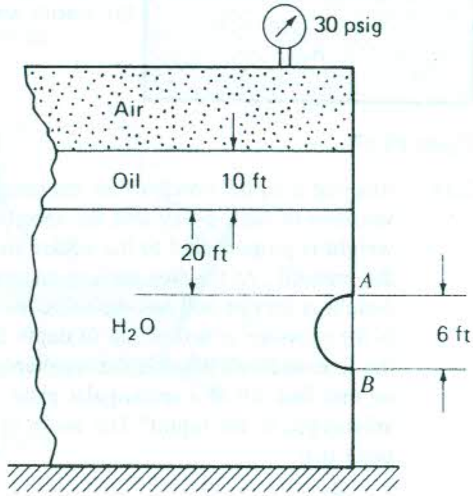


Figure P2.56

2.57 Find the horizontal force from the fluids acting on the plug in Fig. P2.57.

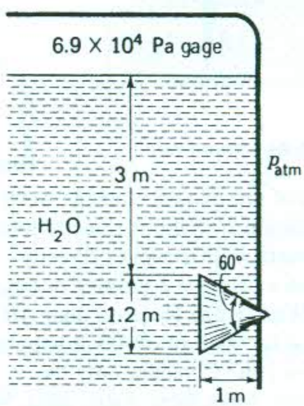


Figure P2.57

2.58 A parabolic gate *AB* is hinged at *A* and latched at *B*. If the gate is 10 ft wide, determine the force components on the gate from the water.

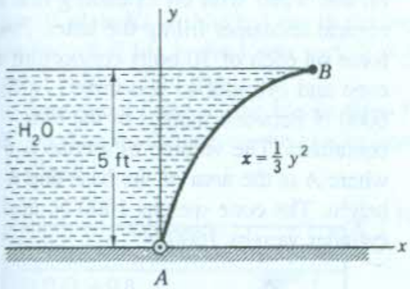


Figure P2.58

***2.59** Consider a wall 10 ft wide and having corrugations (semicircular shapes). What are the resultant horizontal and vertical forces on the wall from the air and water? Give the result per unit width of the wall and for *n* corrugations.

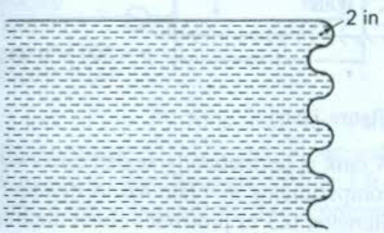


Figure P2.59

2.60 A cylindrical control weir is shown. It has a diameter of 3 m and a length of 6 m. Give the magnitude and direction of the resultant force acting on the weir from the fluids.

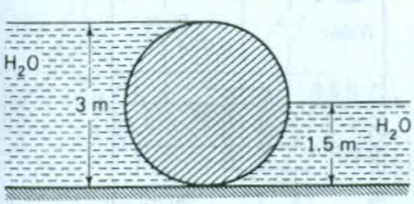


Figure P2.60

2.61 What is the force on the conical stopper from the water?

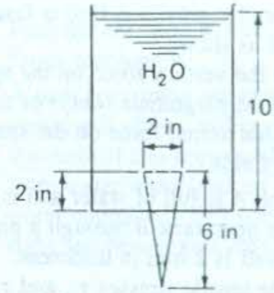


Figure P2.61

2.62 What is the vertical force on the sphere if both sections of the tank are completely sealed from each other?

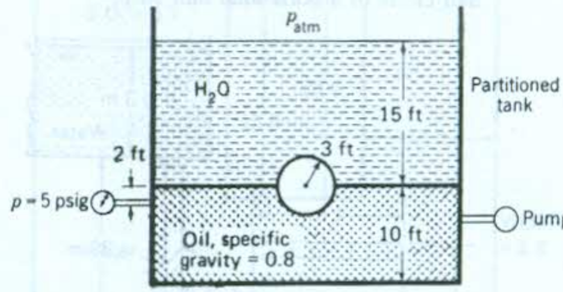


Figure P2.62

2.63 The tank is divided into two independent chambers. Air pressure is present in both sections. A manometer measures the difference between these pressures. A sphere

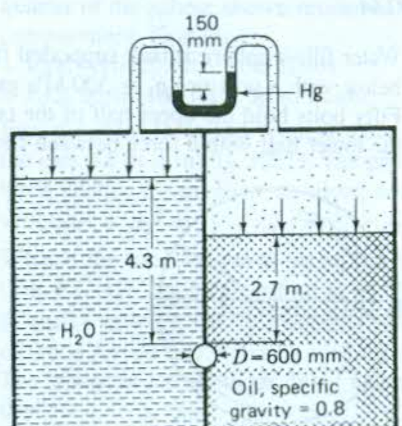


Figure P2.63

of wood (specific gravity is 0.6) is fastened into the wall as shown.

a. Compute the vertical force on the sphere.
b. Compute the magnitude (only) of the resultant horizontal force on the sphere from the fluids.

2.64 A 500-N tank A is full of water and is connected to open tank B through a pipe. If the tank A wall is 2 mm in thickness, determine the tensile stresses τ_{xx} and τ_{yy} from air and water in the tank wall at a point at $y = 3$ m. Also for 40 bolts at the base, compute the force per bolt holding the end plate of the tank. *Hint:* For the stresses, consider two free-body diagrams including a half circle of a horizontal unit strip.

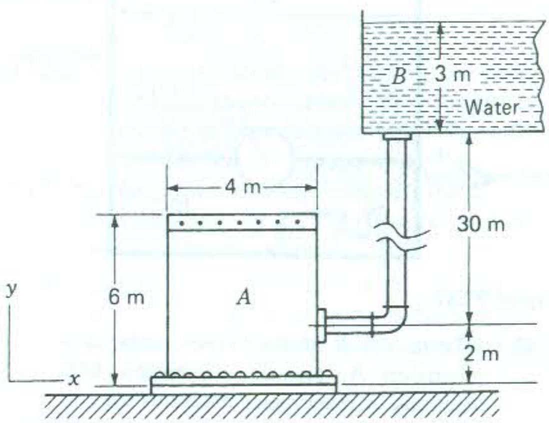


Figure P2.64

2.65 Water fills a spherical tank supported from below with a pressure $p_1 = 300$ kPa gage. Fifty bolts hold the upper half of the tank to the lower half with a force between flanges of 5000 N. What is the force per bolt? Each half of the sphere weighs 2000 N.

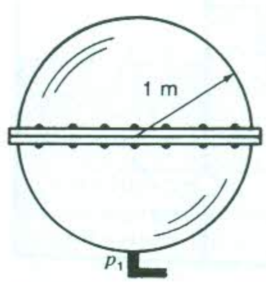


Figure P2.65

2.66 An open-ended 60° conical container is bolted to a cylinder. The cylinder contains oil and water with oil extending into the conical container filling the latter. Find the force on each of 30 bolts connecting the cone and cylinder so that there is a force of 6000 N between flanges of the two containers. The volume of a cone is $\frac{1}{3}Ah$ where A is the area of the base and h is the height. The cone weighs 1000 N, and the cylinder weighs 1600 N.

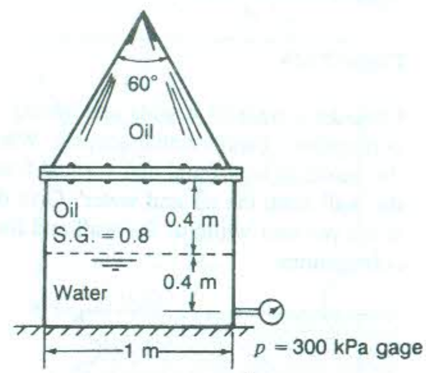


Figure P2.66

2.67 A tank is hermetically sealed into two compartments by plate AB. A cylinder of diameter 0.3 m protrudes above and below the seal AB and is welded to the seal AB. What is the vertical force on the cylinder?

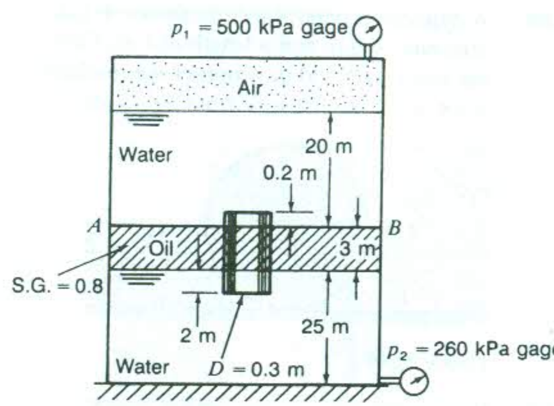


Figure P2.67

2.68 Do Prob. 2.67 for when a hemisphere of diameter 0.3 m is added to the top and bottom of the cylinder and $p_2 = 360$ kPa g.

2.69 A tank is separated into two distinct parts by a stiff plate *EF.* A block *A* fits in the top part and block *B* fits in the lower part. If *A* and *B* are 3 ft long, find the:

a. horizontal force on the blocks from fluids.
b. total vertical force on the blocks from fluids.

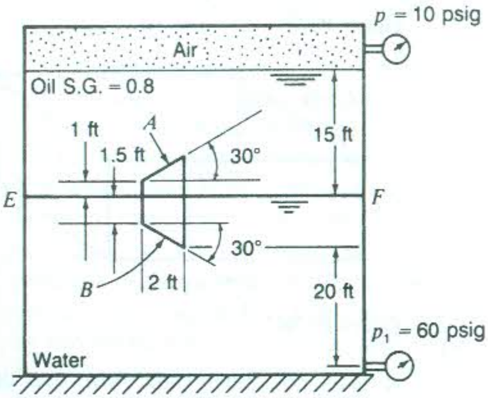


Figure P2.69

2.70 A tank in Fig. P2.70 is made up of three compartments ①, ②, and ③ separated from each other. Triangle *ABC* is 3 ft in length and separates the three compartments.

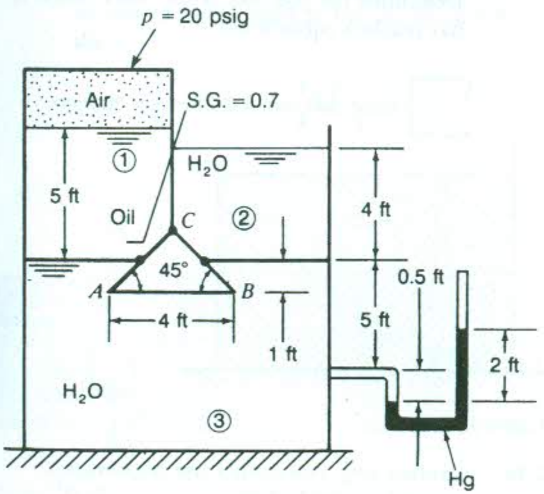


Figure P2.70

Find the net vertical force on *ABC* from the fluids touching it.

2.71 A tank containing water and air under pressure is shown in Fig. P2.71. What are the vertical and horizontal forces on *ABC* from water inside and air outside? Note that water completely fills part of the tank on the right and hence wets *ABC*.

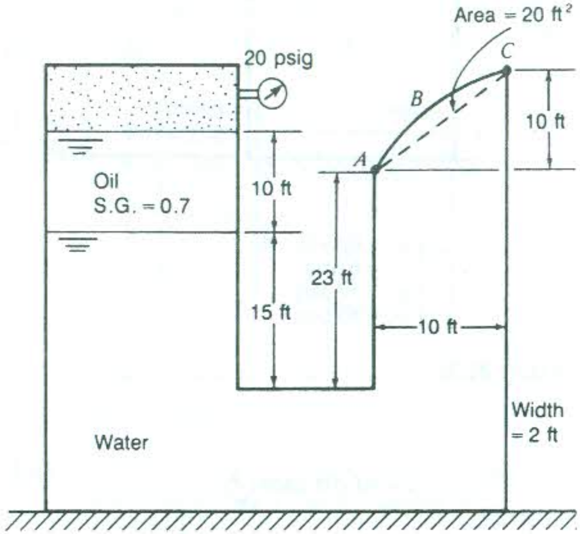


Figure P2.71

2.72 There are four compartments (Fig. P2.72) completely separated from each other. One-quarter of the sphere shown resides in each compartment. Find the

a. total vertical force from fluids.
b. total horizontal force from fluids.

***2.73** Find the shear force and bending moment on the gate *AB* at *A* in Fig. P2.73. The gate has a width of 1 m. *Hint: ds* (along door) $= \sqrt{dx^2 + dy^2} = [1 + (dy/dx)^2]^{1/2}\,dx$.

2.74 What is the total weight of barge and load (Fig. P2.74)? The barge is 6 m in width.

2.75 A wedge of wood having a specific gravity of 0.6 is forced into water by a 150 lb force. The wedge is 2 ft in width. What is the depth *d*?

2.76 A tank is filled to the edge with water. If a cube 600 mm on an edge and weighing 445 N is lowered slowly into the water until

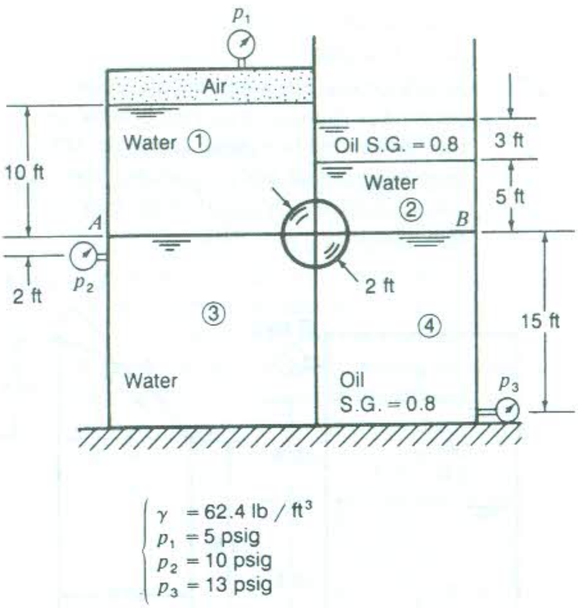


Figure P2.72

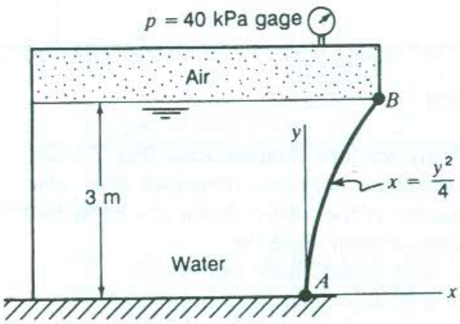


Figure P2.73

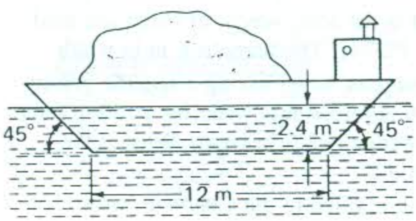


Figure P2.74

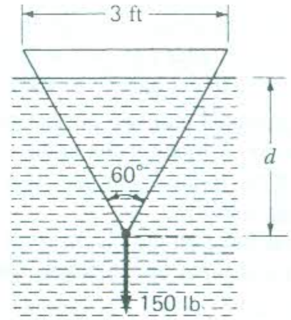


Figure P2.75

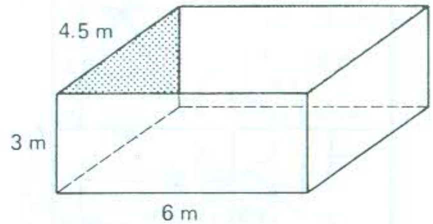


Figure P2.76

it floats, how much water flows over the edge of the tank if no appreciable waves are formed during the action? Neglect effects of adhesion at the edge of the tank.

2.77 A cube of material (Fig. P2.77) weighing 445 N is lowered into a tank containing a layer of water over a layer of mercury. Determine the position of the block when it has reached equilibrium.

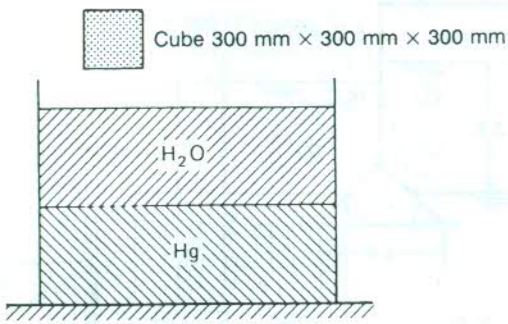


Figure P2.77

2.78 Explain why you cannot use Archimedes' principle in Prob. 2.62.

2.79 In Example 2.12 if 0.28 m^3 of gasoline is lost, what is the weight of the minimum amount of ballast that must be released so as to cause the bathyscaph to start rising? At a depth of 11.3 km, what is the pressure in atmospheres on the outside surface of the cockpit if we take γ of seawater having an average value of 10,150 N/m^3 over the depth? Finally, explain why the bathyscaph was designed using a liquid such as gasoline instead of a gas in the tank, and why the gasoline had to have "contact" with the seawater at B.

2.80 An iceberg has a specific weight of 9000 N/m^3 in ocean water, which has a specific weight of 10^4 N/m^3. If we observe a volume of 2.8×10^3 m^3 of the iceberg protruding above the free surface, what is the volume of the iceberg below the free surface of the ocean?

2.81 A *hydrometer* is a device that uses the principle of buoyancy to determine specific gravity S of a liquid. It is a device weighted by tiny metal spheres to have a total weight W. It has a stem of constant cross section which protrudes through the free surface. It is calibrated by marking the position of the free surface when floating in distilled water ($S = 1$) and by determining its submerged volume V_0. When floated in another liquid, the stem may sit lower or higher at the free surface from this position by distance Δh, as shown to the right in Fig. P2.81. Show that

$$\Delta h = \frac{V_0}{A_s}\frac{S-1}{S}$$

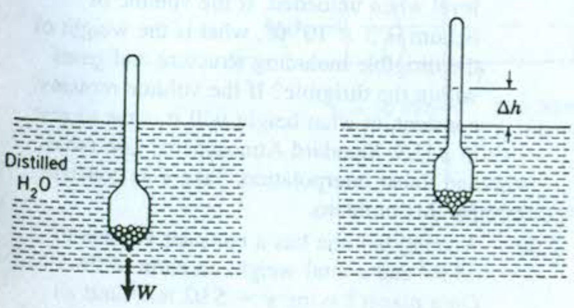


Figure P2.81

where A_s is the cross section of the stem and S is the specific gravity of the liquid. We can thus calibrate the stem to read specific gravity directly.

2.82 A rectangular tank of internal width 6 m is partitioned as shown in Fig. P2.82 and contains oil and water. If the specific gravity of oil is 0.82, what must h be? Next, if a 1000-N block of wood is placed in flotation in the oil, what is the rise of the free surface of the water in contact with the air?

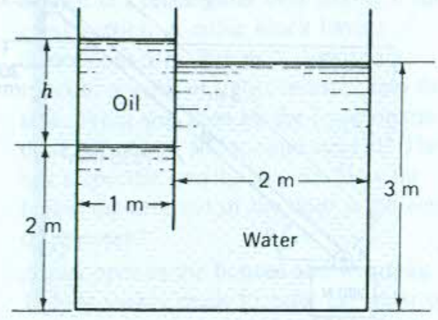


Figure P2.82

2.83 A balloon of 2.8×10^3 m^3 is filled with hydrogen having a specific weight of 1.1 N/m^3

a. What lift is the balloon capable of at the earth's surface if the balloon weighs 1335 N? The temperature is 15°C.

b. What lift is the balloon capable of at 9150 m U.S. Standard Atmosphere, assuming that the volume has increased 5 percent?

2.84 A wooden rod weighing 5 lb is mounted on a hinge below the free surface. The rod is 10 ft long and uniform in cross section, and the support is 5 ft below the free surface. At what angle α will it come to rest when

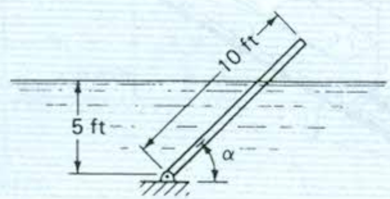


Figure P2.84

allowed to drop from a vertical position? The cross section of the rod is $\frac{3}{2}$ in^2 in area.

2.85 A block of material having a volume of $0.028\ m^3$ and weighing 290 N is allowed to sink in the water. A wooden rod of length 3.3 m and a cross section of 1935 mm^2 is attached to the weight and also to the wall. If the rod weighs 13 N, what will the angle θ be for equilibrium?

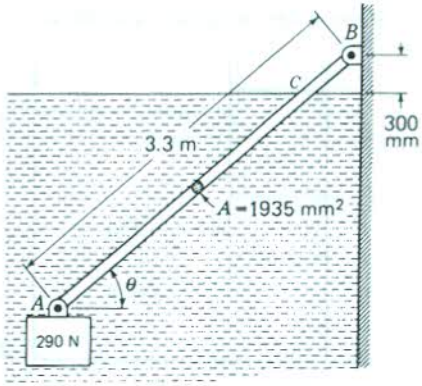


Figure P2.85

2.86 An object having the shape of a rectangular parallelepiped is being pushed slowly down an incline on narrow rails into water. The object weighs 4000 lb, and the coefficient of dynamic friction between the object and incline is 0.4. If hydrostatic pressure is assumed to exist all over the submerged surface of the object, express the force P as a function of x, the distance along the bottom surface submerged in the water, to keep the body moving at a constant slow speed along the incline. Begin calculations when water just touches the *top* surface of the object.

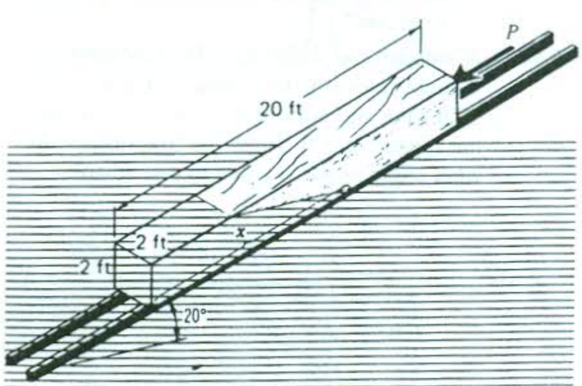


Figure P2.86

2.87 In Prob. 2.86 is there a position x for which there is impending rotation of the object as a result of buoyancy? If so, compute this value of x. The buoyant force as a function of x from the previous solution is $250x - 686$ lb and the force P from this solution is $159.2 - 8.4x$ pounds.

2.88 A hollow cone is forced into the water by a force F. Develop equations from which one may determine e. Neglect the weight of the cone and the thickness of the wall. Be sure to state any assumptions you make.

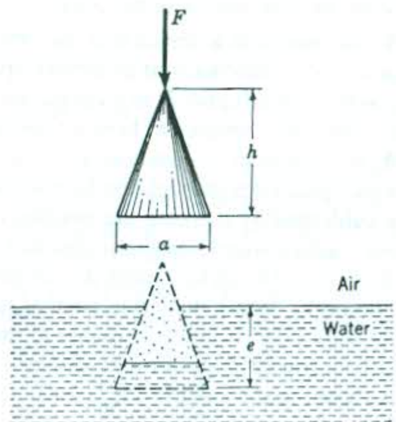


Figure P2.88

2.89 A dirigible has a lift of 130,000 lb at sea level when unloaded. If the volume of helium is $3 \times 10^6\ ft^3$, what is the weight of the dirigible including structure and gases within the dirigible? If the volume remains constant, at what height will it come to rest in a U.S. Standard Atmosphere? Use tables and linear interpolation. Take g as constant for this problem.

2.90 A small balloon has a constant volume of $15\ m^3$ and a total weight on earth of 35.5 N. On a planet having $g = 5.02\ m/s^2$ and an *isothermal* atmosphere with $\rho = 0.250\ kg/m^3$

and p = 10,000 Pa at sea level, what is the maximum load capacity at sea level? If released without this load, at what elevation will it come to rest in this atmosphere? Take g as costant for this problem.

2.91 The outside diameter of the pipe is 250 mm. It is submerged in water in the tank. Find the total force on the pipe from the water in the tank.

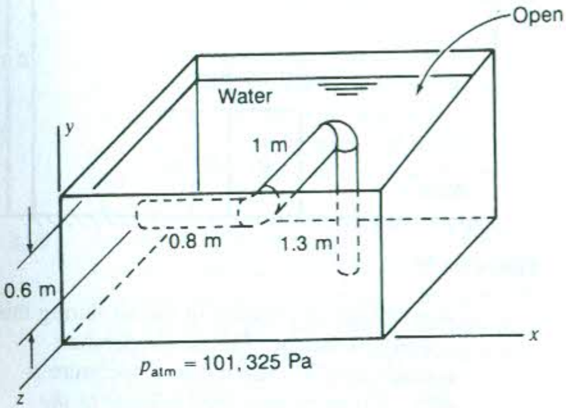


Figure P2.91

***2.92** A pipe system goes through a tank of water. The tank is closed on the top with air above it at a pressure p_1 = 200 kPa gage. Inside the pipe is a static gas with a uniform pressure p_2 = 500 kPa gage.

a. Find the force on the pipe from the static gas on the inside of the pipe.
b. Find the force on the outside surface of the pipe from the water.

Hint: The volume of the frustrum of a cone is

$$\tfrac{1}{3}[A_{\text{base}} + A_{\text{top}} + \sqrt{A_{\text{base}}A_{\text{top}}}](\text{height})$$

2.93 Shown is a rectangular tank having a square cross section. A cubic block having dimensions 1 m × 1 m × 1 m and a specific gravity of 0.9 is inserted into the tank. What will then be the force on the door A from all fluids contacting it? The oil has a specific gravity of 0.65. How far below the centroid of the door is the center of pressure?

2.94 A pail open at the bottom and weighing 10 N is slowly made to enter the water open end first until fully submerged. At what depth will the cylinder no longer return to the free surface from buoyant forces? Explain what happens after this elevation has been exceeded. Water is at 20°C. Air is initially at 20°C. The metal thickness of the

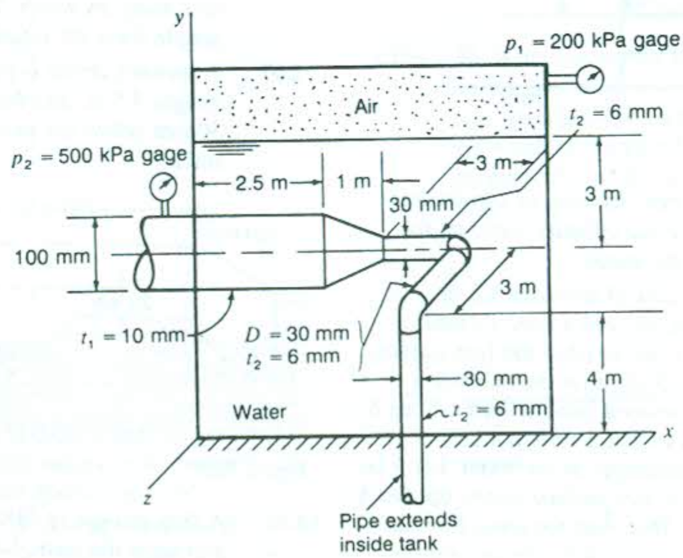


Figure P2.92

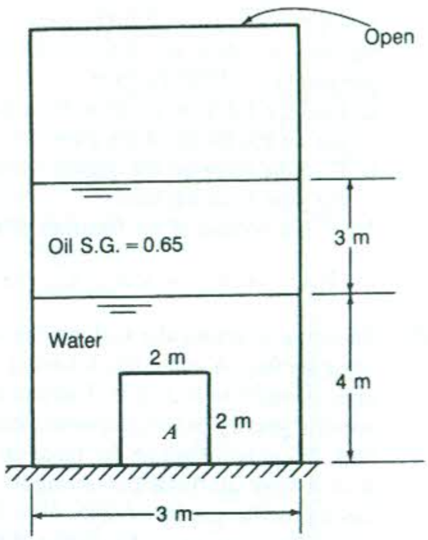


Figure P2.93

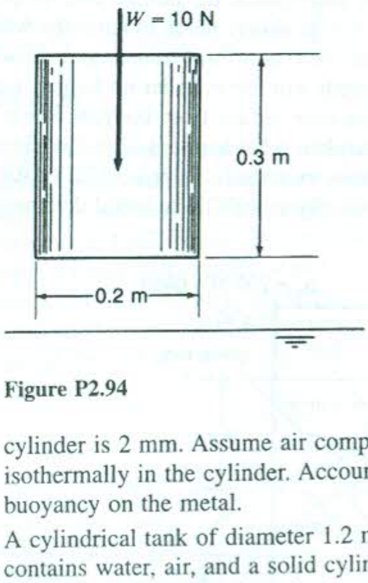


Figure P2.94

cylinder is 2 mm. Assume air compresses isothermally in the cylinder. Account for buoyancy on the metal.

2.95 A cylindrical tank of diameter 1.2 m contains water, air, and a solid cylinder A which initially just touches the free surface. Find the force F to move the cylinder a distance δ downward into the water. Keep δ small enough so that A does not get completely submerged in the water. Let h be the distance the free surface moves up. Get h in terms of δ. Then find the force P on the door B as a function of δ. The pressure of the air initially is $p_1 = 200{,}000$ Pa gage.

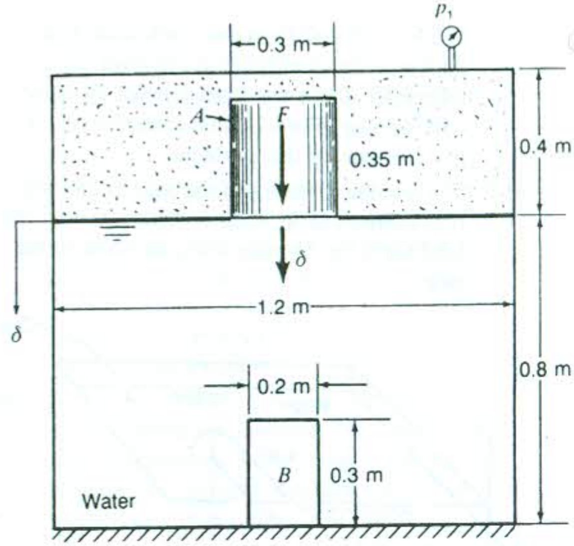


Figure P2.95

Any change in pressure of the air during this action is adiabatic. The air temperature initially is 60°C. The water temperature is 60°C. δ is to be measured *relative to the ground* from a level of the water at initial contact between A and water.

2.96 In Example 2.16 compute the metacentric height for a rotation about the symmetrical axis along its width. What is the righting couple for a 10° rotation about this axis?

2.97 A wooden object is placed in water. It weighs 4.5 N, and the center of gravity is 50 mm below the top surface. Is the object stable?

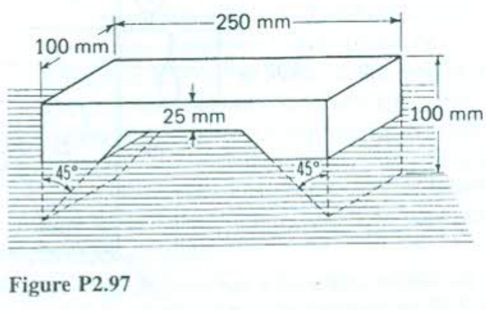


Figure P2.97

2.98 A ship weighs 18 MN and has a cross section at the water line as shown in Fig. P2.98. The center of buoyancy is 1.5 m

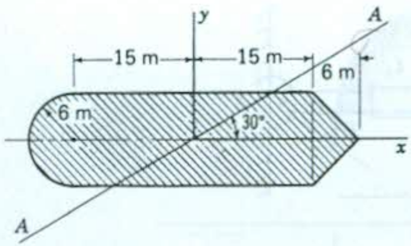


Figure P2.98

below the free surface, and the center of gravity is 600 mm above the free surface. Compute the metacentric heights for the x and y axes. Also determine the metacentric height for axis AA at an angle of 30° as shown.

2.99 A wooden cylinder of length 2 ft, diameter 1 in, and specific weight 20 lb/ft^3 is fastened to a cylinder of metal having a diameter of $\frac{1}{2}$ in, length of 1 ft, and specific weight of 200 lb/ft^3. Is this object stable in water for the orientation shown in Fig. P2.99?

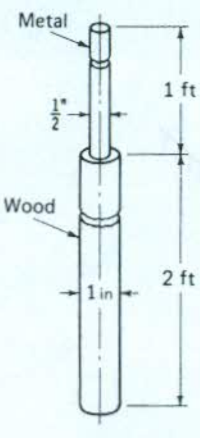


Figure P2.99

2.100 In Prob. 2.99, is there a specific gravity for which the object attains neutral stability? If so, compute this specific gravity.

2.101 A wooden block having a specific gravity of 0.7 is floating in water. A light rod at the center of the block supports a cylinder A

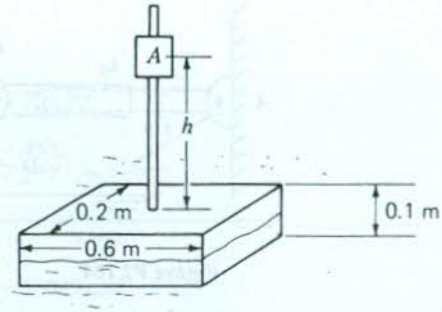


Figure P2.101

whose weight is 20 N. At what height h will there be neutral stability?

2.102 Consider a liquid which when stationary stratifies in such a way that the specific weight is proportional to the square root of the absolute pressure and which at the free surface has a specific weight of value γ_0 (see Fig. P2.48). First, prepare a program that interactively gives the force on a submerged door for a given width of the door b, a given height of the door h, and a given specific weight at the free surface γ. Have the program give the position of the center of pressure. Run your program for the following data:

$$b = 1 \text{ m} \quad h = 5.6 \text{ m}, \quad \text{and} \quad \lambda_0 = 9990 \text{ N/m}^3$$

2.103 Write a program for the resultant force on the gate acted on by water so that the user specifies h in meters, θ in degrees, and the width of the gate L in meters. Let the program also determine the moment about hinge A.

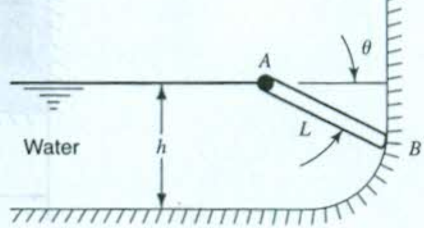


Figure P2.103

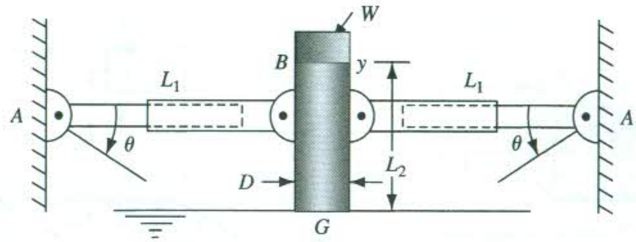


Figure P2.104

2.104 A tube having at one end an inserted disc of weight W and open at the other end is constrained to move vertically by two light rods AB. Write an interactive program giving the depth y the open end G descends into the water before coming to rest. Work the program for the following data:

$L_1 = 2$ ft $\quad L_2 = 8$ in $\quad W = 10$ lb $\quad D = 3$ in

Consider adiabatic compression of the air in the tube. Neglect all weights other than that of the disc.

2.105 A closed light cylindrical tank is floating in a cylindrical receptical supported by water, that is 8 in deep. What is the weight of the cylinder? Now, for specified angles α of the receptical, plot the required force F needed to slowly move the cylinder downward as a function of displacement y from $y = 0$ to $y = 2$ in for angles α of 20°, 30°, and 40°.

2.106 An empty light hemisphere is forced slowly into the water, trapping air as it descends. Treating the air as a perfect gas, plot the approximate force needed to force the hemisphere a distance downward starting from the position of the hemisphere as it just touches the free surface of the water and descending 500 mm thereafter. The air undergoes an adiabatic compression.

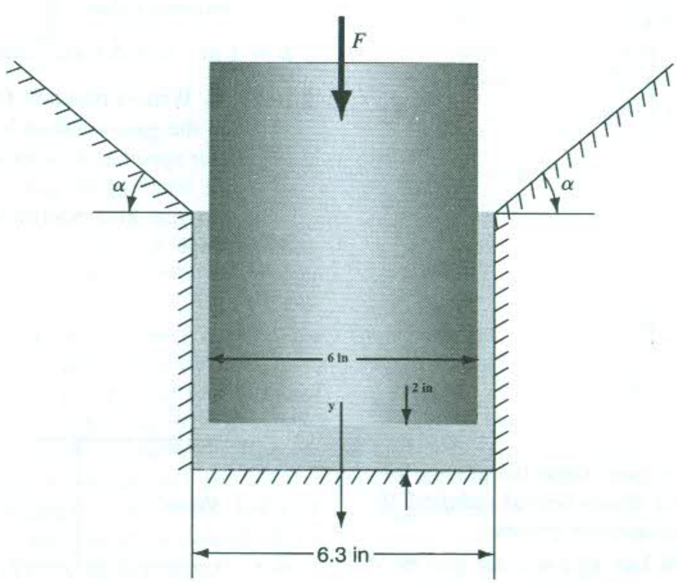


Figure P2.105

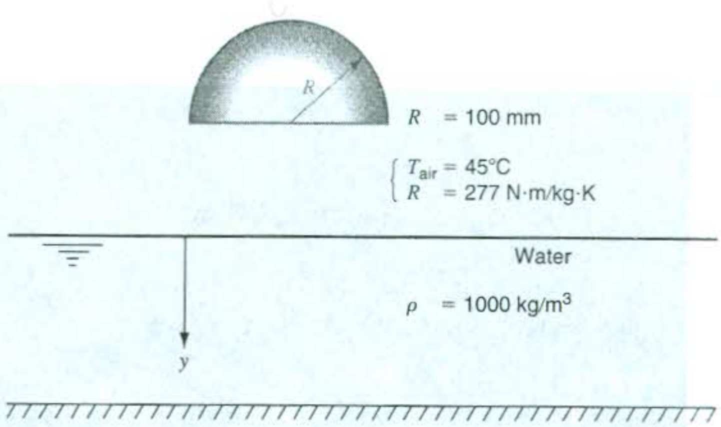


Figure P2.106

2.107 Plot the value of the resultant force on the circular door from the water as a function of height h of the free surface of the water, as the water is allowed to slowly leak from the tank. On the same axis plot the torque from the water about the base A of the door as the water is drained.

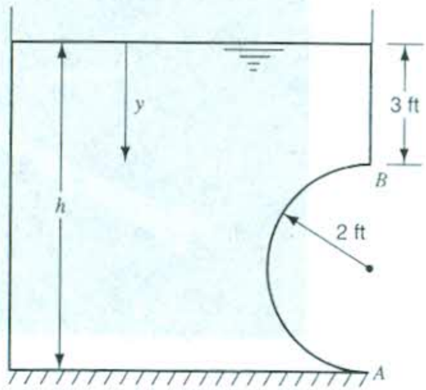
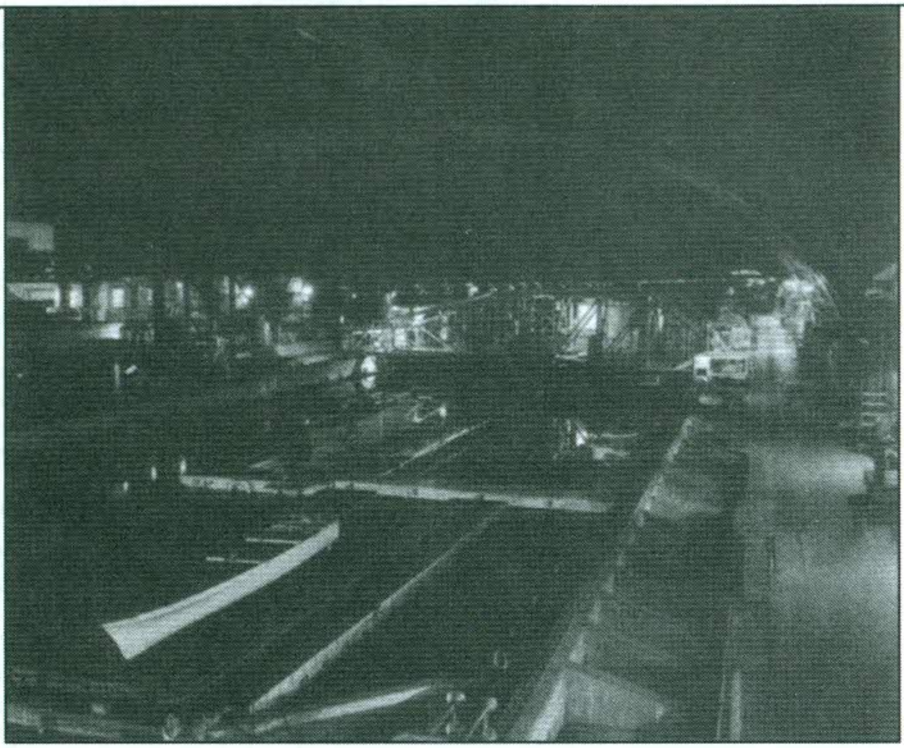


Figure P2.107



Shown is part of a towing tank facility of the Naval Ship Research and Development Center at Carderock, Maryland. The basin comprises three adjoining sections: (1) A deep water section 22 feet deep, 50.20 feet wide, and 889 feet long. (2) A shallow water section 10 feet deep, 50.96 feet wide, and 303 feet long. The depth of water can be varied. A 32 feet by 5 feet fitting out dock is located here. The photograph is taken for the shallow water section. (3) A turning basin in the form of a *J* in which self-propelled models can be allowed to maneuver. The carriage speed can move models up to speeds of 18 knots. In subsequent chapters, towing tanks will be referred to on a number of occasions.

CHAPTER 3

Foundations of Flow Analysis

3.1 THE VELOCITY FIELD

In particle and rigid-body dynamics we are able to describe the motion of each body in a separate and discrete manner. For instance, the velocity of the nth particle of an aggregate of particles moving in space can be specified by the scalar equations

$$\begin{aligned}(V_x)_n &= f_n(t)\\ (V_y)_n &= g_n(t)\\ (V_z)_n &= h_n(t)\end{aligned} \tag{3.1}$$

Note that the identification of a particle is easily facilitated with the use of a subscript. However, in a deformable continuum such as a fluid, there are, for practical purposes, an infinite number of particles whose motions are to be described, which makes this approach unmanageable; so we employ spatial coordinates to help identify particles in a flow. The velocity of all particles in a flow can therefore be expressed in the following manner:

$$\begin{aligned}V_x &= f(x, y, z, t)\\ V_y &= g(x, y, z, t)\\ V_z &= h(x, y, z, t)\end{aligned} \tag{3.2}$$

Specifying coordinates xyz and the time t and using these values in functions f, g, and h in Eq. 3.2, we can directly determine the velocity components of a fluid element at the particular position and time specified. The spatial coordinates thus take the place of the subscript n of the discrete systems studied in mechanics. This is called the *field approach*. If properties and flow characteristics at each position in space remain constant with time, the flow is called *steady flow*. A time-dependent flow, on the other hand, is designated as an *unsteady flow*. The steady-flow velocity field would then be given as

$$\begin{aligned}V_x &= f(x, y, z)\\ V_y &= g(x, y, z)\\ V_z &= h(x, y, z)\end{aligned} \tag{3.3}$$

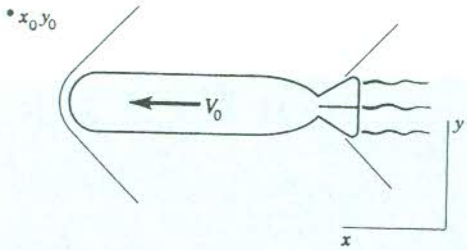


Figure 3.1
Unsteady-flow field relative to xy.

Often, a steady flow may be derived from an unsteady-flow field by simply changing the space reference. To illustrate this, examine the flow pattern created by a torpedo moving near the free surface through initially undisturbed water at constant speed V_0 relative to the stationary reference xy, as shown in Fig. 3.1. It can be seen that this is an unsteady-flow field, as viewed from xyz. Thus, the velocity at position x_0, y_0 in the field, for instance, will at one instant be zero and later, owing to the oncoming waves and wake of the torpedo, will be subjected to a complicated time variation. To establish a steady-flow field, we now consider a reference $\xi\eta$ *fixed* to the torpedo. The flow field relative to such a moving reference is shown in Fig. 3.2. The velocity at fixed position $\xi_0\eta_0$, as seen from the torpedo, clearly does not change with respect to time. This must be true since this position is fixed in an unchanging flow pattern as seen from the torpedo. Note from Fig. 3.2 that the water far upstream of the torpedo has a velocity relative to the torpedo and thus relative to the axes $\xi\eta$, which is equal to $-V_0$. You can now see that the transition from an unsteady flow, relative to reference xy depicted in Fig. 3.1, to a steady flow, relative to reference xy, could have been accomplished by superimposing a velocity $-V_0$ on the entire flow field of Fig. 3.1 to arrive at the steady field of Fig. 3.2. *This may be done any time a body is moving with constant speed through an initially undisturbed fluid.*

Flows are usually depicted graphically with the aid of *streamlines*. These lines are drawn so as to be always tangent to the velocity vectors of the fluid particles in a flow. This is illustrated in Fig. 3.3. For a steady flow the orientation of the streamlines will be fixed. Fluid particles, in this case, will proceed along paths coincident with the streamlines. In unsteady flow, however, an indicated streamline pattern

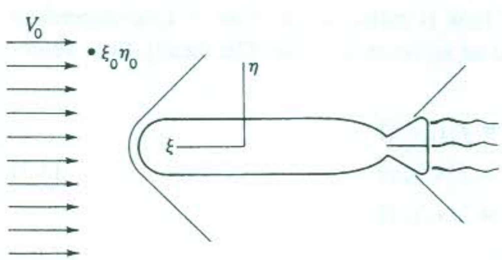


Figure 3.2
Steady-flow field relative to $\xi\eta$.

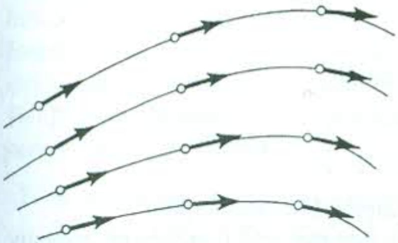


Figure 3.3
Streamlines.

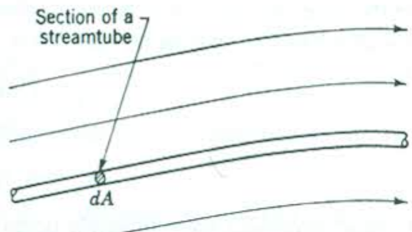


Figure 3.4
Streamtube.

yields only an instantaneous flow representation, and for such flow there will no longer be a simple correspondence between path lines and streamlines.

Streamlines proceeding through the periphery of an infinitesimal area at some time t will form a tube, which is useful in discussions of fluid phenomena. This is called the *streamtube,* which is illustrated in Fig. 3.4. From considerations of the definition of the streamline, it is obvious that there can be no flow through the lateral surface of the streamtube. In short, the streamtube acts like an impervious container of zero wall thickness and infinitesimal cross section. A continuum of adjacent streamtubes arranged to form a finite cross section is often called a *bundle of streamtubes*.

3.2 TWO VIEWPOINTS

In Sec. 3.1 we discussed various general aspects of the velocity field $\mathbf{V}(x, y, z, t)$. Two procedures will now be set forth by which the field may be utilized in computations involving the motion of fluid particles making up the flow. For instance, by stipulating fixed coordinates x_1, y_1, z_1 in the velocity-field functions and letting time pass, we can express the velocity of particles moving by this position at any time. Mathematically, this may be given by the formulation $\mathbf{V}(x_1, y_1, z_1, t)$. Hence, by this technique we express, at a fixed position in space, the velocities of a continuous "string" of fluid particles moving by this position. This viewpoint is sometimes called the *Eulerian* viewpoint.

On the other hand, to study "any one" particle in the flow one must "follow the particle." This means that x, y, z in the expression $\mathbf{V}(x, y, z, t)$ must not be fixed but must vary continuously in such a way as always to locate the particle. This approach is called the *Lagrangian* viewpoint. For any *particular* particle, $x(t)$, $y(t)$, and $z(t)$ become specific time functions which are different, in general, from corresponding time functions for other particles in the flow. Furthermore, the functions $x(t)$, $y(t)$, and $z(t)$ for a particular particle must have particular values $x(0)$, $y(0)$, and $z(0)$ at time $t = 0$ for that particular particle. In most cases, however, we do *not* identify a particular particle in our work, so for any one particle, $x(t)$, $y(t)$, and $z(t)$ are *unspecified* time functions which have the capability, nevertheless, when the form of the time functions and initial positions are chosen, of focusing on any particular particle. Thus we say in this case that

$$\begin{aligned} V_x &= f[x(t), y(t), z(t), t] \\ V_y &= g[x(t), y(t), z(t), t] \\ V_z &= h[x(t), y(t), z(t), t] \end{aligned} \tag{3.4}$$

In fluid dynamics there is ample occasion to employ both techniques.[1]

These considerations do not depend on whether the field is steady or unsteady and should not be confused with the conclusions of Sec. 3.1. You may note that the Eulerian viewpoint was utilized in that section in the discussion of both the steady and unsteady flows about the torpedo.

3.3 ACCELERATION OF A FLOW PARTICLE

We will soon use Newton's law for any one particle in a flow, and we will need the time rate of change of velocity of any one particle in a flow. In using the velocity field we will then have to use the Lagrangian viewpoint. Thus, noting that x, y, z are functions of time, we may establish the acceleration field by employing the chain rule of differentiation in the following way:

$$\mathbf{a} = \frac{d}{dt}\mathbf{V}(x, y, z, t) = \left(\frac{\partial \mathbf{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{V}}{\partial z}\frac{\partial z}{\partial t}\right) + \left(\frac{\partial \mathbf{V}}{\partial t}\right) \tag{3.5}$$

Since x, y, z are coordinates of any one particle, it is clear that dx/dt, dy/dt, and dz/dt must then be the scalar velocity components of any one particle and can be denoted as V_x, V_y, and V_z, respectively. Hence

$$\mathbf{a} = \left(V_x\frac{\partial \mathbf{V}}{\partial x} + V_y\frac{\partial \mathbf{V}}{\partial y} + V_z\frac{\partial \mathbf{V}}{\partial z}\right) + \left(\frac{\partial \mathbf{V}}{\partial t}\right) \tag{3.6}$$

The three scalar equations corresponding to Eq. 3.6 in the three cartesian-coordinate directions are, taking components of the vector $\mathbf{V}$,

$$\begin{aligned} a_x &= \left(V_x\frac{\partial V_x}{\partial x} + V_y\frac{\partial V_x}{\partial y} + V_z\frac{\partial V_x}{\partial z}\right) + \left(\frac{\partial V_x}{\partial t}\right) \\ a_y &= \left(V_x\frac{\partial V_y}{\partial x} + V_y\frac{\partial V_y}{\partial y} + V_z\frac{\partial V_y}{\partial z}\right) + \left(\frac{\partial V_y}{\partial t}\right) \\ a_z &= \left(V_x\frac{\partial V_z}{\partial x} + V_y\frac{\partial V_z}{\partial y} + V_z\frac{\partial V_z}{\partial z}\right) + \left(\frac{\partial V_z}{\partial t}\right) \end{aligned} \tag{3.7}$$

[1] A simple-minded way of thinking of the two viewpoints is to consider a golf tournament where the players are the "particles." If you station yourself as the observer at any particular tee in order to observe the various players coming by this location, you are using the Eulerian viewpoint. On the other hand, if you select your favorite player and move around the course with him/her for purposes of observation, you are using the Lagrangian viewpoint.

Now the acceleration **a** of any one particle is given in terms of the velocity field, the partial spatial derivatives, and the partial time derivative of **V**. But **V** is a function of x, y, z, and t. Hence the acceleration **a** is then given in terms of x, y, z and t and is thus also a field variable.

The acceleration of fluid particles in a flow field may be imagined as the superposition of two effects:

1. In expressions in the first parentheses on the right-hand sides of Eqs. 3.6 and 3.7, the *explicit* time variable t is held constant. Hence, in these expressions at a given time t, the field is assumed to become and remain steady. The particle, under such circumstances, is in the process of changing position in this steady field. It is as a result, undergoing a change in velocity because the velocity at various positions in this field will, in general, be different at any time t. This time rate of change of velocity due to changing position in the field is aptly called the *acceleration of transport,* or *convective acceleration.*
2. The term within the second parentheses in the acceleration equations does not arise from the change of particle position, but rather from the rate of change of the velocity field itself at the position occupied by the particle at time t. It is sometimes called the *local acceleration.*

The differentiation carried out in Eq. 3.6 is called the *substantial,* or *total, derivative.* To emphasize the fact that the time derivative is carried out as one follows the particle, the notation D/Dt is often used in place of d/dt. Hence, the substantial derivative of the velocity is given by $D\mathbf{V}/Dt$. The increased complexity over that which we experienced in mechanics of discrete particles is the price we pay for having, by necessity, brought in spatial coordinates to identify particles in a deformable continuous medium. It should be understood that the substantial derivative is by no means restricted to the velocity field vector. Thus for any vector field **H** associated with a flow we can say:

$$\frac{D\mathbf{H}}{Dt} = \left(V_x \frac{\partial \mathbf{H}}{\partial x} + V_y \frac{\partial \mathbf{H}}{\partial y} + V_z \frac{\partial \mathbf{H}}{\partial z} \right) + \frac{\partial \mathbf{H}}{\partial t}$$

Note that we have, in effect, two vector fields involved in this equation. There is first the field **H** undergoing the substantial derivative, and for any such vector field **H** associated with the flow there is always the fluid velocity field **V** whose components in the above equation facilitate *following any one particle* as one computes the rate of change of **H** for the particle. We have offered several problems with different **H** fields at the end of the chapter.

In many analyses, it is useful to think of a set of streamlines as part of a coordinate system. In such cases the letter s indicates the position of the particle along a particular streamline, and accordingly $\mathbf{V} = \mathbf{V}(s, t)$. Hence, for the acceleration of transport we have $(\partial\mathbf{V}/\partial s)(ds/dt)$, which gives the acceleration that results from the action of the particle's changing position along a streamline. The complete acceleration is then given as

$$\mathbf{a} = V\frac{\partial \mathbf{V}}{\partial s} + \frac{\partial \mathbf{V}}{\partial t} \qquad [3.8]$$

Let us consider the special case of steady flow, where, as we pointed out earlier, there is a fixed streamline pattern and where streamlines are the same as path lines. We can decompose the acceleration of transport vector for such flow into two scalar components by choosing one component a_T tangent to the path and the other component a_N normal to the path in the osculating plane.[2] You will recall from earlier mechanics courses that the acceleration component a_T can be given as

$$a_T = V\frac{dV}{ds} = \frac{1}{2}\frac{dV^2}{ds} \qquad [3.9]$$

and, taking the direction *toward* the center of curvature in the osculating plane as positive, that the other component of acceleration a_N can be given as

$$a_N = \frac{V^2}{R} \qquad [3.10]$$

where R is the radius of curvature. We will have occasion to use acceleration components in the ensuing chapters, particularly Chap. 11.

EXAMPLE 3.1

■ Problem Statement

To illustrate some of the definitions and ideas of Sec. 3.3, we examine a simple two-dimensional flow (see Fig. 3.5) with the upper boundary that of a rectangular hyperbola, given by the equation $xy = K$. Assume that the scalar components of the velocity field are known to be

$$\begin{aligned} V_x &= -Ax \\ V_y &= Ay \qquad A = \text{const} \\ V_z &= 0 \end{aligned} \qquad [a]$$

(Note that the flow is steady.) Determine the streamline pattern and the acceleration field.

■ Strategy

We will first determine the streamline equations, by considering the relation between the streamline slope and the velocity components. This will give us means of getting the acceleration components via the definition of the substantial derivative of the velocity vector field.

[2] The osculating plane at a particular point on a path is the limiting plane formed by the point and two additional points, on the path, ahead and behind, as they are brought ever closer to the particular point. See I. H. Shames, *Engineering Mechanics: Statics and Dynamics*, 4th ed., Prentice-Hall, Englewood Cliffs, NJ, Chap. 11.

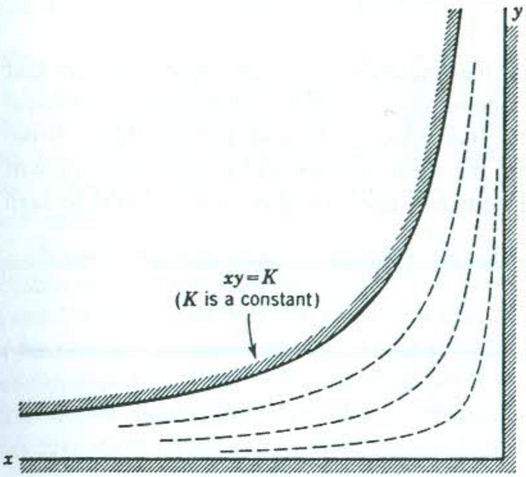


Figure 3.5
Two-dimensional flow showing streamlines.

■ Execution

By definition, the streamlines must have the same slope as the velocity vectors at all points. Equating these slopes, we get

$$\left(\frac{dy}{dx}\right)_{\text{str}} = \frac{V_y}{V_x} = -\frac{y}{x} \qquad \text{[b]}$$

Separating the variables and integrating, we have

$$\ln y = -\ln x + \ln C$$

Hence,

$$xy = C$$

Note that the streamlines form a family of rectangular hyperbolas. The wetted boundaries are part of the family, as is to be expected.

The components of acceleration may now easily be determined. Since this is steady flow, there will be only the acceleration of transport. Employing Eq. 3.7 under these conditions, we get

$$\begin{aligned} a_x &= (-Ax)(-A) + (Ay)(0) + (0)(0) = A^2x \\ a_y &= (-Ax)(0) + (Ay)(A) + (0)(0) = A^2y \\ a_z &= 0 \end{aligned} \qquad \text{[c]}$$

Hence

$$\mathbf{a} = A^2x\mathbf{i} + A^2y\mathbf{j} \qquad \text{[d]}$$

To give the acceleration of a particle at position $x'y'$ at any time, merely substitute x', y' into Eq. d.

■ Debriefing

This example should give us a clear picture of a streamline and the fact that a boundary will always form one of the streamlines. If the flow is inviscid there generally will be nonzero fluid velocity along such a streamline. If the flow is viscous, remember that the fluid velocity along this boundary will have the same velocity as the boundary itself (in this case it would be zero velocity).

EXAMPLE 3.2

■ Problem Statement

Given the velocity field

$$\mathbf{V}(x, y, z, t) = 10x^2\mathbf{i} - 20yx\mathbf{j} + 100t\mathbf{k} \text{ m/s}$$

determine the velocity and acceleration of a particle at position $x = 1$ m, $y = 2$ m, $z = 5$ m, and $t = 0.1$ s.

■ Strategy

We will use the definition of a *field* to get the desired velocity. We will use the definition of the *substantial derivative* to get the desired acceleration.

■ Execution

The velocity is determined by inserting the proper spatial coordinates and time into the vector velocity field to get a specific velocity as follows:

$$\mathbf{V} = (10)(1)\mathbf{i} - (20)(2)(1)\mathbf{j} + (100)(0.1)\mathbf{k} = 10\mathbf{i} - 40\mathbf{j} + 10\mathbf{k} \text{ m/s}$$

To get the acceleration of any one particle, we must use the *Lagrange* viewpoint to establish the acceleration field. Thus

$$\begin{aligned}\mathbf{a}(x, y, z, t) &= \left(V_x\frac{\partial \mathbf{V}}{\partial x} + V_y\frac{\partial \mathbf{V}}{\partial y} + V_z\frac{\partial \mathbf{V}}{\partial z}\right) + \left(\frac{\partial \mathbf{V}}{\partial t}\right)\\ &= [(10x^2)(20x\mathbf{i} - 20y\mathbf{j}) + (-20yx)(-20x\mathbf{j})] + 100\mathbf{k}\\ &= 200x^3\mathbf{i} + (-200x^2y + 400yx^2)\mathbf{j} + 100\mathbf{k} \text{ m/s}^2\end{aligned}$$

For the particle of interest, the acceleration is

$$\begin{aligned}\mathbf{a} &= (200)(1^3)\mathbf{i} + [-200(1^2)(2) + 400(2)(1^2)]\mathbf{j} + 100\mathbf{k}\\ &= 200\mathbf{i} + 400\mathbf{j} + 100\mathbf{k} \quad \text{m/s}^2\end{aligned}$$

■ Debriefing

Note that we used the field approach for both the velocity field and the acceleration field emerging from use of the substantial derivative.

3.4 IRROTATIONAL FLOW

Earlier we presented the velocity field $\mathbf{V}(x, y, z, t)$, permitting us to give the velocity of a particle of fluid anywhere in the flow field. We learned in physics that it is the *relative motion* between *adjacent* atoms and molecules that is related to bonding forces between atoms and molecules. Similarly in fluid flow, it is the *relative motion* between *adjacent* flow particles that is related most simply to stresses. We now examine this relative movement.

We wish to point out first that the word "adjacent" will connote for us particles infinitesimally apart. Accordingly, we have shown in Fig. 3.6 two adjacent particles A and B a distance $\mathbf{dr} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ apart at time t. To aid in the consideration of the relative movement between A and B, we have shown in Fig. 3.7 a rectangular parallelepiped for which $\overline{AB}$ is the diagonal. Now if we can effectively describe the deformation and rotation rates of this rectangular parallelepiped, we can in some way give the relative motion between A and B in terms of these rates. To accomplish

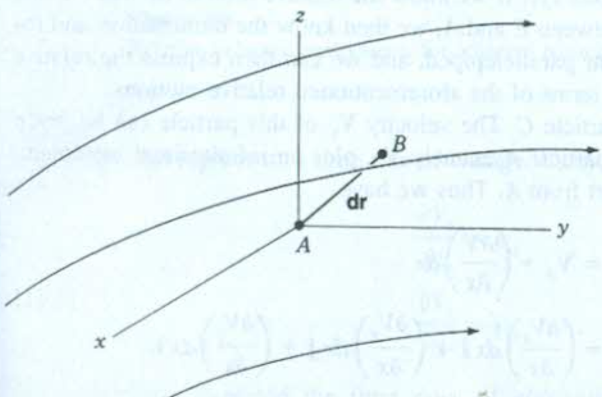


Figure 3.6
Adjacent particles A and B.

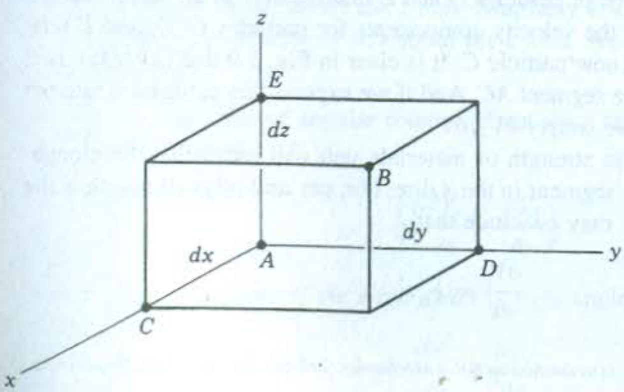


Figure 3.7
Adjacent particles along reference axes.

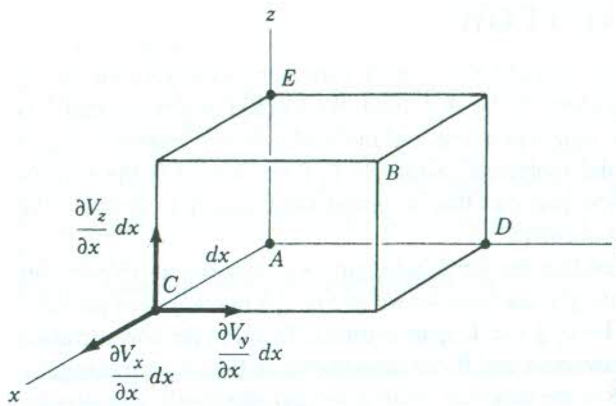


Figure 3.8
Components of $(\mathbf{V}_C - \mathbf{V}_A)$.

this, we have shown three additional particles C, D, and E at corners of the rectangular parallelepiped along axes xyz. If we know the relative motion between C and A, between D and A, and between E and A, we then know the deformation and rotation rates of the rectangular parallelepiped, and we can then express the relative motion between B and A in terms of the aforementioned relative motions.

Hence, we start with particle C. The velocity $\mathbf{V}_C$ of this particle can be given in terms of the velocity of particle A, namely $\mathbf{V}_A$, plus an infinitesimal increment, since C is a distance dx apart from A. Thus we have

$$\mathbf{V}_C = \mathbf{V}_A + \left(\frac{\partial \mathbf{V}}{\partial x}\right) dx$$
$$\therefore (\mathbf{V}_C - \mathbf{V}_A) = \left(\frac{\partial V_x}{\partial x}\right) dx\,\mathbf{i} + \left(\frac{\partial V_y}{\partial x}\right) dx\,\mathbf{j} + \left(\frac{\partial V_z}{\partial x}\right) dx\,\mathbf{k} \qquad [3.11]$$

The relative motion between C and A is $(\mathbf{V}_C - \mathbf{V}_A)$. It will be simplest to consider A as stationary and C as moving. The resulting conclusions will still be general. The components of $(\mathbf{V}_C - \mathbf{V}_A)$ as given by Eq. 3.11 are then shown in Fig. 3.8. We can set forth motion, respectively, of particles D and E relative to A in the same manner. In Fig. 3.9 we have shown the velocity components for particles C, D, and E relative to particle A. Consider now particle C. It is clear in Fig. 3.9 that $(\partial V_x/\partial x)\,dx$ is the rate of elongation of line segment AC. And if we express this elongation rate per *unit original length,* we have simply $\partial V_x/\partial x$.

But from your course in strength of materials you will recall that the elongation of an infinitesimal line segment in the x direction per unit original length is the normal strain ϵ_{xx}.[3] Thus we may conclude that

$$\frac{\partial V_x}{\partial x} = \dot{\epsilon}_{xx}$$

[3]See I. H. Shames and J. Pitarresi *Introduction to Solid Mechanics,* 3rd ed., Prentice-Hall, Englewood Cliffs, NJ, Chap. 3.

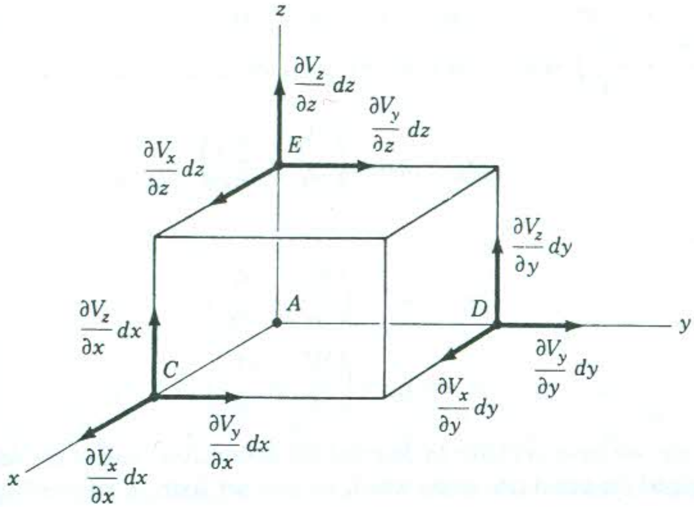


Figure 3.9
Relative velocity components for adjacent particles C, D, and E.

where the dot represents a time rate of change. Similarly we can say that

$$\frac{\partial V_y}{\partial y} = \dot{\epsilon}_{yy}$$

$$\frac{\partial V_z}{\partial z} = \dot{\epsilon}_{zz}$$

Thus we have depicted the time rates of *elongation* per unit original length (normal strain rates) of the sides of the rectangular parallelepiped. Next, we investigate the rate of *angular* change of the sides of the rectangular parallelepiped. Note in examining Fig. 3.9 that the velocity $(\partial V_y/\partial x)\,dx$ divided by dx is the angular velocity of AC about the z axis. Similarly at D, $(-\partial V_x/\partial y)\,dy$ divided by dy is the angular velocity of AD about the z axis. We can make two conclusions at this juncture:

1. The *average* rate of angular rotation about the z axis of the orthogonal line segments AC and AD is

$$\frac{1}{2}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \qquad [3.12]$$

2. *The rate of change of the angle CAD* (a right angle at time t) becomes

$$\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y} \qquad [3.13]$$

The second result, you may recall from strength of materials, where we had $\gamma_{xy} = \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)$, is the time *rate of change* of the *shear angle* γ_{xy} so that

$$\dot{\gamma}_{xy} = \dot{\gamma}_{yx} = \left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right)$$

Similarly,

$$\dot{\gamma}_{xz} = \dot{\gamma}_{zx} = \left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right)$$

$$\dot{\gamma}_{yz} = \dot{\gamma}_{zy} = \left(\frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y}\right)$$

Accordingly, we have available to describe the deformation rate of the rectangular parallelepiped the strain rate terms which we now set forth as follows:[4]

$$\begin{bmatrix} \dot{\epsilon}_{xx} & \frac{\dot{\gamma}_{xy}}{2} & \frac{\dot{\gamma}_{xz}}{2} \\ \frac{\dot{\gamma}_{yx}}{2} & \dot{\epsilon}_{yy} & \frac{\dot{\gamma}_{yz}}{2} \\ \frac{\dot{\gamma}_{zx}}{2} & \frac{\dot{\gamma}_{zy}}{2} & \dot{\epsilon}_{zz} \end{bmatrix} = \text{strain rate tensor} \qquad [3.14]$$

Now experience from solid mechanics and intuition indicates that it is the strain rate tensor part of relative motion that is most simply related to the stress tensor.

We have thus far described two kinds of relative movement between the adjacent particles along coordinate axes. The normal strain rates give the rate of stretching or shrinking of the sides of the associated rectangular parallelepiped, while the shear-strain rates give rate of change of angularity of the edges of the rectangular parallelepiped. What's left of the relative movement must then be rigid-body *rotation*. Thus, the expression

$$\frac{1}{2}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

is actually more than just the average rotation of line segments dx and dy about the z axis—it represents for a deformable medium what may be considered as the

[4]A note to the advanced reader: By using $\gamma/2$ instead of γ, you may have learned in strength of materials that the nine strain terms without dots form a symmetric second-order tensor. Taking time derivatives of each quantity and thereby forming an array of strain rates does not in any way alter the tensor character of the terms.

rigid-body angular velocity ω_z about the z axis.[5] This is,

$$\omega_z = \frac{1}{2}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \qquad [3.15]$$

Similarly for the other axes we have, by permuting indices,

$$\omega_x = \frac{1}{2}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) \qquad [3.16]$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) \qquad [3.17]$$

$$\therefore \boldsymbol{\omega} = \frac{1}{2}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\mathbf{i} + \frac{1}{2}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\mathbf{j} + \frac{1}{2}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\mathbf{k} \qquad [3.18]$$

Had we used a different coordinate system, we would have arrived at formulations which have a different form from Eqs. 3.15 to 3.18, but they would all pertain to the angular motion of fluid elements. Since the angular motion of fluid elements is a physical action not dependent on man-made coordinate systems, we have devised a vector operator called the *curl*[6] which when operating on a vector field **V** portrays twice the angular velocity. Thus Eq. 3.18 becomes

$$\boldsymbol{\omega} = \tfrac{1}{2}(\mathbf{curl\ V}) \equiv \tfrac{1}{2}\boldsymbol{\nabla} \times \mathbf{V} \qquad [3.19]$$

Note that Eq. 3.19 alludes to no particular coordinate system. Like the divergence operator and the gradient operator, the curl operator takes on a particular form when carried out in a particular coordinate system.[7] For instance, for cartesian coordinates we see from Eq. 3.18 that

$$\mathbf{curl\ A} \equiv \boldsymbol{\nabla} \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k} \qquad [3.20]$$

[5]The expression given by Eq. 3.15 is the *average* angular velocity of two orthogonal vanishingly small line segments dx and dy about the z axis. One can show that it is *also* the average angular velocity about the z axis of *all line segments* in the vanishingly small region dv. The "rigid body" interpretation obtains from the conclusion that if the fluid element in dv were imagined to become frozen at time t with the surrounding fluid made to simultaneously disappear, the frozen element would have the above angular velocity ω_z about the z axis at time t.

[6]The mathematical definition of the curl operator is given as

$$\mathbf{curl\ B} = -\lim_{\Delta V \to 0}\left[\frac{1}{\Delta V}\oiint_S \mathbf{B} \times \mathbf{dA}\right]$$

where ΔV is any volume in space and S is the surface enclosing the volume.

[7]It is to be pointed out that there are straightforward general methods for forming the various vector operators for orthogonal coordinate systems. These may be found in mathematics books dealing with vector analysis.

We will not at this time evaluate the curl operator on other coordinate systems. It should be pointed out the curl can be used on any continuous vector field, and the physical interpretation of the resulting curl vector so formed will depend on the particular field operated on. The physical picture of rotation of an element is thus restricted to the curl of the velocity field, but understanding this particular case will help you interpret the curl of other fields.

At this time, we define *irrotational* flows as those for which $\boldsymbol{\omega} = \mathbf{0}$ at each point in the flow. *Rotational* flows are those where $\boldsymbol{\omega} \neq \mathbf{0}$ at points in the flow. For irrotational flow, we require that

$$\boxed{\begin{aligned} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} &= 0 \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} &= 0 \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} &= 0 \end{aligned}} \qquad [3.21]$$

From Eq. 3.19 it becomes clear that another criterion for irrotationality, and the one we will use, is

$$\boxed{\textbf{curl V} = \mathbf{0}} \qquad [3.22]$$

Finally we point out that $2\boldsymbol{\omega}$ is often called the *vorticity* vector.

3.5 RELATION BETWEEN IRROTATIONAL FLOW AND VISCOSITY

We now discuss some conditions under which we can expect rotational and irrotational types of flows. A development of rotation in a fluid particle in an initially irrotational flow would require shear stress to be present on the particle surface. It will be recalled that shear stress on a surface may be evaluated for parallel flows by the relation $\tau = \mu(\partial V/\partial n)$. Thus the shear stress in such flows and in more general flows will depend on the viscosity of the fluid and the manner of spatial variation of velocity (or the so-called velocity gradient) in the region. For fluids of small viscosity, such as air, irrotational flow will then persist in regions where large velocity gradients are not encountered. This may very often be over a great part of the flow. For instance, for an airfoil section moving through initially undisturbed air (Fig. 3.10), the fluid motion relative to the airfoil is that of an irrotational flow over most of the field. However, it is known that no matter how small the viscosity, real fluids "stick" to the surface of a solid body. Thus at point *A* on the airfoil the fluid velocity must be zero relative to the

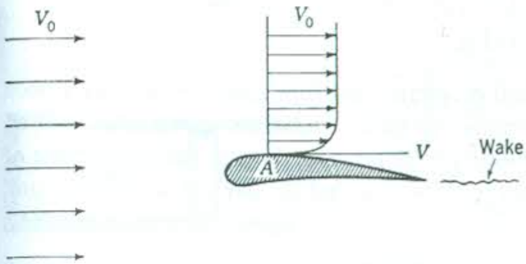


Figure 3.10
Velocity profile shows large velocity gradients near airfoil.

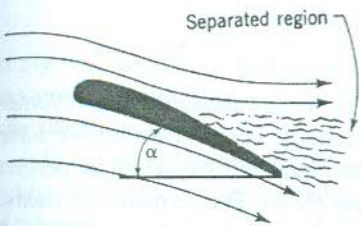


Figure 3.11
Flow separation for airfoil. The angle of attack is α.

airfoil, and at a comparatively short distance away it is almost equal to the free-stream velocity V_0. This is illustrated in the velocity profile of the diagram. Thus one sees that there is a thin region adjacent to the boundary where sizable velocity gradients must be present. Here, despite low viscosity, shear stresses of consequential magnitude are present, and the flow becomes rotational. This region adjacent to the solid boundary is called the *boundary layer*. It is fortunate, however, that much of the main flow is very often little affected by the flow conditions in the boundary layer, so that irrotational analysis may be carried out over a large part of the problem.

Another rotational-flow region may be found behind the trailing edge of the airfoil, where flows of different velocities from the upper and lower surfaces come into contact. Here again, large velocity gradients are present and consequently a rotational flow is present over a region behind the airfoil. This region is often called the *wake*.

Finally, we examine a condition called *separation*,[8] where the fluid flow cannot follow the boundary smoothly, as illustrated in Fig. 3.11 in the case of the airfoil at high angle of attack. Inside the separated regions we can again expect rotational flow.

In the flow shown in Fig. 3.11, it may be that the flow downstream of the separation point has regions of relatively small velocity gradients (hence small shear stress), where the flow is rotational. In the complete absence of further viscous action this rotation would persist indefinitely, so one may admit with good reason the theoretical possibility of frictionless rotational flow.

[8]The boundary layer and the separation process will be discussed at length in Chap. 12.

3.6 BASIC AND SUBSIDIARY LAWS FOR CONTINUOUS MEDIA

Now that means for describing fluid properties and flow characteristics have been established, we turn to the considerations of the interrelations among scalar, vector, and tensor quantities that we have set forth. Experience dictates that in the range of engineering interest four *basic laws* must be satisfied for any continuous medium. These are:

1. Conservation of matter (continuity equation).
2. Newton's second law (momentum and moment-of-momentum equations).
3. Conservation of energy (first law of thermodynamics).
4. Second law of thermodynamics.

In addition to these general laws, there are numerous *subsidiary laws*, sometimes called *constitutive* relations, that apply to specific types of media. We have already discussed two subsidiary laws, namely, the equation of state for the perfect gas and Newton's viscosity law for certain viscous fluids. Furthermore, for elastic solids there is the well-known Hooke's law, which you studied in strength of materials.

3.7 SYSTEMS AND CONTROL VOLUMES

In employing the basic and subsidiary laws, either one of the following modes of application may be adopted:

1. The activities of each and every given mass must be such as to satisfy the basic laws and the pertinent subsidiary laws.
2. The activities in each and every volume in space must be such that the basic laws and the pertinent subsidiary laws are satisfied.

In the first instance the laws are applied to an identified quantity of matter called the *system*. A system may change shape, position, and thermal condition but must *always entail the same matter*. For example, one may choose the steam in an engine cylinder (Fig. 3.12) after the cutoff[9] to be the system. As the piston moves, the volume of the system changes but there is no change in the quantity and identity of mass.

For the second case, a definite volume, called the *control volume*, is designated in space, and the boundary of this volume is known as the *control surface*.[10] The amount and identity of the matter in the control volume may change with time, but the shape of the control volume to be used in this text is fixed.[11] For instance, to

[9]No further addition of steam takes place after cutoff during the expansion stroke of the steam engine.

[10]In some thermodynamics texts the term *closed system* corresponds to our *system* and *open system* corresponds to our *control volume*.

[11]Some problems can be solved by employing a control volume of variable shape. However, in this text the control volume will always have a fixed shape.

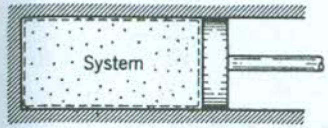


Figure 3.12
A system.

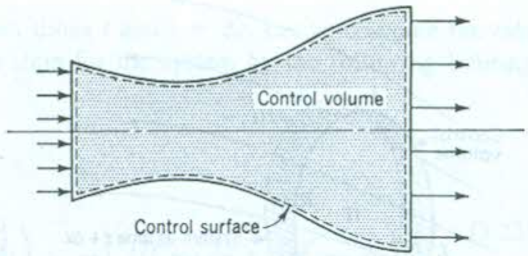


Figure 3.13
Control volume for the inside of a nozzle.

study flow through a nozzle, one could choose, as a control volume, the interior of the nozzle as shown in Fig. 3.13. We note that the control volume and the system can be infinitesimal.

In rigid-body mechanics it was the system approach (at that time called the *free-body diagram*) that was invariably used, since it was easy and direct to identify the rigid body, or portions thereof, in the problem and to work with each body as a discrete entity. However, since infinite numbers of particles having complicated motion relative to each other must be dealt with in fluid mechanics, it will often be advantageous to use control volumes in certain computations.

3.8 A RELATION BETWEEN THE SYSTEM APPROACH AND THE CONTROL-VOLUME APPROACH

In Sec. 3.2 we presented two viewpoints involving vector fields associated with a velocity field. These viewpoints allow us either to observe particles moving by a fixed position in space or to follow any one particle. We will now consider these viewpoints for *aggregates* of fluid elements constituting a finite mass where, in following the aggregate as per the Lagrange viewpoint, we are using the system approach. On the other hand, in stationing ourselves and observing in a finite region of space as per the Eulerian viewpoint, we are adopting the control-volume approach. We will now be able to relate the system approach and the control-volume approach for certain fluid and flow properties which we next describe.

In thermodynamics one usually makes a distinction between those properties of a substance whose measure depends on the amount of mass of the substance present and those properties whose measure is independent of the amount of mass of the substance present. The former are called *extensive* properties; the latter are called *intensive* properties. Examples of extensive properties are weight, momentum, volume, and energy. Clearly, changing the amount of mass directly changes the measure of these properties, and it is for this reason that we think of extensive properties as directly associated with the material itself. For each extensive variable such as

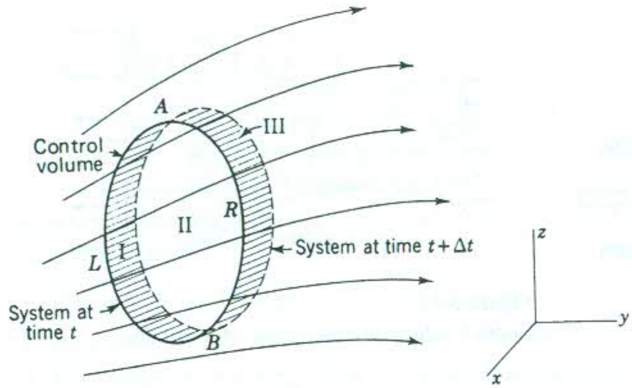


Figure 3.14
Simplified view of a moving system.

volume V and energy E, one can introduce by *distributive measurements* the corresponding intensive properties, namely, volume per unit mass v and energy per unit mass e, respectively. Thus we have $V = \iiint v\rho\, dv$[12] and $E = \iiint e\rho\, dv$. Clearly, v and e do not depend on the amount of matter present and are hence the intensive quantities related to the extensive properties V and E by distributive measure. Also, such quantities are termed *specific,* i.e., specific volume and specific energy, and are generally denoted by lowercase letters. Furthermore, such properties as temperature and pressure are by their *mass-independent nature* already in the category of the intensive property. Thus any portion of a metal bar at uniform temperature T_0 also has the same temperature T_0. Nor does the pressure of 1 ft^3 of air in a 10-ft^3 tank at uniform pressure p_0 differ from the pressure of 3 ft^3 of air in the tank. It is with *extensive* properties that we will now relate the system approach with the control-volume approach.

Consider next an arbitrary flow field $\mathbf{V}(x, y, z, t)$ as seen from some reference xyz wherein we observe a *system* of fluid of finite mass at times t and $t + \Delta t$, as shown in a highly idealized manner in Fig. 3.14 by the full line curve and the dashed line curve, respectively. The streamlines correspond to those at time t. In addition to this system, we will consider that the volume in space occupied by the system at time t is a *control volume fixed* in position and shape in xyz. Hence, at time t our system is identical to the fluid inside our control volume, shown by the full line curve. Let us now consider some arbitrary extensive property N of the fluid for the purpose of relating the rate of change of this property for the system with the variations of this property associated with the control volume. The distribution of N per unit mass will be given as η, such that $N = \iiint \eta\rho\, dv$ with dv representing an element of volume.

To do this, we have divided up the overlapping systems at time $t + \Delta t$ and at time t into three regions, as you will note in Fig. 3.14, where region II

[12] In this text we use v to represent specific volume and dv to represent the volume of a fluid element. Although the same letter is used in both terms, there should be no confusion if the terms are taken in context.

is common to the system at both times t and $t + \Delta t$. Let us compute the rate of change of N with respect to time for the system by the following limiting process:

$$\left(\frac{dN}{dt}\right)_{\text{system}} = \frac{DN}{Dt}$$

$$= \lim_{\Delta t \to 0}\left[\frac{\left(\iiint\limits_{\text{III}} \eta\rho\, dv + \iiint\limits_{\text{II}} \eta\rho\, dv\right)_{t+\Delta t} - \left(\iiint\limits_{\text{I}} \eta\rho\, dv + \iiint\limits_{\text{II}} \eta\rho\, dv\right)_{t}}{\Delta t}\right] \quad [3.23]$$

We may use the rule that the sum of the limits equals the limit of the sums to rearrange the equation above in the following manner:

$$\frac{DN}{Dt} = \lim_{\Delta t \to 0}\left[\frac{\left(\iiint\limits_{\text{II}} \eta\rho\, dv\right)_{t+\Delta t} - \left(\iiint\limits_{\text{II}} \eta\rho\, dv\right)_{t}}{\Delta t}\right]$$

$$+ \lim_{\Delta t \to 0}\left[\frac{\left(\iiint\limits_{\text{III}} \eta\rho\, dv\right)_{t+\Delta t}}{\Delta t}\right] - \lim_{\Delta t \to 0}\left[\frac{\left(\iiint\limits_{\text{I}} \eta\rho\, dv\right)_{t}}{\Delta t}\right] \quad [3.24]$$

Each one of the limiting processes above will now be considered separately. In the first one, we see on noting that $(\iiint_{\text{II}} \eta\rho\, dv)$ is a function of time that we have here by definition a partial time derivative of this function of time. And as $\Delta t \to 0$, the volume II becomes that of the control volume and the subscript II is replaced by the subscript CV. Also, as $\Delta t \to 0$, the time derivative is taken at time t. Accordingly, we can say that

$$\lim_{\Delta t \to 0}\left[\frac{\left(\iiint\limits_{\text{II}} \eta\rho\, dv\right)_{t+\Delta t} - \left(\iiint\limits_{\text{II}} \eta\rho\, dv\right)_{t}}{\Delta t}\right] = \frac{\partial}{\partial t}\iiint\limits_{\text{CV}} \eta\rho\, dv \quad [3.25]$$

In the next limiting process of Eq. 3.24, we can consider the integral $(\iiint_{\text{III}} \eta\rho\, dv)_{t+\Delta t}$ to approximate the amount of property N that has crossed part of the control surface, which we have shown diagrammatically as ARB in Fig. 3.14 during the time Δt, so the ratio $(\iiint_{\text{III}} \eta\rho\, dv)_{t+\Delta t}/\Delta t$ approximates the average rate of efflux of N across ARB during the interval Δt. In the limit as $\Delta t \to 0$, this ratio becomes the *exact* rate of *efflux* of N through the control surface. Similarly, in considering the last limiting process of Eq. 3.24, we can consider for flows with continuous-flow characteristics and properties that the integral $(\iiint_{\text{I}} \eta\rho\, dv)_t$ approximates the amount of N that has passed *into* the control volume during Δt through the remaining portion of the control surface, which we have shown diagrammatically in Fig. 3.14 as ALB. In the limit, the ratio $(\iiint_{\text{I}} \eta\rho\, dv)_t/\Delta t$ then becomes the *exact* rate of *influx* of N into the control volume

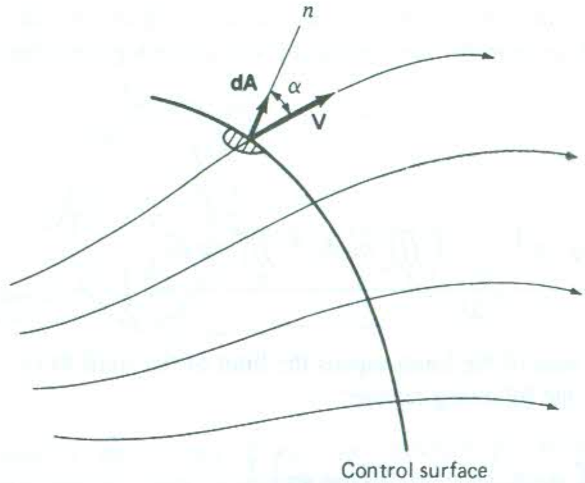


Figure 3.15
Interface dA at control surface at time t.

at time[13] t. Hence, the last two integrals of Eq. 3.24 give the *net* rate of *efflux* of N from the control volume at time t as

$$\lim_{\Delta t \to 0}\left[\frac{\left(\iiint_{\mathrm{III}} \eta\rho\, dv\right)_{t+\Delta t}}{\Delta t}\right] - \lim_{\Delta t \to 0}\left[\frac{\left(\iiint_{\mathrm{I}} \eta\rho\, dv\right)_{t}}{\Delta t}\right] = \text{Net efflux rate of } N \text{ from } CV \qquad [3.26]$$

We thus see that by these limiting processes, we have equated the rate of change of N for a *system* at time t with the sum of two things:

1. The rate of change of N inside the control volume having the shape of the system at time t (Eq. 3.25).
2. The rate of efflux of N through the control surface at time t (Eq. 3.26).

We will now express Eq. 3.26 in a more compact, useful form. In this regard, consider Fig. 3.15, where we have a steady-flow velocity field and a portion of a control surface. An area **dA** on this surface has been shown. Now this area is also the interface of fluid that is just touching the control surface at the time t shown in the diagram. In Fig. 3.16 we have shown that interface of fluid at time $t + dt$. Note that the interface has moved a distance $V\,dt$ along a direction tangent to the streamline at that point. The volume of fluid dv that occupies the region swept out by dA in time dt thus forming a streamtube is

$$dv = (V\,dt)(dA\cos\alpha)$$

[13]Hence it is *minus* the *efflux* of N through ALB.

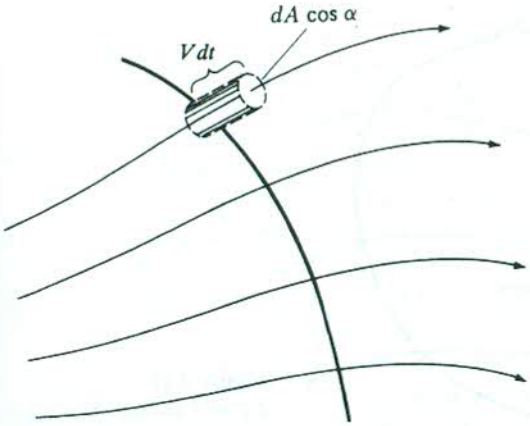


Figure 3.16
Interface dA at control surface at time $t + dt$.

Using the definition of the dot product, this becomes

$$dv = \mathbf{V} \cdot \mathbf{dA}\, dt$$

It should be apparent that dv is the volume of fluid that has crossed dA of the control surface in time dt. Multiplying by ρ and dividing by dt then gives the instantaneous rate of mass flow of fluid, $\rho\mathbf{V} \cdot \mathbf{dA}$, leaving the control volume through the indicated area dA.

The *efflux rate* of N through the control surface can be given approximately as[14]

$$\text{Efflux rate through CS} \approx \iint_{ARB} \eta(\rho\mathbf{V} \cdot \mathbf{dA})$$

Note next that for fluid *entering* the control volume (see Fig. 3.17) the expression $\rho\mathbf{V} \cdot \mathbf{dA}$ must be negative because of the dot product. Hence, the *influx rate* expression of N through the control surface requires a negative sign to make the result the positive value that we know must exist. Hence, we have

$$\text{Influx rate through CS} \approx -\iint_{ALB} \eta(\rho\mathbf{V} \cdot \mathbf{dA})$$

The approximate *net efflux* rate of N is then

$$\begin{aligned}\text{Net efflux rate} &\approx \text{efflux rate on } ARB - \text{influx rate on } ALB\\ &= \iint_{ARB} \eta(\rho\mathbf{V} \cdot \mathbf{dA}) - \left[-\iint_{ALB} \eta(\rho\mathbf{V} \cdot \mathbf{dA})\right]\end{aligned}$$

[14]Considering the units of the expression $\eta(\rho\mathbf{V} \cdot \mathbf{dA})$, we get

$$\eta\left(\frac{\text{units of } N}{\text{mass}}\right)\rho\mathbf{V} \cdot \mathbf{dA}\left(\frac{\text{mass}}{\text{unit time}}\right)$$

which is the efflux of N per unit time through $\mathbf{dA}$.

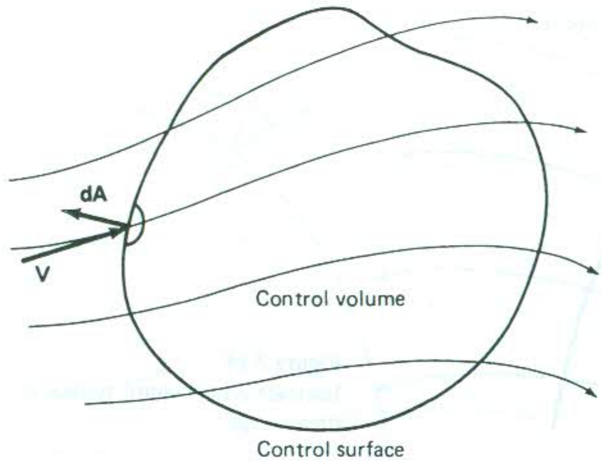


Figure 3.17
Control surface showing influx of mass.

In the limit as $\Delta t \to 0$, the approximations become exact, so we can express the right side of the equation above as $\oiint_{CS} \eta(\rho \mathbf{V} \cdot \mathbf{dA})$, where the integral is a closed surface integral over the entire control surface. Thus Eq. 3.26 can now be given as

$$\text{Net efflux rate of } N \text{ from CV} = \oiint_{CS} \eta(\rho \mathbf{V} \cdot \mathbf{dA}) \qquad [3.27]$$

It is to be pointed out that the development of Eq. 3.27 was made for simplicity for a steady-flow velocity field. However, it also holds for unsteady flow, since unsteady effects are of second order for this development. Now using Eqs. 3.27 and 3.25 for the various limiting processes, we can go back to Eq. 3.23 and state that

$$\boxed{\frac{DN}{Dt} = \oiint_{CS} \eta(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \iiint_{CV} \eta \rho \, dv} \qquad [3.28]$$

This is called the *Reynolds transport* equation.[15] This equation permits us to change from a system approach to a control-volume approach.

You will note in the development that the velocity field was measured relative to some reference *xyz* and the control volume was *fixed* in this reference. This makes it clear that the fluid velocity **V** in the equation above *is in effect measured relative to the control volume*. Furthermore, you will recall from mechanics that the time

[15] Although the Reynolds transport equation has been carefully developed from a mathematical point of view, it does have a rather straightforward physical interpretation. We can illustrate this most simply by considering your classroom as the control volume and the system consisting of all the students in the classroom at any time *t*. Let *N* be the mass of the system. After the bell has rung for the end of the class period, there will be, at time *t*, an efflux rate of mass through the doorways (part of the control surface) with a resulting rate of change of mass inside the classroom. The Reynolds transport equation requires that $dN/dt = 0$ at any time *t* since we are not destroying students nor are we creating students. Thus, the efflux rate of mass plus the rate of change of mass inside at this time *t* clearly should be zero. (Would you have it any other way?)

rate of change of a vector quantity depends on the reference from which the change is observed. This is an important consideration for us here, since N (and η) can be a vector quantity (as, for example, momentum). Since the system moves in accordance with the velocity field given relative to xyz in our development, we see that the time rate of change of N is observed also from the xyz reference. Or, a more important conclusion, *the time rate of change of N is in effect observed from the control volume. Thus all velocities and time rates of change of Eq. 3.28 are those seen from the control volume.* Since we could have used a reference xyz having any arbitrary motion in the development above, it means that our control volume can have any motion whatever. Equation 3.28 will then instantaneously be correct if we measure the time derivatives and velocities relative to the control volume, no matter what the motion of the control volume may be. Finally, it can be shown that for an *infinitesimal control volume,* and an *infinitesimal system,* Eq. 3.28 reduces to an identity. This will explain why the system and control-volume equations as developed in subsequent chapters become redundant for infinitesimal considerations.

In Chaps. 4 and 5 we formulate the control-volume approach for the basic laws mentioned earlier by starting in each case with the familiar system formulation and extending it with the aid of the Reynolds transport equation to the control-volume formulation. As you do this several times in the next Chap. 4, you will develop a greater physical feel for the Reynolds transport equation, which may seem at this time "artificial." Perhaps the realization that all human efforts to explain nature analytically are artificial may be of some comfort. Two additional "artificialities" will now be presented to permit us to use the basic laws, soon to be developed, with greater effect.

3.9 ONE- AND TWO-DIMENSIONAL FLOWS

In every analysis a hypothetical substance or process is set forth which lends itself to mathematical treatment while still yielding results of practical value. We have already discussed the continuum concept. Now, simplified flows are set forth, which, when used with discretion, will permit the use of highly developed theory on problems of engineering interest.

One-dimensional flow is a simplification where all properties and flow characteristics are assumed to be expressible as functions of *one space coordinate* and *time*. The position is usually the location along some path or conduit. For instance, a one-dimensional flow in the pipe shown in Fig. 3.18 would require that the velocity, pressure, and so forth be constant over any given cross section at any given time, and vary only with s at this time t.

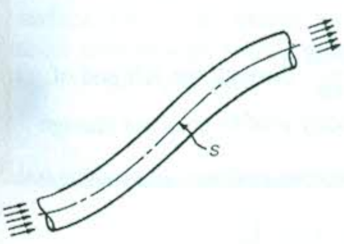


Figure 3.18
One-dimensional (1-D) flow.

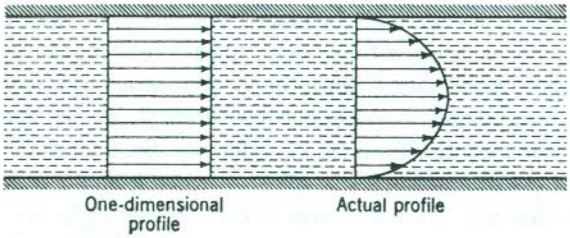


Figure 3.19
Comparison of 1-D flow and actual flow.

In reality, flow in pipes and conduits is never truly one dimensional, since the velocity will vary over the cross section. Shown in Fig. 3.19 are the respective velocity profiles of a truly one-dimensional flow and that of an actual case. Nevertheless, if the departure is not too great or if average effects at a cross section are of interest, one-dimensional flow may be assumed to exist. For instance, in pipes and ducts this assumption is often acceptable where

1. Variation of cross section of the container is not too excessive.
2. Curvature of the streamlines is not excessive.
3. Velocity profile is known not to change appreciably along the duct.

Two-dimensional flow is distinguished by the condition that all properties and flow characteristics are functions of two cartesian coordinates, say, x, y, and time, and hence do not change along the z direction at a given instant. All planes normal to the z direction will, at the given instant, have the same streamline pattern. The flow past an airfoil of infinite aspect ratio[16] or the flow over a dam of infinite length and uniform cross section are mathematical examples of two-dimensional flows. Actually, in a real problem a two-dimensional flow is often assumed over most of the airfoil or dam, and "end corrections" are made to modify the results properly.

EXAMPLE 3.3

■ Problem Statement

Consider a viscous, steady flow through a pipe (Fig. 3.20). We will learn in Chap. 11 that the velocity profile forms a paraboloid about the pipe centerline, given as

$$V = -C\left(r^2 - \frac{D^2}{4}\right) \text{ m/s} \qquad \text{[a]}$$

where C is a constant.

a. What is the flow rate of mass $\dfrac{DM}{Dt}$ through the left end of the control surface, shown dashed?

b. What is the flow rate of kinetic energy $\dfrac{DKE}{Dt}$ through the left end of the control surface? Assume that the velocity profile does not change along the pipe.

[16]A wing of constant cross section and infinite length.

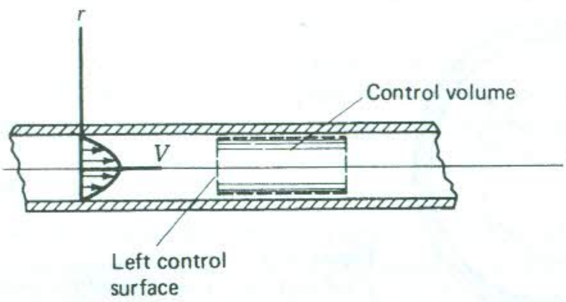


Figure 3.20
Steady viscous flow in a pipe.

■ Strategy

We shall consider in the cross section of the pipe a concentric ring of infinitesimal thickness. Because the velocity depends only on the variable r, the velocity will be constant through the ring. The integration of the mass flow through the rings covering the cross section will now be simple.

Next, using the intensive property for kinetic energy, and then including it in the integral for the mass flow, we will get on integration the kinetic energy flow rate in the pipe.

■ Execution

In Fig. 3.21, we have shown a cross section of the pipe. Using infinitesimal circular rings, we can say, noting that **V** and **dA** are colinear but of opposite sense,

$$\frac{DM}{Dt} = \iint \rho \mathbf{V} \cdot \mathbf{dA} = \rho \int_0^{D/2} C\left(r^2 - \frac{D^2}{4}\right) 2\pi r\, dr$$

$$\frac{DM}{Dt} = 2\pi\rho C\left[\frac{r^4}{4} - \frac{D^2}{4}\frac{r^2}{2}\right]_0^{D/2}$$

$$\frac{DM}{Dt} = -\frac{\rho C \pi D^4}{32} \quad \text{kg/s} \tag{b}$$

We now turn to the flow of kinetic energy through the left end of the control surface. The kinetic energy for an element of fluid is $\frac{1}{2} dm V^2$. This corresponds to an infinitesimal amount of an extensive property. To get η, the corresponding intensive property, we divide by dm and use v for V to get

$$\eta = \tfrac{1}{2}v^2 \tag{c}$$

We accordingly wish to compute

$$\iint \eta(\rho \mathbf{V} \cdot \mathbf{dA}) = \iint (\tfrac{1}{2}v^2)\{\rho \mathbf{V} \cdot \mathbf{dA}\}$$

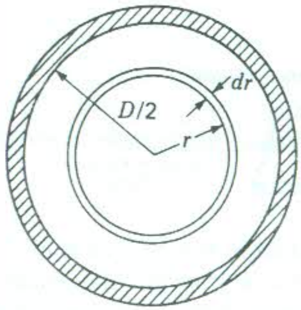


Figure 3.21
Cross section of pipe with infinitesimal ring of fluid.

Employing Eq. a for V, and noting again that **V** and **dA** are collinear but of opposite sense, we get

$$\frac{DKE}{Dt} = \iint \eta(\rho \mathbf{V} \cdot \mathbf{dA}) = \int_0^{D/2} \frac{1}{2} C^2\left(r^2 - \frac{D^2}{4}\right)^2 \left\{\rho\left[C\left(r^2 - \frac{D^2}{4}\right) 2\pi r\, dr\right]\right\}$$

$$\frac{DKE}{Dt} = \rho C^3 \pi \int_0^{D/2} \left(r^2 - \frac{D^2}{4}\right)^3 r\, dr$$

$$\frac{DKE}{Dt} = \frac{\rho C^3 \pi D^8}{2048} \text{ N} \cdot \text{m/s} \qquad \text{[d]}$$

where we could have facilitated the integration by making a change of variable for $\left(r^2 - \frac{D^2}{4}\right)$ to a single variable—say ξ.

■ Debriefing

We have demonstrated the setting up of integrals for computing two flow rates of extensive properties to get DM/Dt and $D(KE)/Dt$ through what could be part of a control surface. We will be making similar calculations starting with Chap. 4 and continuing through the book for integrals stemming from the Reynolds transport equation.

EXAMPLE 3.4

■ Problem Statement

For Example 3.3, assume a *one-dimensional* model with the same mass flow. Compute the kinetic energy flow through a section of the pipe for this flow.

■ Strategy

Using a constant axial velocity component V_{av} times the cross section area, we will get, on including the mass density, the mass flow rate for this axial velocity for a one-dimensional flow. Setting the mass flow rate developed in Example 3.3 for an actual velocity profile, equal to that of the one-dimensional case, we will determine the proper value of the aforementioned constant velocity required for the one-dimensional simplification. Using this constant velocity, we will determine by ordinary multiplication the flow rate of kinetic energy for the one-dimensional model.

■ Execution

We first proceed to compute the constant velocity needed to achieve a mass flow rate in a one-dimensional flow in the pipe equal to the actual mass flow rate in the pipe as developed in Example 3.3 (see Eq. b). Thus, equating these mass flow rates,

$$-(V_{\text{av}})\left(\frac{\rho\pi D^2}{4}\right) = -\frac{\rho C D^4 \pi}{32}$$
$$\therefore\ V_{\text{av}} = \frac{CD^2}{8} \quad \text{m/s} \qquad \text{[a]}$$

The kinetic energy flow for the one-dimensional model is then

$$\iint \frac{V^2}{2}(\rho \mathbf{V}\cdot d\mathbf{A}) = -\frac{\rho}{2}\left(\frac{CD^2}{8}\right)^3\left(\frac{\pi D^2}{4}\right)$$
$$= -\frac{\rho C^3 D^8 \pi}{4096} \quad \text{N}\cdot\text{m/s} \qquad \text{[b]}$$

We now define the *kinetic-energy correction* factor α as the ratio of the actual flow of kinetic energy through a cross section to the flow of kinetic energy for a one-dimensional model for the same mass flow. That is

$$\alpha = \frac{KE \text{ flow for section}}{KE \text{ flow for 1-D model}} \qquad \text{[c]}$$

For the case at hand, we have from Eq. b of this example and Eq. d of Example 3.3

$$\alpha = \frac{-\rho C^3 \pi D^8/2048}{-\rho C^3 \pi D^8/4096} = 2 \qquad \text{[d]}$$

The factor α exceeds unity, so there is an underestimation of kinetic energy flow for a one-dimensional model. We will have more to say about this point later in the text.

■ Debriefing

This example gives us the opportunity to assess the degree of error incurred using the one-dimensional model for the single case of kinetic energy flow. Clearly, this kind of error must at times be taken into account for other variables when modeling flows for the purpose of simplifying calculations of problems.

HIGHLIGHTS

When we are dealing with a finite set of particles, we can identify any one particle by using a subscript. Then, incorporating this into a time function, we can easily describe the velocity of any one particle. This is exactly what we did in your dynamics course where we used such notation as, for example, $(V(t))_n$. In the case of a fluid, where countless particles are involved, it is clear that a different approach must be used. Here, instead of using a subscript to identify any one particle, we use the spatial coordinates and the time to identify any one particle. And we incorporate these coordinates into a function to give the velocity for any one particle. Thus, we use the notation $\mathbf{V}(x, y, z, t)$ where the position of any one particle as well as the velocity of this particle can be specified. This is called the **field approach.**

We demonstrated that the field approach can be used very creatively. First, in $\mathbf{V}(x, y, z, t)$ we can pick fixed coordinates which we denote as (x_0, y_0, z_0, t) and allow t to progress. The resulting velocity formulation $\mathbf{V}(x_0, y_0, z_0, t)$ then conveys the velocity as time progresses of a string of particles as they pass by the chosen fixed point. This very useful approach is called the **Eulerian viewpoint.** On the other hand, we can imagine following any one particle in the flow as time progresses. For this use, the spatial coordinates must vary with time in such a way as to always locate the chosen particle at any time t. This is called the **Lagrangian viewpoint.** Why is this useful? In the dynamics of a single particle, Newton's law applies to this particle as it is followed. We are doing the same thing here for a fluid wherein we are in the presence of countless particles requiring the use of a field approach to manage this.

Let us then straightway go to Newton's law for a fluid. We must use the Lagrangian viewpoint to focus on any one particle in the flow. We treat the spatial coordinates as certain time functions varying in such a manner as to follow any one particle. However, we shall not specify the time functions but realize that at any later time they could be specified for any particular particle. We are thus keeping the discussion open-ended at this point. We will need the acceleration of any one particle. We thus take the time derivative of $\mathbf{V}$ using the familiar **chain rule** of differential calculus. We get

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \left(\frac{\partial \mathbf{V}}{\partial x}\frac{dx}{at} + \frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{V}}{\partial z}\frac{dz}{dt}\right) + \frac{\partial \mathbf{V}}{\partial t}$$

It should be clear that to follow any one particle we require $\frac{dx}{dt} = V_x, \frac{dy}{dt} = V_y, \frac{dz}{dt} = V_z$, namely the velocity components of the particle.

The resulting formulation, called the **substantial derivative** or the **total derivative,** has the notation D replacing d,

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt} = \left(V_x \frac{\partial \mathbf{V}}{\partial x} + V_y \frac{\partial \mathbf{V}}{\partial y} + V_z \frac{\partial \mathbf{V}}{\partial z}\right) + \frac{\partial \mathbf{V}}{\partial t}$$

The first bracketed expression gives the acceleration resulting from the particle being in the process of changing position in a velocity field. This velocity field is mathematically held constant by virtue of the fact that we are holding t constant. During the computation, we are allowing the particle to be in the process of moving. We are thus in the process of changing the position of the particle in this steady flow field. Because the velocity is a varied function of position (albeit a steady velocity field), the particle is hence in the process of accelerating. This acceleration is aptly called the **acceleration of transport.** For the last expression, we are mathematically holding the spatial coordinates stationary and are getting the acceleration contribution by virtue of our allowing the velocity field to be in the process of varying with time.

To apply the preceding concept, we will remind you of two simple definitions. The **system** is an identified aggregate of matter whose mass is constant but whose shape may be changing arbitrarily. A **control volume** is an identified fixed volume in space wherein there may be flow through the boundary, called the **control surface,** and where the amount of mass inside can be changing. Also, we define an extensive property N for a body as one which depends for its value on the amount of mass of the body. We then presented an extremely useful equation in this chapter called the **Reynolds transport equation** that we will use in Chapters 4 and 5. Noting that η is N per unit volume, we have

$$\frac{DN}{Dt} = \oiint_{\text{CS}} \eta(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \iiint_{\text{CV}} \eta(\rho \, dv)$$

In essence, this theorem relates the time derivative of N of a system at time t, as one follows it (much like you did in dynamics), with the time rate of change of N determined by focusing on a control volume corresponding to the volume occupied by the system at time t. That is, we are relating a Lagrangian viewpoint for a system with an Eulerian viewpoint using the control volume, which is the boundary of the system at time t. How is the latter step, which perhaps is the least familiar to you, accomplished? At time t we have the rate of flow of N passing through the control surface, and we add to this the rate of change of N inside. In this way, we account for the total rate of change of N as we look at the control volume. Since the system has the identical volume as the control volume at time t and since the control volume entails identically the same matter as the system at time t, one would intuitively expect the same rates of change of N from both viewpoints at time t. However, since we proved the relation earlier we do not have to depend on intuition, although it is nice when it can be applied.

3.10 CLOSURE

In this chapter we have laid the foundation for the handling of fluid flow. Specifically, we have presented (1) kinematical procedures and concepts which enable us to describe the motion of fluids including the concept of irrotationality; (2) the four basic laws which will form the basis for our calculating the motion and flow characteristics of fluids; (3) the system and control-volume viewpoints by which we can apply these laws effectively to physical problems, and (4) the Reynolds transport equation relating the system approach to the control-volume approach or, in other words, relating the Eulerian and the Lagrangian viewpoints. In Chapters 4 and 5, we will develop these basic laws for both finite systems and finite-control volumes in a very general form. And, in solving problems in those chapters, we will make liberal usage of the one-and two-dimensional flow models.

*3.11 COMPUTER EXAMPLE

COMPUTER EXAMPLE 3.1

■ Computer Problem Statement

You are making plans for a sailing regatta. The starting point is at A (see Fig. C3.1). You plan to make only one tack to get to the buoy at B as shown in the diagram. The speed of the boat depends on the orientation of the sail and the wind velocity, the latter as seen from the boat. We shall assume that the skipper has set the sail to achieve the maximum speed for any velocity direction of the sailboat as measured by the angle α shown in the diagram. We will estimate that the velocity of the boat is given as

$$V_i = 6.3 - 0.055\alpha_i \quad \text{knots with } \alpha_i \text{ in degrees}$$

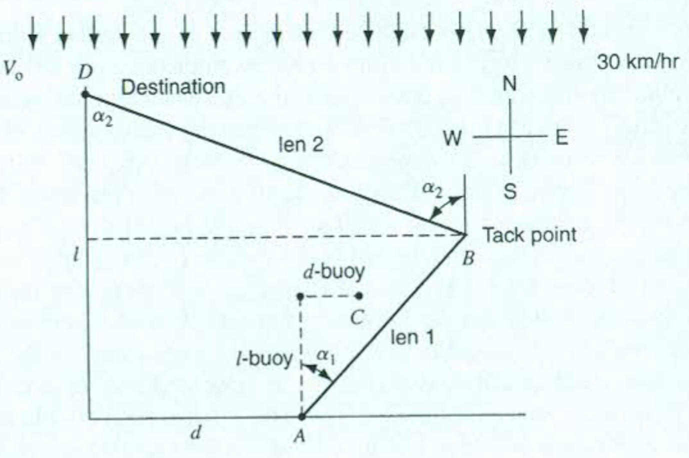


Figure C3.1

The range of each α_i is from 10 degrees to 45 degrees. Write an interactive program for the skipper for the time of passage from A to B asking for

What distance in feet is the starting point south of the destination?
What distance in feet is the starting point east of the destination?
How far north of the starting point is the buoy you have to go around?

Using the command `[a, b] = min` find the minimum value of "`a`" from each column, and "`b`" will then be the corresponding row for the matrix. This gives minimum time. Do the same for the "`c,d`" matrix. This will get the minimum time, the angle α_i, and the distance before tacking to win the race.

Use the program for the following data:
Starting point is 2900 ft south of destination.
Starting point is 800 ft east of destination.
Buoy is 1000 ft north of starting point.
Buoy is 500 ft east of starting point.

■ Strategy

We will use matrices to store values of every possible combination of distance before tacking `(len1)` and angle of departure `(a1)`. Since we have every possible combination, we can use these to determine which combination of initial angle and distance to tack makes for the fastest time to the finish, based on the starting and ending point locations and the location of the buoy, b (which we must go around). Once the minimum time is pinpointed in the total time matrix, its location can be used to determine which "`len1`" and "`a1`" were used in determining it. These two values will tell us all we need to know to win the race!

■ Execution

```
clear all;
%Putting this at the beginning of the program ensures
%values don't overlap from previous programs.

con=pi./180;
%This is the constant for converting from degrees to
%radians.

len=input('What distance in feet is the starting
point south of the destination?\n');
%This is the total distance the starting point (a) is
%south of the ending point (d).

dist=input('What distance in feet is the starting
point east of the destination?\n');
%This is the total distance the starting point (a) is
%east of the ending point (d).
```

```
l_buoy=input('How far north of the starting point is
the buoy you have to go around?\n');
d_buoy=input('How far east of the starting point is
the buoy you have to go around?\n');

l1_vector=linspace(sqrt(l_buoy.^2+d_buoy.^2), 2.*(sqrt
(l_buoy.^2+d_buoy.^2)));
%We choose our tack to be after the buoy (obviously)
%but before twice the distance, ab, to the buoy.

a1_vector=linspace(atan(d_buoy./l_buoy).*(180./pi), 89
,length(l1_vector));
%The initial angle must be chosen to at least clear
%the buoy. We also want the size of this array to be
%the same size as the vector "l1_vector".

for i=1:length(l1_vector);

for j=1:length(a1_vector);

len1 (i, j)=l1_vector (i);
%This makes a matrix out of a vector by making rows
%of each value in the vector "l1_vector".

a1 (i,j)=a1_vector(j);
%This makes a matrix out of a vector by making
%columns of each value in the vector "a1_vector".

end

end

a2=(atan((dist+len1.*sin(a1.*con))./(len-
len1.*cos(a1.*con)))).*(180./pi);
%This is the equation solving for alpha2 (in
%degrees).

len2=sqrt((dist+len1.*sin(a1.*con)).^2+(len-
len1.*cos(a1.*con)).^2);
%This is the equation for the distance between the
%tack and the ending point.

v1=6.3-.055.*a1;
%This is the equation for the velocity in knots
%between the beginning point and the tack.

t1=len1./(v1.*1.6878);
%Since we know the velocity and the distance of
%travel we can determine the time it takes. The
```

```
%1.6878 is to convert the velocity from knots to
%ft/sec.

v2=6.3-.055.*a2;
%This is the equation for the velocity in knots
%between the tack and the ending point.

t2=len2./(v2.*1.6878);
%This is the time between the tack and the ending
%point.

time=t1+t2;
%Once we add "t1" and "t2" then we can find the
%minimum value for the total time and find the "a1"
%and "l1" that correspond to this minimum time.

[a,b]=min(time);
%When executed, "a" will be the minimum value from
%each column of matrix "time" and "b" will be the
%corresponding row of the matrix that the value was
%found in.

[c,d]=min(a);
%When executed, "c" will be the minimum value of the
%row vector created immediately above and "d" will be
%the corresponding column that it was found in.

fprintf('\nThe minimum amount of travel time is:
%4.2f minutes.\n\n',c./60);
fprintf('The value of a1 that will give us the
minimum time is: %4.2f\n\n', a1(b(d),d));
fprintf('The value of len1 that will give us the
minimum time is: %4.2f\n\n',len1(b(d),d));
fprintf('Therefore, continue a course of %4.2f
degrees for %4.2f feet before
tacking. \n\n',a1(b(d),d), len1 (b(d),d));

x1=linspace(dist,dist+len1(b(d),d).*sin(a1(b(d),d).*c
on));
y1=tan((90-a1(b(d),d)).*con).*x1-tan((90-
a1(b(d),d)).*con).*dist;
x2=linspace(0,len2(b(d),d).*sin(a2(b(d),d).*con));
y2=-tan((90-a2(b(d),d)).*con).*x2+len;
axis([0(dist+len1(b(d),d).*sin(a1(b(d),d).*con))
+1000 0 len]);
plot(x1,y1,'b',x2,y2,'g');
hold on;
plot(x1(1)+d_buoy,y1(1)+l_buoy,'o');
```

```
grid;
title ('The Best Course To Take To Win The Race!!');
text(x1(1)+100,y1(1)+100,`Point A: The starting
point');
text(x1(100),y1(100),`"Tacking!"');
text(x2(1),y2(1)-100,`Point C: The ending point');
text(x1(1)+d_buoy-500,y1(1)+1_buoy,`Buoy');
%All this just gives us a plot of the course we must
take to win the race and labels things appropriately
(see Fig. C3.2).
```

■ Debriefing

In MATLAB, matrices of values can be easily manipulated without the need of loop iteration. The only reason we even used loops in this problem was to generate the matrices. Once you have matrices they can be added, subtracted, and multiplied very easily without loops. If you want random values, MATLAB has an intrinsic function "`rand`" and "`randn`" which will generate a `n x n` matrix as easily as "`rand (n, n)`" and there is no limit on "`n`"!

In world class 30 m racing more careful computations are made in deciding the racing strategy. The initial direction that the yacht takes is very important for determining the route to take and where to take the tacks.

■ Computer Output

```
EDU>> mp14a
What distance in feet is the starting point south of the
destination?
2900
What distance in feet is the starting point east of the
destination?
800
How far north of the starting point is the buoy you have
to go around?
1000
How far east of the starting point is the buoy you have
to go around?
500
The minimum amount of travel time is: 7.44 minutes.
The value of a1 that will give us the minimum time is:
26.57
The value of len1 that will give us the minimum time is:
1118.03
Therefore, continue a course of 26.57 degrees for
1118.03 feet before tacking.
EDU>>
```

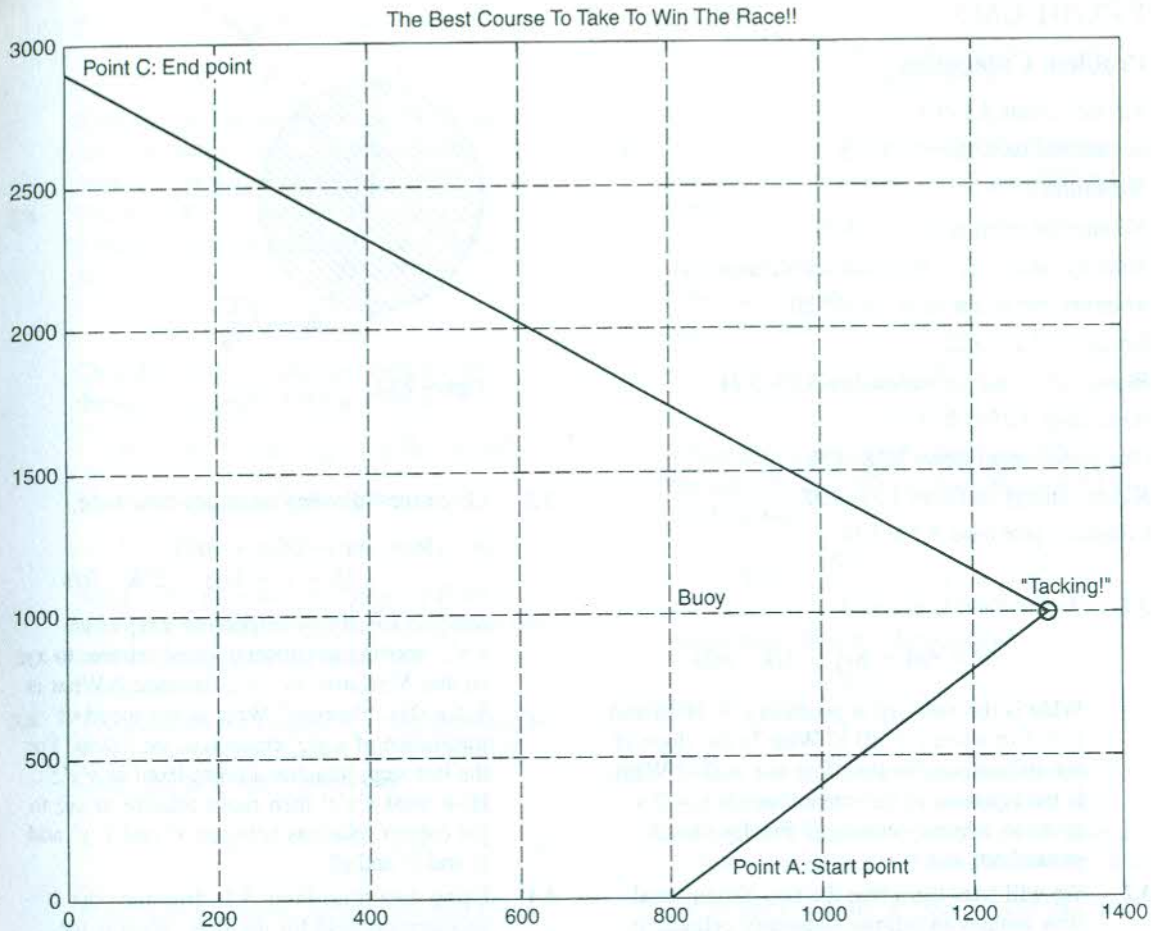

The Best Course To Take To Win The Race!!
Point C: End point
"Tacking!"
Buoy
Point A: Start point
3000
2500
2000
1500
1000
500
0
0
200
400
600
800
1000
1200
1400

Figure C3.2

PROBLEMS

Problem Categories

Velocity fields 3.1–3.3
Substantial derivatives 3.4–3.8
Streamlines 3.9–3.13
Noninertial references 3.14–3.15
Velocity field with cylindrical coordinates 3.16
Rotation and strain rates 3.17–3.20
Gradients 3.21–3.22
Rotationality and irrotationality 3.23–3.24
Basic laws 3.25–3.27
One-dimensional flows 3.28–3.30
Kinetic energy in flows 3.31–3.32
Computer problems 3.33–3.34

3.1 A flow field is given as

$$\mathbf{V} = 6x\mathbf{i} + 6y\mathbf{j} - 7t\mathbf{k} \quad \text{m/s}$$

What is the velocity at position $x = 10$ m and $y = 6$ m when $t = 10$ s? What is the slope of the streamlines for this flow at $t = 0$ s? What is the equation of the streamlines at $t = 0$ s up to an arbitrary constant? Finally, sketch streamlines at $t = 0$ s.

3.2 We will later learn that the two-dimensional flow around an infinite stationary cylinder is given as follows, using cylindrical coordinates:

$$V_r = V_0 \cos\theta - \frac{\chi \cos\theta}{r^2}$$

$$V_\theta = -V_0 \sin\theta - \frac{\chi \sin\theta}{r^2}$$

where V_0 and x are constants. (Note that there is no flow in the z direction.) What is the slope (dy/dx) of a streamline at $r = 2$ m and $\theta = 30°$? Take $V_0 = 5$ m/s and $\chi = \frac{5}{4}\,\text{m}^3/\text{s}$. Show that at $r = \sqrt{\chi/V_0}$ (i.e., on the boundary of the cylinder) the streamline must be tangent to the cylinder wall. *Hint:* What does this imply about normal component V_N at the boundary?

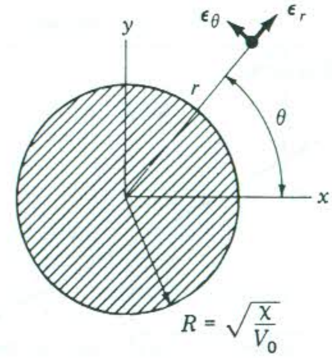


Figure P3.2

3.3 Given the following *unsteady*-flow field,

$$\mathbf{V} = 3(x - 2t)(y - 3t)^2\mathbf{i} + (6 + z + 4t)\mathbf{j} + 25\mathbf{k} \quad \text{ft/s}$$

can you specify by inspection a reference $x'y'z'$ moving at constant speed relative to xyz so that $\mathbf{V}$ relative to $x'y'z'$ is *steady*? What is $\mathbf{V}$ for this reference? What is the speed of translation of $x'y'z'$ relative to xyz? *Hint:* For the last step, imagine a point fixed in $x'y'z'$. How must $x'y'z'$ then move relative to xyz to get correct relations between x' and x, y' and y, and z' and z?

3.4 Using data from Prob. 3.1, determine the acceleration field for the flow. What is the acceleration of the particle at the position and time designated in Prob. 3.1?

3.5 Given the velocity field

$$\mathbf{V} = 10\mathbf{i} + (x^2 + y^2)\mathbf{j} - 2yx\mathbf{k} \quad \text{ft/s}$$

what is the acceleration of a particle at position (3, 1, 0) ft?

3.6 Given the velocity field

$$\mathbf{V} = (6 + 2xy + t^2)\mathbf{i} - (xy^2 + 10t)\mathbf{j} + 25\mathbf{k} \quad \text{m/s}$$

what is the acceleration of a particle at (3, 0, 2) m at time $t = 1$ s?

3.7 A flow of charged particles (a plasma) is moving through an electric field $\mathbf{E}$ given as

$$\mathbf{E} = (x^2 + 3t)\mathbf{i} + yz^2\mathbf{j} + (x^2 + z^2)\mathbf{k} \quad \text{N/C}$$

The velocity field of the particles is given as

$$\mathbf{V} = 10x^2\mathbf{i} + (5t + \sqrt{y})\mathbf{j} + t^3\mathbf{k} \quad \text{m/s}$$

If the charge per particle is 10^{-5} C, what is the time rate of change of force on any one particle as it moves through the field?

3.8 The force **F** on a particle with electric charge q moving through a magnetic field **B** is given as

$$\mathbf{F} = q\mathbf{V} \times \mathbf{B}$$

Consider a flow of charged particles moving through a magnetic field **B** given as

$$\mathbf{B} = (10 + t^2)\mathbf{i} + (z^2 + y^2)\mathbf{k} \quad \text{W/m}^2$$

where the velocity field is given as

$$\mathbf{V} = (20x + t^2)\mathbf{i} + (18 + zy)\mathbf{j} \quad \text{m/s}$$

What is the time rate of change of **F** for a flow particle with charge 10^{-5} C? Do not take time to multiply out terms in final computation.

3.9 The equation for streamlines corresponding to a two-dimensional doublet (to be studied in Chap. 11) is given in meters as

$$x^2 + y^2 - \frac{\chi}{C}y = 0 \qquad \text{[a]}$$

where χ is a constant for the flow and C is a constant for a streamline. What is the direction of the velocity of a particle at position $x = 5$ m and $y = 10$ m? If $V_x = 5$ m/s, what is V_y at the point of interest?

3.10 In Prob. 3.9, it should be apparent from analytic geometry that the streamlines represent circles. For a given value of χ and for different values of C, along what axis do the centers of the aforementioned circles lie? Show that all circles go through the origin. Sketch a system of streamlines.

3.11 In Example 3.1, what is the equation of the streamline passing through position $x = 2$, $y = 4$? Remembering that the radius of curvature of a curve is

$$R = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|}$$

determine the acceleration of a particle in a direction normal to the streamline and toward the center of curvature at the aforementioned position.

3.12 We are given the following family of curves representing streamlines for a two-dimensional source (Chap. 11):

$$y = Cx \qquad (1)$$

where C is a constant for each streamline. Also we know that

$$|\mathbf{V}| = \frac{K}{\sqrt{x^2 + y^2}} \qquad (2)$$

where K is a constant for the flow. What is the velocity field $\mathbf{V}(x, y, z)$ for the flow? That is, show that

$$V_x = \frac{Kx}{x^2 + y^2} \qquad V_y = \frac{Ky}{x^2 + y^2}$$

Suggestion: Start by showing that

$$|\mathbf{V}| = V_x\sqrt{1 + \left(\frac{V_y}{V_x}\right)^2} \quad \text{and} \quad \frac{V_y}{V_x} = C = \frac{y}{x}$$

3.13 A *path line* is the curve traversed by any one particle in the flow and corresponds to the *trajectory* as employed in your earlier course in particle mechanics. Given the velocity field

$$\mathbf{V} = (6x)\mathbf{i} + (16y + 10)\mathbf{j} + (20t^2)\mathbf{k} \quad \text{m/s}$$

what is the path line of a particle which is at (2, 4, 6) m at time $t = 2$ s? *Suggestion:* Form dx/dt, dy/dt, and dz/dt. Integrate: solve for constants of integration; then eliminate the time t to relate xyz in a single equation.

3.14 Consider a velocity field $\mathbf{V}(x, y, z, t)$ as measured from reference xyz. The reference xyz is moving relative to another reference XYZ with an angular velocity $\boldsymbol{\omega}$ and a translational velocity $\dot{\mathbf{R}}$ and has, in addition, an angular acceleration $\dot{\boldsymbol{\omega}}$ and a translational acceleration $\ddot{\mathbf{R}}$. From your earlier dynamics course, you may have learned that the acceleration of a particle relative to XYZ (that is, $\mathbf{a}_{XYZ}$) is given as

$$\mathbf{a}_{XYZ} = \mathbf{a}_{xyz} + \ddot{\mathbf{R}} + 2\boldsymbol{\omega} \times \mathbf{V}_{xyz} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})$$

where $\mathbf{a}_{xyz}$ and $\mathbf{V}_{xyz}$ are taken relative to xyz. We have the following data at an instant:

$$\mathbf{V} = 10x\mathbf{i} + 30xy\mathbf{j} + (3x^2z + 10)\mathbf{k} \quad \text{m/s}$$
$$\boldsymbol{\omega} = 10\mathbf{i} \quad \text{rad/s}$$
$$\dot{\mathbf{R}} = 0 \quad \text{m/s}$$
$$\ddot{\mathbf{R}} = 16\mathbf{k} \quad \text{m/s}^2$$
$$\dot{\boldsymbol{\omega}} = 5\mathbf{k} \quad \text{rad/s}^2$$

What is the acceleration relative to xyz and XYZ, respectively, of a particle at

$$\boldsymbol{\rho} = 3\mathbf{i} + 3\mathbf{k} \quad \text{m}$$

at the instant of interest?

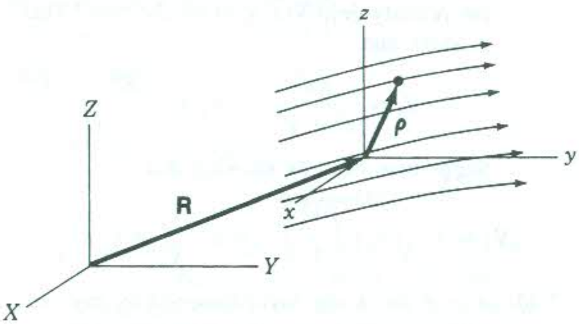


Figure P3.14

3.15 Think up and discuss a few situations where the formulations developed in Prob. 3.14 would be of use.

3.16 Consider a steady two-dimensional inviscid flow about a cylinder of radius a. Using cylindrical coordinates, we can express the velocity field of a nonviscous incompressible flow in the following manner,

$$\mathbf{V}(r, \theta) = -\left(V_0 \cos\theta - \frac{a^2 V_0}{r^2}\cos\theta\right)\boldsymbol{\epsilon}_r + \left(V_0 \sin\theta + \frac{a^2 V_0}{r^2}\sin\theta\right)\boldsymbol{\epsilon}_\theta$$

where V_0 is a constant and $\boldsymbol{\epsilon}_r$ and $\boldsymbol{\epsilon}_\theta$ are unit vectors in the radial and transverse directions, respectively, as shown in the diagram. What is the acceleration of a fluid particle at $\theta = \theta_0$ at the *boundary* of the cylinder whose radius is a? *Suggestion:* Use path coordinates. *Hint:* What must V_r be at the boundary?

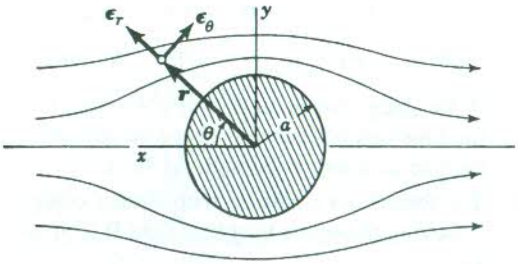


Figure P3.16

3.17 Given the following velocity field

$$\mathbf{V} = 10x^2y\mathbf{i} + 20(yz + x)\mathbf{j} + 13\mathbf{k} \quad \text{m/s}$$

what is the strain rate tensor at (6, 1, 2) m?

3.18 In Prob. 3.17, what is the total angular velocity of a fluid particle at (1, 4, 3) m?

3.19 Given the velocity field

$$\mathbf{V} = 5x^2y\mathbf{i} - (3x - 3z)\mathbf{j} + 10z^2\mathbf{k} \quad \text{m/s}$$

compute the angular velocity field $\boldsymbol{\omega}(x, y, z)$.

3.20 A flow has the following velocity field:

$$\mathbf{V} = (10t + x)\mathbf{i} + yz\mathbf{j} + 5t^2\mathbf{k} \quad \text{ft/s}$$

What is the angular velocity of a fluid element at $x = 10$ ft, $y = 3$ ft, and $z = 5$ ft? Along what surface is the flow always irrotational?

3.21 Show that any velocity field $\mathbf{V}$ expressible as the gradient of a scalar ϕ must be an irrotational field.

3.22 If $\mathbf{V} = \textbf{grad}\ \phi$, what irrotational flow is associated with the function

$$\phi = 3x^2y - 3x + 3y^2 + 16t^3 + 12zt$$

Read Prob. 3.21 before proceeding.

3.23 Is the following flow field irrotational or not?

$$\mathbf{V} = 6x^2y\mathbf{i} + 2x^3\mathbf{j} + 10\mathbf{k} \quad \text{ft/s}$$

3.24 Explain why in a capillary tube the flow is virtually always rotational.

3.25 What were the basic laws and subsidiary laws that you used in your course in strength of materials?

3.26 In the studies of rigid-body mechanics, how was conservation of mass ensured? Also, was conservation of energy a law independent and apart from Newton's laws? Explain the reason for your answer.

3.27 Have we placed any restrictions on the motion of a control volume? Can it have material other than fluid inside or passing through?

3.28 A fluid is moving along a curved circular pipe such that the pressure, velocity, and so forth are uniform at each section of the pipe and are functions of the position s along the centerline of the pipe and time. How would we classify this flow in the light of our discussion in this chapter? If the flow properties were also functions at a section of the radial distance r from the centerline in addition to s and t, would this then be a two-dimensional flow? Why?

3.29 In Example 3.3, compute the linear momentum flow through a cross section of the control volume. Recall that the linear momentum of a particle is $m\mathbf{V}$.

3.30 In Prob. 3.29 find a momentum correction factor which would be the ratio for the actual momentum flow to that of the one-dimensional model of the flow for the same mass flow. In the previous problem, we got the result

$$\iint V(\rho\mathbf{V}\cdot d\mathbf{A}) = -\frac{\rho C^2 \pi D^6}{192}$$

Do not consult Example 3.3 while doing this problem.

3.31 In Example 3.3, compute the kinetic energy flow through one face of the control surface if it is moving to the left at a speed of V_0 relative to the ground.

3.32 In Chap. 11, we discuss the simple *vortex* where in cylindrical coordinates

$$V_r = 0 \qquad V_z = 0$$
$$V_\theta = \frac{\Lambda}{2\pi r}$$

Λ is a constant called the *strength* of the vortex. Draw the streamlines for the simple vortex. What is the mass flow and kinetic energy flow through the surface shown in the diagram?

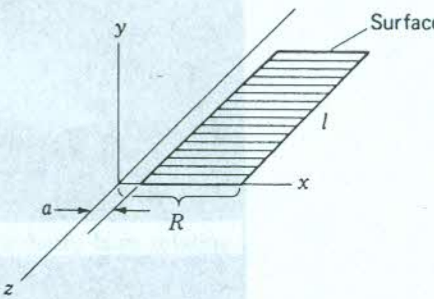


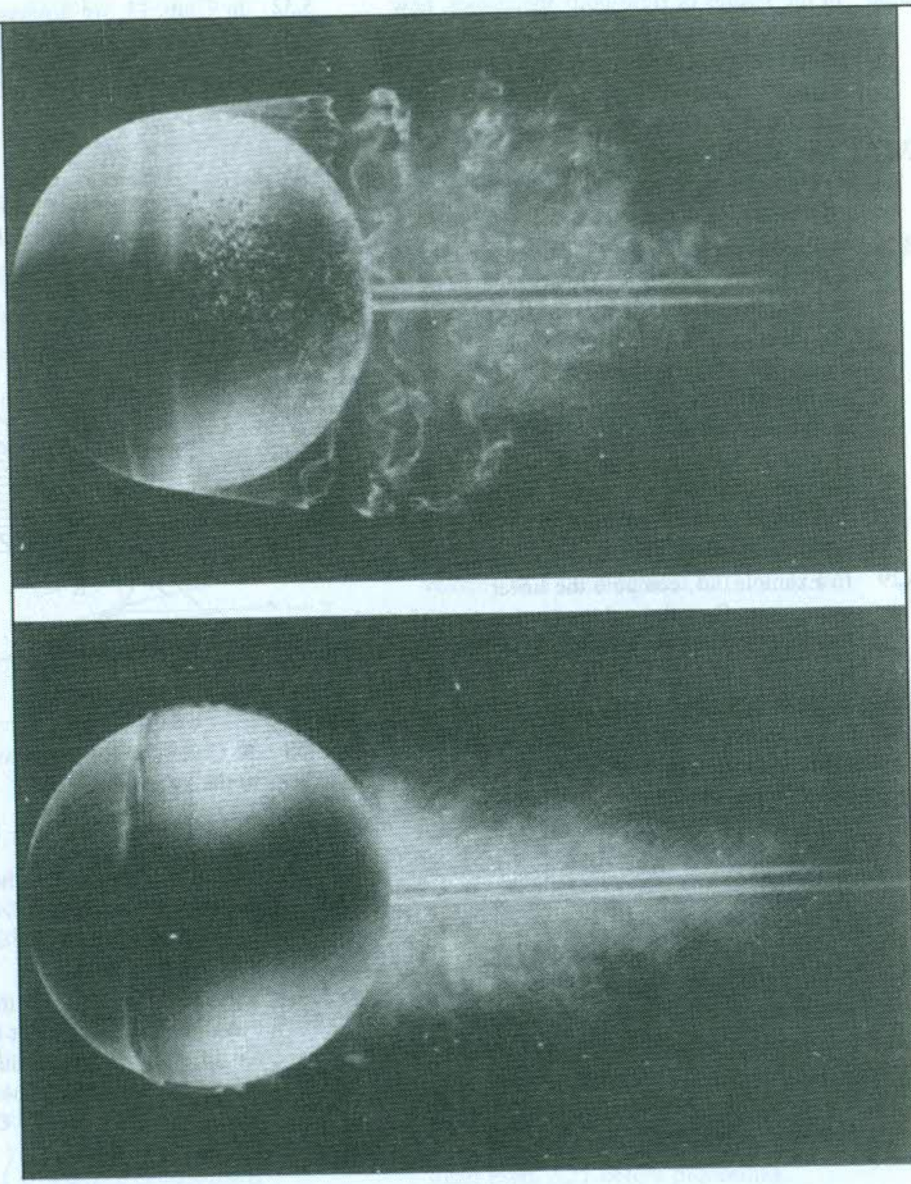
Figure P3.32

3.33 Given the following velocity field parallel to the xy plane

$$\mathbf{V} = (3t^2x^3 + t^{1/2})\mathbf{i} + (\ln y)(t^{3/2})\mathbf{j} \quad \text{ft/sec}$$

with t in seconds, plot the path of a fluid particle starting from (2, 5) ft at time $t = 0$. Observe the particle for 20 seconds, one second at a time.

3.34 In Problem 3.9, plot the streamlines for a two-dimensional doublet for which $\chi = 10\ \text{m}^3$ for different constant values of C (which identifies the contour lines of the velocity field). Use the values of C equal to 2, 5, 8, and 12.



Flows over spheres illustrating separation from laminar and from turbulent boundary layers.
(*Courtesy Dr. Henry Werlé, Onera, France.*)

The top photo shows a flow of water with dye inserted in the boundary layer. There is a laminar boundary layer showing a separation ahead of the equator and remaining laminar for almost one radius before becoming turbulent. In the second photo a wire hoop is placed ahead of the equator to *trip* the boundary layer into turbulent flow. It now separates further rearward than if it were laminar. Drag is dramatically reduced. It is for this reason that for some airfoils, a string of small vortex generator blades is to be found. This last photo was made with air bubbles in water.

CHAPTER 4

Basic Laws for Finite Systems and Finite Control Volumes I

Continuity and Momentum

4.1 INTRODUCTION

In Chaps. 2 and 3 fluid properties and flow characteristics have been described with the use of the field concept. We will now develop two of the basic laws relating these quantities in various forms: Conservation of mass and methods of momentum. In each development, the more familiar finite system approach will first be undertaken, and then the formulations will be extended to the case of the finite control volume. The chapter will be presented in three parts, dealing with the topics of:

A. Conservation of mass

B. Linear momentum

C. Moment of momentum

In this chapter and in Chap. 5 we will develop many of the basic equations which form the basis for much of the analytical work in the text. Consequently, this is a vital chapter.

PART A
CONSERVATION OF MASS

4.2 CONTINUITY EQUATION

Recall that a *system*, by our definition, always entails the same quantity of matter. Thus, by employing the definition of a system properly, i.e., by keeping M, the mass, constant, we find the conservation of mass is in this instance automatically ensured.

In using the *control volume* (Fig. 4.1) for the handling of flow problems, it is clear that matter is *not* identified and that there is not the simple and direct manner

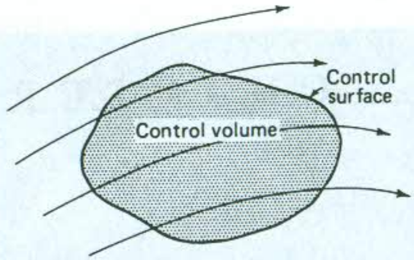


Figure 4.1
Flow through a control volume.

for ensuring conservation of mass as was the case in previous studies of discrete particles and rigid bodies where a systems approach was used.

To go from the systems approach to the control-volume approach here, we employ the Reynolds transport equation (3.28), where

1. The extensive property N is, for our case, M, the mass of a fluid system.
2. The quantity η is unity for our case, since $M = \iiint_V \rho\, dv$.

Since the mass M of any system is constant as noted above, we can then say at any time t on using the Reynolds transport equation that

$$\frac{DM}{Dt} = 0 = \oiint_{\text{CS}} (\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \iiint_{\text{CV}} \rho\, dv$$

Since we can choose a system of any shape at time t, the relation above is then valid for any control volume at time t. Furthermore, as we pointed out in Chap. 3, this control volume may have any motion whatever provided that the velocity $\mathbf{V}$ and the time derivative $\partial/\partial t$ are measured relative to the control volume. To interpret this equation most simply, we can rewrite it as follows:

$$\boxed{\oiint_{\text{CS}} (\rho \mathbf{V} \cdot \mathbf{dA}) = -\frac{\partial}{\partial t} \iiint_{\text{CV}} \rho\, dv} \tag{4.1}$$

That is, *the net efflux rate of mass through the control surface equals the rate of decrease of mass inside the control volume*. In this way we account for the mass entering or leaving any volume that we may choose in the flow at any time. Equation 4.1 and its simplified forms are called *equations of continuity*.

If the flow is steady relative to a reference fixed to the control volume, all fluid properties, including the density at any fixed position in the reference, must remain invariant with time. Since we are dealing with control volumes of fixed shape, the right side of Eq. 4.1 can be written in the form $\iiint (\partial\rho/\partial t)\, dv$, and it is clear that this integral is zero. Hence we can state that *any steady flow* involving one or many species of fluids must satisfy the equation

$$\oiint_{\text{CS}} (\rho \mathbf{V} \cdot \mathbf{dA}) = 0 \tag{4.2}$$

Next, consider the case of *incompressible* flow involving only a *single species* of fluid in the domain of our control volume. In this case, ρ is constant at all positions in the domain and for all time even if the velocity field is unsteady. The right side of Eq. 4.1 vanishes then, and on the left side of this equation we can extract ρ from under the integral sign. We may then say that

$$\oiint_{CS} (\mathbf{V} \cdot \mathbf{dA}) = 0 \qquad [4.3]$$

Thus, for any incompressible flow involving a single species of fluid, conservation of mass reduces to conservation of volume.

You should not be intimidated by the rather forbidding integration operations given in the preceding equations. These equations should be thought of as precise mathematical language for conservation of mass equivalent to, but more precise than, the statement made after Eq. 4.1. From these general formulations, we can now develop a useful special equation.

Let us, for example, consider the very common situation in which fluid enters some device through a pipe and leaves the device through a second pipe, as shown diagrammatically in Fig. 4.2. The control surface we have chosen is indicated by a dashed line. We assume that the flow is *steady* relative to the control volume and that the inlet and outlet flows are *one-dimensional*. Applying Eq. 4.2 for this case, we get

$$\oiint_{CS} (\rho\mathbf{V} \cdot \mathbf{dA}) = \iint_{A_1} (\rho\mathbf{V} \cdot \mathbf{dA}) + \iint_{A_2} (\rho\mathbf{V} \cdot \mathbf{dA}) = 0$$

where A_1 and A_2 are, respectively, the entrance and exit areas. Upon noting that the velocities are normal to the control surfaces at these areas and employing the previously discussed sign convention of an outward directed normal for the representation of area, the equation above becomes

$$\oiint_{CS} (\rho\mathbf{V} \cdot \mathbf{dA}) = -\iint_{A_1} \rho V\, dA + \iint_{A_2} \rho V\, dA = 0$$

With ρ and V constant at each section as a result of the one-dimensional restriction for the inlet and outlet flows, we get for this equation

$$-\rho_1 V_1 \iint_{A_1} dA + \rho_2 V_2 \iint_{A_2} dA = 0$$

Integrating, we get

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \qquad [4.4]$$

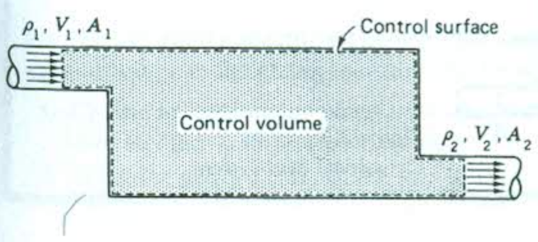


Figure 4.2
Control volume for device with 1-D inlet and outlet.

which is a simple relation known by every high-school physics student. The main purpose in going through the development above was to see how a particular algebraic equation for a simple case is fashioned from the general formulation. Note that $\rho_1 V_1 A_1$ is the mass flow and can be given as ρQ where Q is the volume flow rate.

In Example 4.1, we shall illustrate the use of the continuity equation as well as a format that we encourage the student to follow.

EXAMPLE 4.1

■ Problem Statement

Water flows through a large tank having a diameter of 5 ft (see Fig. 4.3a). The velocity of water relative to the tank is given as

$$V = 6.25 - r^2 \text{ ft/s}$$

What is the *average* velocity of the water leaving by the smaller pipe which has an inside diameter of 1 ft?

■ Strategy

Our first step is to choose a control volume which on part of its surface is a known velocity of flow and on the remainder of its surface is information that will lead to the desired average velocity via conservation of mass considerations. Thus, in Fig. 4.3b, the control surface chosen is normal to the flow into the tank with a known velocity profile, and the control surface slices the flow in the pipe away from the large cylinder.

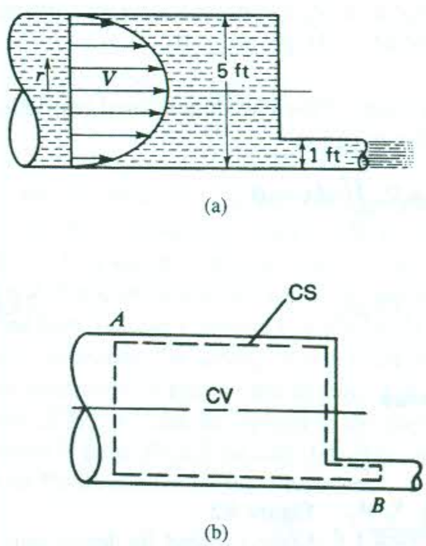


Figure 4.3
Cylindrical duct system.

Next we shall go to the general *continuity equation*, which we state as

$$\oint\!\!\!\oint_{CS} (\rho \mathbf{V} \cdot \mathbf{dA}) = -\frac{\partial}{\partial t} \iiint_{CV} \rho \, dv \qquad \text{[a]}$$

To insert data into this equation and solve for the desired velocity, we will have to make simplifications of the flow. The following simplifications that we can reasonably make without creating too much of a departure from reality are presented next. These assumptions are a vital part of the analysis, and for this reason we box them in. Thus,

1. Steady flow.
2. Incompressible flow.
3. One dimensional flow at *B*.

How do these assumptions affect Eq. a? Note that assumption 1 permits us to delete the time derivative term on the right side of Eq. a. Assumption 2 allows us to take ρ as a constant, and this term can then be canceled out of Eq. a. Finally, assumption 3 permits us to express the volume flow through the control surface cutting the pipe simply as $V_B A_B$ with V_B as a constant and hence the desired average velocity in the pipe.

■ Execution

We can now write Eq. a as follows:

$$-\int_0^{2.5} (6.25 - r^2)(2\pi r)\, dr + V_B \frac{\pi(1^2)}{4} = 0 \qquad \text{[b]}$$

Note that the first minus sign above obtains because the velocity vector and the area vector at *A* are colinear and of *opposite* direction. Integrating and solving for V_B we get

$$-(2\pi)\left(6.25\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^{2.5} + V_B \frac{\pi}{4} = 0 \qquad \text{[c]}$$

$$\therefore V_B = 78.125 \text{ ft/s}$$

The one-dimensional model at *B* then gives us the desired average velocity in the small pipe.

■ Debriefing

We have followed a specific set of steps that will characterize many of the problems to follow in the examples and in the homework. They are:

1. Choose an appropriate control volume involving the desired unknown data and the appropriate known data.
2. Write the general form of the key equation to be used.

3. Simplify the flow by making reasonable assumptions.
4. Insert data using the simplifying assumptions. Solve.
5. Stop and assess whether your formulations have made sense.

In later problems, there will be more than one basic equation to be used, and there may be more than one control volume. In all cases, the control volumes must be chosen with care and with clear purpose.

Later you will find that the conservation of mass consideration will occur in problems *after* other basic laws have been formulated. In such cases, assumptions of the flow will have already been made. We then suggest that if a simple form of continuity such as $\rho_1 V_1 A_1 = \rho_2 V_2 A_2$ is applicable, you use it directly without going through the detailed discussion such as that of Example 4.1 for which only conservation of mass was the prime consideration. We shall illustrate this in subsequent problems.

EXAMPLE 4.2

■ Problem Statement

Water flows into a cylindrical tank (Fig. 4.4) at a steady speed of 20 ft/s through pipe 1 and leaves through pipes 2 and 3 at rates of 8 ft/s and 10 ft/s, respectively. At 4 we have an open air vent. First determine the rate of change of h of the free surface. Then determine the average speed of airflow through the air vent. The following data apply:

$$D_1 = 3 \text{ in} \qquad D_2 = 2 \text{ in} \qquad D_3 = 2.5 \text{ in} \qquad D_4 = 2 \text{ in}$$

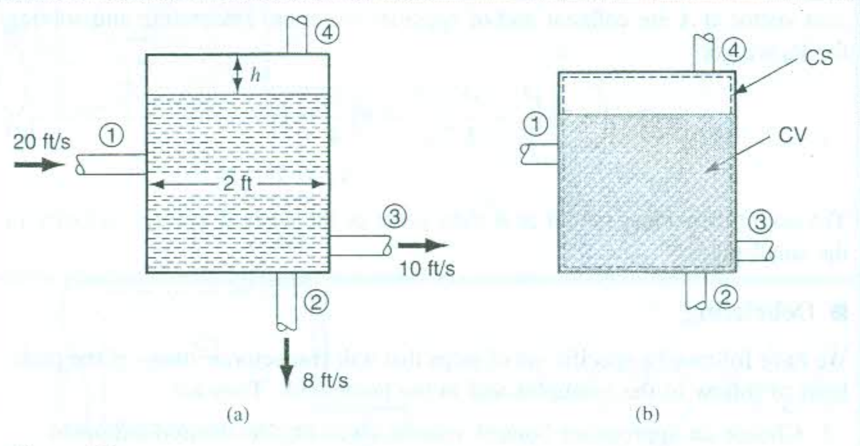


Figure 4.4
Two species problem.

■ Strategy

We will use as a control volume the interior of the tank (Fig. 4.4b), wherein we have two species of fluids. We will use the conservation of mass for each species. In each case we go to the following general equation:

$$\oiint_{CS} (\rho \mathbf{V} \cdot \mathbf{dA}) = -\frac{\partial}{\partial t} \iiint_{CV} \rho\, dv$$

We now present assumptions for working with this equation separately for air and water.

1. Flows of air and water are incompressible.
2. Flows are 1-D in and out of the control volume.
3. The free surface remains a plane horizontal surface.

■ Execution

We insert data into the equation for the water while employing assumptions 1 and 2 for the flows on the left side of the equation and assumption 3 for the time rate of change of mass on the right side of the equation. We get

$$-(\rho)(20)\left(\pi\frac{(3/12)^2}{4}\right) + (\rho)(8)\left(\pi\frac{(2/12)^2}{4}\right) + (\rho)(10)\left(\pi\frac{(2.5/12)^2}{4}\right) = -(\rho)(\pi)\left(\frac{2^2}{4}\right)\left(\frac{dh}{dt}\right)$$

Solving for dh/dt gives us

$$\frac{dh}{dt} = 0.1484 \text{ ft/s}$$

Now we return to the conservation of mass equation for the second of our two species, namely air. We use assumptions 1 and 2 for the flow of air in the vent and assumption 3 for the rate of change of the mass of air inside the control volume. Thus we have

$$(\rho_{\text{air}})(V_{\text{air}})\left(\pi\frac{(2/12^2)}{4}\right) = (\rho_{\text{air}})\left(\frac{\pi 2^2}{4}\right)\left(\frac{dh}{dt}\right)$$

Inserting the preceding result for dh/dt and canceling ρ_{air}, we can then determine the average velocity of airflow in the vent. Thus:

$$V_{\text{air}} = 21.37 \text{ ft/s}$$

■ Debriefing

This problem is very useful since it involves two species of incompressible flows wherein it was most expedient to use the same control volume when considering each species. We could do this because with no mixing we knew that for this same control volume each species had to separately satisfy the appropriate continuity equation. Furthermore, the same distance h was the controlling factor for the mass of air and the mass of water in this control volume.

EXAMPLE 4.3

■ Problem Statement

Water flows downward in a 2 ft pipe at the rate of 50 ft^3/s. It enters a conical section with porous walls (see Fig. 4.5a) that allow an outflow of water in a radial direction all around the conical surface. This outflow has a velocity that varies linearly from zero at A to 3 ft/s at B. What is the volume outflow rate emanating from the exit at B?

■ Strategy

We shall choose the interior of the cone section as the control volume (Fig. 4.5b). We will make some calculations to get the radial velocity of the water as a function of y, the distance downward from top A, and we will need the inside radius of the cone section, also as a function of y. The obvious assumptions that will be used are:

1. Steady flow
2. Incompressible flow
3. Axisymmetric radial flow

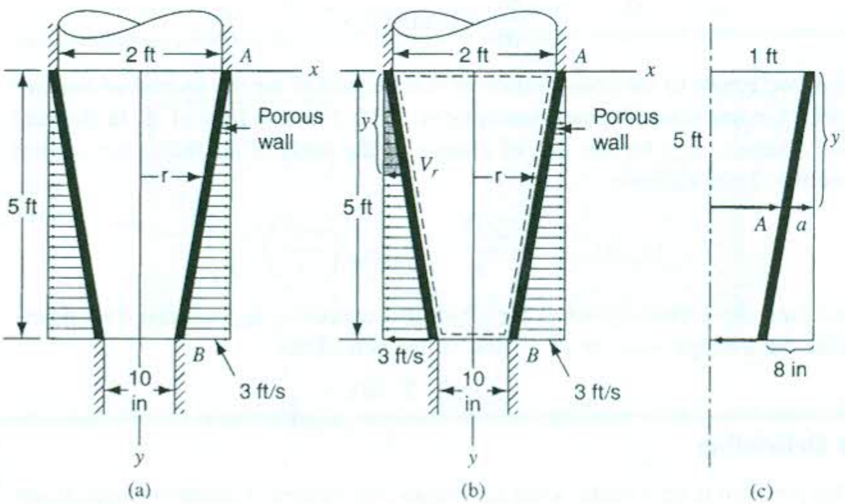


Figure 4.5
Flow in a porous duct.

The *continuity equation* will not have the volume integral because of assumption 1, and the density ρ will cancel out as a constant because of assumption 2, leaving us to work with the equation

$$\oiint_{CS} \mathbf{V} \cdot \mathbf{dA} = 0$$

which conserves volume. We will now expand the surface integral using assumption 3.

■ Execution

Going over the entire control surface to evaluate the above surface integral, we get

$$-50 + Q_B + \int_0^5 (V_r)(2\pi r)(dy) = 0 \qquad \text{[a]}$$

We can express V_r as a function of r by using similar triangles and looking at the linear velocity profile by using the left side of Fig. 4.5b. Thus we can say

$$\frac{y}{5} = \frac{V_r}{3} \qquad \text{Hence } V_r = 0.6y$$

As for r, we can look at line AB where at the top $r = 1$ ft. Going down we must subtract from this (see Fig. 4.5c), the distance a found using similar triangles. We note first that $\frac{y}{5} = \frac{a}{7/12}$. Hence, $a = \left(\frac{y}{5}\right)\left(\frac{7}{12}\right)$. We then have for r

$$r = 1 - \left(\frac{y}{5}\right)\left(\frac{7}{12}\right) = 1 - \left(\frac{7}{60}\right)y$$

Substituting for r and V_r in Eq. a and rearranging terms, we get

$$-50 + \int_0^5 (0.6y)(2\pi)\left(1 - \frac{7}{60}y\right)dy = -Q_B$$

Integrating and solving for Q_B,

$$Q_B = 50 - (0.6)(2\pi)\left[\frac{y^2}{2} - \frac{7}{60}\frac{y^3}{3}\right]_0^5$$

$$Q_B = 21.20 \text{ cfs}$$

■ Debriefing

If we wanted the downward velocity at exit B, we most likely would have hypothesized a one-dimensional flow there. Considering the region near exit B, it should be apparent that the radial flow nearby and the converging cross section of the cone section will complicate the flow pattern, compromising a one-dimensional downward exit flow assumption. In this problem, details of the velocity field were not needed since all we wanted was the exit volume flow.

PART B
LINEAR MOMENTUM

4.3 SYSTEM ANALYSIS

Let us now consider a finite fluid system moving in a flow. *Newton's law* says that

$$\mathbf{F}_R = \frac{d}{dt_{\text{system}}}\left[\iiint_M \mathbf{V}\, dm\right] = \frac{d\mathbf{P}}{dt_{\text{system}}} \tag{4.5}$$

where $\mathbf{F}_R$ is the resultant external force acting on the system, and $\mathbf{V}$ and the time derivative are taken for an *inertial reference*. You will recall that $\mathbf{P}$ is the linear momentum vector. Since we are following the system, we may express Eq. 4.5 as follows:

$$\mathbf{F}_R = \frac{D}{Dt}\iiint_M \mathbf{V}\, dm = \frac{D\mathbf{P}}{Dt} \tag{4.6}$$

We will distinguish between two types of forces which combine to give the resultant force $\mathbf{F}_R$. Recall from Chap. 1 that force distributions acting on the boundary of the system are called *surface-force* distributions or *surface tractions,* denoted as $\mathbf{T}(x, y, z, t)$, and given as force per unit area on the boundary surfaces. Force distributions acting on the material inside the boundary are called *body-force* distributions, denoted as $\mathbf{B}(x, y, z, t)$, and given as force per unit mass at a point. For example, gravity is the most common body-force distribution, and for gravity, $\mathbf{B} = -g\mathbf{k}$. We can now rewrite Eq. 4.6 as follows:

$$\boxed{\oint\!\!\!\oint_S \mathbf{T}\, dA + \iiint_V \mathbf{B}\rho\, dv = \frac{D\mathbf{P}}{Dt}} \tag{4.7}$$

We have thus expressed Newton's law for a finite system. Of greater interest to us now in fluid mechanics is the control-volume approach, which can be readily set forth with the help of the Reynolds transport equation (3.28).

4.4 CONTROL VOLUMES FIXED IN INERTIAL SPACE

We will consider linear momentum $\mathbf{P}$ as the extensive property to be considered in the Reynolds transport equation (3.28). The quantity η used in this equation now becomes momentum per unit mass, which is simply $\mathbf{V}$, the velocity of fluid elements. This is readily verified by noting that $\mathbf{P} = \iiint_V \mathbf{V}(\rho\, dv)$. We then have

$$\frac{D\mathbf{P}}{Dt} = \oint\!\!\!\oint_{\text{CS}} \mathbf{V}(\rho\mathbf{V}\cdot d\mathbf{A}) + \frac{\partial}{\partial t}\iiint_{\text{CV}} \mathbf{V}(\rho\, dv) \tag{4.8}$$

It was pointed out in Chap. 3 that the velocities and time derivatives must be those seen from the control volume. If the control volume is *fixed in inertial space,* then the derivative on the left side is taken for an inertial reference and we may then use Newton's law (Eq. 4.7) to replace this term to form the desired *linear momentum equation.*[1]

$$\boxed{\oiint_{CS} \mathbf{T}\,dA + \iiint_{CV} \mathbf{B}\rho\,dv = \oiint_{CS} \mathbf{V}(\rho\mathbf{V}\cdot\mathbf{dA}) + \frac{\partial}{\partial t}\iiint_{CV} \mathbf{V}(\rho\,dv)} \qquad [4.9]$$

Since the system and control volume have the same shape at time t, the surface-force distribution $\mathbf{T}$ is now the total-force distribution acting on the control surface, and the body-force distribution $\mathbf{B}$ is now the total-force distribution acting on the fluid inside the control volume. *This equation then equates the sum of these force distributions with the rate of efflux of linear momentum across the control surface plus the rate of increase of linear momentum inside the control volume.* For steady flow and negligible body forces, as is often the case, the equation above becomes

$$\oiint_{CS} \mathbf{T}\,dA = \oiint_{CV} \mathbf{V}(\rho\mathbf{V}\cdot\mathbf{dA}) \qquad [4.10]$$

It should be kept in mind that the momentum equation is a vector equation. The scalar-component equations in the orthogonal x, y, and z directions may then be written by simply taking the components of the vectors $\mathbf{V}$, $\mathbf{T}$, and $\mathbf{B}$. Thus

$$\begin{aligned}
\oiint_{CS} T_x\,dA + \iiint_{CV} B_x\rho\,dv &= \oiint_{CS} V_x(\rho\mathbf{V}\cdot\mathbf{dA}) + \frac{\partial}{\partial t}\left(\iiint_{CV} V_x\rho\,dv\right)\\
\oiint_{CS} T_y\,dA + \iiint_{CV} B_y\rho\,dv &= \oiint_{CS} V_y(\rho\mathbf{V}\cdot\mathbf{dA}) + \frac{\partial}{\partial t}\left(\iiint_{CV} V_y\rho\,dv\right) \qquad [4.11]\\
\oiint_{CS} T_z\,dA + \iiint_{CV} B_z\rho\,dv &= \oiint_{CS} V_z(\rho\mathbf{V}\cdot\mathbf{dA}) + \frac{\partial}{\partial t}\left(\iiint_{CV} V_z\rho\,dv\right)
\end{aligned}$$

In using Eq. 4.11, one selects directions for the inertial reference axes xyz so that positive directions of the velocities V_x, V_y, and V_z, as well as the surface and body forces T_x and B_x, and so on, are established. It must be kept in mind that this sign consideration is independent of the sign of $\mathbf{V}\cdot\mathbf{dA}$, a scalar whose sign obtains from the dot product and is independent of the reference orientation. To exemplify this, examine a portion of a device shown in Fig. 4.6, where the control surface

[1]As with the Reynolds' transport equation, we must not let the imposing mathematical appearance of Eq. 4.9 obscure its rather simple physical interpretation. In short, we can say here that the resultant force on the material in a given domain "drives" linear momentum through the control surface and, in addition, changes linear momentum inside the control surface. The imposing mathematical formulation of this simple physical picture allows us to precisely apply data to properly carry out the formulation in given circumstances.

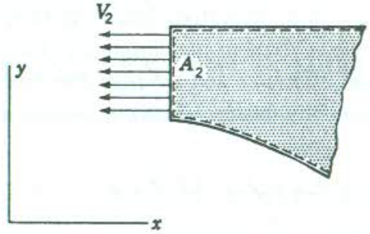


Figure 4.6
1-D flow out of control volume.

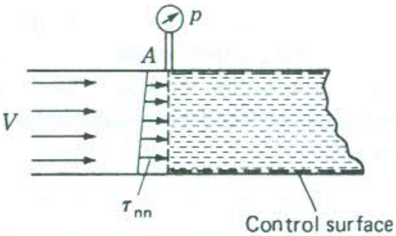


Figure 4.7
Parallel flow.

extends over the outlet area. A one-dimensional flow normal to the outlet area is indicated. The mass flow for the control surface at the outlet in Eq. 4.8 is simply $+\rho_2 V_2 A_2$, since $\mathbf{V}_2$ and $\mathbf{A}_2$ are oriented in the *same* direction. On the other hand, the velocity component V_x in the surface integral is given by the term $-V_2$, since it is directed in the negative direction of the selected coordinate system. Thus the portion of the surface integration over the outlet area is $(-V_2)(+\rho_2 V_2 A_2)$.

We should also like to point out that, for a *parallel flow* (i.e., the streamlines are straight and parallel as shown in Fig. 4.7), we will prove later in the text that the normal stress τ_{nn} on a control surface perpendicular to the flow will have a value equal to the pressure p measured by the gage shown, plus a *hydrostatic* increase in pressure as one goes down into the flow from A of the pressure gage. For such flows in small cross sections, as in pipes, where p may be reasonably high, we can usually neglect the hydrostatic increase with depth. We then have a uniform traction force distribution of magnitude p over this surface.

The momentum equations developed in this section are both very general and of great importance in fluid mechanics. In Sec. 4.5, a number of problems have been undertaken in detail to help explain the meaning of the terms and the manner of use of these equations.

4.5 USE OF THE LINEAR MOMENTUM EQUATION FOR THE CONTROL VOLUME

We have thus far developed very general formulations for the law of conservation of mass and Newton's law as applied to control volumes. From the general continuity equation, we developed simpler specialized equations, one of which was probably quite familiar to you. For most problems, it is advisable to go directly to the proper continuity equation unless for instructional purposes you wish to begin with the general case. However, in the case of linear momentum we have not developed any of the common specialized forms of this equation, such as the thrust formula of a nozzle, since we feel that in view of the complexity of the momentum equation, you should develop the simpler equations yourself as they are needed for particular problems. Doing this will give you a greater awareness of the limitations of your results that are imposed by the simplifications and idealizations employed in reaching the working equations. It is the experience of the author that overreliance on specialized formulas in this area, coupled with unclear specifications of control volumes, is often the source of serious errors by engineers both in and out of school.

Since the linear momentum equation is primarily a relation between forces and velocities, you should choose a control volume that will involve forces and velocities in an economical manner which will contribute to the solution of the problem. Generally, you will expose for consideration in the problems the *reaction* to the force you are seeking. That is, you may want the force *on* a pipe or vane *from water*, but using your fluid mechanics you will actually first seek the force *from* the pipe or vane *on the water*. Just as when you selected several free-body diagrams in your statics course, you may be required to select several different control volumes to have enough independent equations to carry through to the solution. It is extremely important to designate each control volume carefully and to denote clearly the particular control volume for which each equation is written. Furthermore, and most importantly, as in free body diagrams, you must include *all* force systems involved with materials in the control volume. Not to do this is a cardinal error!

Finally, one must be reminded that the linear momentum equation 4.9 was developed from $\mathbf{F} = m\mathbf{a}$, so it has the limitation that the control volume, relative to which the velocities in the equation are measured, must be *fixed* in inertial space.

We will now examine the use of the linear momentum and continuity equations in the following examples, which you are urged to study carefully.

In Example 4.4 we shall use *absolute pressures* and thereby compute the force from the *internal flow* on the reducing elbow of the problem. We shall then point out that by using *gage pressures* instead of absolute pressure, we can get the *combined force* on the elbow from internal flow as well as from the force of the atmosphere on the outside surface of the elbow. Those that accept this approach intuitively need not go to Example 4.5 where the aforementioned procedure is justified.

EXAMPLE 4.4

■ Problem Statement

Determine the forces coming onto the reducing elbow shown in Fig. 4.8a from the steady flow of water inside the elbow. The average values of the flow characteristics at the inlet and the outlet are known, as is the geometry of the reducing elbow.

■ Strategy

We shall use a control volume constituting the entire interior of the reducing elbow, and thus encompassing all the fluid inside the elbow. This control volume is shown in Fig. 4.8b. All the forces acting on the fluid that is inside the control must be accounted for. We will use the *linear momentum equation* and the *continuity equation* for this control volume. These equations will enable us to relate known quantities of the flow at the inlet and outlet, as well as the known weight of the fluid inside, with the resultant force components R_x and R_y coming

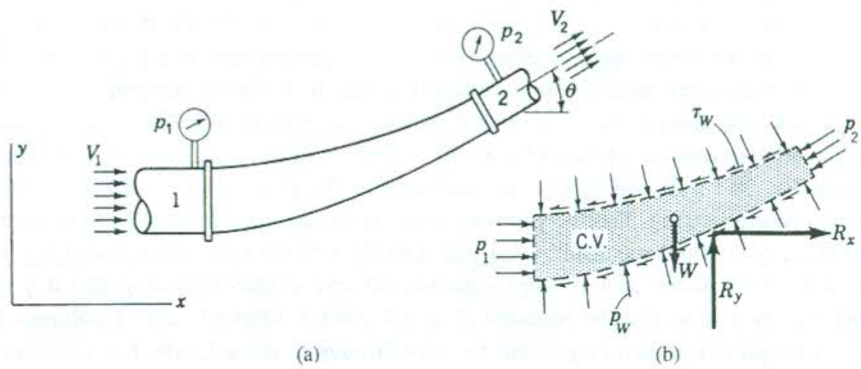


Figure 4.8
Flow through reducing elbow.

from the wall of the elbow *onto* the fluid in contact with the wall. It should be clear that we seek the *reaction* to this latter force. That is, we want the force components from the flow onto the elbow. A structural designer will have to know these force components since quite often appreciable forces can be developed on pipe fittings like our reducing elbow from rapidly moving liquids. These forces must be resisted safely by the building structure supporting the pipe system.

The following are the assumptions that we shall make to simplify the problem.

1. Steady flow.
2. Incompressible flow.
3. One-dimensional, parallel flow enters at 1 and leaves at 2.

■ Execution

Following the steps stipulated in the debriefing section of Example 4.2, we first state the general form of the **linear momentum equation:**

$$\oiint_{CS} \mathbf{T}\, dA + \iiint_{CV} \mathbf{B}\rho\, dv = \oiint_{CS} \mathbf{V}(\rho\mathbf{V}\cdot \mathbf{dA}) + \frac{\partial}{\partial t}\iiint_{CV} \mathbf{V}(\rho\, dv)$$

Next we simplify the equation in light of the flow model we have proposed. First we note that the last expression is zero by virtue of assumptions 1 or 2. Now we examine the other expressions using separate horizontal and vertical components of the equation. In doing so we keep the control volume and its surface under close scrutiny. Thus, considering forces first, note that we neglect hydrostatic pressure variation at the entrance and exit of the control surface and have taken uniform absolute pressures p_1 and p_2 over these sections, respectively, in accordance with assumption 3.

The x and y components of the resultant force on the fluid may be expressed as

$$\oiint_{CS} T_x\,dA + \iiint_{CV} B_x\rho\,dv = p_1A_1 - p_2A_2\cos\theta + R_x$$
$$\oiint_{CS} T_y\,dA + \iiint_{CV} B_y\rho\,dv = -p_2A_2\sin\theta - W + R_y \quad \text{[a]}$$

where R_x and R_y are the net force components of the reducer wall on the fluid. R_x and R_y, being unknown, have been selected positive.

We examine the linear momentum flow through the control surface. The surface integration need be carried out only at the inlet and outlet surfaces of the control volume since $\mathbf{V}\cdot\mathbf{dA}$ is zero at the walls. (Why?) The normal components of velocity at the inlet and outlet surfaces are seen to equal V_1 and V_2, respectively. By virtue of the one-dimensionality assumption 3, the efflux rate of linear momentum may then be expressed as

$$\oiint_{CS} \mathbf{V}(\rho\mathbf{V}\cdot\mathbf{dA}) = \mathbf{V}_1(-\rho_1V_1A_1) + \mathbf{V}_2(\rho_2V_2A_2) \quad \text{[b]}$$

The scalar components of Eq. b in the x and y directions are given as

$$\oiint_{CS} V_x(\rho\mathbf{V}\cdot\mathbf{dA}) = -V_1(\rho_1V_1A_1) + (V_2\cos\theta)(\rho_2V_2A_2)$$
$$\oiint_{CS} V_y(\rho\mathbf{V}\cdot\mathbf{dA}) = (V_2\sin\theta)(\rho_2V_2A_2) \quad \text{[c]}$$

The **continuity equation** for this control volume meanwhile, using assumptions 1 and 3, may be stated in a simple form. Thus

$$\rho_1V_1A_1 = \rho_2V_2A_2 \quad \text{[d]}$$

Now we substitute the preceding results into the linear momentum equations in the x and y directions. Thus we get

$$p_1A_1 - p_2A_2\cos\theta + R_x = (V_2\cos\theta - V_1)\rho_1V_1A_1$$
$$-p_2A_2\sin\theta - W + R_y = (V_2\sin\theta)\rho_1V_1A_1 \quad \text{[e]}$$

We may now solve for R_x and R_y. Changing the sign of these results will then give the force components *on the elbow* from the fluid. Using the symbols K_x and K_y for these components, we get

$$K_x = p_1A_1 - p_2A_2\cos\theta - \rho_1V_1A_1(V_2\cos\theta - V_1)$$
$$K_y = -p_2A_2\sin\theta - W - \rho_1V_1A_1(V_2\sin\theta) \quad \text{[f]}$$

Now for thin-walled devices such as the elbow of this example, using *gage pressures* for p_1 and p_2, we get *both* the force on the elbow from internal flow as well as the force on the elbow from air outside. In Example 4.5 we shall justify this assertion.

■ Debriefing

Note that just like in your course in statics where you had to include *all* the external forces when drawing a free body diagram, you must do likewise for applying the momentum equation to a control volume. And, of course, it is for the very same reason: all external forces affect the momentum of the flow inside the control volume. Furthermore, note that the steps taken in this problem are identical to those taken in Example 4.1. Next, we ask are the assumptions reasonable? The one-dimensional flow assumption at the ends of the control volume should not be of great concern since there is no great sudden curvature nor sudden change of diameter there. Also, we shall later learn that for parallel flow (streamlines parallel) there is little change in pressure at a section of pipe flow; generally only small hydrostatic variations in pressure are present. This is usually neglected except for very large diameter pipes. In a building, the assumption of uniform pressure that we used at the ends of the elbow should not be of concern.

We will see in Example 4.5 that if the wall of the reducing elbow is *thin,* and if there is *uniform atmospheric pressure* outside, we can get the force from inside water *and* outside air using the inside control volume while simply using *gage pressures* when computing K_x and K_y. When similar conditions prevail in other problems, we can shorten and simplify calculations via this procedure. However, when these conditions do *not* prevail we must make *separate* calculations for the flow inside the device and for conditions outside the device. The outside could conceivably be a hydrostatic pressure field or even a different outside flow! For the former we would then deal with a *curved surface* submerged in the *hydrostatic pressure field*—a problem we dealt with in Chap. 2.

EXAMPLE 4.5

■ Problem Statement

In Example 4.4, compute the force from atmospheric pressure acting on the outside surface of the reducing elbow. Show that if *gage* pressures are used for p_1 and p_2 in the linear momentum equation for internal flow, then the total force from *both* internal and external fluids will automatically be determined.

■ Strategy

We will examine the outside surface of the reducing elbow in the x and y directions. The force components in these directions will be determined using the hydrostatics of Chap. 2. We will denote them as $(K_x)_{\text{air}}$ and $(K_y)_{\text{air}}$ respectively. In doing this, we will have to picture what the *projected* areas of the outside reducer elbow surface are in the x and y directions. We will then get the force component from the air on the outside surface, which we will finally add to the respective force components from the internal flow determined in Example 4.4.

■ Execution

We first examine the force on the outside that is directed in the x direction. This is a simple statics calculation of a uniform pressure on a curved surface. We get

$$(K_x)_{\text{air}} = -p_{\text{atm}}A_1 + p_{\text{atm}}A_2 \cos\theta \qquad \text{[a]}$$

Similarly in the y direction we have

$$(K_y)_{\text{air}} = p_{\text{atm}}A_2 \sin\theta \qquad \text{[b]}$$

Using Eqs. a and b, and Eq. f of Example 4.4, we compute the *total* force *on* the elbow from the water inside and the air outside. Collecting terms we then get

$$\begin{aligned}(K_x)_{\text{total}} &= (p_1 - p_{\text{atm}})A_1 - (p_2 - p_{\text{atm}})A_2 \cos\theta - \rho_1 V_1 A_1 (V_2 \cos\theta - V_1)\\ (K_y)_{\text{total}} &= -(p_2 - p_{\text{atm}})A_2 \sin\theta - W - \rho_1 V_1 A_1 (V_2 \sin\theta)\end{aligned} \qquad \text{[c]}$$

We may now use gage pressures in the above equations.

$$\begin{aligned}(K_x)_{\text{total}} &= (p_1)_{\text{gage}} - (p_2)_{\text{gage}} A_2 \cos\theta - \rho_1 V_1 A_1 (V_2 \cos\theta - V_1)\\ (K_y)_{\text{total}} &= -(p_2)_{\text{gage}} A_2 \sin\theta - W - \rho_1 V_1 A_1 (V_2 \sin\theta)\end{aligned} \qquad \text{[d]}$$

■ Debriefing

Compare Eq. f of Example 4.4 with Eq. d of Example 4.5. Clearly, you could have arrived at Eq. d directly had you used *gage pressures* at the outset of Problem 4.4, when considering the internal flow of fluid through the reducing elbow. However, this is not a correct use of the linear momentum equation. It is a *formal* step that we will take, when we can do so, to get the desired total force.

EXAMPLE 4.6

■ Problem Statement

In Fig. 4.9, water flows through a double exit elbow for which $V_1 = 5$ m/s and $V_2 = 10$ m/s. The inside volume of the elbow is 1 m^3. What are the vertical and horizontal force components from air and water on the elbow? Note that the pressure in the free jets is atmospheric.

■ Strategy

The obvious control volume to use is the interior of the elbow (see Fig. 4.10) since in the linear momentum equation only the desired unknowns and easily computable forces will appear. And, by using gage pressures, we will get the totality of forces from inside and outside the control surface. We make the following assumptions for simplifying the linear momentum equation:

1. Steady flow.
2. Incompressible flow.
3. 1-D flows passing through the control surface.
4. The elbow has thin walls.

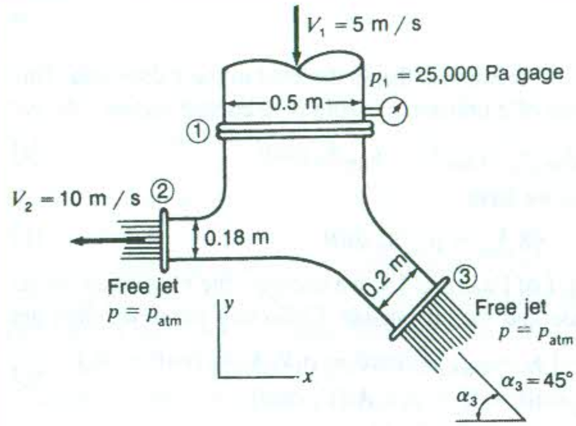


Figure 4.9
Elbow with multiple openings.

■ Execution

We start with the *continuity equation* for the control volume. Making full use of assumptions 1, 2, and 3, we can go directly to a working equation which is

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 + \rho_3 V_3 A_3$$

We have, on canceling out the ρ's and inserting data,

$$5\left(\frac{\pi(5)^2}{4}\right) = 10\left(\frac{\pi(0.18)^2}{4}\right) + V_3\left(\frac{\pi(0.2)^2}{4}\right) \qquad \therefore\ V_3 = 23.15 \text{ m/s}$$

We now go to the *linear momentum equation.*

$$\oiint_{CS} \mathbf{T}\, dA + \iiint_{CV} \mathbf{B}\, \rho\, dv = \oiint_{CS} \mathbf{V}(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \iiint_{CV} \mathbf{V} \rho\, dv$$

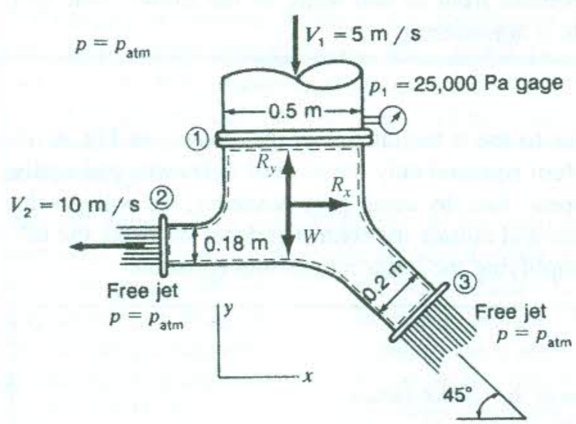


Figure 4.10
Elbow with control volume.

Inserting data, we get for the x component of force from the elbow on water

$$R_x = -(10)\left[(1000)(10)\left(\frac{\pi(0.18)^2}{4}\right)\right] + (23.15)(0.707)\left[(1000)(23.15)\left(\frac{\pi(0.2)^2}{4}\right)\right]$$

We then get

$$R_x = 9539 \text{ N} \qquad K_x = -9539 \text{ N}$$

For the y component we have

$$R_y - 25{,}000\left(\frac{\pi(0.5)^2}{4}\right) - (1)(9806) = \left(5\left[(1000)(5)\left(\frac{\pi(0.5)^2}{4}\right)\right]\right) + (-23.15)(0.707)\left[(1000)(23.15)\left(\frac{\pi(0.2)^2}{4}\right)\right]$$

Hence,

$$R_y = 7720 \text{ N} \qquad K_y = -1720 \text{ N}$$

■ Debriefing

In looking at the assumptions more critically, it can be said that the weakest one is the one-dimensional assumption of flows through the exit portions of the control surface. The flow in the elbow proper can be expected to be quite complex away from the inlet, and if the outlets are not reasonably far from the center of the elbow, the flow in the outlets will share some of this complexity. If the lengths of these outlet extensions (not specified in the problem) are small, the results will have to be judged as less accurate.

EXAMPLE 4.7

■ Problem Statement

A jet of water in Fig. 4.11 issues from a nozzle at a speed of 6 m/s and strikes a stationary flat plate oriented normal to the jet. The cross-sectional area of the jet is 645 mm^2. What is the total horizontal force on the plate from the fluids in contact with it?

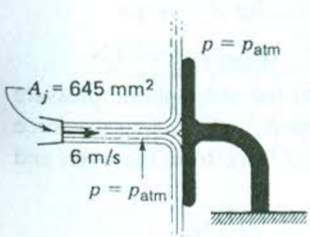


Figure 4.11
Jet strikes a flat plate.

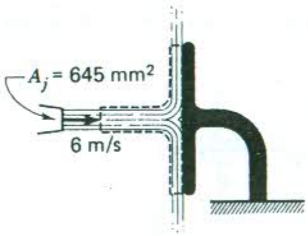


Figure 4.12
Control volume for the jet on the plate.

■ Strategy

We shall use a control surface that coincides with the wetted area of the plate and cuts the jet away from the plate as well as the water moving out radially from the plate (see Fig. 4.12) By using the *linear momentum equation* for this control volume, we will bring into play the horizontal force from the plate onto the water, the reaction to which we seek. As for the air acting on the nonwetted part of the plate we shall use simple statics. We now make the following assumptions:

1. Steady, incompressible flow.
2. 1-D flow out of the nozzle.
3. Free jet entering the control volume $p = p_{\text{atm}}$.
4. Exiting sheet of water is at atmospheric pressure.

■ Execution

We start with the **linear momentum equation** in the x direction. Thus:

$$R_x + \iiint_{CV} B_x \rho \, dv = \oiint_{CS} V_x(\rho \mathbf{V} \cdot \mathbf{dA})$$

We have deleted the time derivative expression because of assumption 1, and we shall treat ρ as a constant. Taking A_P as the wetted area of the plate and noting that the projected area of the control surface in the x direction on which atmospheric pressure acts is A_P, we have for the above equation

$$p_{\text{atm}} A_P + R_x = -V_x^2 \rho A_j$$

where A_j is the cross-sectional area of the jet. Solving for R_x, we get

$$R_x = -p_{\text{atm}} A_P - (6)^2(1000)(645 \times 10^{-6}) = -p_{\text{atm}} A_P - 23.2 \text{ N}$$

The net horizontal force directly on the plate from the atmospheric pressure clearly is $-p_{\text{atm}} A_P$. Taking the reaction to the above R_x, and adding the force from the atmospheric pressure, we then have the total force from the water and air on the plate.

$$(K_x)_{\text{total}} = p_{\text{atm}} A_P + 23.2 - p_{\text{atm}} A_P = 23.2 \text{ N}$$

■ Debriefing

It should be noted that had we used gage pressures in the calculation, we would have automatically included the force stemming from the atmospheric pressure on the nonwetted surface of the plate. Also, we could have used a control surface similar to what is to be used in Example 4.8. Finally, note that for this problem we did not have to assume no friction or gravity effects as we will for vane problems. Why?

EXAMPLE 4.8

■ Problem Statement

A steady stream of water in Fig. 4.13a is directed against a stationary curved trough. What is the horizontal force on the trough from the water wetting the trough and the atmosphere on the rest of the trough?

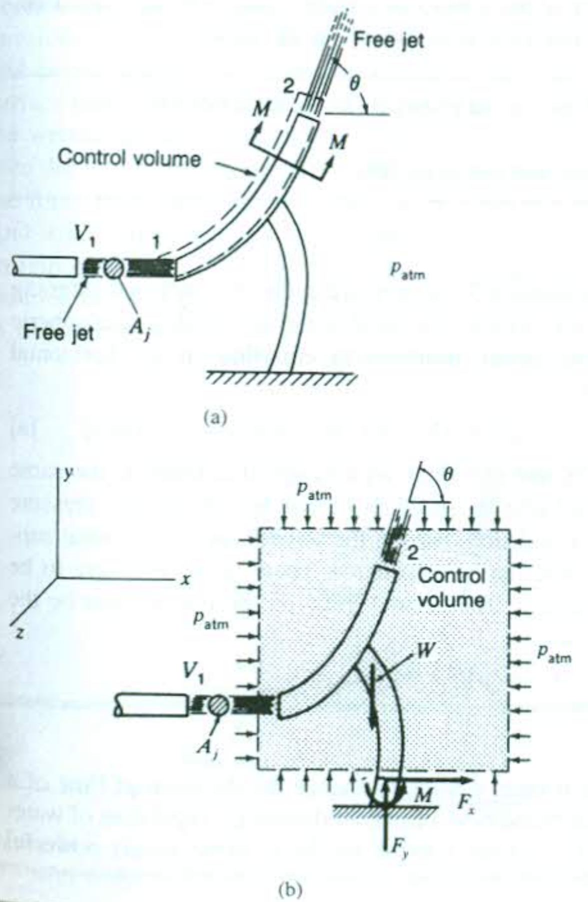


Figure 4.13
Jet acting on a trough.

■ Strategy

Rather than the control volume in Fig. 4.13a, we will choose as a control volume the region that surrounds the trough entirely, as shown in Fig. 4.13b, and cuts through the support as well as the incoming and outgoing stream of water. The forces acting on the control volume are:

Surface tractions	F_x, F_y, and M, which are the components of the resultant force system at the cut through the metal support. Also, the forces in the incoming and outgoing stream which can be considered as free jets through whose cross sections we have atmospheric pressure.
Body forces	The total weight of all the material inside the control volume (this includes water, trough, and part of the support system).

We will use the component of the *linear momentum equation* in the x direction. For its use we make the following assumptions for the flow:

1. Neglect the action of friction and gravity on the speed of the flow.
2. Incompressible flow.
3. 1-D flow at the entrance and exit of the flow.

■ Execution

Note that as a result of assumption 3, the pressure in the incoming and outgoing jets is uniform over the cross section and equal to the surrounding atmospheric pressure. Accordingly, the **linear momentum equation** in the horizontal direction can be given as

$$F_x = -(\rho V_1 A_j)(V_1) + (\rho V_1 A_j)(V_1 \cos\theta) = -(\rho V_1^2 A_j)(1 - \cos\theta) \qquad [a]$$

where, from assumption 1, the exit fluid velocity-speed is taken as the same value as the incoming fluid velocity speed. Note that the atmospheric pressure gives a zero force in the x direction, leaving the force F_x at the cut metal support as the only traction force to be considered. There is no need here to be concerned with gage pressures. The desired force on the trough must be the reaction to F_x. Thus we have:

$$K_x = (\rho V_1^2 A_j)(1 - \cos\theta)$$

■ Debriefing

This problem shows that it takes a force to change the direction of flow of a fluid. Surely, one can understand how a fire hose directing a rapid flow of water against a person, as used for crowd control, can be so devastatingly powerful in knocking down and pushing the person, since he or she is unwittingly changing the direction of the flow of the jet. In a turbomachine, such as a jet engine, force is developed in changing the direction of flow of combustion products flowing through the engine, and from this force comes torque—a vital aspect of the functioning of the engine. (We will look at the jet engine in Example 4.10.) Next we ask about the reasonableness of assumption 3 in the strategy. For short, smooth troughs we can expect no appreciable change in speed as proposed. We will be more authoritative on this matter when we learn Bernoulli's equation and can more fully appreciate this assumption. Finally, note that we again take the now familiar series of steps listed in the debriefing of Example 4.2.

Before leaving the trough problem we wish to show the other simple control volume that may be used effectively. In Fig. 4.14, a control volume is shown enveloping a jet of water while cutting it normal to the velocity at the entrance and exit of the jet, just as we have done in Examples 4.7 and 4.8. The control surface partly lies at the surface of contact between the water and the trough (i.e., the wetted surface). Thus this control surface will expose the traction force R_x onto the water from the trough. In this way, we can get the reaction force K_x (i.e., the force from the water onto the trough) from the linear momentum equation. And using gage pressures we will get $(K_x)_{\text{total}}$, the force from air and water. As an important exercise using the indicated control volume, show in a simple calculation that you get the same horizontal force on the trough from water and air as in Example 4.8.

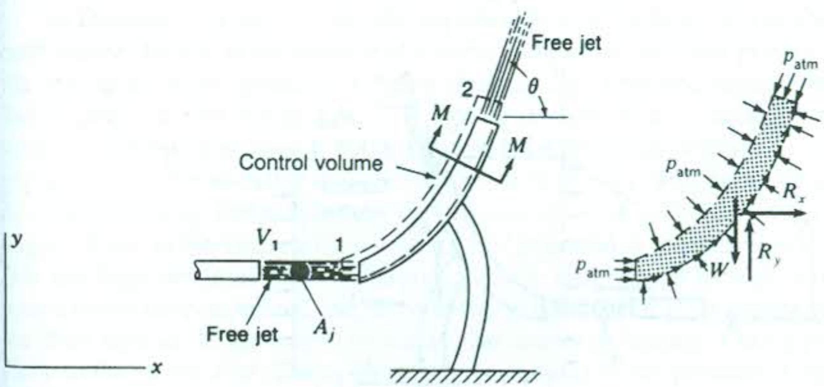


Figure 4.14
Stationary trough.

EXAMPLE 4.9

■ Problem Statement

Consider that the trough of Example 4.8 is *moving* with a constant speed V_0 relative to the ground, as shown in Fig. 4.15. What is the horizontal force acting on the trough, and what is the power being developed by the water on this-trough?

■ Strategy

We shall select the same kind of control volume as in Example 4.8, but now we shall attach this control volume to the trough. The reference *xyz* being fixed to the control volume is then moving at a constant speed V_0 relative to the ground reference *XYZ*, and hence is an inertial reference. This permits us to use the *linear momentum equation* for this control volume as presented in this chapter. The assumptions to be used are identical to those stated in the Example 4.8. Indeed, we may even use the results of Example 4.8 simply by using velocities, as seen from the moving control volume, in place of the corresponding velocities, as seen from the stationary control volume used in the example. Finally, the desired power on the trough from the jet stems from the product of the trough velocity times the horizontal force computed using the linear momentum equation in the *x* direction as well as the continuity equation.

■ Execution

Using the appropriate components of the relative velocity vector $\mathbf{V}_1 - \mathbf{V}_0$, we can now modify the **linear momentum equation** directly from the preceding problem. Thus we have

$$F_x = (V_1 - V_0)[-(V_1 - V_0)(\rho)(A_j)] + (V_1 - V_0)\cos\theta[(V_1 - V_0)(\rho)(A_2)] \quad [a]$$

The **continuity equation** for the control volume is

$$(V_1 - V_0)(\rho)(A_j) = (V_1 - V_0)(\rho)(A_2) \quad [b]$$

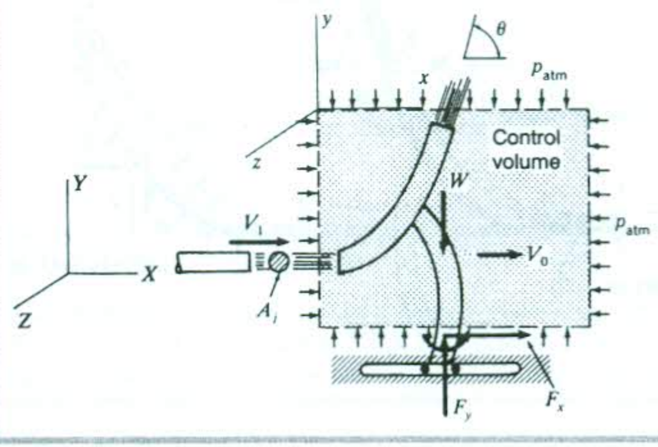


Figure 4.15
Moving trough.

Upon using Eqs. b and a, the solution for the thrust F_x can be stated as

$$F_x = -(V_1 - V_0)^2(\rho A_j)(1 - \cos\theta) \qquad [c]$$

Finally, to get the power all we need do is multiply F_x above by the velocity V_0 of the trough.

■ Debriefing

The key step for this problem was to use velocities measured from the control volume. Furthermore, it may come as a surprise to the student that this simple problem, while illustrating the case of a control volume moving at constant speed relative to inertial space, is also the beginning example of a turbomachine.

In Example 4.9 we considered a device which we will call a simple *turbomachine*. Note that for this device the motion of an *unconfined* fluid is altered in such a way that the propulsive thrust is created on the device. As it moves, power is developed on the device from energy supplied by the jet of water. In more complex turbomachines, such as turbojets and ram jets, fluid is made to undergo certain processes so as to achieve a flow pattern which gives rise to a propulsive thrust. For these devices, burning fuel supplies the requisite energy for maintaining the necessary flow to accomplish this task. Other turbomachines have different missions from that of propulsion. Steam turbines, for example, are devices in which the flow pattern is arranged so as to develop a torque on a rotor and, in this way, to drive a generator for electric power. In addition we have rotary pumps, torque converters, fluid couplings, centrifugal compressors, and so on. All of these devices are considered apart from the familiar *reciprocating machine* in that during the process that the fluid undergoes in a turbomachine the fluid is at no time "trapped," or confined, by the machine, as is the case with reciprocating machines such as a diesel engine, where the fluid is confined in cylinders during most of the action.

In Example 4.10, we will consider the calculation of the thrust of a turbojet aircraft engine. Hence, as an example of a turbomachine, we will now present a simple description of its operation. A highly idealized and simplified representation of this engine is shown in Fig. 4.16. It is moving through air at a constant subsonic speed V_1. Air enters at speed V_1 relative to the plane and is slowed down while gaining pressure in the so-called *diffuser* section (we will study diffusers in Chap. 10). A *compressor* then further increases the pressure of the air as it flows along in the engine. Then, in the combustion chambers, fuel is burned in the airstream to maintain this high pressure. The combustion products then expand through a *turbine* which drives the compressor, and, in so doing, gives up some of its energy. Finally, the fluid expands in the *nozzle* section so that ideally on leaving it has a pressure close to that of the atmosphere. The buildup and decay of the pressure as the fluid flows through the machine results in traction force distributions on the interior surfaces of the engine that combine to give a powerful forward thrust which is transmitted to the plane.

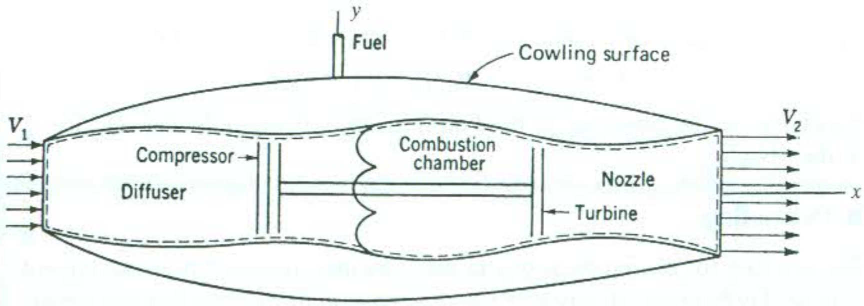


Figure 4.16
Simple sketch of a jet engine.

EXAMPLE 4.10

■ Problem Statement

Using Fig 4.16, determine the formula giving the thrust of the jet engine. The exit velocity is known from other calculations to be V_2 relative to the engine. Furthermore, it is known that $(1/N)$ kilograms of fuel is burned during operation per kilogram of incoming air. N is a number dependent on the particular engine under consideration. (See Fig. 4.17 for a cutaway view of an early jet engine.)

■ Strategy

We will select as a control volume the interior of the jet engine (Fig. 4.16) where the control surface is identified by the closed dashed line. Although not shown, this control surface *cuts through* the supports of the compressor, combustion chamber, and so forth going to the inside of the engine cowling. Such a control volume clearly "exposes" for calculations the force distribution

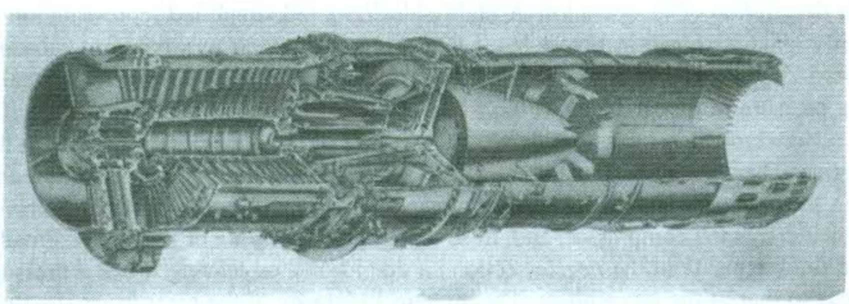


Figure 4.17
Cutaway view of an early jet engine. (*Courtesy Curtiss Wright Corp.*)

on the fluid from the wall of the jet engine, as well as the traction forces transmitted from the wall to the elements interior to the wall, and thus inside the control volume (such as the compressors, combustion chambers, etc.). The reaction to the totality of these forces is then the thrust that the engine develops.

The assumptions that we shall make to simplify the problem are these:

1. Steady flow.
2. 1-D flow of air into the control surface and 1-D flow of combustion products out of the control surface.
3. Inlet and exhaust pressures are at atmospheric pressure.

■ Execution

With ρ_1 and N known, the rate of efflux of mass from the control volume is given by the **continuity equation** using the simplifying assumptions in the following manner:

$$\rho_1 V_1 A_1 + \frac{1}{N}(\rho_1 V_1 A_1) = \rho_2 V_2 A_2$$
$$\therefore \rho_2 V_2 A_2 = \left(1 + \frac{1}{N}\right)\rho_1 V_1 A_1 \qquad \text{[a]}$$

The **linear momentum equation** in the x direction is next examined:

$$\iiint_{CV} B_x \rho\, dv + \oiint_{CS} T_x\, dA = \oiint_{CS} V_x(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t}\iiint V_x(\rho\, dv)$$

Assumption 1 eliminates the time derivative while 2 simplifies the surface integrations. Using p_{atm} in the traction force calculations as per assumption 3, we then have with $B_x = 0$,

$$p_{\text{atm}} A_1 - p_{\text{atm}} A_2 + R_x = (\rho_2 V_2^2 A_2) - (\rho_1 V_1^2 A_1) \qquad \text{[b]}$$

Solving for R_x, replacing $\rho_2 V_2 A_2$ using Eq. a, and taking the reaction to R_x gives the thrust from the working fluid:

$$K_x = \rho_1 V_1 A_1 \left[V_1 - \left(1 + \frac{1}{N}\right) V_2 \right] + p_{\text{atm}} A_1 - p_{\text{atm}} A_2 \qquad \text{[c]}$$

Note that the linear momentum flow of the entering fuel has no component in the x direction and hence does not appear in Eqs. b and c.

If the pressure on the cowling surface is close to atmospheric pressure, the use of statics on the cowling surface gives the following force due to air:

$$(K_x)_{\text{air}} = -p_{\text{atm}} A_1 + p_{\text{atm}} A_2 \qquad \text{[d]}$$

Adding Eqs. c and d will include inside and outside forces with the exception of outside shear tractions in accordance with our conclusions in Example 4.5.

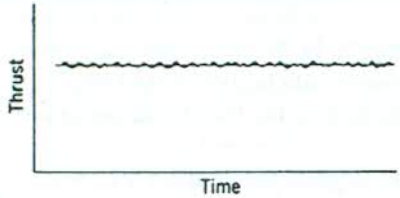


Figure 4.18
Thrust variation.

Since the shear drag is not usually "charged" against the engine, it may be stated approximately that[2]

$$\text{Thrust} = \rho_1 V_1 A_1 \left[V_1 - \left(1 + \frac{1}{N}\right) V_2 \right] \quad \text{[e]}$$

■ Debriefing

There was the tacit assumption of steady flow in the analysis. Actually, this is not the case in regions around the compressor blades and turbine blades. However, these departures from steady flow, if taken into account accurately, would superpose on the constant thrust computed earlier a high-frequency small-amplitude variation having a mean time average variation of zero over time intervals which are large compared with the periods of these variations. This is shown in Fig. 4.18. In computing the thrust for purposes of evaluating performance of an aircraft, these variations are insignificant, so the assumption of steady flow throughout yields with greatest ease the desired average thrust. However, in vibration studies, it is the variation that is important, since even small disturbances can cause large stresses in parts if resonance is reached for these parts. Clearly, a more precise study would be needed for such problems.

EXAMPLE 4.11

■ Problem Statement

Fig. 4.19a shows the cross section of a sluice gate, which is a device used for controlling the flow of water in a channel. Determine the force per unit width of the gate for the data given.

■ Strategy

It is clear that the *linear momentum equation* should be used along with the *continuity equation* for a control volume chosen to permit a simple formulation. This control volume must expose the desired unknown force for use in

[2]Here is an instance where care must be used in examining flows inside the control volume and flow on the outside of the body. Overly glib use of gage pressures must be guarded against at this stage of your studies.

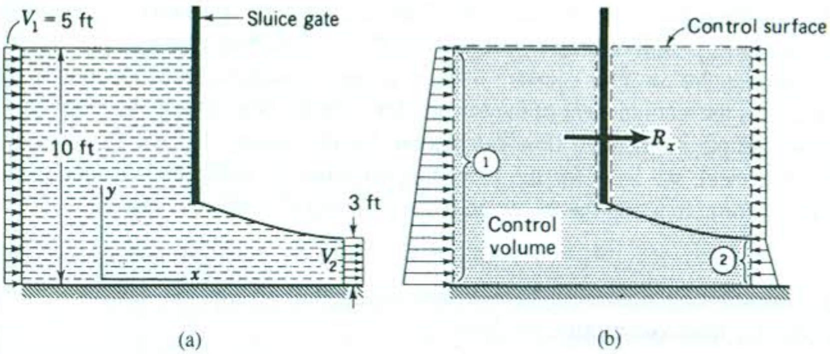


Figure 4.19
Flow by a sluice gate.

the momentum equation. Such a control volume is proposed in Fig. 4.19b, wherein the control surface is along the wetted surface of the gate on one side and wherein there is atmospheric pressure on the opposite side. The control surface upstream and downstream of the gate are taken far enough from the gate so as to approach 1-D flow at these locations shown in Fig. 4.19a. The net traction force on the control surface from the gate on the water on one side and the air on the other side is denoted as R_x. The traction force distribution on face 1 of the control surface (see Fig. 4.19b) is the superposition of atmospheric pressure plus a uniformly increasing hydrostatic pressure. At the other end of the control surface, we have again atmospheric pressure, plus on part 2 of this surface we have a uniformly increasing hydrostatic pressure starting at the free surface of the exiting flow. The force we are seeking is the reaction to the force R_x.

For this purpose, we shall make the following assumptions to form a simple working equation from the general linear momentum equation.

1. At the inlet and outlet we assume a uniform pressure p_{atm} on which is superposed a hydrostatic pressure variation.
2. Steady flow.
3. Incompressible flow.
4. 1-D velocity at entrance and exit.
5. Zero shear stress at bed of channel.

■ Execution

The **linear momentum equation** in the x direction is first stated as follows:

$$\iiint_{CV} B_x \rho \, dv + \oiint_{CS} T_x \, dA \equiv \oiint_{CS} V_x(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \iiint_{CV} V_x(\rho \, dv)$$

The condition $B_x = 0$ and assumption 2 kill the first and last expressions, while assumption 1 permits a simple calculation of the traction integral at inlet and outlet. Assumption 3 allows for constant ρ and γ in the calculation, while assumption 5 eliminates the x component of the traction force on the bed surface. Assumption 4 allows simple momentum flow calculation. Finally, noting that the atmospheric effects cancel, we have for the preceding equation on using hydrostatics from Chap. 2 for computing the forces on plane submerged surfaces 1 and 2,

$$\gamma(y_1)_c A_1 - \gamma(y_2)_c A_2 + R_x = \rho V_2^2 A_2 - \rho V_1^2 A_1 \qquad \text{[a]}$$

Substituting numerical values and noting that V_2 can be determined as $V_1(A_1/A_2)$ from **continuity,** we have

$$(62.4)(5)(10) - (62.4)(1.5)(3) + R_x = (1.938)(16.67)^2(3) - (1.938)(5^2)(10)$$

$$R_x = -1708 \text{ lb/ft of width} \qquad \text{[b]}$$

Therefore, the force on the gate is then 1708 lb/ft of width.

■ Debriefing

One might question the use of hydrostatics at the inlet and exit of the control volume. Even if the flow is not really one-dimensional there, the fact that the flow is essentially parallel permits the use of hydrostatics as we shall later see in Chap. 8, the pipe flow chapter. Intuitively, we can argue, meanwhile, that because of the parallel flow, there will be no acceleration vertically, and hence a hydrostatic pressure variation is to be taken as close to the actual case. Also, with a comparatively slow flow of a fluid with low viscosity, such as water, our own everyday experience would argue for very small friction at the ground.

All the examples up to this time in this chapter have been steady flow problems. Unsteady flow problems require the inclusion of the expression $\partial/\partial t[\iiint \mathbf{V}(\rho\, dv)]$. This requires knowledge of the velocity field inside the entire control volume and not just at the control surface as is the case for steady flow problems. This negates one of the key advantages of the integral approach which allowed up to now the calculation of useful information with a minimum knowledge of the flow details. In Example 4.12 we look at an interesting unsteady flow problem which, as you will see, submits to a rather simple solution.

EXAMPLE 4.12

■ Problem Statement

A fighter plane is being refueled in flight (see Fig. 4.20) at the rate of 150 gal/min of fuel having a specific gravity of 0.68. What additional thrust does the plane need to develop to maintain the constant velocity it had just before the hookup? The inside diameter of the flexible pipe is 5 in. The fluid pressure in the pipe at the entrance to the plane is 4 lb/in^2 gage. Do not consider the mechanical forces on the plane directly from the flexible pipe itself.

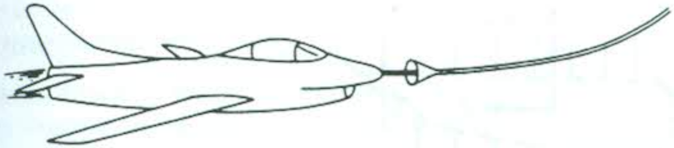


Figure 4.20
Fighter plane being refueled in flight.

■ Strategy

We shall use a hypothetical control volume supported by the plane to represent the actual gas tank as well as the duct leading from the inlet to the gas tank (see Fig. 4.21). A reference will be attached to the plane. And we will assume that the plane and the reference are moving at constant speed relative to the ground, making them an inertial control volume and an inertial reference, respectively. But note this: The flow in the tank is *unsteady* since gasoline is constantly accumulating inside the tank. To determine the additional thrust needed from the jet engine to maintain constant speed once pumping gasoline has started, we will use the *linear momentum equation* for this control volume. We will assume atmospheric pressure over the entire outside control surface area, except at the cut supports and the gasoline duct inlet. Because the horizontal projection of this area is virtually zero, we can throw out the horizontal force contribution from atmospheric pressure. This will most easily be accomplished by using gage pressure. We shall make the following additional assumptions for the flow in the control volume:

1. 1-D flow at entrance.
2. Incompressible flow.
3. *Average* velocity in the tank in the x direction is *zero* relative to xyz and is constant in the duct leading to the tank.

■ Execution

The general **linear momentum equation** in the x direction is first stated.

$$\oiint_{CS} T_x\,dA + \iiint_{CV} B_x\rho\,dv = \oiint_{CS} V_x(\rho\mathbf{V}\cdot\mathbf{dA}) + \frac{\partial}{\partial t}\iiint_{CV} V_x(\rho\,dv)$$

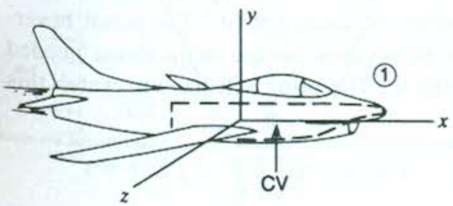


Figure 4.21
Fighter plane showing a hypothetical gasoline tank system.

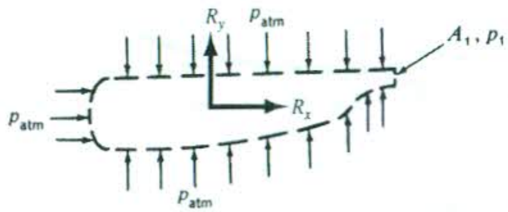


Figure 4.22
Outer surface of gas tank system.

Since V_x has been assumed to average zero value in the tank (assumption 3), we can delete the last expression even though the flow technically is unsteady. There is no body force contribution in the x direction. The velocity entering at 1 is given as follows:

$$V_1 = \frac{150(\text{gal/min})\left(0.002228 \dfrac{\text{ft}^3/\text{s}}{\text{gal/min}}\right)}{\left[\dfrac{\pi}{4}\left(\dfrac{5}{12}\right)^2(\text{ft}^2)\right]}$$

$$\therefore\ V_1 = 2.45 \text{ ft/s}$$

Hence entering numerical data, using Fig 4.22, into the linear momentum equation we get, noting assumptions 1 and 2,

$$R_x - (4)\left(\frac{\pi 5^2}{4}\right) = -(2.45)\left[-\frac{62.4}{g}(0.68)(2.45)\frac{\pi 5^2}{(4)(144)}\right]$$

$$\therefore\ R_x = 79.6 \text{ lb}$$

The force on the plane from the gasoline is then

$$\boxed{K_x = -79.6 \text{ lb}}$$

■ Debriefing

By using a hypothetical tank in place of a complex storage region which might have included portions of the wing interior and regions of the fuselage, we see how we can tremendously simplify a problem by a creative use of control volume geometry. With regard to the steady flow assumption, we are not on as solid a ground. There is to be expected sloshing of gasoline back and forth in the tank, whatever its shape. Actually, a more complex tank geometry might mitigate this action to some extent.[3] The result nevertheless will mean that there will be superposed on the main thrust needed for the gasoline exchange some variation. The pilot will have to watch this carefully.

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[3]here will be an unsteady contribution in the vertical direction as the gasoline rises in the tank.

EXAMPLE 4.13

■ Problem Statement

Water flows through a frustrum-shaped nozzle in Fig. 4.23. It weighs 500 N. The inlet diameter to the nozzle is 1 m and the outlet diameter is 0.3 m. The average velocity of flow into the nozzle is 2 m/s. If a clamping force of 3000 N is required to insure that there will be no leaking, compute the minimum stress needed in each of the 10 bolts to properly hold the cone and the pipe together. The diameter of each bolt is 20 mm.

■ Strategy

We shall choose a control surface that cuts the bolts, and is at the interface between the frustrum and the pipe cutting the inflow of water at the top and cutting the outflow of water at the bottom, as shown in Fig. 4.24. Notice that at the top there is a pressure from the incoming water which we shall determine using the U-tube. There is also shown forces exposed by cutting the bolts. Finally, there is the total force of 3000 N coming from the pipe onto the frustrum. Everywhere else on the control surface we have shown atmospheric

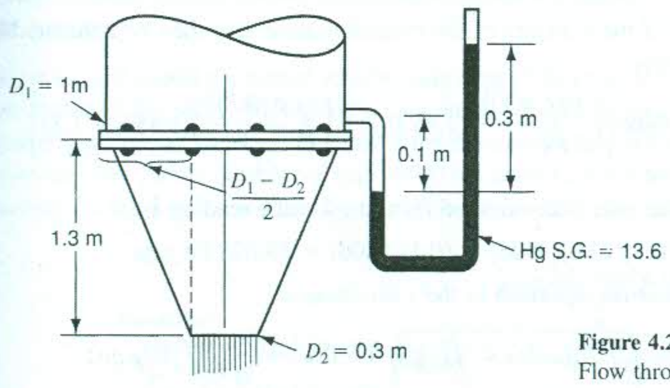


Figure 4.23
Flow through a frustum.

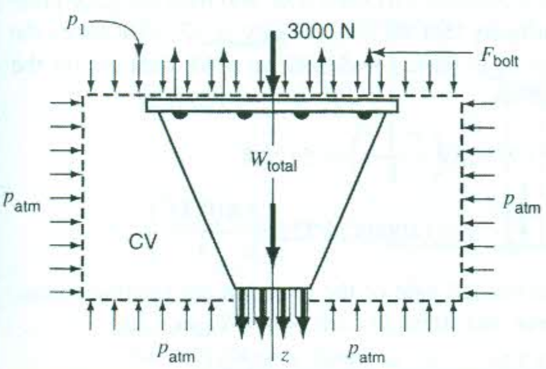


Figure 4.24
Control volume to be used.

pressure. We will use the z component of the *linear momentum equation*. In doing so, we will use the following assumptions:

1. Steady flow.
2. 1-D flow entering and leaving the control volume.
3. Incompressible flow.

■ Execution

We have to calculate next the weight of the water inside the frustrum using the data given. If the frustrum was extended to form a cone of height h, we could easily determine this height by similar triangles as follows (see Fig. 4.23):

$$((D_1 - D_2)/2)/1.3 = (D_1/2)/h$$
$$h = 1.857 \text{ m}$$

We can then use the formula for the volume of a right circular cone:

$$\text{Volume} = \left(\frac{1}{3}\right)(\text{area of the base})(\text{height})$$

The total weight of the contents of the control volume can then be computed in the following way:

$$W_{\text{total}} = 500 + 9806\left[\left(\frac{1}{3}\right)\left(\frac{\pi(1)^2}{4}\right)(1.857) - \left(\frac{1}{3}\right)\left(\frac{\pi(0.3)^2}{4}\right)(1.857 - 1.3)\right]$$
$$= 5138.6 \text{ N}$$

The pressure at the inlet is determined from the U-tube reading as

$$p_1 = (0.3)(13.6)(9806) - (0.1)(9806) = 39{,}028 \text{ Pa gage}$$

The **linear momentum equation** in the z direction is

$$\oint\!\!\!\oint_{CS} T_z\, da - \iiint_{CV} B_z(\rho\, dv) = \oint\!\!\!\oint_{CS} V_z(\rho\mathbf{V}\cdot d\mathbf{A}) + \frac{\partial}{\partial t}\iiint_{CV} V_z(\rho\, dv)$$

Now, first take F_B as the force transmitted by each bolt, and then use gage pressures. Finally, note from **continuity** that the exit velocity is $(D_1/D_2)^2$ times the inlet velocity of 2 m/s so that $V_2 = (\frac{1}{.3})^2(2) = 22.222$ m/s. We then get for the above linear momentum equation

$$10\,F_B - 3000 - (39{,}028)\left(\frac{\pi(1^2)}{4}\right) - 5138.6$$
$$= (1000)(2)^2\left(\frac{\pi(1)^2}{4}\right) - (1000)(22.222)^2\left(\frac{\pi(0.3)^2}{4}\right)$$

where the first three terms on the left side of the equation are traction forces. We can now solve for the force and stress in each bolt. We get

$$F_{\text{bolt}} = 703.3 \text{ N} \qquad \tau_{zz} = 2.239 \times 10^6 \text{ Pa}$$

■ Debriefing

How good is the 1-D flow assumption at the inlet and the outlet? At the inlet, one should expect a good 1-D representation of the velocity profile except for some radial flow at the outer region of the frustum. We can consider this to be the case because the pipe flow reaching the frustum will have the expected turbulent flow, wherein the velocity profile is close to uniform across the section (as we will later see when we study pipe flow). However, at the end of the frustum, we can expect a radial component of velocity the amount of which will depend on the value of the frustum angle. In homework problems 4.62 and 4.63 we have asked you to examine the effect on thrust stemming from the radial velocity component appearing in a nozzle. Generally, except in extreme cases, it is not large, and so our assumption is reasonable at the outlet.

EXAMPLE 4.14

*■ Problem Statement

What is the total vertical force on the elbow in Fig. 4.25 from the oil flowing through it and the water on the outside surface?

■ Strategy

First we will consider a control volume consisting of the entire interior of the elbow as shown in Fig. 4.26. We cannot simply employ the linear momentum equation using gage pressures to get the total force from inside and outside the elbow in one step. We cannot do this because we do not have the needed uniform pressure acting on the outside surface as was the case in other problems that we have

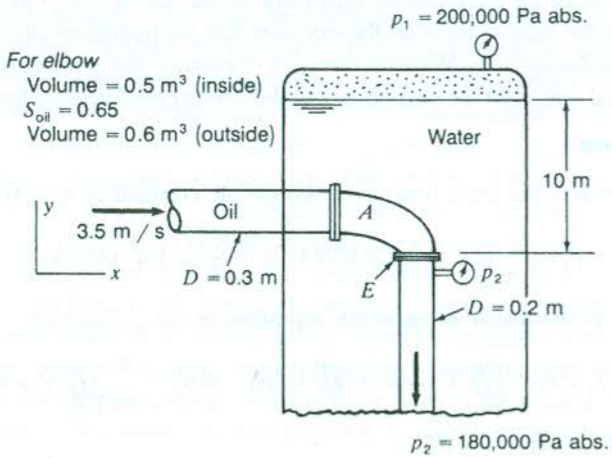


Figure 4.25
Two fluid elbow problem.

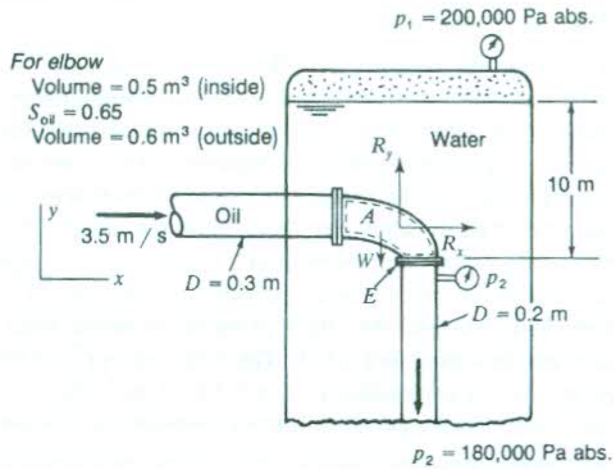


Figure 4.26
Control volume inside elbow.

previously tackled. Instead, we will use the hydrostatics of Chap. 2 for the outside surface, and then we will superpose results. For starters, we shall need the mass flow of the oil. Using the given data and this mass flow, we will then go to the *linear momentum equation* in the vertical direction to determine the vertical force contribution from the oil. We will use absolute pressures. Next, we will imagine first that the elbow has closed ends and that it is *removed* from the pipes and placed in the tank surrounded by water. For this hypothetical, submerged body, we will use *Archimedes' principle* to get a buoyant force on this body. Then, to isolate from this buoyant force the force coming only from the *lateral surface* of the elbow, we will subtract the vertical force on the *exit area* that we hypothetically inserted so we could use Archimedes. We need now only superpose the two forces, one from the inside and one from the outside, to get the desired total vertical force.

■ Execution

We first compute the mass flow using the simple **continuity equation.**

$$\rho_2 V_2 A_2 = \rho_1 A_1 V_1 = (0.65)(3.5)\left(\frac{\pi(0.3)^2}{4}\right) = 1608 \text{ kg/s}$$

We now go to the **linear momentum equation** in the y direction.

$$\oiint_{CS} T_y\, dA + \iiint_{CV} B_y \rho\, dv = \oiint_{CS} V_y(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t}\iiint_{CV} V_y \rho\, dv$$

Now inserting values, we get

$$R_y + (180{,}000)\left(\frac{\pi(0.2)^2}{4}\right) - (0.65)(9806)(0.5) = -(1608)\left[\left(\frac{0.3}{0.2}\right)^2(3.5)\right]$$

where the last bracket expression is the ratio of inlet to outlet diameters squared times the inlet velocity, thereby giving us the outlet velocity in accord with continuity. We get for R_y

$$R_y = -3734 \text{ N} \qquad \text{Hence } (K_y) = 3734 \text{ N}$$

Now going to the outside surface, we can say from **hydrostatics,**

$$(K_y)_{\text{water}} = (9806)(0.6) - [(200{,}000) + (9806)(10)]\left(\frac{\pi(0.2)^2}{4}\right) = -3478 \text{ N}$$

where the second extended expression is the force from water and compressed air pressure acting on the outlet face of the isolated, hypothetical, solid, nozzle that we placed in the water to permit us to get the buoyant force over the rest of the nozzle. The total force is then

$$(K_y)_{\text{total}} = 3734 - 3478 = 256 \text{ lb.}$$

■ Debriefing

This problem is instructive in that it clearly illustrates the limit of using gage pressures to get the total force on a body having flow inside the control surface, wherein we do *not* have uniform pressure on the outside of the control surface. We were able, in this case, to use our results from hydrostatics of a submerged body. Conceivably, one could have a flow of fluid about the outside of the control surface. Then we would have to deal with dynamics of flow in this region, a subject that we will be concerned with in the remainder of the text.

4.6 A BRIEF COMMENT

You will note in thinking back over these sample problems that what we have been doing is simplifying the flows through carefully selected control volumes so that with the integral forms of the linear momentum and continuity equations we could solve for certain resultant forces or certain average velocities. This procedure is no different from what you followed in strength of materials, where you assumed certain behavior of the material in beams and shafts, namely, that plane sections remain plane, to compute bending and torsional stresses. To analyze the behavior of the material in beams and shafts more accurately, we would have to consider the basic laws and pertinent subsidiary laws in *differential* form with the view to integrating them to fit the boundary condition of the problem (this is done in the theory of elasticity). You were perhaps not distressed in your strength of material course, since the assumption of plane sections was made only once by the text. From there on very few additional assumptions had to be made to handle the problems, whereas in the work of the present chapter you are required to make assumptions for each problem. Lacking experience, you may be ill at ease in doing this. Also it may seem "unscientific" to you.

The thing to do is to make the most reasonable assumptions you can that permit you to reach answers that according to your judgment will be reasonably close to reality. The scientific aspect to your analysis will lie in whether there is consistency between your computations and the flow models you have chosen, according to the basic and subsidiary laws. (This is why you will be requested to state your assumptions prominently.) On perhaps a much finer scale, this is how all analytic studies proceed. Thus you see that the analytical investigator can use imagination and intuition in such undertakings. The quality and success of the work can usually be assessed by observing how the results, predicted by the use of fundamental laws as applied to the models, check with what is observed and measured in the physical world for reasonably similar conditions.

You are therefore encouraged to proceed boldly with the problems and to try each time to gage the success of your analysis, using your instructor as a guide. If you consider your results studiously each time, you will develop more confidence and precision in carrying out problems.

In Parts 2 and 3 of the text we consider *differential* forms of the laws, and using finer and more realistic models, we are able to learn more accurately what the flows are really like in certain situations.

PART C
MOMENT OF MOMENTUM

4.7 MOMENT OF MOMENTUM FOR A SYSTEM

Consider a finite system of fluid as shown in Fig. 4.27. An element dm of the system is acted on by a force $\mathbf{dF}$ and has a linear momentum $dm\,\mathbf{V}$. From Newton's law, we can say that

$$\mathbf{dF} = \frac{D}{Dt}(\mathbf{V}\,dm)$$

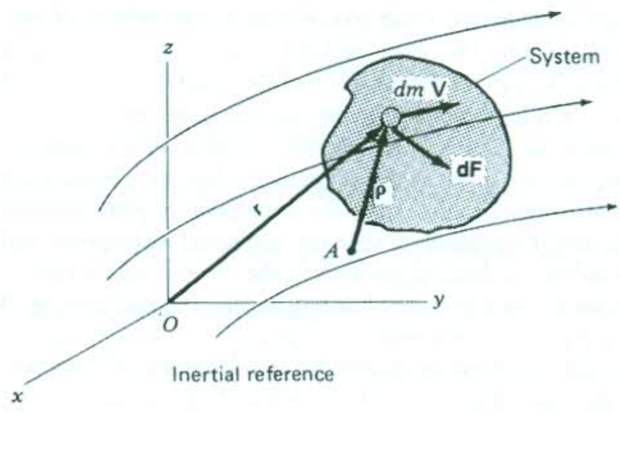


Figure 4.27
Mass dm in a finite system.

Now take the cross product of each side of the equation using the position vector $\mathbf{r}$. Thus,

$$\mathbf{r} \times \mathbf{dF} = \mathbf{r} \times \frac{D}{Dt}(dm\,\mathbf{V}) \tag{4.12}$$

Consider next the following operation:

$$\frac{D}{Dt}(\mathbf{r} \times dm\,\mathbf{V}) = \frac{D\mathbf{r}}{Dt} \times dm\,\mathbf{V} + \mathbf{r} \times \frac{D}{Dt}(dm\,\mathbf{V}) \tag{4.13}$$

Note that $D\mathbf{r}/Dt = \mathbf{V}$, so that the first expression on the right side is zero, since $\mathbf{V} \times \mathbf{V} = \mathbf{0}$. The second expression on the right side of the equation is identical to the right side of Eq. 4.12. Accordingly, we can now write Eq. 4.12 as follows:

$$\mathbf{r} \times \mathbf{dF} = \frac{D}{Dt}(\mathbf{r} \times dm\,\mathbf{V}) \tag{4.14}$$

This equation equates the moment of the total force on an element dm about the origin of an inertial reference with the time rate of the moment about the origin of the linear momentum viewed from the inertial reference. It is a trivial matter to prove that instead of the origin we could have chosen any fixed point A in the inertial reference about which to take moments. We would have arrived at Eq. 4.14 with $\boldsymbol{\rho}$, the position vector from A to dm, replacing $\mathbf{r}$.

We now integrate the expressions in Eq. 4.14 over the entire system, using O as the fixed point. Thus,

$$\int \mathbf{r} \times \mathbf{dF} = \iiint_{M} \frac{D}{Dt}(\mathbf{r} \times \mathbf{V})\,dm \tag{4.15}$$

The mass of the system is fixed so that the limits of the integration on the right side of Eq. 4.15 are fixed, permitting us to interchange the integration operation with that of the substantial derivative. Accordingly, we may say that

$$\int \mathbf{r} \times \mathbf{dF} = \frac{D}{Dt}\left(\iiint_{M} \mathbf{r} \times \mathbf{V} dm\right) = \frac{D\mathbf{H}}{Dt} \tag{4.16}$$

where $\mathbf{H}$ is the moment about a fixed point A in inertial space of the linear momentum of the system as seen from the inertial reference.[4] The integral on the left side of the equation represents the total moment about point A of the external forces acting on the system and may be given in terms of the traction force and the body force as follows.[5]

$$\int \mathbf{r} \times \mathbf{dF} = \oiint_{S} \mathbf{r} \times \mathbf{T}\,dA + \iiint_{V} \mathbf{r} \times \mathbf{B}\rho\,dv$$

[4]Equation 4.16 is the same as $\mathbf{M} = \dot{\mathbf{H}}$ derived in particle mechanics for any system of particles. It is to be pointed out that $\mathbf{H}$ is also termed the *angular* momentum.

[5]Note that the moments of the internal forces cancel out because of Newton's third law.

We may now give the *moment-of-momentum equation* for a finite system as follows:

$$\boxed{\oiint_{S} \mathbf{r} \times \mathbf{T}\, dA + \iiint_{V} \mathbf{r} \times \mathbf{B}\rho\, dv = \frac{D\mathbf{H}}{Dt}} \qquad [4.17]$$

As in the case of linear momentum we find the finite-control-volume approach extremely useful, so we now use Eq. 4.17 to formulate the equation for moment of momentum of a finite control volume.

4.8 CONTROL-VOLUME APPROACH FOR THE MOMENT-OF-MOMENTUM EQUATION FOR INERTIAL CONTROL VOLUMES

We may easily express the moment-of-momentum equation by considering $\mathbf{H}$ to be the extensive property in the Reynolds transport equation. Since $\mathbf{H} = \iiint_{\text{sys}} (\mathbf{r} \times \mathbf{V})\rho\, dv$, the quantity η then becomes $\mathbf{r} \times \mathbf{V}$ for this case. Thus

$$\frac{D\mathbf{H}}{Dt} = \oiint_{CS} (\mathbf{r} \times \mathbf{V})(\rho\mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t}\iiint_{CV} (\mathbf{r} \times \mathbf{V})(\rho\, dv) \qquad [4.18]$$

By substituting Eq. 4.17 into Eq. 4.18, we are limiting the reference for which the resulting equation is valid to that of an inertial reference XYZ, so the control volume is inertial for this equation. Also, since the system and the control volume occupy the same space at time t we can interpret

$$\iint_{S} \mathbf{r} \times \mathbf{T}\, dA \qquad \text{and} \qquad \iiint_{V} \mathbf{r} \times \mathbf{B}\rho\, dv$$

in the resulting equation to be, respectively, the total moment about some point A in XYZ of the surface-force distribution on the control surface and the total moment about point A in XYZ of the body-force distribution throughout the material inside the control volume. We then have the desired *moment-of-momentum equation* for an inertial control volume.

$$\boxed{\begin{aligned}\oiint_{CS} \mathbf{r} \times \mathbf{T}\, dA + \iiint_{CV} \mathbf{r} \times \mathbf{B}\rho\, dv &= \oiint_{CS} (\mathbf{r} \times \mathbf{V})(\rho\mathbf{V} \cdot \mathbf{dA}) \\ &\quad + \frac{\partial}{\partial t}\iiint_{CV} (\mathbf{r} \times \mathbf{V})(\rho\, dv)\end{aligned}} \qquad [4.19]$$

The terms on the right side represent the efflux of moment of momentum through the control surface plus the rate of increase of moment of momentum inside the control volume where both quantities are observed from the control volume.

Again we ask the student not to be intimidated by this rather impressive-looking equation. It has a simple physical connotation whereby we have the total moment vector of all body and traction forces for a control volume, wherein this moment is

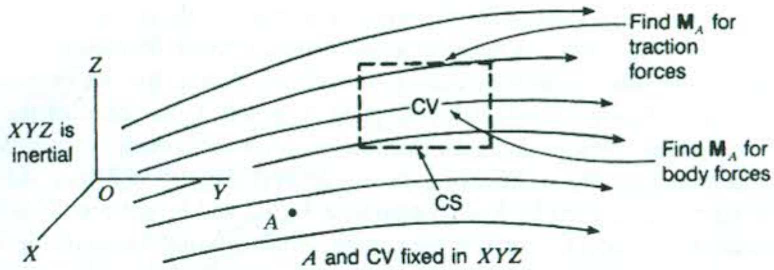


Figure 4.28
Elements entering moment-of-momentum equation.

taken about some convenient point A fixed in inertial space. This total moment drives angular momentum about A through the control surface and changes angular momentum about A inside the control volume. The complex mathematical apparatus employed allows us to apply numerical data precisely and directly for a specific problem involving the moment-of-momentum principle.

Thus for Eq. 4.19, we choose a useful control volume (see Fig. 4.28) and a useful fixed point A with both A and the control volume fixed in inertial space. As indicated in the diagram we must then find the moment $\mathbf{M}_A$ of traction forces and body forces and equate their sum with the efflux rate of angular momentum about A flowing through the control surface plus the rate of change of angular momentum about A inside the control volume.

A more specific situation is illustrated in Fig. 4.29a, showing a cantilevered pipe with water flowing through it. We desire the stresses in the pipe cross section

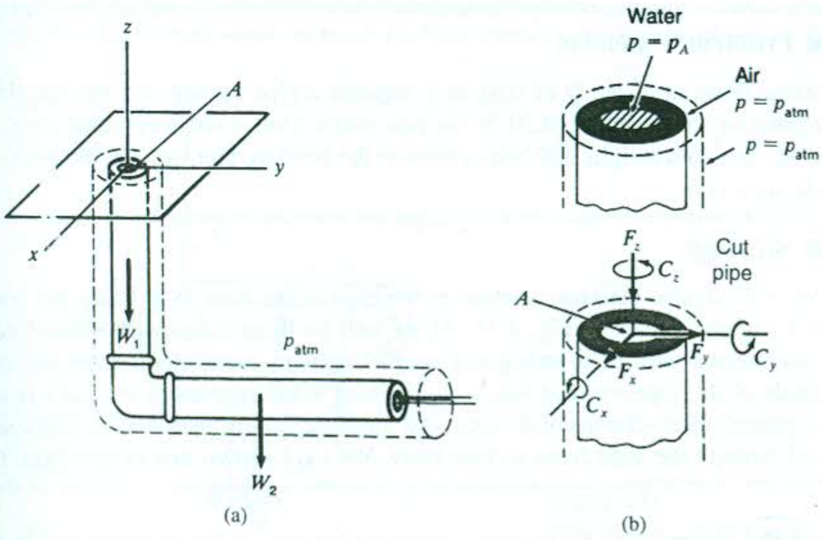


Figure 4.29
Cantilevered pipe with outside control volume.

at the base of the pipe system. For this purpose a control volume has been chosen enveloping the pipe system on the outside and cutting through the pipe and incoming water at the base, as well as cutting through the exiting free jet of water. A reference *xyz* is shown with the origin at point A, which is the center of the pipe section at the base. Now examine the traction forces on the control surface. Where we cut the pipe (see Fig. 4.29b) we expose a general force system from the wall onto the pipe section given by force components F_x, F_y, and F_z and couple-moment components C_x, C_y, and C_z. Also on the control surface cutting the incoming water there is pressure p_A. Over the rest of the control surface, including the exiting free jet, there is atmospheric pressure. The body forces are the total weights W_1 and W_2 of pipe and water in the two pipe lengths.

If we want F_x, F_y, and F_z, we can employ the linear momentum equations (clearly C_x, C_y, and C_z will not appear). If we are interested in computing the couple moments C_x, C_y, and C_z, we can use the moment-of-momentum principle. We could for this purpose use *any fixed* point in space for taking moments of forces and angular momentum of flows. Any such point will yield the same couple-moment components C_x, C_y, and C_z.[6] If we are not interested in the unknown forces F_x, F_y, and F_z, it is wisest to choose the origin as fixed point A. This will eliminate the unknown forces F_x, F_y, and F_z from the moment-of-momentum equation since they will all go through point A as will the force from pressure p_0 on the flow at the base. Finally, the force from p_{atm} will act over a projected area which equals the area enclosed by the pipe section at the base. The force from p_{atm} also has a line of action through point A.

Example 4.18 will reiterate some of the comments and illustrate the use of the moment-of-momentum equation for a control volume.

EXAMPLE 4.15

■ Problem Statement

Water flows at a rate Q of 0.01 m^3/s through a pipe having two right-angled elbows as shown in Fig. 4.30. If the pipe interior has a cross-sectional area of 2580 mm^2 and weighs 300 N/m, compute the bending moments on the base of the pipe at A.

■ Strategy

We will expose the cross section of the pipe at the base at A using the control volume shown in Fig. 4.31. There will be three orthogonal shear-force components and three orthogonal couple-moment components from the cut made of the pipe itself at this section. From solid mechanics we can expect a general three-dimensional force and moment system that will be transmitted through the pipe cross section there. We have shown this in the diagram.

[6] Remember from mechanics that the couple-moment is a free vector so that the moment of this vector is the same for any point in space.

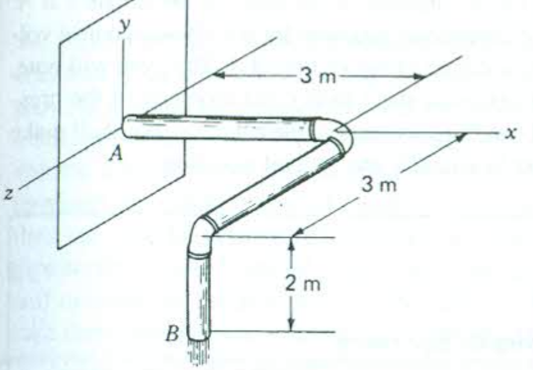


Figure 4.30
Steady flow through a pipe with right-angled elbows.

As for the rest of the control surface, we have the following traction force distributions:

1. At the entrance, we have pressure from water about to enter acting on the water at the very entrance and hence at the control surface.
2. On the rest of the control surface, we have atmospheric pressure including at the exit flow section where we will assume that we have a free jet.

Finally, as body forces, we have the sums of weights of the separate pipes plus the respective weights of water the sum given as W_i. We will take the lengths of the pipes and their respective enclosed water as shown in Fig. 4.30. This geometry will include approximately the weights of the elbows and the water in the elbows. Note that the resultant force from the atmospheric pressure on the control surface cancels everywhere except at the base section A, which is enclosed by the

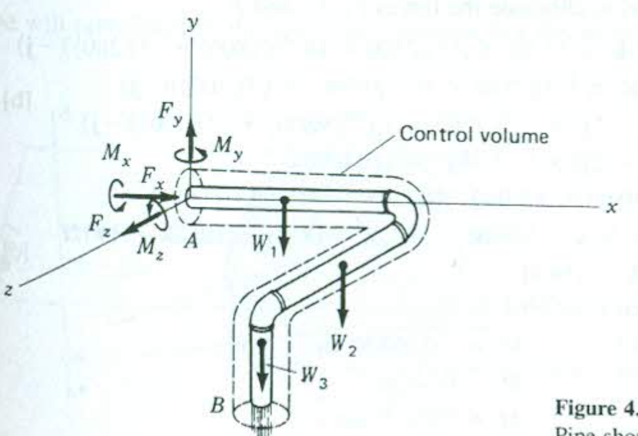


Figure 4.31
Pipe showing control volume.

pipe itself, and hence this force is directed horizontally through the center of the section there. This is also true for the pressure of the water at the entrance at A.

We will use the *moment-of-momentum equation* for our chosen control volume which we will take about the center of the section at A. This, you will note, eliminates the moments of the unknown shear forces and moments of the pressures, leaving only the desired bending moments in the equation. We shall make the following assumptions next to simplify the general equation.

1. Steady flow.
2. Incompressible flow.
3. Free jet leaving the pipe.
4. 1-D flow entering and leaving the pipe system.

Execution

The average velocity V in the pipe is

$$V = \frac{Q}{A} = \frac{0.01}{2580 \times 10^{-6}} = 3.88 \text{ m/s} \qquad [a]$$

The **moment-of-momentum equation** for point A is next stated as

$$\oiint_{CS} \mathbf{r} \times \mathbf{T}\, dA + \iiint_{CV} \mathbf{r} \times \mathbf{B}\rho\, dv = \oiint_{CS} (\mathbf{r} \times \mathbf{V})(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \iiint_{CV} (\mathbf{r} \times \mathbf{V})(\rho\, dv)$$

The last integral vanishes because of assumption 1. The use of assumption 3 simplifies the surface integral on the right side of the equation. Using weights of pipe and water at the centers of gravity corresponding to the geometric centers of the pipes (see assumption 2), we formulate our working equation about the origin at A so as to eliminate the forces F_x, F_y, and F_z:

$$\begin{aligned}(M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k}) &+ (1.5\mathbf{i}) \times [(3)(2580 \times 10^{-6})(9806) + (3)(300)](-\mathbf{j}) \\ &+ (3\mathbf{i} + 1.5\mathbf{k}) \times [(3)(2580 \times 10^{-6})(9806) + (3)(300)](-\mathbf{j}) \\ &+ (3\mathbf{i} + 3\mathbf{k} - 1\mathbf{j}) \times [(2)(2580 \times 10^{-6})(9806) + (2)(300)](-\mathbf{j}) \\ &= (3\mathbf{i} + 3\mathbf{k} - 2\mathbf{j}) \times (-3.88\mathbf{j})[(0.01)(1000)]\end{aligned} \qquad [b]$$

Carrying out the products, we may reach the following equation:

$$\begin{aligned}M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k} &- 1464\mathbf{k} - 2928\mathbf{k} + 1464\mathbf{i} - 1952\mathbf{k} + 1952\mathbf{i} \\ &= -116.4\mathbf{k} + 116.4\mathbf{i}\end{aligned} \qquad [c]$$

The torques are then calculated as

$$M_x = -3300 \text{ N} \cdot \text{m}$$
$$M_y = 0 \text{ N} \cdot \text{m}$$
$$M_z = 6230 \text{ N} \cdot \text{m}$$

These are the desired torques from the wall onto the pipe at A.

■ Debriefing

We point out that with short pipes and many elbows our results would be less accurate. Also, note that we could have used any other stationary point to employ the moment-of-momentum equation, but this would have brought into play unknown shear forces F_i. To work the problem, we could use the same control volume applied now to the linear momentum equation. We would then have determined the entire system of bending moments and shear forces. You would then use your knowledge of structural mechanics to check the stresses in this pipe section to see if they were excessive. Also, in the design of buildings, forces and moments of the kind we have discussed can be large, and we must often take them into account in the design of the building itself.

In many practical problems we usually employ only a single scalar component of Eq. 4.19 at any time, which means that we are taking moments of forces and momenta about an *axis* rather than a point. It will then be easiest to use cylindrical coordinates (Fig. 4.32) with the z direction along the axis shown as AA.[7] The term $\mathbf{r} \times \mathbf{V}$ may be replaced by $\bar{r}V_\theta$, where $\bar{r}$ is the radial distance from the axis to a particle and V_θ is the transverse component of the velocity of the particle. You will remember from mechanics that V_θ is so directed that it is at right angles to $\bar{r}$ and with $\bar{r}$ forms a plane normal to the axis (plane I in Fig. 4.32). Similarly $\mathbf{r} \times \mathbf{T}$ is replaced by $\bar{r}T_\theta$ and $\mathbf{r} \times \mathbf{B}$ is replaced by $\bar{r}B_\theta$. This scalar component of the moment-of-momentum equation then becomes

$$\boxed{\begin{aligned}&\oiint_{CS} \bar{r}T_\theta \, dA + \iiint_{CV} \bar{r}B_\theta \rho \, dv \\ &\quad = \oiint_{CS} (\bar{r}V_\theta)(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \iiint_{CV} (\bar{r}V_\theta)(\rho \, dv)\end{aligned}} \qquad [4.20]$$

We will now illustrate the use of this equation in a simple example.

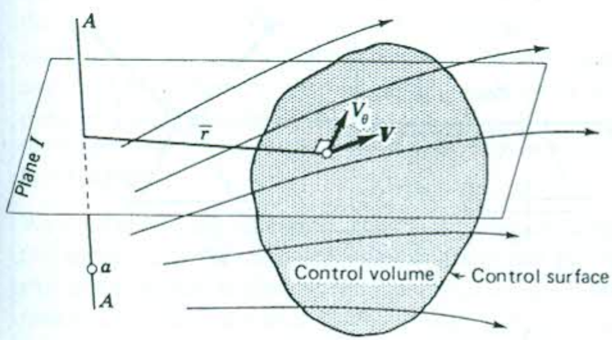


Figure 4.32
For computation of moment of momentum about axis AA.

[7]Although for simplicity we have shown the axis to be vertical it can have any orientation.

EXAMPLE 4.16

■ Problem Statement

A lawn sprinkler shown in Fig. 4.33 is held stationary by you acting on the outside surface of the rotor while water is flowing through it. What torque must you exert if the flow through the sprinkler is q cubic feet per second? Give your result in terms of q, exit rotor cross section A_2, L, and ρ.

■ Strategy

We shall choose as a control volume the exterior of the rotor arm of the sprinkler where this control surface cuts through the support and the inlet water flow as shown in the diagram. Your resisting torque is applied at the interface between you and the rotor surface. It will be easiest if we use a component of the moment-of-momentum equation taken along the axis MM, which is the axis of rotation of the system. We make the following assumptions to reach our working equation:

1. Steady flow.
2. Incompressible flow.
3. 1-D flow at exits.

■ Execution

We start with the general component of the **moment-of-momentum equation.**

$$\oiint_{CS} \bar{r}T_\theta\, dA + \iiint_{CV} \bar{r}B_\theta \rho\, dv = \oiint_{CS} \bar{r}V_\theta(\rho \mathbf{V}\cdot d\mathbf{A}) + \frac{\partial}{\partial t}\iiint_{CV} \bar{r}V_\theta(\rho\, dv)$$

The volume integrals vanish because of assumption 1 and because gravity is parallel to the axis. From continuity and assumption 3, the efflux velocity is $V_2 = q/2A_2$. The rate of efflux of mass at each nozzle is $\rho q/2$. Also, the transverse component V_θ of the efflux velocity can be seen to be $V_2 \cos\alpha$

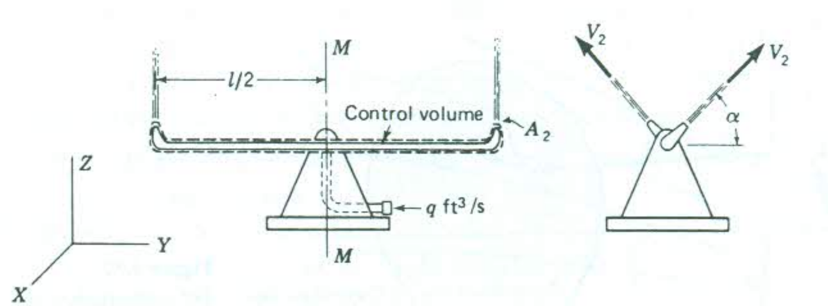


Figure 4.33
Lawn sprinkler held stationary.

and the corresponding arm $\bar{r} = l/2$. The moment-of-momentum equation then gives us

$$M_\theta = 2\left[\underbrace{\frac{l}{2}}_{\bar{r}}\underbrace{\left(\frac{q}{2A_2}\cos\alpha\right)}_{V_\theta}\right]\underbrace{\left(\rho\frac{q}{2}\right)}_{\text{mass efflux}} = \frac{\rho l q^2 \cos\alpha}{4A_2} \quad \text{[a]}$$

where M_θ is the torque from you to hold the rotor stationary. Note that the entering water having zero moment arm about the rotation axis does not appear in Eq. a.

■ Debriefing

We see in this simple problem how the moment-of-momentum equation can be put to good use when fluid moving inside the control volume and/or fluid passing through the control surface has angular momentum about a common axis. This is particularly useful for turbomachines such as jet engines, rotary pumps, water wheels, and the torque converters of your car. Here, in addition to axial flow of fluid, there is rotational motion as well about the axis of the turbomachine. We can use for a suitable control volume the linear momentum equation for determining thrust as was the case in Example 4.10 for the jet engine, and now we can use the moment-of-momentum equation to calculate torque associated with the flow. We have powerful tools for making such vital calculations.

EXAMPLE 4.17

■ Problem Statement

Water enters a pipe at *A* (see Fig. 4.34) and issues out of an open slot in pipe *CD*. This slot is 8 ft in length, and is so shaped on the inside that a sheet of water of uniform thickness ¼ in issues out radially from the pipe as shown in the diagram. The exit velocity of this sheet varies linearly along the pipe discharging the 2 ft^3/s of water entering the pipe at *A*. Determine the torque about axis *MM* from the flow of water on the inside of the pipe and the atmospheric pressure on the outside of the pipe.

■ Strategy

We shall choose as our control volume the *interior* of the pipe, wherein the control surface cuts the incoming water at *A* and cuts the outgoing water along the slot at the outside surface of the pipe. Because the flow in the slot has angular momentum about the axis *MM*, we shall use the component of the *general angular momentum equation* about this axis to bring the desired torque into

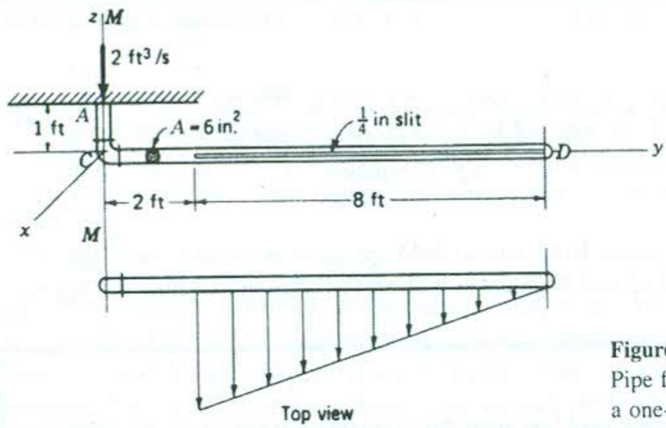


Figure 4.34
Pipe flow with efflux not a one-dimensional flow.

our calculations. We shall simplify this equation by making the following assumptions:

1. Steady flow.
2. Incompressible flow.
3. Sheet of exiting water is at pressure p_{atm}.
4. 1-D flow entering the pipe.

We shall have to carefully consider the continuity equation for the chosen control volume.

■ Execution

As a first step we determine the velocity of the exiting flow as a function of y. We start with the equation of a straight line for V. Thus

$$V = my + b$$

We subject this to the conditions that

$$\text{at} \quad y = 10 \text{ ft} \qquad V = 0 \text{ ft/s}$$
$$\text{at} \quad y = 2 \text{ ft} \qquad V = V_0 \text{ ft/s}$$

where V_0 is as yet undetermined. We find that

$$m = -\frac{V_0}{8} \qquad b = \frac{5}{4}V_0$$

Hence

$$V = -\frac{V_0}{8}y + \frac{5}{4}V_0 \qquad \text{[a]}$$

To determine V_0 we use **conservation of mass** for our chosen control volume.

$$-(2)(\rho) + \int_2^{10} \rho V\left(\frac{1/4}{12}\right) dy = 0$$

$$-2\rho + \frac{\rho}{48}\int_2^{10}\left(-\frac{V_0}{8}y + \frac{5}{4}V_0\right) dy = 0 \tag{b}$$

$$-2\rho + \frac{\rho}{48}\left[V_0\left(-\frac{y^2}{16} + \frac{5}{4}y\right)\Big|_2^{10}\right] = 0$$

$$\therefore V_0 = 24 \text{ ft/s}$$

Hence Eq. a becomes

$$V = -3y + 30 \text{ ft/s} \tag{c}$$

Next we go to the **moment-of-momentum equation** about axis MM:

$$\oiint_{CS} \bar{r}T_\theta\, dA + \iiint_{CV} \bar{r}B_\theta\rho\, dv = \oiint_{CS} \bar{r}V_\theta(\rho\mathbf{V}\cdot d\mathbf{A}) + \frac{\partial}{\partial t}\iiint_{CV} \bar{r}V_\theta(\rho\, dv)$$

Clearly the expressions involving volume integrals are zero since gravity is parallel to the axis MM and since the flow is steady (1) and/or incompressible (2). We then have

$$\underbrace{(p_{\text{atm}})(8)\left(\frac{1/4}{12}\right)(6)}_{\text{at exit slot}} + \underbrace{T_{MM}}_{\text{from pipe wall}} = -\int_2^{10}[y(-3y+30)](\rho)(-3y+30)\left(\frac{1/4}{12}\right) dy$$

The minus sign on the right side obtains because V_θ here for cylindrical coordinates is negative. Note that both the traction force at the entrance and the entering velocity are parallel to MM, thus yielding zero moment about axis MM for that part of the control surface. Solving for T_{MM} we have

$$T_{MM} = -\frac{1.938}{48}\int_2^{10}(30^2y - 180y^2 + 9y^3)\, dy - p_{\text{atm}}\left[\frac{1/4}{12}8\right](6)$$
$$= -248 - p_{\text{atm}} \text{ ft-lb}$$

Taking the reaction we get the torque from the water on the pipe. That is,

$$(T_{\text{pipe}})_{MM} = 248 + p_{\text{atm}} \text{ ft-lb} \tag{d}$$

At this point we can include the torque on the pipe from *both water inside and air outside* by simply using gage pressures in the above formulation as was discussed earlier for linear momentum. We thus get for the *total* torque

$$(T_{MM})_{\text{total}} = 248 \text{ ft-lb}$$

■ Debriefing

Here is a case where along part of the control surface the issuing fluid is not that of a one-dimensional flow, requiring the use of straightforward calculus. Note that the triple integral notation and the closed surface notation simply informed us of the kind of integration called for and should not have intimidated anyone since setting up such integrals was a procedure practiced in earlier calculus courses. Thus, in the first surface integral, we considered first the uniform atmospheric acting on the curved surface of the outside boundary of the slot, and we used simple statics for this torque contribution. This allowed us next to determine T_{MM} the torque from rest of the control surface, namely the pipe wall.

EXAMPLE 4.18

■ Problem Statement

A simple turbomachine called the *Pelton water wheel* is shown in Fig. 4.35a. A single jet of water issues out of a nozzle and impinges on a system of buckets attached to a wheel. The runner, which is the assembly of buckets and wheel, has a radius r measured to the center of the buckets. The shape of the buckets is shown in Fig. 4.35b, where a section of the bucket is shown. Note that the jet is split into two parts by the bucket and rotated almost 180 degrees. If Q cubic feet per second flow from the nozzle, and the runner is loaded by a generator to run at a constant speed of ω radians per second, what is the torque on the water wheel?

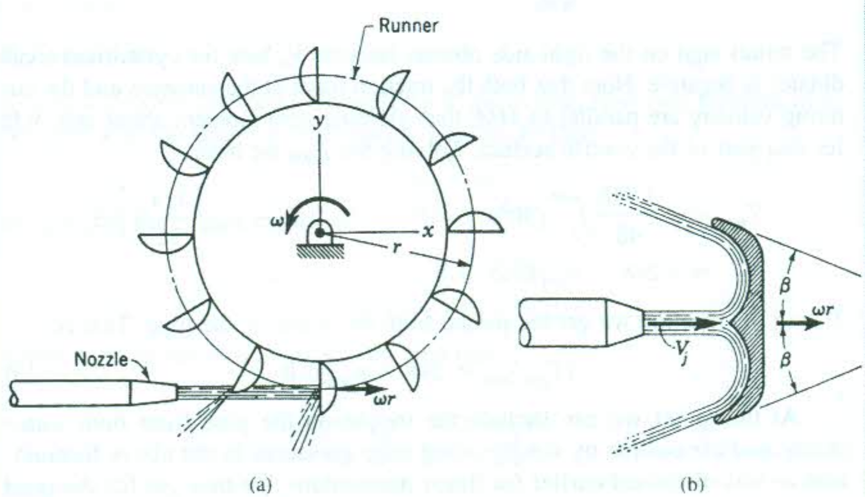


Figure 4.35
Pelton water wheel.

■ Strategy

We shall choose as a control volume a stationary region which completely encloses the water wheel and cuts through the axle on both sides of the wheel. This has been shown in Fig. 4.36. We have atmospheric pressure everywhere along the control surface except where it cuts the axle, and it is here that we develop torque T about the axis of the wheel. We shall employ the *moment-of-momentum equation* for this control volume about this axis. We state this equation in general form next and with certain forthcoming assumptions we shall formulate a working equation.

$$\oint\!\!\!\oint_{CS} \bar{r}T_\theta \, dA + \iiint_{CV} \bar{r}B_\theta \, dv = \oint\!\!\!\oint_{CS} (\bar{r}V_\theta)(\rho\mathbf{V}\cdot\mathbf{dA}) + \frac{\partial}{\partial t}\iiint_{CV} (\bar{r}V_\theta)(\rho\, dv) \qquad \text{[a]}$$

The assumptions are:

1. There is negligible change of speed of the water along a bucket from friction and gravity (as in the earlier trough problems).
2. We can use a fixed average flow geometry of the incoming and outgoing jets corresponding to the water hitting only one bucket when this bucket is moving in its lowest position.
3. We will neglect torque from the weight of water inside the control volume.
4. We will assume steady flow in the fixed geometry of assumption 2.

We are now ready to simplify the above equation.

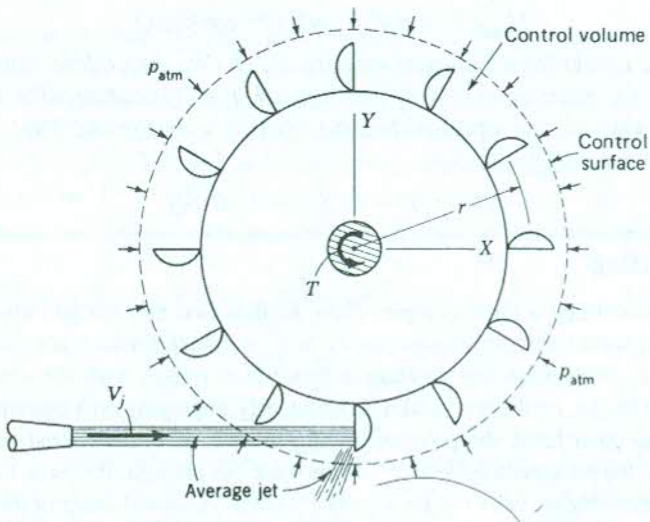


Figure 4.36
Control volume for moment of momentum.

■ Execution

Because of assumptions 3 and 4, we can delete the volume integrals in Eq. a. The only torque about the wheel axis stems from the traction force on the cut axle, so we replace the first surface integral by the resultant torque about the axle. We denote it as M_{shaft}. Next, we examine the flow of angular momentum for the jet. First, the rate of incoming mass flow is $-\rho Q$ whose meaning you may verify from its dimensions and whose minus sign stems from the implied dot product. So the linear momentum is then $-V_j(\rho Q)$ with V_j the velocity of the jet. Finally, the moment of linear momentum for the incoming water is $-rV_j\rho Q$. To ascertain the corresponding quantity for the outgoing flow we remind you to use velocity *relative to the ground*. Hence, we will use simple kinematics learned in dynamics. That is, the velocity of the outgoing jet relative to the ground will be the superposition of the velocity of the jet relative to the bucket, $(V_j - \omega r)$, plus vectorially the velocity of the bucket relative to the ground. In the transverse direction, this velocity component V_θ of the jet relative to the ground (see Fig. 4.35b) is

$$V_\theta = [-(V_j - \omega r)\cos\beta] + \omega r$$

Hence, the angular momentum of the flow leaving the control volume is then

$$r[\omega r - (V_j - \omega r)\cos\beta]\rho Q$$

where the square bracketed expression is clearly V_θ, a constant for our model, and ρQ is the mass flow. Now, we insert the incoming and outgoing angular momenta into the **angular momentum equation** and solve for M_{shaft}. Thus,

$$M_{\text{shaft}} = -rV_j\rho Q + r[\omega r - (V_j - \omega r)\cos\beta]\rho Q$$

Collecting terms we get

$$M_{\text{shaft}} = -r(V_j - \omega r)(1 + \cos\beta)\rho Q$$

This is the torque from the wheel onto the water. (We are dealing with the dynamics of the water up to now as has been the usual procedure.) The torque T from the water on the wheel is then the reaction to our result. Thus we have the sought for information as

$$T = r(V_j - \omega r)(1 + \cos\beta)\rho Q$$

■ Debriefing

We have simplified a fairly complex flow. Realize first that the jet, which has a slight downward trajectory due to gravity, acts on a bucket which is continuously changing its orientation and position in space as it rotates with the wheel. This means that the jet, on the one hand, is continuously impinging on a changing target and, on the other hand, the point of impingement is at a continuously changing location of the jet trajectory. Finally, if this were not enough, the jet at times will be simultaneously partially hitting adjacent buckets. A careful study of the process would indeed be very difficult. Assumption 2 has permitted us to make a reasonable calculation for an approximate average torque. The result will be more accurate if the radius of the wheel is large and the length of the free jet short.

HIGHLIGHTS

We shall first review certain vital topics from Chap. 3. Recall that an extensive property (which in general we shall denote as N) depends for value on the amount of mass m in the system. Measured per unit mass, it is denoted as η and is what we call an *intensive property,* independent of the amount of mass involved. Note that η can be computed at times as N/m. Also recall that for the Lagrange viewpoint we follow the system and denote the time variation with the notation D/Dt, the so-called substantial or total derivative. On the other hand, the Euler viewpoint focuses action at a fixed domain in space. We developed the Reynolds transport equation in Chap. 3 to relate the time variation of a property N from the Lagrange viewpoint to that stemming from the Euler viewpoint. That is

$$DN/Dt = \oiint_{S} \eta(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \iiint_{V} \eta(\rho\, dv)$$

On the left side we are following the system at time t, and on the right side we focus on the stationary volume that the system occupied at time t. Specifically, on the right side we are accounting for the time variation of N by adding the rate of flow of N at time t through the boundary of the stationary volume (the control surface) plus the rate of change of N at time t inside the control surface (the control volume). This is, strictly speaking, a mathematical relation and could be used for any extensive property associated with a movement of mass. However, we will attach physical meaning to this equation as we develop the integral formulations for two basic laws in what follows.

Conservation of Mass. The extensive property that we use here is m, the mass of the system. The corresponding intensive property is simply determined by dividing by m so that $m/m = 1$, or unity. Inserting unity for η in the Reynolds transport equation still gives us a mathematical relation for a specific N. Now comes the key step. We first state the conservation of mass equation for an aggregate as $Dm/Dt = 0$. Substituting this relation into the left side of the transport equation means that the satisfaction of the resulting equation is now necessary (and indeed sufficient) for conservation of mass. Thus a general integral form of the conservation of mass equation is:

$$\oiint_{CS} (\rho \mathbf{V} \cdot \mathbf{dA}) = -\frac{\partial}{\partial t} \iiint_{CV} (\rho\, dv)$$

where we have used unity for η and have used CS and CV for the respective integral limits. We can now present a simple physical explanation for the above formulation. For a control volume, the net rate of influx/efflux of mass flow through the control surface must equal the rate of increase/decrease of mass of fluid inside the control volume. Anybody drinking a coke out of a can knows this!

Linear Momentum Equation. Now we choose for N the vector $m\mathbf{V}$, namely the linear momentum of the system. To get η, we divide by the mass m, so that $\boldsymbol{\eta} = \mathbf{V}$. Next we go back to dynamics for Newton's law, $\mathbf{F} = \frac{D}{Dt}(\mathrm{m}\mathbf{V})$. We shall spell out $\mathbf{F}$ in terms of the body force distribution $\mathbf{B}(x, y, z)$ and the traction force distribution $\mathbf{T}(x, y, z)$, as indicated below for use in Newton's law.

$$\mathbf{F} = \oiint_{CS} \mathbf{T} \cdot \mathbf{dA} + \iiint_{CV} \mathbf{B}(\rho\, dv) = \frac{D}{Dt}(m\mathbf{V})$$

If we next insert this law into the left side of the transport equation, we get a general form of the linear momentum equation which we now state as

$$\oiint_{CS} \mathbf{T} \cdot \mathbf{dA} + \iiint_{CV} \mathbf{B}(\rho\, dv) = \oiint_{CS} \mathbf{V}(\rho\mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t}\iiint_{CV} \mathbf{V}(\rho\, dv)$$

This equation is now the integral form of the linear momentum equation because we have inserted Newton's law into the Reynolds transport equation. As a physical interpretation of the linear momentum equation, we can say that the total force from those on the control surface plus those on matter inside the control volume drives linear momentum through the control surface and causes a time rate of change of linear momentum inside the control volume. This should form a vivid picture for a student attempting to enter a sold-out, ongoing rock concert.

Moment-of-Momentum Equation. The extensive quantity that we now go to is $m\,(\mathbf{r} \times \mathbf{V})$. For the intensive quantity we divide by m so that now $\boldsymbol{\eta} = (\mathbf{r} \times \mathbf{V})$. Again, going to dynamics, we have for the moment-of-momentum equation using body force $\mathbf{B}$ and traction force $\mathbf{T}$ distributions to get the moment about a fixed point in inertial space,

$$\oiint_{CS} (\mathbf{r} \times \mathbf{T})\, dA + \iiint_{CV} (\mathbf{r} \times \mathbf{B})\rho\, dv = \frac{D}{Dt}(\mathbf{r} \times m\mathbf{V})$$

Next, we substitute the above formulation into the left side of the transport equation and at the same time use the preceding formulation for η. We then get the integral form of the moment-of-momentum equation because we have included Newton's angular momentum equation in the Reynolds transport equation. Thus

$$\oiint_{CS} (\mathbf{r} \times \mathbf{T})\, dA + \iiint_{CV} (\mathbf{r} \times \mathbf{B})\rho\, dv = \oiint_{CS} (\mathbf{r} \times \mathbf{V})(\rho\mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t}\iiint_{CV} (\mathbf{r} \times \mathbf{V})(\rho\, dv)$$

The physical interpretation for the integral form of the moment-of-momentum equation can be given as follows. The total moment of all the forces acting on the control surface and acting on the interior of the control volume, all taken about a fixed point in inertial space, drives angular momentum about this point through the control surface and causes a rate of change of the angular momentum about this point inside the control volume.

We point out that for the linear momentum and the moment-of-momentum integral formulations we must use control volumes fixed in inertial space because of this requirement on Newton's law, from which the integral forms are derived. Finally, remember that the velocities and time derivatives used in all the integral forms of equations presented here must be seen as looking out from the chosen control volumes.

Finally, we offer a word of encouragement to students who may have been alarmed by the complex appearance of the integral forms of the basic laws just presented. There need be no undo concern on this matter. The various integrals are standard mathematical notation which are rarely used at full strength as you may already have noticed from the examples. There will almost always be opportunity to make simplifying assumptions which will generally reduce the integrals to simple forms. In particular, notice that the expression $\eta(\rho\mathbf{V} \cdot \mathbf{dA})$ seems to appear everywhere. You should have no trouble, using dimensions, to deduce that it is simply the flow rate of N through an infinitesimal area dA of the control surface.

We shall continue in exactly the same way to formulate the integral form of the first law of thermodynamics in Chap. 5.

4.9 CLOSURE

In this important chapter we presented two of the basic laws for finite systems and finite control volumes. And just as you were required to draw carefully considered free body diagrams in your sophomore mechanics courses, it should be patently clear that the same care has to be exercised in drawing appropriate control volumes. Now, in addition, you should list assumptions made in constructing the flow model you are using to represent the actual problem. All told we presented a rather formal attack on each problem. You are urged, at least until full mastery has been achieved, to follow a similar formal, methodical approach.

In Chap. 5 we will present the first law of thermodynamics and from it the very useful Bernoulli equation. We will merely introduce the second law of thermodynamics and leave its use to a later chapter when it will be more directly needed.

The Reynolds transport equation will once again be featured in the development of the first law of thermodynamics. You should emerge from Chaps. 5 and 6 with a good feel and sound appreciation of the Reynolds transport equation.

*4.10 COMPUTER EXAMPLES

COMPUTER EXAMPLE 4.1

■ Computer Problem Statement

A jet of water having a velocity of 10 m/s and a diameter of 50 mm is directed against a trough in a horizontal plane, as shown in Fig. C4.1. The trough can have an angle shown as θ with line *A-A* which is normal to the jet. The water splits up along the trough such that the volume flow rates Q_1 and Q_2 are related as $Q_1 = Q_2 \cos^2 \theta$. We will neglect friction and gravity effects on the speed of the water at all times. Plot the algebraic sum of the force components at the hinge *B* for θ going from 0° in steps of 2 degrees. Also, what is the torque at *B*? The trough is held in equilibrium by *B* at each setting of θ.

■ Strategy

In Fig. C4.2, we choose a control volume that is at the interface between the water and the trough surface while cutting the pin at *B* and the incoming and the exiting jets. We will use the *linear momentum* and *angular momentum equations* in simple forms at each stationary angular setting of the trough.

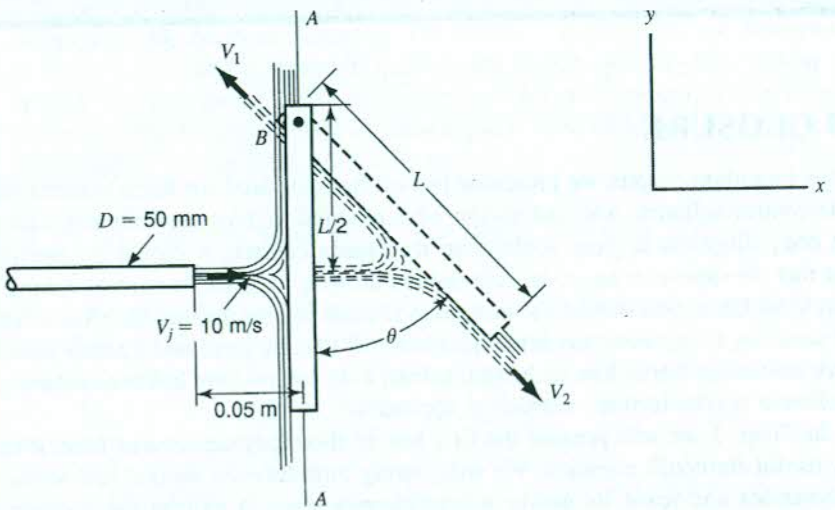


Figure C4.1

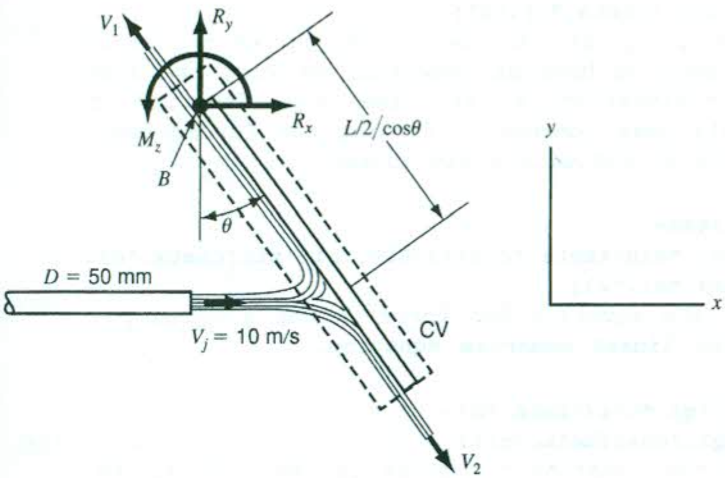


Figure C4.2

■ Execution

```
clear all;
%Putting this at the beginning of the program ensures
%values don't overlap from previous programs.

c=pi./180;
%"c" is for conversion. Matlab's intrinsic functions
%only work in radians.

theta=eps:2:60;
%You can see by a simple calculation that when theta
%equals 60 degrees, the stream of water will lose
%contact with the trough. "eps" is the smallest
%number in Matlab's memory greater than zero. We use
%it because we don't want to start with zero in our
%series. If we started with zero we would divide by
%zero later and get an error message. So here we are
%going from zero (essentially) to 60 degrees in steps
%of 2 degrees (eps=2.22*10^-16).

q1=.019635./(1+1./(cos(theta.*c).^2));
%This is the volume flow to the top at all times as a
%function of theta. This was solved using
%conservation of mass.
```

```
q2=q1./(cos (theta.*c).^2);
%We know how q1 and q2 are related so we can solve
%for q2 once we have q1. Now that we know the flow
%going to either end at all times as a function of
%theta all that remains is finding the forces and
%moments involved with these flows.

fx=-196.3495-
(10000.*q1.*sin(theta.*c))+(10000.*q1.*sin(theta.*c).
/cos(theta.*c).^2);
%This is the equation for force in the x direction
%using the linear momentum equation.

fy=10000.*q1.*cos(theta.*c)-
(10000.*q1./cos(theta.*c));
%This is the equation for force in the y direction
%using the linear momentum equation.

total_force=fx+fy;
%The total force is obviously just the force in the x
%direction plus the force in the y direction.

plot(theta,total_force);
grid;
xlabel('Theta(degrees)');
ylabel('Total force (Newtons)');
title ('Total force on the trough vs. Theta');
%All this just gives us the plot that we want.

torque_a=883.573;
```

■ Debriefing

The plot of total force vs. angle is shown in Fig. C4.3. Torque at the base of the trough will not change throughout the rotation of the trough. Since the flows toward the end of the trough have no moment arm, the only flow that will cause a torque is the constant incoming flow. The force from this flow will not change throughout, and the moment arm will not change during rotation either. It will always remain length/2, which in this case is 4.5 m. The constant torque value of 883.573 was determined in the following way:

```
4.5*10*(1000*10*(pi/4)*0.05^2)=883.573
```

where 4.5 equals the moment arm when the trough is 9 m long, 10 is the velocity of the incoming water, 1000 is the density of the water, and the remaining terms give the area of the incoming water stream.

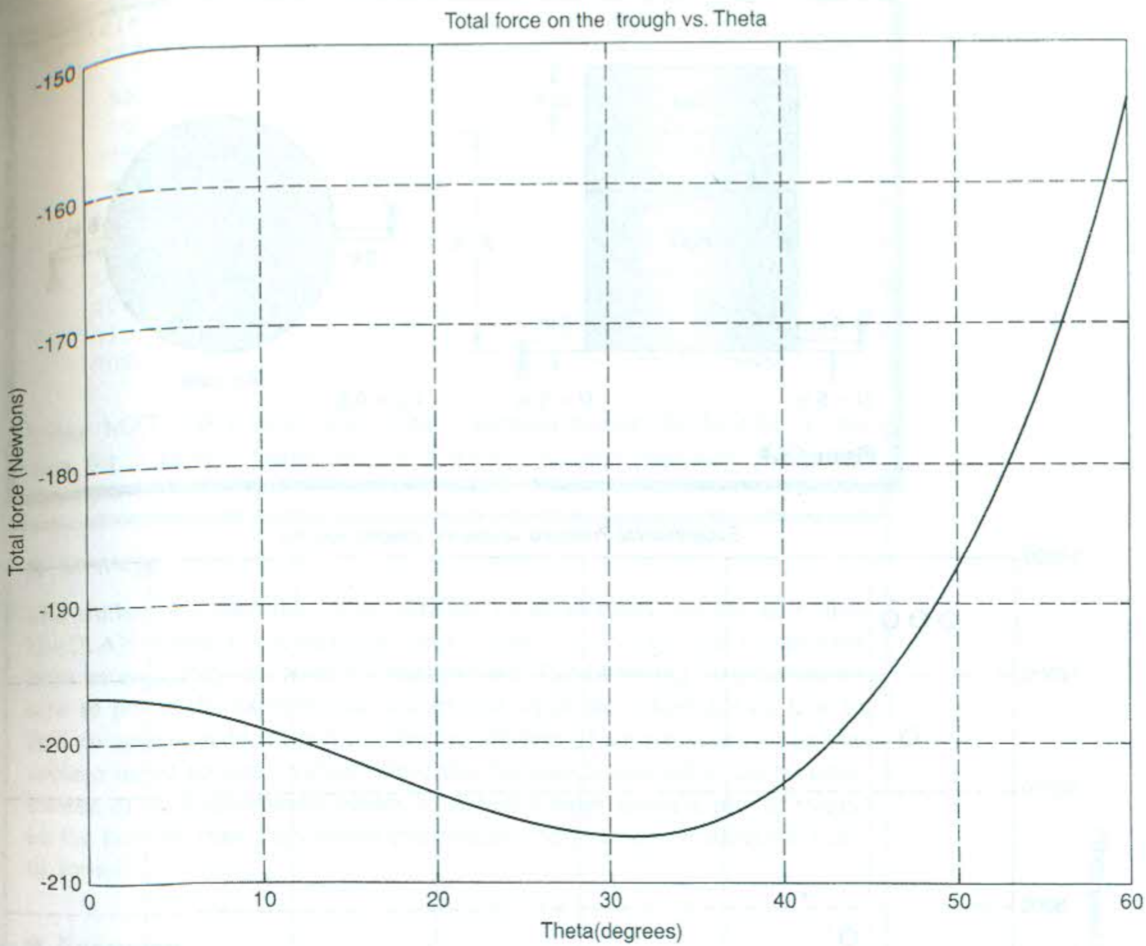


Figure C4.3

COMPUTER EXAMPLE 4.2

■ Computer Problem Statement

A tank containing water and a gas is shown in Fig. C4.4. The gas has been found to have the following 10 experimental values of specific volume and pressure. These data are also shown in Fig. C4.5.

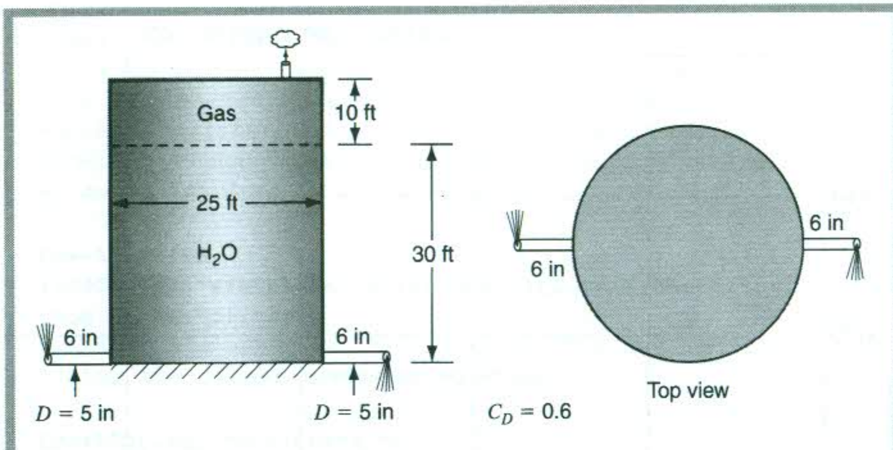


Figure C4.4

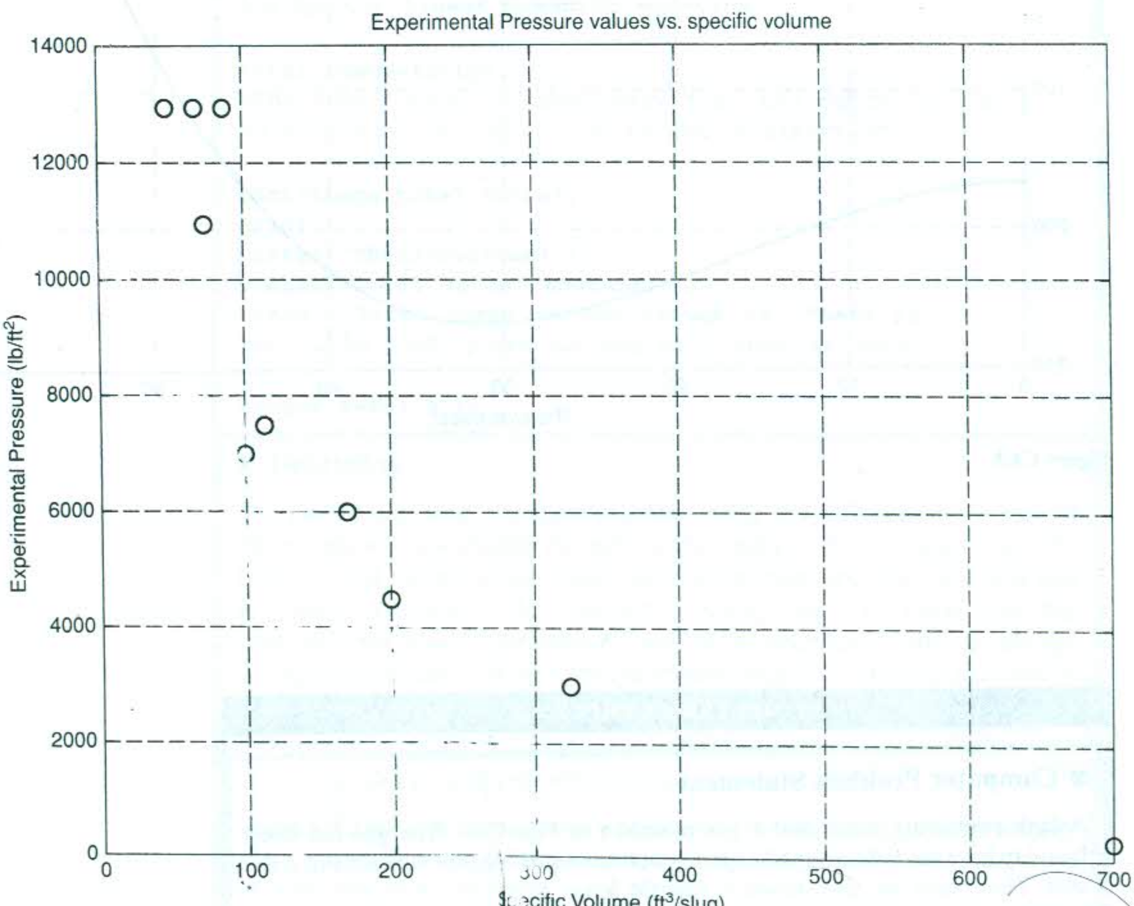


Figure C4.5

Specific volume (ft ^3/ slug)	Pressure (b/ft ^2 gage)
50.00	13000.00
70.00	13000.00
75.00	11000.00
90.00	13000.00
100.00	7000.00
115.00	7500.00
170.00	6000.00
200.00	4500.00
325.00	3000.00
700.00	300.00

Using MATLAB find an approximate formula having the best-fit of this data for a more accurate use in a torque vs. time problem (Computer Example 4.3).

■ Strategy

The student will take the values obtained experimentally and put them into MATLAB in arrays "pressure" and "spec_vol". Once the values have been entered, they can be easily manipulated. The following program shows how to put in the experimental values and, once the values are in, how to find an approximate formula for the best-fit plot of air pressure vs. specific volume based on those values. Once this formula is derived it can be used, instead of the experimental values, to obtain a more accurate plot of torque on the tank vs. time than would theoretically be available without the best-fit formula.

■ Execution

```
clear all;
%Putting this at the beginning of the program ensures
%values don't overlap from previous programs.
disp ('Give the values for specific volume and
pressure starting with the first');
disp ('experimental values and ending with the last
values.');
%This prompts the user of the program to enter the
%values for pressure and specific volume obtained
%experimentally.

for i=1:10;

spec_vol(i)=input('What is the specific volume?');
pressure(i)=input('What is the pressure reading?');
```

```
end
%This is how we are taking the experimental values
%and entering them into Matlab. Now we have 10 values
%in the array "pressure" and 10 values in the array
%"spec_vol". Using only these 10 ordered pairs of
%values we can use Matlab to determine any number of
%other ordered pairs between the ones we obtained
%experimentally (interpolate) and we can also
%determine best-fit polynomials to any order.

n=input('What order polynomial do you want to try to
fit your data?');
%"n" can be any integer.

coef=polyfit(spec_vol,pressure,n);
%This gives an array of "n+1" coefficients of an n-
%order polynomial that gives a best-fit line through
%the points in the arrays "spec_vol" and "pressure"
%(an "n" order polynomial will go through every point
%in a plot where there are "n+1" points). A 3rd
%or 4th order polynomial will give a pretty accurate
%best-fit representation of the data points in this
%case.

disp('The coefficients of your polynomial
in descending order are as follows:');
for i=1:n+1;
  fprintf('Coefficient %1.0f',i);
  fprintf('%4.10f\n',coef(i)');
end
%This displays the coefficients of your n-order best-
%fit polynomial.

best_fit_pressure=0;
for i=1:n+1;
best_fit_pressure=best_fit_pressure+coef(i).*spec_vol.^(n
+1-i);
end
%Now we are taking the coefficients of our best-fit
%polynomial (from the array "coef") and making an
%equation of a curve through our data points with
%them.

global new_spec_vol new_pressure
%The global command lets us use the specified
%variable in any other M-file where the variables are
%declared "global".
```

```
new_spec_vol=linspace(min(spec_vol),max(spec_vol),400);
%"new_spec_vol" defines a finer grain of "spec_vol"
%values for the interpolation, 400 in this case.

new_pressure=interp1(spec_vol,best_fit_pressure,new_s
pec_vol,'spline');
%The arrays "spec_vol" and "best_fit_pressure"
%contain the data values we are using and
%"new_spec_vol"(finer grain defined above) contains
%the new points for which the above Matlab "interp1"
%function computes 400(or any number) interpolated
%"new_pressure" values. Basically what we are doing
%here is filling in the best-fit plot with more
%ordered pairs between the relatively few we had
%originally. We started with 10 ordered pairs of
%"experimental" values. Then we found a best-fit
%line that went through the data values. Now we are
%interpolating 400 ordered pairs along the best-fit
%line to give a smooth plot of torque when we are all
%done. You can see at the end of the above formula
%the term 'spline'. All 'spline' means is that we
%are interpolating points along the best-fit curve
%with a 3rd order polynomial(cubic spline) instead a
%straight line. The 3rd order polynomial is computed
%so that it provides a smooth curve between the two
%points and a smooth transition from the 3rd order
%polynomial between the previous pair of points. The
%choice of linear or cubic spline interpolation is up
%to you and should be based on the anticipated
%curve's behavior. For linear interpolation, simply
%delete the "spline" term.
```

■ Debriefing

What we have done in this problem is take the 10 initial values for specific volume and pressure and with them obtain a best-fit plot with a much finer mesh of values (from 10 data points to 400). This process could save hours in lab time. This best-fit plot is going to be used in Computer Example 4.3 to determine the torque on the tank through the "`global`" function.

■ Computer Output

```
EDU>> mp8
What is the specific volume for the fluid at atmospheric
pressure(ft^3/slug)? 400
Give the values for specific volume and pressure starting
with the first experimental values and ending with
the last values.
```

```
What is the specific volume?50
What is the pressure reading?13000
What is the specific volume?70
What is the pressure reading?13000
What is the specific volume?75
What is the pressure reading?11000
What is the specific volume?90
What is the pressure reading?13000
What is the specific volume?100
What is the pressure reading?7000
What is the specific volume?115
What is the pressure reading?7500
What is the specific volume?170
What is the pressure reading?6000
What is the specific volume?200
What is the pressure reading?4500
What is the specific volume?325
What is the pressure reading?3000
What is the specific volume?700
What is the pressure reading?300
What order polynomial do you want to try to fit your data?4
The coefficients of your polynomial in descending order
are as follows:
Coefficient 1      0.0000000840
Coefficient 2      -0.0003740542
Coefficient 3      0.3772869940
Coefficient 4      -137.4055747169
Coefficient 5      19740.5545569677
EDU>>
```

COMPUTER EXAMPLE 4.3

■ Computer Problem Statement

In Computer Example 4.2, we took pressure-specific volume data for 10 experiments for some gas, and we formed a "best-fit" curve for this data. Now, in this problem, we shall show how to use the best-fit curve to estimate the torque vs. time on the cylinder as the fluid descends, undergoing a pressure change of 60 psi abs to 20 psi abs. We will assume that the smallest pressure given in the data corresponds to the initial height of the free surface of 30 ft.

■ Strategy

In this problem, the values in the arrays `"new_spec_vol"` and `"new_pressure"` are going to be used (instead of the experimental values), to obtain a more accurate plot of torque on the tank vs. time than would theoretically be available without the best-fit formula.

■ Execution

```
global new_spec_vol new_pressure;
%The global command will let us use the specified
%variable in any other M-file where the variables are
%declared "global". Since we have declared the
%variables "new_spec_vol" and "new_pressure" global
%in the last problem we can use the values assigned
%to these variables in this problem.

spec_vol_atm=input('Spec. Vol. for the fluid you are
using at atm. press.(ft^3/slug)?\n\n\n');

a=find(new_pressure<=8640&new_pressure>=2880);
%We only want the values of the interpolated pressure
%that fall between 60 psig and 20 psig (8640 lb/ft^2
%and 2880 lb/ft^2) because this is the range we want
%to plot the torque on the tank for. The "find"
%function in Matlab returns an array of the indices
%of all the non-zero elements. The non-zero elements,
%in this case, would be those elements between 2880
%lb/ft^2 and 8640 lb/ft^2.

for i=1:length(a)

torque_pressure(i)=new_pressure (a(i));
torque_sv(i)=new_spec_vol(a(i));

end
%This puts the values of "new_pressure" that fall
%between 60 psig and 20 psig into the array
%"torque_p" and the "new_spec_vol" values that
%correspond to the "new_pressure" values into an
%array "torque_sv". We now have all the information
%we need from the original experimental values that
%were obtained in order to make a plot of the torque
%on the tank vs. time when the fluid pressure goes
from %60 psig to 20 psig.

mass=19634.954./spec_vol_atm;
%This is the mass of fluid present in the tank; the
%total volume of the tank divided by the specific
%volume of fluid at the standard atmospheric
%conditions. Since we are using specific volume and
%not volume we need to know the mass of fluid present
%in the tank initially. We will assume that the tank
%was initially filled with nothing but fluid to
```

```
%determine the mass of fluid present throughout the
%problem as the mass of fluid will not change (closed
%system). We will also assume that when the tank was
%empty it was at atmospheric pressure. Using these
%two assumptions we know a volume and the specific
%volume that corresponds to it. We can then
%determine the mass of fluid.

volume=mass.*torque_sv;
%This is how the volume in the tank changes while the
%specific volume is changing for 60 psig to 20 psig.

height=40-volume./490.874;
%Since we know how the volume will be changing
%throughout the problem and we know the area of a
%cross-section of the tank (490.874 ft^2), we then know
%how the height will be changing throughout the
%problem. We need this information to use Bernoulli's
%equation for water. We subtract the volume/area from 40
%since we want the height off the bottom of the tank to
%the free surface to be used in Bernoulli's equation.

velocity=sqrt (1.03199.*torque_pressure+64.4.*height);
%This is the velocity of the water leaving the bottom
%of the tank as a function of the pressure of the
%fluid and the height of the water.

torque=25.194.*velocity.^2.*.13635;
%This is the torque on the tank from simplifying the
%moment of momentum equation where .13635 ft^2 is the
%total area where the water comes out of the bottom
%of the tank.

volumechange=diff (volume);
%This is the volume change for each height step and
%it will be constant. Since it is constant, in the
%next step it is fine to just use the first value in
%the array.

time=volumechange (1)./(velocity.*.6.*.13635);
%This is the time for the volume of water to be
%discharged out of the bottom of the tank for each
%velocity.

time1=linspace(0,sum(time),length(torque));
%We want to plot the total time since the
%commencement of draining vs. torque on the tank.
%Therefore, we make a linearly spaced array with
%values ranging from time equal to zero through the
%sum of all the times. We also want this array to be
```

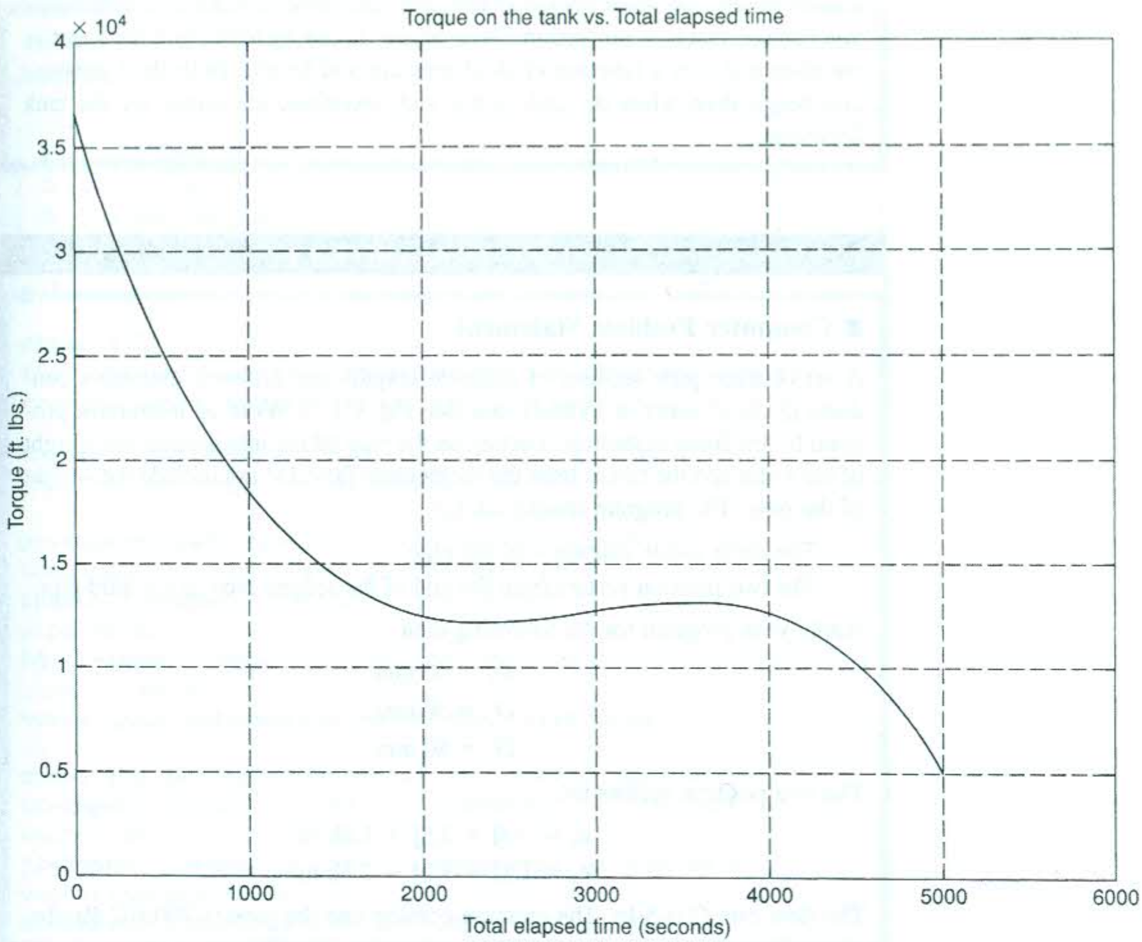


Figure C4.6

```
%the same length as the "torque" array so
%"length (torque)" is the number of elements in
%this array.

plot(time1,torque);
grid;
xlabel('Total elapsed time (seconds)');
ylabel('Torque (ft. lbs.)');
title('Torque on the tank vs. Total elapsed time');
%This just gives us the plot that we want.
```

■ Debriefing

You can see by looking at the plot (Fig. C4.6) that as time goes by and the tank drains, the torque on the tank goes down. Looking at the equation for torque, you can see that it is proportional to velocity^2, and by looking at the velocity equation that it is a function of fluid pressure and height. Both fluid pressure and height drop when the tank drains and, therefore, the torque on the tank decreases.

COMPUTER EXAMPLE 4.4

■ Computer Problem Statement

A set of three pipe sections of different lengths and different diameters conducts Q cfs of water at a steady rate (see Fig. C4.7). Write an interactive program for the force at the base A acting on the pipe taking into account the weight of the water and the forces from the momentum flow. Do not include the weight of the pipe. The program should ask for

The three inside diameters of the pipe.

The two position vectors from the end of the second pipe to the third pipe.

Apply your program for the following data:

$$D_1 = 60 \text{ mm}$$
$$D_2 = 50 \text{ mm}$$
$$D_3 = 80 \text{ mm}$$

The two position vectors are

$$\mathbf{r}_2 = -3\mathbf{i} + 2.5\mathbf{j} + 1.8\mathbf{k} \text{ m}$$
$$\mathbf{r}_3 = 1.5\mathbf{i} + 3.2\mathbf{j} - 2.5\mathbf{k} \text{ m}$$

The flow rate Q is 5 l/s. The pressure coming into the pipe is 790,802 Pa abs.

■ Strategy

We choose the interior of the pipe system for the control volume, and we use the gage pressure so as to include the force from the outside atmosphere. We

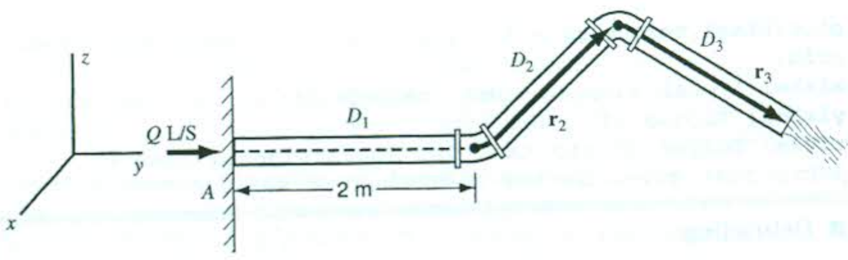


Figure C4.7

will use the **linear momentum equation** for which we make the following assumptions:

1. Steady flow.
2. Incompressible flow.
3. 1-D flow throughout.

■ Execution

```
clear all;
%Putting this at the beginning of the program ensures
%values don't overlap from previous programs.

volume_flow=input('\nWhat is the volume flow rate
through the pipes in liters/sec?\n');

pressure=input('\nWhat is the absolute pressure at
the inlet of the pipes in Pascals?\n');
length(1)=input('\nWhat is the length of the first
pipe(meters)?\n');
id(1)=input('\nWhat is the inside diameter for the first
pipe in meters?\n');
%This gets information about the first pipe.

disp('The position vector of the end of pipe 2:');
i2=input('\nWhat is the "i" component of the unit
vector(meters)?\n');
j2=input('\nWhat is the "j" component of the unit
vector(meters)?\n');
k2=input('\nWhat is the "k" component of the unit
vector(meters)?\n');
id(2)=input('\nWhat is the inside diameter for the
second pipe in meters?\n');
%This gets information about the second pipe.
disp('The position vector of the end of pipe 3');
i3=input('\nWhat is the "i" component of the unit
vector(meters)?\n');
j3=input('\nWhat is the "j" component of the unit
vector(meters)?\n');
k3=input('\nWhat is the "k" component of the unit
vector(meters)?\n');
id(3)=input('\nWhat is the inside diameter for the
third pipe in meters?\n');
%This gets information about the third pipe.

area(1)=(pi./4.)*id(1).^2;
area(2)=(pi./4.)*id(2).^2;
```

```
area(3)=(pi./4.)*id(3).^2;
flow_velocity(1)=volume_flow.*.001./area(1);
flow_velocity(2)=volume_flow.*.001./area(2);
flow_velocity(3)=volume_flow.*.001./area(3);
%The volume flow is multiplied by .001 because we are
%converting from liters/sec. to m^3/sec.

vector1=[0 length(1) 0];
vector2=[i2 j2 k2];
vector3=[i3 j3 k3];
dis_vec2=vector2-vector1;
dis_vec3=vector3-vector2;
%Using the position vectors, we can get the
%displacement vectors of pipes 2 and 3.

length(2)=sqrt(sum(dis_vec2.^2));
length(3)=sqrt(sum(dis_vec3.^2));
%This is the familiar length equation for a vector in
%computer language form(the length of pipe #1 is
%already known since it is co-linear with the "y"
%axis).

unit_vector3i=dis_vec3(1)./length(3);
unit_vector3j=dis_vec3(2)./length(3);
unit_vector3k=dis_vec3(3)./length(3);
unit_vector3=[unit_vector3i unit_vector3j
unit_vector3k];
%A unit vector corresponding to the direction of the
%displacement vector for pipe 3 must be made since
%the direction of the efflux flow is important in
%analyzing the forces involved. Also, the orientation
%of the third pipe (given by this unit vector) will
%affect the direction in which the atmospheric
%pressure will act on the control volume.

body_force(1)=area(1).*length(1).*9806;
body_force(2)=area(2).*length(2).*9806;
body_force(3)=area(3).*length(3).*9806;
%9806 has the units N/m^3 and is the weight per cubic
%meter of water. Since, for this problem, the pipes
%are considered weightless and are external to our
%chosen control volume, the weight of the water is
%the only body force we need to worry about. These
%forces would obviously act in the -k direction in
%this case.

force_atm=101325.353.*area(3).*unit_vector3;
%This gives a 1x3 vector of the force of the
```

```
%atmosphere acting on the end of pipe 3. It
%obviously depends on the unit vector describing the
%direction of the third pipe.

momentum_flow3=1000.*flow_velocity(3).^2.*area(3)
.*unit_vector3;
%This gives a 1x3 vector of momentum flow out of the
%end of pipe 3 and depends on the unit vector that
%describes the direction of the third pipe.

forcei=force_atm(1)+momentum_flow3(1);

forcej=-pressure.*area(1)+force_atm(2)-
1000.*flow_velocity(1).^2.*area(1)
+momentum_flow3(2);

forcek=sum(body_force)+force_atm(3)
+momentum_flow3(3);
%These equations give all of the forces ON the
%control volume FROM the pipes. We want the reaction
%to these forces, i.e., the forces on the pipes from
%the fluid flowing within.

force_pipei=-forcei;
force_pipej=-forcej;
force_pipek=-forcek;
%These are the sought after forces, i.e., the
%reaction forces to the forces determined above.

disp('The forces acting on the assembly of pipes, in
Newtons, are:');
fprintf('\n\nIn the "i" direction: %4.4f\nIn the "j"
direction: %4.4f\n',force_pipei,force_pipej)
fprintf('In the "k" direction:
%4.4f\n\n\n',force_pipek);
%This displays the sought after information in the
%Matlab Command Window.
```

■ Debriefing

You have developed your own software for a modest problem. This may be the kind of assignment you might have when you graduate. Certainly, if you have repetitive projects with varying data and geometry, you might want to develop useful software for yourself.

■ Computer Output

```
EDU>> mp9
```

```
What is the volume flow rate through the pipes in
liters/sec?
5

What is the absolute pressure at the inlet of the pipes
in Pascals?
790000

What is the length of the first pipe(meters)?
2

What is the inside diameter for the first pipe in
meters?
.06

The position vector of the end of pipe 2:

What is the component of the "i" unit vector(meters)?
-3

What is the component of the "j" unit vector(meters)?
2.5

What is the component of the "k" unit vector(meters)?
1.8

What is the inside diameter for the second pipe in
meters?
.05

The position vector of the end of pipe 3

What is the component of the "i" unit vector(meters)?
1.5

What is the component of the "j" unit vector(meters)?
3.2

What is the component of the "k" unit vector(meters)?
-2.5

What is the inside diameter for the third pipe in meters?
.08

The forces acting on the assembly of pipes, in Newtons, are:

In the "i" direction: -369.4977
In the "j" direction: 2185.0369
In the "k" direction: -79.1465
EDU>>
```

COMPUTER EXAMPLE 4.5

■ Computer Problem Statement

Do Computer Example 4.4, this time getting the bending moments at the base of the pipe system and including the weights of the pipes. Develop software for an interactive program for this calculation and use this software for a specific set of data given in Computer Example 4.4. Take $Q = 10$ cfs.

■ Strategy

In this problem we will take advantage of the fact that the first pipe is limited to lie collinear with the y axis. Because of this, flow entering the first pipe will not cause a moment because the moment arm has zero length. The flow exiting the three-pipe arrangement is arbitrary in direction and magnitude, so, generally, there will be a torque from this flow. Also, there will be moments around the base from the body forces acting on the pipe material and the liquid flowing on the inside of the pipes, and these moments must be accounted for. Weight per foot values for pipes can be found in Appendix IV-D of author's *Introduction to Solid Mechanics, Third Edition,* Prentice Hall Inc., 2000.

■ Execution

```
clear all;
%Putting this at the beginning of the program ensures
%values don't overlap from previous programs.

volume_flow=input('\nWhat is the volume flow rate
through the pipes(liters/sec)?\n');

length(1)=input('\nWhat is the length of the first
pipe (meters?)\n');
id(1)=input('\nWhat is the inside diameter for the
first pipe(meters)?\n');
weight(1)=input('What is the weight per meter of the
first pipe (Newtons)?');
%This gives us information about the first pipe.

disp('The position vector of the end of pipe 2:');
i2=input('\nWhat is the component of the "i" unit
vector(meters)?\n');
j2=input('\nWhat is the component of the "j" unit
vector(meters)?\n');
k2=input('\nWhat is the component of the "k" unit
vector(meters)?\n');
id(2)=input('\nWhat is the inside diameter for the
second pipe(meters)?\n');
weight(2)=input ('What is the weight per meter of the
```

```
second pipe(Newtons)?');
%This gives us information about the second pipe.
disp('The position vector of the end of pipe 3');
i3=input('\nWhat is the component of the "i" unit
vector(meters)?\n');
j3=input('\nWhat is the component of the "j" unit
vector(meters)?\n');
k3=input ('\nWhat is the component of the "k" unit
vector(meters)?\n');
id(3)=input('\nWhat is the inside diameter for the third
pipe(meters)?\n');
weight(3)=input('What is the weight per meter of the
third pipe (Newtons)?');
%This gives us information about the third pipe.

area(1)=(pi./4.)*id(1).^2;
area(2)=(pi./4.)*id(2).^2;
area(3)=(pi./4.)*id(3).^2;
flow_velocity(1)=volume_flow.*.001./area(1);
flow_velocity(2)=volume_flow.*.001./area(2);
flow_velocity(3)=volume_flow.*.001./area(3);
%The volume flow is multiplied by .001 because we
%are converting from liters/sec. to m^3/sec.

vector1=[0 length(1) 0];
vector2=[i2 j2 k2];
vector3=[i3 j3 k3];
dis_vec2=vector2-vector1;
dis_vec3=vector3-vector2;
%Using the position vectors obtained earlier, we can
%get the displacement vectors of pipes 2 and 3.

length(2)=sqrt(sum(dis_vec2.^2));
length(3)=sqrt(sum(dis_vec3.^2));
%This is the familiar length equation for a vector in
%computer language form (the length of pipe #1 is
%already known since it is co-linear with the "y"
%axis).

unit_vector3i=dis_vec3(1)./length(3);
unit_vector3j=dis_vec3(2)./length(3);
unit_vector3k=dis_vec3(3)./length(3);
unit_vector3=[unit_vector3i unit_vector3j
unit_vector3k];
%A unit vector corresponding to the displacement
%vector for pipe 3 must be made since the direction
%of the efflux flow is important in analyzing the
%forces involved.
```

```
body_force(1)=area(1).*length(1).*9806+length(1).*
weight(1);
bf_vector1=[0 0 -body_force(1)];
body_force(2)=area(2).*length(2).*9806+length(2).*
weight(2);
bf_vector2=[0 0 -body_force(2)];
body_force(3)=area(3).*length(3).*9806+length(3).*
weight(3);
bf_vector3=[0 0 -body_force(3)];
%9806 is in N/m^3 and is the weight per cubic meter
%of water. All these forces would obviously act in
%the -k direction and we have made vectors out of
%them corresponding to this.

momentum_flow3=1000.*flow_velocity(3).^2.*area(3).*
unit_vector3;
%This gives a 1x3 vector of momentum flow out of the
%end of pipe 3 corresponding to "unit_vector3".

moment_arm1=.5.*vector1;
moment_arm2=vector1+.5.*dis_vec2;
moment_arm3=vector2+.5.*dis_vec3;
%These give the moment arms to be used in the body
%force moment calculations.

moment_pipe1=cross(moment_arm1,bf_vector1);
%This is the moment caused by gravity acting on pipe
%1 and the water it contains.

moment_pipe2=cross(moment_arm2,bf_vector2);
%This is the moment caused by gravity acting on pipe
%2 and the water it contains.

moment_pipe3=cross(moment_arm3,bf_vector3);
%This is the moment caused by gravity acting on pipe
%3 and the water it contains.

moment_efflux=cross(vector3,momentum_flow3);
%This is the moment caused by the flow of water
%leaving pipe 3.

momenti=-moment_pipe1(1)-moment_pipe2(1)-
moment_pipe3(1)+moment_efflux(1);

momentj=-moment_pipe1(2)-moment_pipe2(2)-
moment_pipe3(2)+moment_efflux(2);

momentk=-moment_pipe1(3)-moment_pipe2(3)-
moment_pipe3(3)+moment_efflux(3);
```

```
%Summation of moments in each direction from the
%angular momentum equation.
disp('The total moment acting at the base of the
pipe, in Newton-meters, is:');
fprintf('\n\nIn the "i" direction: %4.4f\nIn the "j"
direction: %4.4f\n',momenti,momentj);
fprintf('In the "k" direction: %4.4f\n\n\n',momentk);
%This displays the sought after information in the
%Matlab Command Window.
```

■ Debriefing

This program demonstrates how MATLAB's intrinsic functions can really make short work of operations such as cross products. In Fortran, the programmer has to write a program to do a cross product, but in MATLAB all you have to do is reference an intrinsic function. Also, remember that the moment arm must always go *from* the point where you want the moment (the base of the piping arrangement in this case) *to* the force which induces the moment.

COMPUTER EXAMPLE 4.6

■ Computer Problem Statement

A gun is firing experimental rubber bullets each of mass m_i against a set of three hard walls as shown in Fig. C4.8. The initial bullet velocity is V_0. Three inelastic impacts take place, each having different coefficients of restitution α_i. If N rubber bullets are fired per second, what are the average forces normal to each wall? What is the final bullet velocity? Set up an interactive computer program and let it run for the following data. Neglect friction and consider the path of the bullets to be coplanar and parallel to the gun barrel. What is the force on the wall system from the flow of rubber balls.?

$$V_0 = 500 \text{ m/s} \qquad N = 100 \qquad m = 50 \text{ grams}$$
$$\alpha_0 = 60° \qquad \alpha_1 = 20° \qquad \alpha_2 = 75° \qquad \alpha_3 = 18°$$
$$\varepsilon_1 = 0.8 \qquad \varepsilon_2 = 0.6 \qquad \varepsilon_3 = 0.4$$

■ Strategy

We will assume that the collisions with the walls are frictionless as is the action of the air on the bullets moving through the air between collisions. Furthermore we will consider that the bullets hit all walls in a sequential manner. Finally, when the motion of a bullet is known, we will choose a control volume to encompass the flow of bullets, and we will replace the discrete system of bullets by a hypothetical flowing continuum. We will then use the **linear momentum equation** to get a time average of the desired force on the wall system.

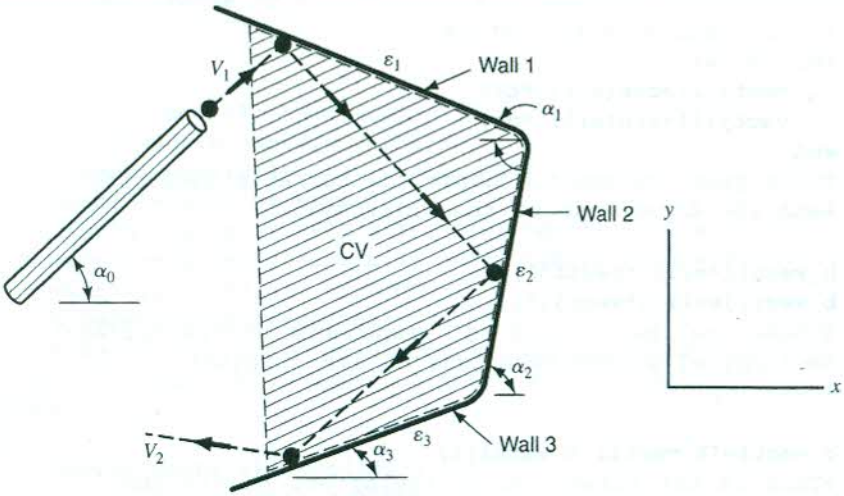


Figure C4.8

■ Execution

```
clear all;
%Putting this at the beginning of the program ensures
%values don't overlap from previous programs.

c=pi./180;
%This is the conversion constant that must be used
%because the intrinsic functions of Matlab are not
%compatible with degrees but only with radians.

v(1)=input('What is the initial speed of the
bullet(s) from the gun (m/sec)?\n');
n=input('How many bullets are fired per second?\n');
m=input('What is the mass of each bullet(kg) ?\n');
%Getting the mass of each bullet and the number of
%bullets per second we get the mass flow.

a(1)=input('What is the angle of the barrel of the
gun to the positive x-axis?\n');
a(2)=input('What is the angle of the first wall to
the positive x-axis?\n');
a(3)=input('What is the angle of the second wall to
the positive x-axis?\n');
a(4)=input('What is the angle of the third wall to
the positive x-axis?\n');
e=input('What are the coefficients of restitution [e1
e2 e3]?\n');
```

```
%These input statements take information about the
%bullet and wall properties.
for i=1:4;
    vecti(i)=cos(a(i).*c);
    vectj(i)=sin(a(i).*c);
end
%This gets information about the barrel's direction
%and the directions of the three walls.

b_vect1i=v(1).*vecti(1);
b_vect1j=v(1).*vectj(1);
%These are the "i" and "j" components of the bullet
%multiplied by the magnitude of its original
%velocity.

b_vect1=[b_vect1i b_vect1j];
%This is the bullet vector coming out of the gun.

w_vect1=[vecti(2) vectj(2)];
%This vector describes the position of the first
%wall.

v_tan(1)=dot(b_vect1,w_vect1);
%Using Matlab's intrinsic "dot" function we can get
%the component of the bullet's velocity that is
%tangential to the wall.

v_norm(1)=sqrt(v(1).^2-v_tan(1).^2);
%Using Pythagorean's theorem we can get the component
%of the bullet's velocity that is normal to the wall.

v_norm(2)=-e(1).*v_norm(1);
%This is the equation for the coefficient of
%restitution solved for the final velocity
%of an object when the object strikes an object with
%infinite mass (the walls in this case). This gives
%the bullet's velocity leaving the wall based on the
%coefficient of restitution. Since the walls are
%assumed friction-less, there will be no change
%in the tangential velocity of the bullet.

b_vect2i=v_norm(2).*sin(a(2).*c)+v_tan(1).*cos(a(2).*c);
b_vect2j=v_norm(2).*cos(a(2).*c)+v_tan(1).*sin(a(2).*c);
%These are the "i" and "j" components of the bullet
%coming off the first wall.

b_vect2=[b_vect2i b_vect2j];
%Vector describing the bullet leaving the first wall.
```

```
v(2)=sqrt(sum(b_vect2.^2));
%magnitude of velocity of the bullet off the first
%wall.

w_vect2=[vecti(3) vectj(3)];
v_tan(2)=dot(b_vect2,w_vect2);
v_norm(2)=sqrt(v(2).^2-v_tan(2).^2);
v_norm(3)=-e(2).*v_norm(2);
b_vect3i=v_norm(3).*sin(a(3).*c)+v_tan(2).*cos(a(3).*c);
b_vect3j=v_norm(3).*cos(a(3).*c)+v_tan(2).*sin(a(3).*c);
b_vect3=[b_vect3i b _vect3j];
v(3)=sqrt(sum(b_vect3.^2));
%These are the same equations as above and involve
%the interaction of the bullet(s) and the second
%wall.

w_vect3=[vecti(4) vectj(4)];
v_tan(3)=dot(b_vect3,w_vect3);
v_norm(3)=sqrt(v(3).^2-v_tan(3).^2);
v_norm(4)=-e(3).*v_norm(3);
b_vect4i=v_norm(4).*sin(a(4).*c)+v_tan(3).*cos(a(4).*c);
b_vect4j=v_norm(4).*cos(a(4).*c)+v_tan(3).*sin(a(4).*c);
b_vect4=[b_vect4i b_vect4j];
v(4)=sqrt(sum(b_vect4.^2));
%These are the same equations as above and involve
%the interaction of the bullet(s) and the third wall.

unitvector=[b_vect4i./v(4) b_vect4j./v(4)];
%This is a unit vector describing the exiting
%bullet(s) trajectory.

force_wallsi=-m.*n.*(b_vect4i-b_vect1i);
%This is the force in the "i" direction on the wall
%assembly from the bullets. The minus sign is because
%we are taking the reaction force to the force on the
%bullets from the walls.

force_wallsj=-m.*n.*(b_vect4j-b_vect1j);
%This is the force in the "j" direction on the wall
%assembly from the bullets.

fprintf('\n\nMagnitude of the exiting bullet(s)
velocity in m/sec: %4.4f\n',v(4));

fprintf('Unit vector of exiting bullet(s):
%4.4fi%4.4fj\n\n', unitvector(1), unitvector(2));

fprintf('The force acting on the walls in the "i"
direction: %4.4f Newton.\n',force_wallsi);
```

```
fprintf('The force acting on the walls in the "j"
direction: %4.4f Newton.\n',force_wallsj);
%These print statements display the information that
%we want.
```

■ Debriefing

In this problem we used a vector approach to describe the bullet's trajectory and the walls involved. Obviously, some arrangements of walls would not be compatible with the bullet striking the three walls in order, as was the assumption at the beginning of the problem. Because of this, it is up to the operator of the program to take a look at the walls involved and use the program for an arrangement of walls that makes sense.

■ Computer Output

```
What is the initial speed of the bullet(s) from the gun
(m/sec)?
500
How many bullets are fired per second?
100
What is the mass of each bullet (kg)?
.05
What is the angle of the barrel of the gun to the
positive x-axis?
60
What is the angle of the first wall to the positive
x-axis?
-20
What is the angle of the second wall to the positive
x-axis?
75
What is the angle of the third wall to the positive
x-axis?
18
What are the coefficients of restitution? Enter them
like this: [e1 e2 e3]?
[.8 .6 .4]

Magnitude of the exiting bullet(s) velocity in m/sec:
438.4938

Unit vector of exiting bullet(s): -0.8715i-0.4905j

The force acting on the walls in the "i" direction:
1[illegible].0652 Newton
The force acting on the walls in the "j" direction:
3[illegible]4.0392 Newton
```

PROBLEMS

Problem Categories

Conservation of Mass 4.1–4.16
Linear Momentum 4.17–4.66
Moment of Momentum 4.67–4.83
Computer Problems 4.84–4.89

Starred Problems

4.66

4.1 A flow of 0.3 m^3/s of water enters a rectangular duct. Two of the faces of the duct are porous. On the upper face water is added at a rate shown by the parabolic curve; on the front face a portion of the water leaves at a rate determined linearly by the distance from the end. The maximum values of both rates are given in cubic meters per second per unit length along the duct. What is the average velocity V of the water leaving the end of the duct if it is 0.3 m long and has a cross section of 0.01 m^2?

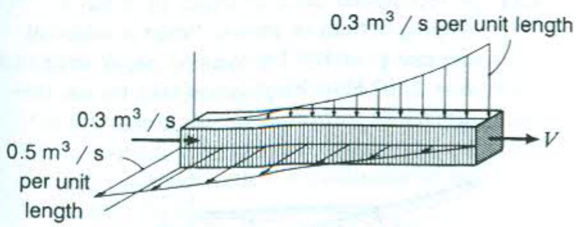


Figure P4.1

4.2 In Prob. 4.1, determine the position along the duct where the average velocity of flow along the duct is at maximum.

4.3 Hot steel is being rolled at a rolling mill. The steel emerging from the rollers is 10 percent more dense than before entering. If the steel is being fed at the rate of 0.2 m/s, what is the speed of the rolled material? There is a 9 percent increase in the width of the steel.

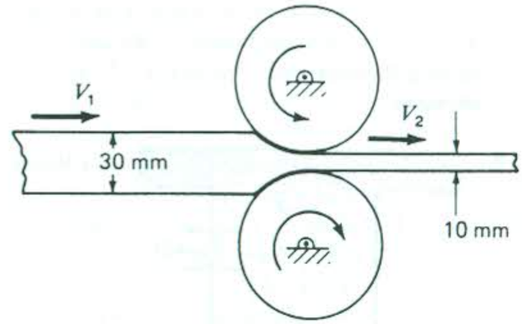


Figure P4.3

4.4 Shown is a device into which 0.3 m^3/s of water is admitted at the axis of rotation and directed out radially through three identical channels whose exit areas are each 0.05 m^2 perpendicular to the direction of flow relative to the device. The water leaves at an angle of 30° relative to the device as measured from a radial direction, as is shown in the diagram. If the device rotates clockwise with a speed of 10 rad/s relative to the ground, what is the magnitude of the average velocity of the fluid leaving the vane as seen from the ground?

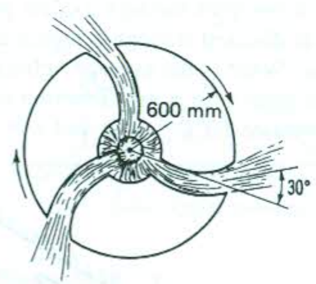


Figure P4.4

4.5 If the device in Prob. 4.4 has a clockwise angular acceleration of 5 rad/s^2 at the instant that the other data are given and the rate of increase of influx is 0.03 m^3/s^2, what is the magnitude of the acceleration of the water leaving the device relative to the ground? Take the fluid as completely incompressible for the calculations.

4.6 Water is forced into the device at the rate of 0.1 m^3/s through pipe A, while oil of specific gravity 0.8 is forced in at the rate of 0.03 m^3/s through pipe B. If the liquids are incompressible and form a homogeneous mixture of oil globules in water, what is the average velocity and density of the mixture leaving through pipe C having a 0.3-m diameter?

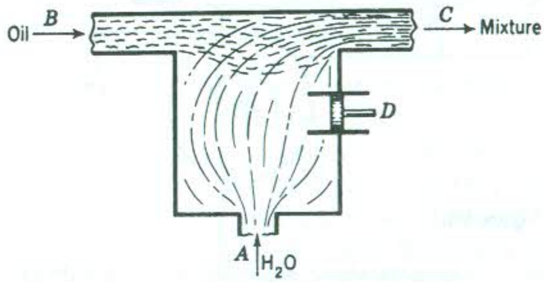


Figure P4.6

4.7 In Prob. 4.6 the piston at D having a 150-mm diameter moves at the rate of 0.3 m/s to the left. What is the average velocity of the fluid leaving at C?

4.8 Do Example 4.3 for the case where the radial velocity varies parabolically from zero at A to 3 ft/s at B.

4.9 5 ft³/s of water goes through a 12-in pipe and then is directed through a region around a 60° cone. What is the average velocity in the region from C to E as a function of η and δ? Evaluate for $\delta = 2$ in and $\eta = 16$ in.

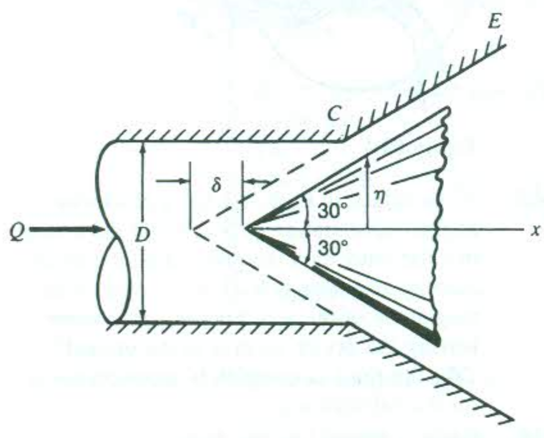


Figure P4.9

4.10 Water is flowing in at ① into a rectangular tank A with a length of 5 ft and a width of 5 ft. The rate of flow Q_1 at ① is 5 ft³/s. At the instant of interest, $h_1 = 15$ ft and water is flowing into tank B through ③ at the rate of 4 ft³/s. At this instant, h_2 is 12 ft. If the free surface in tank B is dropping at the rate of 0.2 ft/s, what is the flow Q_2 at ② at the instant of interest? Tank B is of length 8 ft and has the same width as tank A. Also, what is the rate at which h_1 is changing value?

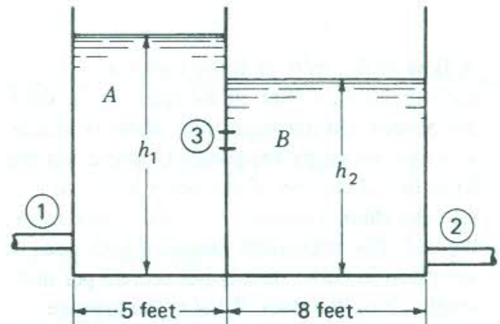


Figure P4.10

4.11 A rectangular ditch of width 10 m has a sloping bottom as shown. Water is added at the rate Q of 100 l/s. What is dh/dt when $h = 1$ m? How long does it take for the free surface to go from $h = 1$ m to $h = 1.2$ m?

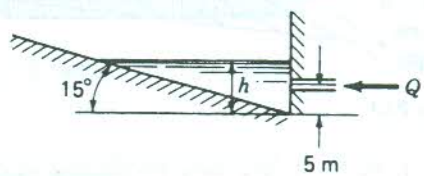


Figure P4.11

4.12 Suppose in Prob. 4.11 that there is influx q normal to the sloping bottom of 2 l/m² of the bottom. What is dh/dt and the time Δt for h to go from 1 to 1.2 m? *Hint:*

$$\int \frac{x\,dx}{a + bx} = \frac{1}{b^2}[a + bx - a\ln(a + bx)]$$

4.13 A rectangular cube a meters on an edge is moving in space at a very high velocity V m/s. The box is designed to capture solar dust particles inside the box as they strike the box. Because the speed of the box is much faster than the speed of the dust particles, we can assume that the latter are stationary. If there are n dust particles per unit volume, and if N represents the number of dust particles inside the box, what is the rate of accumulation of dust particles in the box (dN/dt)? What is the total number of dust particles ΔN collected during a time interval Δt seconds?

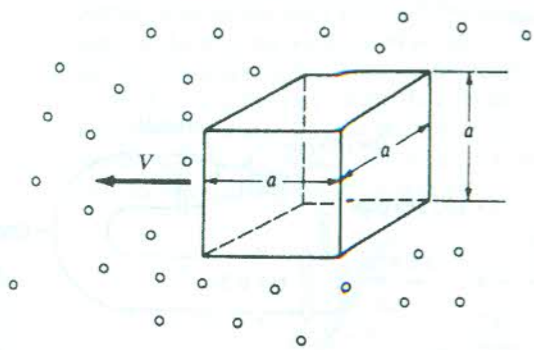


Figure P4.13

4.14 Do Prob. 4.13 using a sphere of radius a as the dust collector. Start by considering a strip of length $a\, d\theta$, then integrate to cover surface of impact. What general simple rule can you now state in words for a collector of any shape? *Hint:*

$$\int \sin\theta \cos\theta \, d\theta = \frac{\sin^2\theta}{2}$$

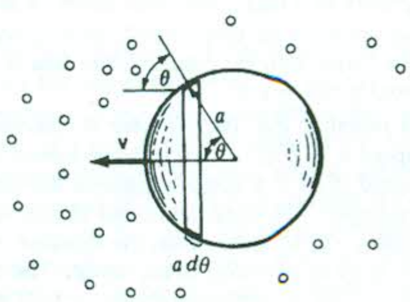


Figure P4.14

4.15 A Kingsbury thrust bearing consists of a number of small pads whose bottom surface is inclined to the horizontal. One such pad A is shown in detail. Oil moves under the pad; and because of the flow under the pad, there is a vertical thrust developed. The velocity profile of the oil relative to the base is shown at the inlet to the pad. If we do not consider side leakage, what is the average speed relative to the ground of the oil on leaving the region under the pad? V_0 is the average velocity of the pad relative to the stationary base.

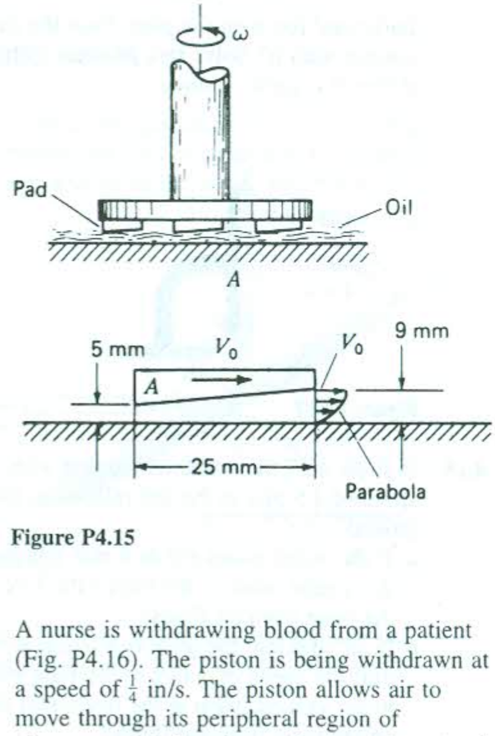


Figure P4.15

4.16 A nurse is withdrawing blood from a patient (Fig. P4.16). The piston is being withdrawn at a speed of $\frac{1}{4}$ in/s. The piston allows air to move through its peripheral region of clearance with the glass cylinder at the rate of 0.001 in^3/s. What is the average speed of blood flow in the needle? Choose as a control volume the region just to the right of the piston to the tip of the needle.

4.17 A jet of water issues from a nozzle at a speed of 6 m/s and strikes a stationary flat plate oriented normal to the jet. The exit area of the nozzle is 645 mm^2. What is the total

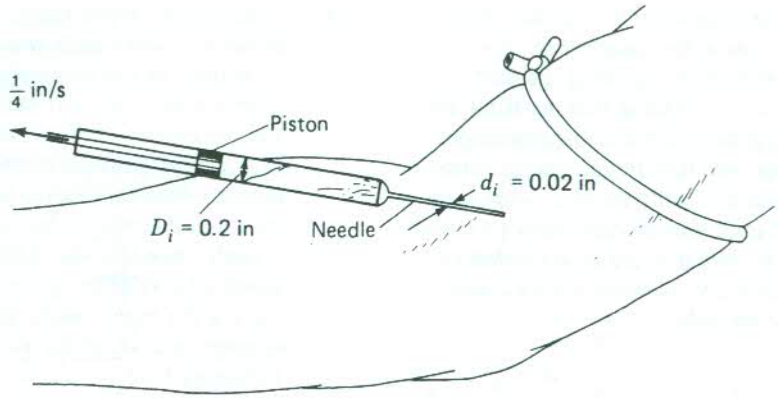


Figure P4.16

horizontal force on the plate from the fluids in contact with it? Solve this problem using two different control volumes.

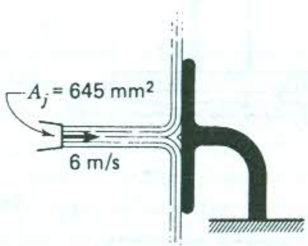


Figure P4.17

4.18 In Prob. 4.17 the nozzle is moving with a speed of 1.5 m/s to the left relative to the ground.

a. If the water issues out at 6 m/s relative to the nozzle, what is the horizontal force on the plate from all fluids?

b. If, in addition, the plate is moving at a uniform speed of 3 m/s to the right relative to the ground, what is the horizontal force on the plate from the fluids?

Use only one control volume in each case.

4.19 For Example 4.9 determine the value of V_0 and θ for maximum power.

4.20 What is the force on the elbow-nozzle assembly from the water and air? The water issues out as a free jet from the nozzle. The interior volume of the nozzle elbow assembly is 0.1 m^3.

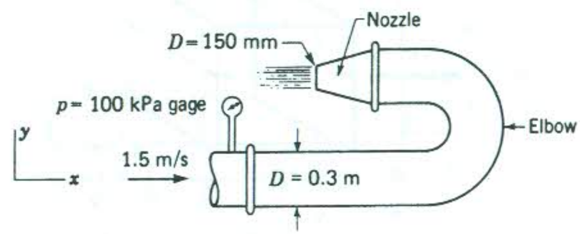


Figure P4.20

4.21 Find the horizontal force on the device of Prob. 4.6 if the oil enters at a pressure of 1.4×10^5 Pa gage, the water at 1.2×10^5 Pa gage, and the mixture leaves at a pressure of 1.0×10^5 Pa gage. Pipe B has a diameter of 0.5 m, pipe A has a diameter of 0.3 m, and pipe C has a diameter of 0.3 m. From Prob. 4.6 we have $\rho_c = 955.5$ kg/m^3 and $V_c = 1.914$ m/s.

4.22 A dredging operation delivers 5000 gal/min of a mixture of mud and water having a specific gravity of 3 into a stationary barge. What is the force on the barge which tends to separate the barge from the dredger? The area of the nozzle exit is 1 ft^2.

4.23 A trough in Fig. P4.23 moves at constant speed $u = 2$ m/s. A jet of water having a speed of $V_j = 6$ m/s impinges on the trough as shown. The water leaves the trough in three places. At the exit nozzle, the speed of water V_1 is 10 m/s relative to the trough. The area $A_1 = 0.02$ m^2 while the area $A_j = 0.08$ m^2. Twice as much water leaves at B than leaves

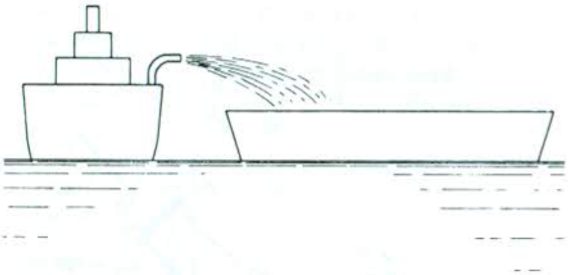


Figure P4.22

at C. Compute thrust on the trough. Use a control volume that does not cut the trough support. Assume no friction and no affect of gravity on the *unconfined* flow in the trough itself. However, the exit nozzle flow results in a different fluid exit velocity because in the nozzle the flow is *confined* and squirts out at a higher velocity relative to the trough.

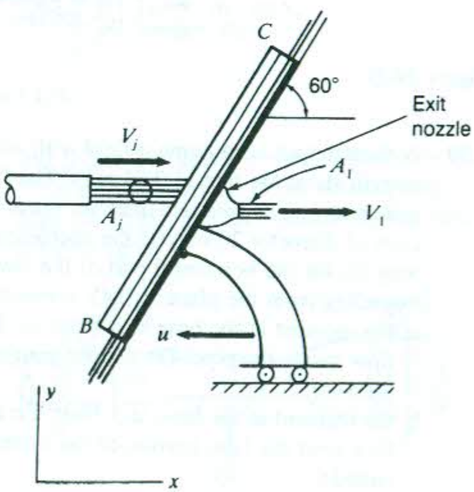


Figure P4.23

4.24 Do Prob. 4.23 using a control volume that cuts the trough support.

4.25 We are looking down from above on a large tank of water which is connected to a 12-in horizontal pipe. The water, once in the pipe, has a speed of 5 ft/s before reaching the end of a second thin pipe, AB, through which water is pumped at a speed of 25 ft/s. The pressure p_1 in the main stream at the position shown is 5 psig, and at A the high speed jet emerges as a free jet. At about 3 ft from A the two flows are thoroughly mixed. If we neglect friction at the walls of the 12-in pipe, what is the pressure p_2?

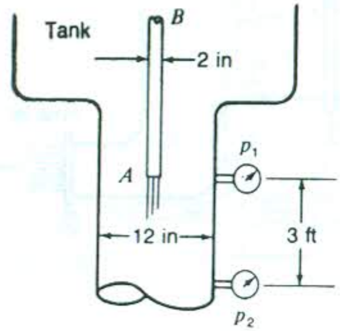


Figure P4.25

4.26 What is the dynamic force (i.e., excluding gravity) on the flat plate from the water on one side and air on the other? Water is at 10°C.

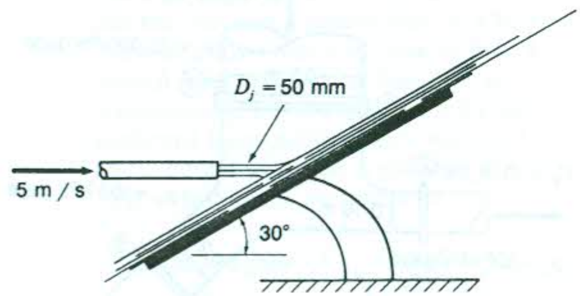


Figure P4.26

4.27 Water flows at a steady rate through the device shown. The following data apply:

$$p_1 = 20 \text{ psig}$$
$$V_1 = 10 \text{ ft/s}$$
$$D_1 = 15 \text{ in}$$
$$D_2 = 8 \text{ in}$$
$$D_3 = 4 \text{ in}$$
$$V_2 = 20 \text{ ft/s}$$

What is the horizontal thrust from water and air?

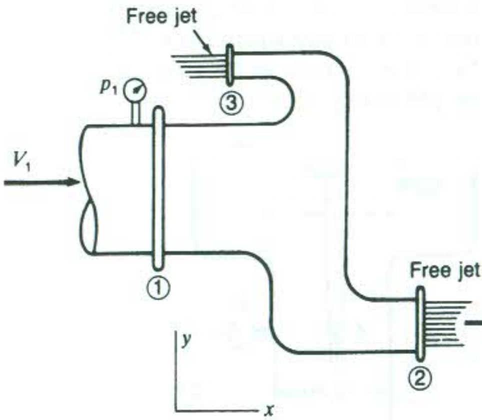


Figure P4.27

4.28 Water is moving steadily through a double-exit elbow for which $V_1 = 5$ m/s. The inside volume of the elbow is 1 m^3. Find the vertical and horizontal forces from air and water on the elbow. Take $V_2 = 10$ m/s.

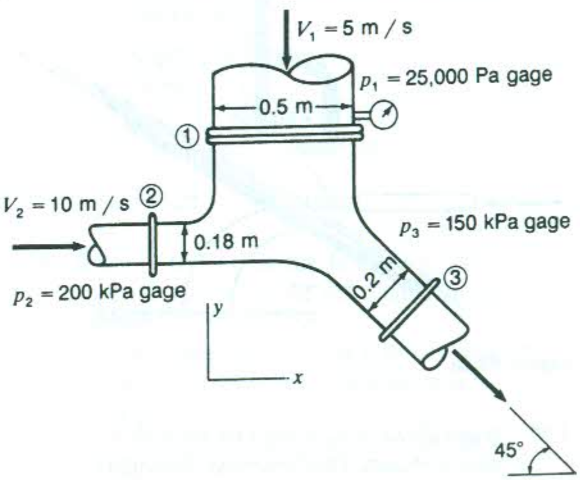


Figure P4.28

4.29 Water is flowing through a reducing elbow. A pipe welded to the reducing elbow passes through the reducing elbow and carries a steady flow of oil.

a. Find the horizontal force component from the *water* on the *elbow*.
b. Find horizontal force from *air* on the elbow.
c. Find horizontal force from *oil* on the elbow.
d. Give total horizontal force on the elbow from water, air, and oil.

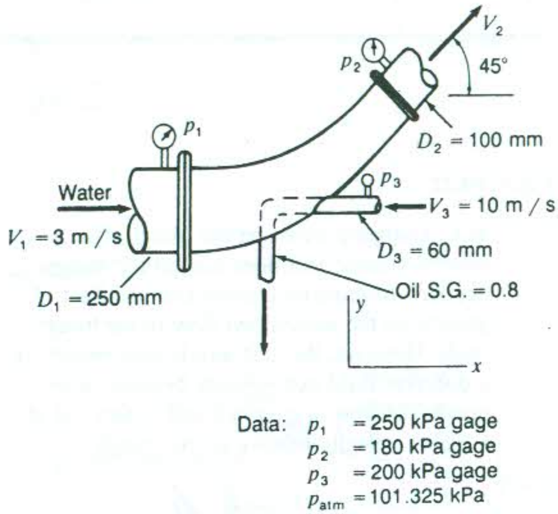


Figure P4.29

4.30 A dustcropper is spraying a field with an insecticide at the rate of 0.01 m^3/s. The fluid is coming out as *free jets* from six openings each of diameter 20 mm. If the coefficient of drag C_D for the horizontal part of the device extending from the plane is 0.45, compute:

a. the moment at the base at A from the fluid flow inside the pipe. Do not use gauge pressure.
b. the moment at the base at A from the air flow over the base portion of the system outside.
c. the total moment at A.

Note: We will later learn that for the drag force F_D we have $F_D = \frac{1}{2}C_D A \rho V^2$ where A is the frontal projected area of the surface of the base portion in the direction of the velocity and ρ is the density of the air.

4.31 30 l/s of water at 10°C enter a jet pump at A at a pressure $p_1 = 300{,}000$ Pa gage. Oil is sucked in at C at the rate of 1 L/s. The oil has a specific gravity of 0.65. A thoroughly mixed flow of water and oil leave at B at a pressure p_2 of 150,000 Pa gage. The dimensions of D_1

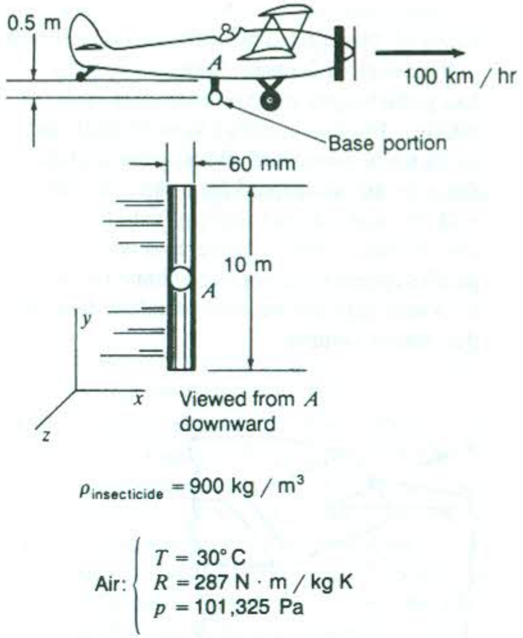


Figure P4.30

and D_2 are 200 and 250 mm, respectively. What is the horizontal thrust on the pump from water, oil, and air? Density of the water is 999.7 kg/m^3.

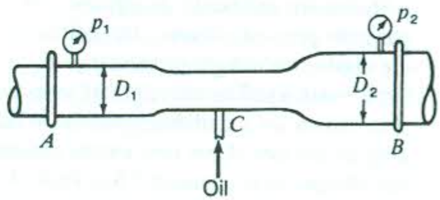


Figure P4.31

4.32 A *vertical* system in Fig. P4.32 conducts water from a large reservoir at the rate of 1 m^3/s. At the tee at B, $\frac{1}{3}$ m^3/s goes to the left and $\frac{2}{3}$ m^3/s goes to the right. Pipe EB weighs 1 kN/m, pipe AB weighs 0.6 kN/m, and pipe BC weighs 0.8 kN/m. Find the *total* vertical and horizontal forces on the pipes from fluid flow and air as well from gravity on water and pipe. Free jets are at A and C. $p_E =$ 390.4 kPa gage.

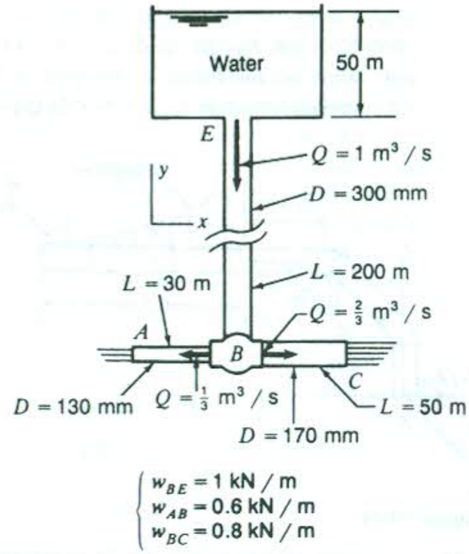


Figure P4.32

4.33 Water is flowing in a circular duct of diameter 1 m. The velocity profile is paraboloidal with a volume flow of 5000 L/s. The water flows over a 90° cone to form a conical sheet of water that then exits to the atmosphere. If $\delta = 0.1$ m, what is the total horizontal force on the ground supports A, B, and C from the cylinder E and cone F? Neglect hydrostatic pressure at G.

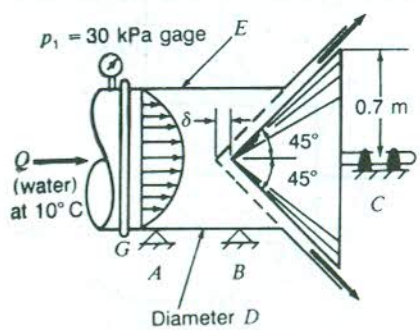


Figure P4.33

4.34 Five-hundred liters of water per second flow through the pipe shown in Fig. P4.34. The flow exits through a rectangular area of length 0.8 m and width of 40 mm. The velocity profile is that of a parabola. The pipe weighs 1000 N/m and has an inside diameter of 250 mm. What are the forces on the pipe at A? The entering pressure p_1 is 100 kPa gage.

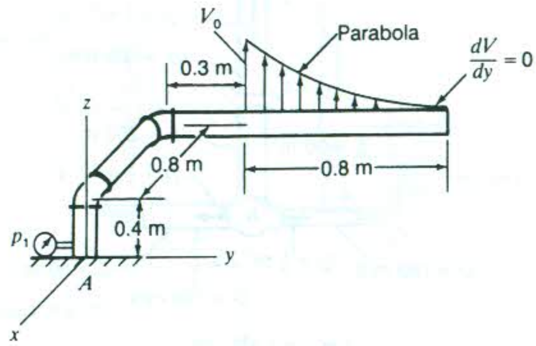


Figure P4.34

4.35 A control volume in Fig. P4.35 is moving at constant speed V_0. There are n stationary particles per unit volume uniformly distributed, each of mass m. Compute the flow of mass through left face A_1 realizing that the formulation $\iint_{A_1}(\rho \mathbf{V} \cdot \mathbf{dA})$ must be adjusted from a continuum approach to a flow of distinct separate particles. Do the same for linear momentum and kinetic energy. *Hint:* Use an approach similar to Fig. P4.14. Show that $\iint \rho \mathbf{V} \cdot \mathbf{dA} \equiv V_0 mnA_1$.

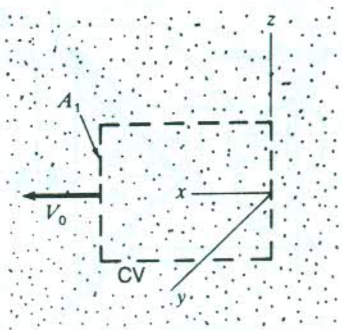


Figure P4.35

4.36 A light plane is moving through a hail storm at a speed of 200 km/h. The windshield shown has an area of 0.25 m^2 and its normal unit vector $\boldsymbol{\epsilon}_n$ is

$$\boldsymbol{\epsilon}_n = 0.2\mathbf{i} + 0.25\mathbf{j} + 0.947\mathbf{k}$$

The hail element has a mass of 1 mg and slides off the windshield after *plastic* collision with negligible friction. There are $n = 500$ hail particles per unit volume—that is, $n = 500/\text{m}^3$. The hail particles have vertical speed of 10 km/h downward. What is the average force on the windshield from the collisions with the hail? Do not include body force contributions. *Hint:* Choose a rectangular parallelepiped for a control volume of small thickness with the windshield as one face of the control volume.

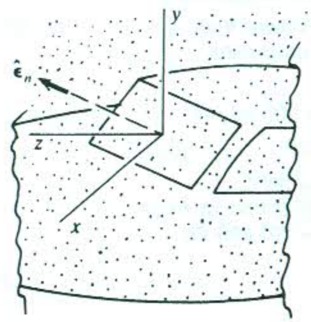


Figure P4.36

4.37 A double-wedge airfoil is moving at very high speed V_0 through stationary dust particles in outer space, each particle of mass m. The particles are uniformly distributed with n particles per unit volume. The collisions are elastic with angle of incidence $\beta = (\pi/2 - \alpha)$ equal to the angle of reflection β as shown for a colliding particle. What is the drag on the airfoil per unit length assuming a two-dimensional approach? See Prob. 4.35 before doing this problem and use a control volume such as is shown in Fig. P4.37.

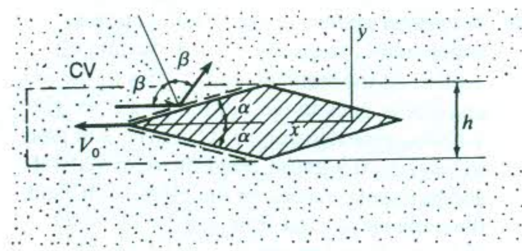


Figure P4.37

4.38 You examined in particle mechanics an inelastic collision of a particle with a large rigid body. Recall that the coefficient of restitution ϵ is related to the velocity of approach component normal to the surface of a rigid body as well as to the velocity of rebound normal to the surface. The following simple relation was presented:

$$\epsilon = -\frac{(V_N)_{\text{rebound}}}{(V_N)_{\text{approach}}}$$

Do the preceding problem with a coefficient of restitution ϵ for all collisions.

4.39 A spherical communications satellite is moving in outer space with a speed 20 or more times the speed of sound (i.e., a high Mach number). Molecules of mass m move at the speed of sound. Because of the speed disparity between satellite and molecule, we will assume molecules are stationary and that the satellite moves with speed V_0 hitting molecules in front of it. We will assume that the collisions are elastic so that the angle of incidence α equals the angle of reflection β (remember optics?). Using a control volume fixed to the satellite as shown, compute the *drag*. The molecules each have a mass m. There are n molecules per unit volume. Make use of strips shown, integrating from $\theta = 0$ to $\theta = \pi/2$. The radius of the sphere is R. Because we do not have a continuum, we have no pressure as such; only the collision of discrete molecules with the satellite surface. See Prob. 4.35 before proceeding.

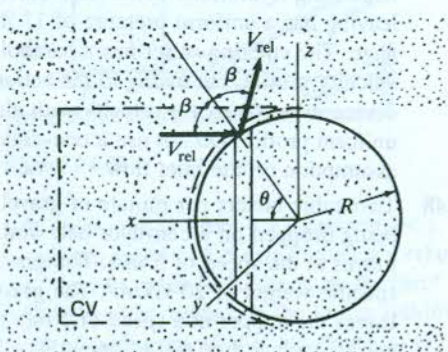


Figure P4.39

4.40 A light attack boat is leaving an engagement at full speed. To help in the process, a battery of four 50-calibre machine guns is fired to the rear continuously. The muzzle velocity of the guns is 1000 m/s and the rate of firing for each gun is 3000 rounds per minute. Each bullet weighs 0.5 N. The ship weighs 440 kN. What is the additional thrust from the guns when the boat has attained constant speed of 40 knots? Neglect the change of total mass of the boat and its contents.

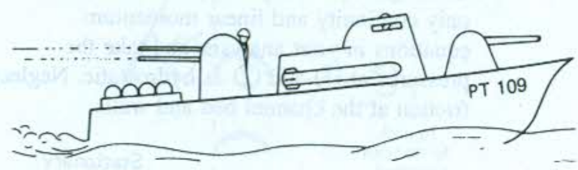


Figure P4.40

4.41 A jet plane is on the runway after touching down. The pilot puts into play movable vanes so as to achieve a reverse thrust from his two engines. Each engine takes in 40 kg of air per second. The fuel-to-air ratio is 1 to 40. If the exit velocity of the combustion products is 800 m/s relative to the plane, what is the total reverse thrust of the airplane if it is moving at the instant of interest at a speed of 150 km/h? The deceleration of the plane will not be great, so little error is incurred if you consider

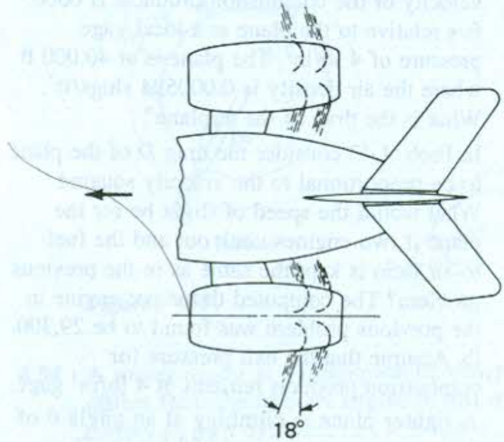


Figure P4.41

your control volume to be an inertial control volume. The exit jets are close to atmospheric pressure.

4.42 Water flows over a dam into a rectangular channel of width b. Beyond the bottom of the dam, the depth of flow h_1 is 3 ft. As you may have seen yourself on viewing rapid channel flow, the water changes elevation to height h_2 as it passes through a highly disturbed region called the *hydraulic jump*. If we assume one-dimensional flows at ① and at ②, what is the height h_2? The velocity V_1 is 25 ft/s. Use only continuity and linear momentum equations in your analysis, and take the pressures at ① and ② as hydrostatic. Neglect friction at the channel bed and walls.

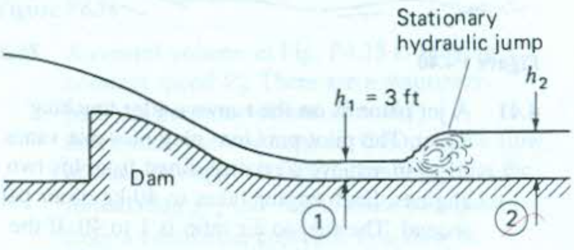


Figure P4.42

4.43 An airplane is moving at a constant speed of 600 m/h. Each of its four jet engines has an inlet area of 10 ft^2 and an exit area of 3 ft^2. The fuel-to-air ratio is 1 to 40. The exit velocity of the combustion products is 6000 ft/s relative to the plane at a local gage pressure of 4 lb/in^2. The plane is at 40,000 ft where the air density is 0.000594 slugs/ft^3. What is the drag of the airplane?

4.44 In Prob. 4.43 consider the drag D of the plane to be proportional to the velocity squared. What would the speed of flight be for the plane if two engines conk out and the fuel-to-air ratio is kept the same as in the previous problem? The computed thrust per engine in the previous problem was found to be 29,300 lb. Assume that the exit pressure for combustion products remains at 4 lb/in^2 gage.

4.45 A fighter plane is climbing at an angle θ of 60° at constant speed of 950 km/h. The plane takes in air at a rate of 450 kg/s. The fuel-to-air ratio is 1 to 40. The exit speed of the combustion products is 1825 m/s relative to the plane. If the plane changes to an inclination θ of 20°, what will be the speed of the plane when it reaches uniform speed? The new engine settings are such that the same amount of air is taken in and the exhaust speed is the same relative to the plane. The plane weighs 130 kN. The drag force is proportional to the speed squared of the plane. The exhaust jet is at the outside pressure.

4.46 A train is to take on water on the run by scooping up water from a trough. The scoop is 1 m wide and skims off a 25-mm layer of water. If the train is moving at 160 km/h, how much water does it take on per second, and what is the drag on the train due to this action?

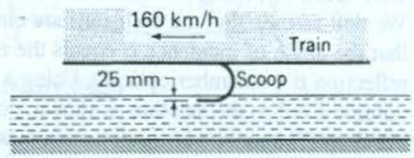


Figure P4.46

4.47 Consider Example 4.1. Determine the horizontal force on the device from inside and outside fluids if, on entering, the water has a uniform pressure of 25 lb/in^2 gage (we are neglecting hydrostatic variations) and on leaving has a uniform pressure of 13 lb/in^2 gage. The entering water has a paraboloidal velocity profile, but owing to the action in the device it has on exit a velocity with almost a uniform profile. Do not use a one-dimensional assumption on the inlet flow.

4.48 Two cubic meters per minute of gravel is being dropped on a conveyor belt which moves at the speed of 5 m/s. The gravel has a specific weight of 20 kN/m^3. The gravel leaves the hopper at a speed of 1 m/s and then has an average free fall of height h = 2 m. What torque T is needed by the conveyor to

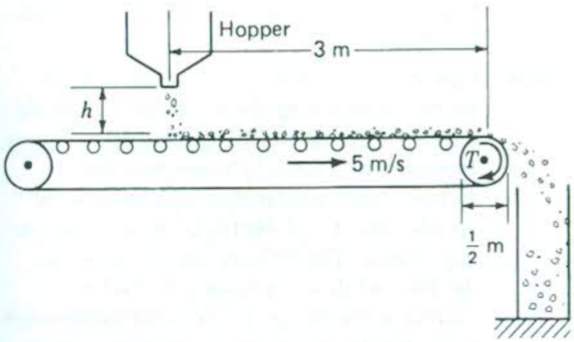


Figure P4.48

do the job? Neglect friction of rollers. *Hint:* Does the vertical motion of the gravel enter into your calculations?

4.49 In Prob. 4.48 compute the total vertical force from the gravel onto the conveyor. *Hint:* You will recall from mechanics that $V = \sqrt{V_0^2 + 2gh}$ for a free fall from V_0 initial speed.

4.50 Water issues from a large tank through a 1300-mm² nozzle at a velocity of 3 m/s relative to the cart to which the tank is attached. The jet then strikes a trough which turns the direction of flow by an angle of 30°, as is shown in Fig. P4.50. Assuming steady flow, determine the thrust on the cart which is held stationary relative to the ground by the cord.

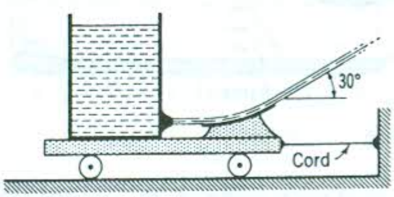


Figure P4.50

4.51 a. If the trough in Prob. 4.50 is moving at a uniform speed of 1.5 m/s to the right relative of the cart, what is the thrust on the cart?

b. If the cart is moving to the left at a uniform speed of 9 m/s, what is the thrust on the cart?

4.52 A jet of air from a 50-mm nozzle impinges on a series of vanes on a turbine rotor. The turbine has an average radius r of 0.6 m to the vanes and rotates at a constant angular speed ω. What are the transverse force and the torque on the turbine if the air has a constant specific weight of 12 N/m³? The velocities given in Fig. P4.52 are relative to the ground.

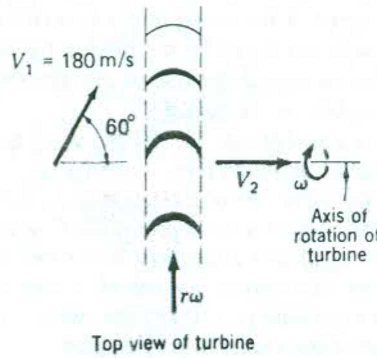


Figure P4.52

4.53 In Prob. 4.52 the angle of the blade at the left side is 45°. What must the speed ω of the turbine be to admit the air most smoothly? What is the power developed by the turbine? The torque on the turbine rotor is 40.4 N · m from Problem 4.52.

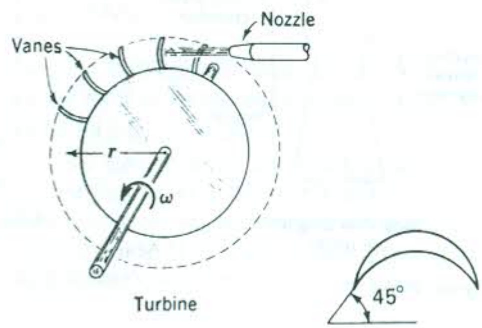


Figure P4.53

4.54 A *rocket engine* is a turbomachine which differs from the turbojet engine in that it carries its own oxidizer; i.e., it is not air-breathing. An oxidizer such as nitric acid and a fuel such as aniline are burned in the combustion chamber to attain a pressure of about 2×10^3 kPa. This then expands out to a lower pressure, which is usually close to the atmosphere on leaving the nozzle. (Since the flow will be supersonic on exit, it does not have necessarily the same pressure as the surroundings, as was the case of the subsonic free jets we have been using in this chapter.) The thrust of the rocket motor is attributed to the force developed by the fluid in the rocket thrust chamber above that of the surrounding atmosphere of the rocket.

If a rocket (see Fig. P4.54) using nitric acid and aniline on a test stand has an oxidizer flow rate of 2.60 kg/s and a fuel flow of 0.945 kg/s (hence a propellant flow rate of 3.545 kg/s), and if the flow leaves the nozzle at 1900 m/s through an area of 0.0119 m^2 with a pressure of 110 kPa abs, what is the thrust of the rocket motor? Neglect momentum of entering fluids.

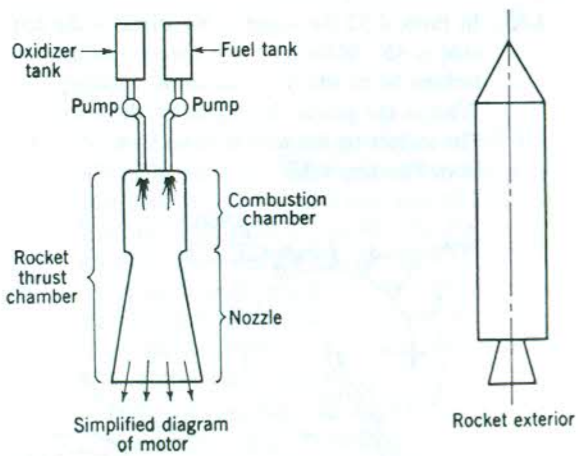


Figure P4.54

4.55 A rocket has a propellant flow rate of 11.40 kg/s. The exit area is 0.0335 m^2, and the exhaust velocity is 2000 m/s relative to the nozzle at the pressure of 101.4 kPa. What is the test-stand thrust of the motor (a) at sea level and (b) in an atmosphere equivalent to standard atmosphere at 9150 m? (Read the first paragraph of Prob. 4.54 before doing this problem.)

4.56 Figure P4.56 shows a supersonic *ram-jet engine*. It performs the same function as the turbojet discussed in Example 4.10. However, it is more efficient than the turbojet when operated at high supersonic speeds, and it is deceptively more simple in appearance. The diffuser section slows up the flow while compressing it. Fuel is burned in the stream in the combustion zone, and the products of combustion then expand down to some pressure p_e coming out of the nozzle. Because the fluid leaves at supersonic speed, the pressure p_e is not necessarily that of the ambient pressure around the jet.

Assume that the ram jet is moving at a speed V_1 and that the effective inlet area is A_1. Assume that w_f pound-mass of fuel is burned per unit time by the system and that the exit velocity is V_2 relative to the nozzle. If we disregard skin friction on the outside surface of the engine cowling, what is the thrust developed by the ram jet?

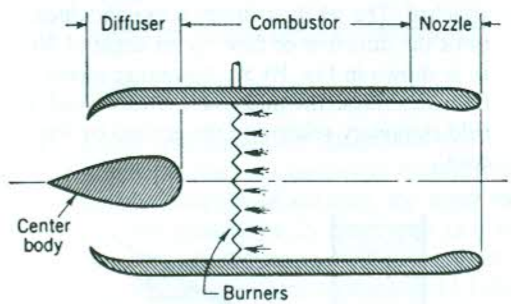


Figure P4.56

4.57 A jet of water of area A_j of 2 in^2 and speed V_j of 60 ft/s impinges on a trough which is moving at a speed u of 10 ft/s. If the water divides so that two-thirds goes up and one-third goes down, what is the force on the vane?

4.58 A locomotive snow remover is clearing snow from a track as seen from the top view. The snow is 2 ft above the top of the tracks and has an average density of 20 lbm/ft^3. If the

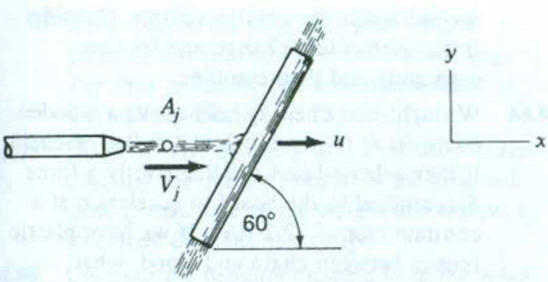


Figure P4.57

locomotive is going 30 ft/s at a steady rate, estimate the thrust needed by the vehicle to remove the snow.

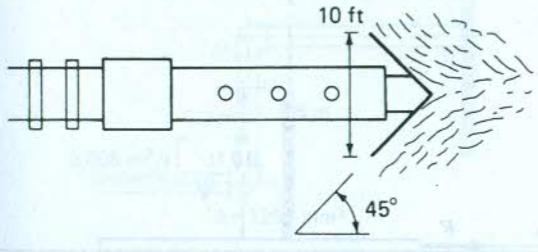


Figure P4.58

4.59 Water issues out of a triangular nozzle as a 10-mm sheet at a speed of 10 m/s. The pressure at the gage is 20 kPa gage. If the triangular nozzle weighs 500 N, what is the average force per bolt connecting A and B? The initial tension in each bolt developed by turning the nut is 50 N. There are 14 bolts.

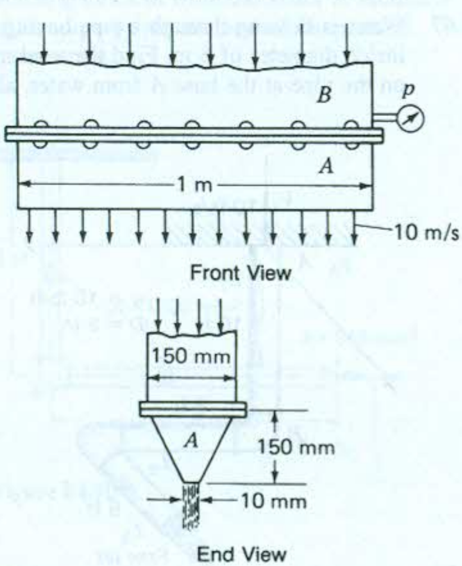


Figure P4.59

4.60 Water is pumped into a tank in Fig. P4.60 at the rate of 0.03 m³/s. The exit area of the pipe jet is 2000 mm², and the outside diameter of the pipe itself is 57.3 mm. The inside diameter of the tank is 1.2 m. When the water is 0.6 m above the exit of the pipe, estimate the upward force required to hold up the tank not including the weight of the tank itself. Assume the water discharges into the tank as a free jet at the hydrostatic pressure in the tank. Be sure to state other assumptions of your analysis clearly.

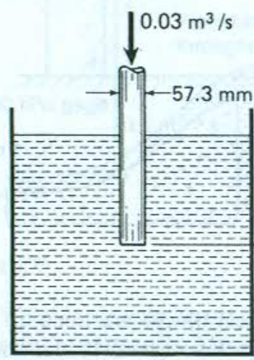


Figure P4.60

4.61 In Prob. 4.60 estimate what the rate of change of the force is for the configuration given.

4.62 In computing the thrust for rockets, ram jets, etc., we have assumed one-dimensional flow for the fluid leaving the nozzle. Actually, the flow issues out of the nozzle in a somewhat radial manner. If the exit speed is of constant magnitude V_e relative to the nozzle, what is

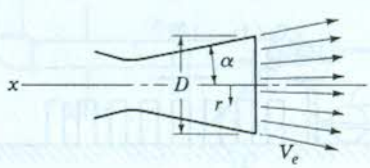


Figure P4.62

the proper expression for the linear momentum flow in the x direction across the exit of the nozzle, using this flow model?

4.63 Using the model you have established in Prob. 4.62 for exit flow in a nozzle, recompute the thrust of the rocket at sea level in Prob. 4.55 for the angle $\alpha = 20°$. Retain the assumption that the exit pressure p_e is uniform across the jet. *Hint:* Use the result $r = (D \tan\theta)/(2 \tan\alpha)$ in your calculations for ρ.

4.64 Water is flowing over a dam (Fig. P4.64). Upstream the flow has an elevation of 12 m and has an average speed of 0.3 m/s, while at a position downstream the water has a fairly uniform elevation measured as 1 m. If the width of the dam is 9 m, find the horizontal force on the dam.

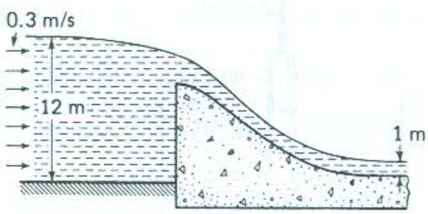


Figure P4.64

4.65 A row of identical blocks are lined up as shown. Each block weighs 5 lb and has a coefficient of dynamic friction with the ground of 0.3. A bulldozer moving at a constant speed V_0 of 10 ft/s is going to move these blocks to the right. If the impacts are completely inelastic (completely plastic), calculate the *average* force developed by the bulldozer as a function of time after the first block is touched. *Hint:* Consider a stationary control volume encompassing all the blocks. Then consider the average change in linear momentum per second inside the control volume. Consider linear momentum change and friction separately and then combine.

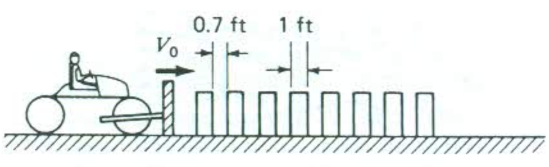


Figure P4.65

***4.66** Wrought-iron chain is held above a wooden board so as to just touch at $t = 0$. The chain is then released and simultaneously a force F is applied to the board to accelerate at a constant rate of 32.2 ft/s². If we have plastic impact between chain and wood, what function of time must F be to do the task? The coefficient of dynamic friction between the board and the floor is 0.3. The chain weighs 10 lb/ft. The board weighs 5 lb. Use stationary control volume for wrought iron.

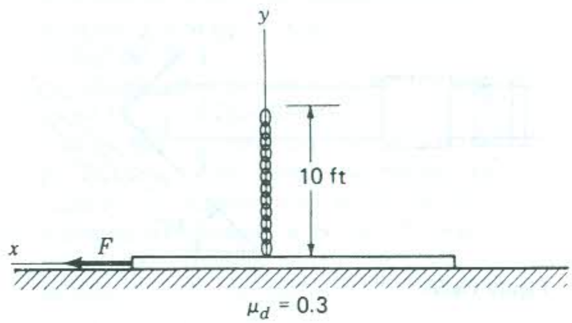


Figure P4.66

4.67 Water is flowing through a pipe having an inside diameter of 6 in. Find the total moment on the pipe at the base A from water, air, and

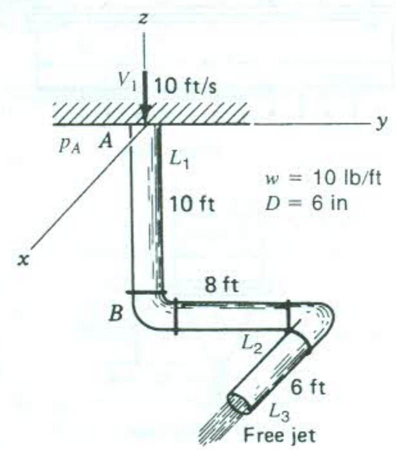


Figure P4.67

the weight of the pipe. The pipe weighs 10 lb/ft. The pressure at *A* is 10 lb/in^2 gage. The flow is steady. Use a control volume as was used in Example 4.15.

4.68 Do Prob. 4.67 by using first a control volume covering the interior volume of the pipe and then, to take care of the weight of the pipe, a free-body diagram of the pipe.

4.69 Compute the bending moment from the water at point *E* of the pipe system containing water using the method of moment of momentum. The flow is steady.

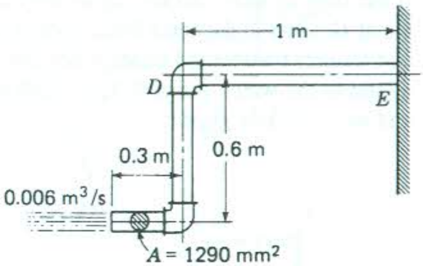


Figure P4.69

4.70 For a steady flow of water, compute the bending moment from the water at section *A* of the pipe by the method of moment of momentum.

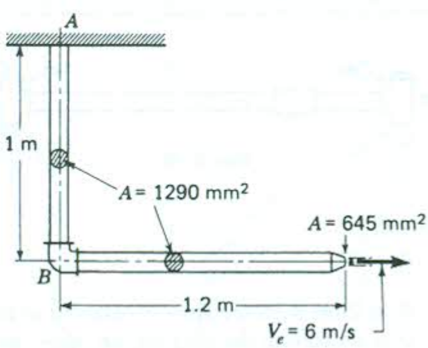


Figure P4.70

4.71 Water is flowing steadily through the 200-mm pipe (Fig. P4.71).

a. Find the moment components on the base of the pipe at *A* from water, air, and pipe weight.

b. Find the force components at the base of the pipe at *A* from water, air, and pipe weight.

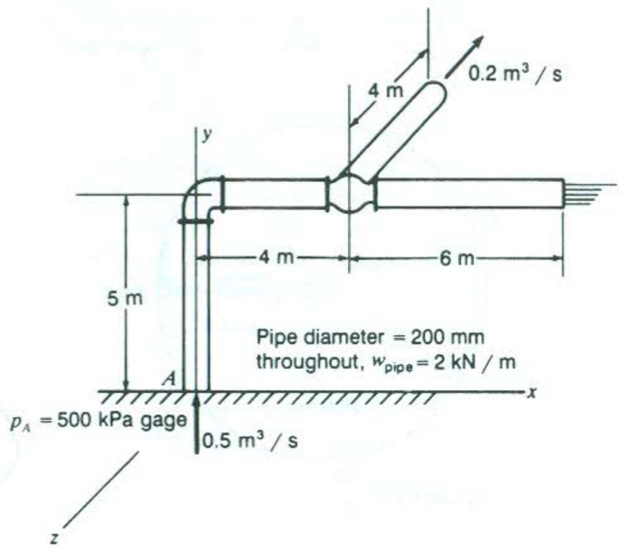


Figure P4.71

4.72 Water is steadily flowing through a pipe at the rate *Q* of 0.2 m^3/s. The pipe weighs 0.3 kN/m. The pressure at *A* is 10 kPa gage.

a. What are the total bending moments and twisting moments in the pipe at the base *A* of the pipe system?

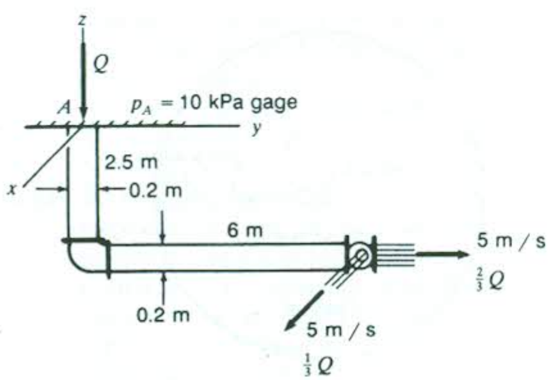


Figure P4.72

b. What are the shear forces and axial force in the pipe at A due *only* to the *water* and *air*. The diameter of the pipe is 0.2 m.

4.73 A platform is shown which can rotate about axis MM. A jet of water is directed out from the center of the platform while it is stationary and strikes a vane at the periphery of the platform. The vane turns the jet 90° as shown. What is the torque developed about MM?

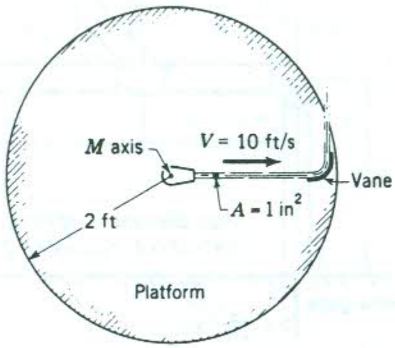


Figure P4.73

4.74 A jet of water (Fig. P4.74) 645 mm² in cross section is directed out at a speed of 3 m/s at a 90° vane positioned 0.6 m from the center of the platform from which the jet issues as shown. It then strikes a 90° vane to attain a direction of motion parallel to its original direction. If the platform is stationary, what is the torque about MM as a result of this action?

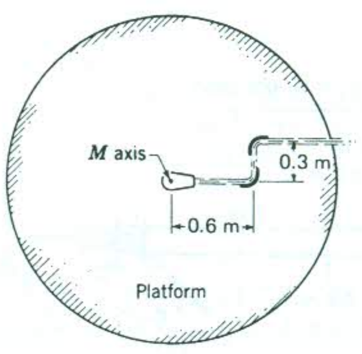


Figure P4.74

4.75 In Example 4.16 the following data apply: $q = 5$ L/s, $\alpha = 20°$, $l = 300$ mm, $A_e = 600$ mm². Find the angular speed ω of the arm for zero frictional torque.

4.76 In Prob. 4.75 find ω for steady rotation if there is a resisting torque T_f due to bearing friction and windage given as $0.08\omega^2$ newton-meters, with ω in radians per second.

4.77 Consider Prob. 4.4. What is the steady-state rotational speed if there is a constant resisting torque of 30 N · m?

4.78 If the flow of water divides up equally at the tee at D, what is the total force system (force and torque) transmitted through section C owing to the water and air? $A_e = 1290$ mm² and $p_C = 70$ kPa gage.

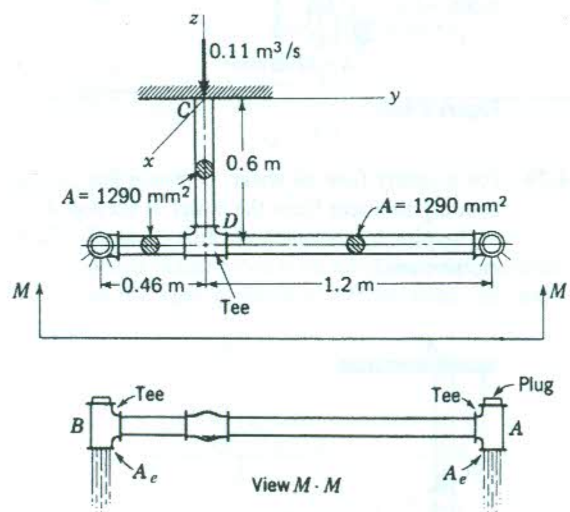


Figure P4.78

4.79 If in Prob. 4.78 the flow is reversed at the tee at B by putting the plug on the other side, what is the total force system (force and torque) transmitted through section C owing to the water?

4.80 A rotor is held stationary while 5 L/s of water enters at C and flows out through three channels, each of which has a cross-sectional area of 1800 mm^2. What is the angular speed ω of the rotor 2 s after its release? Let us assume that there is no frictional resistance to rotation about a vertical axis z coming out of the paper to you at C. The moment of inertia about z, I_{zz}, for the rotor plus water is 10 kg · m^2 (that is, $\iiint \bar{r}^2 \rho \, dv = 10$ kg · m^2). Use a stationary control volume.

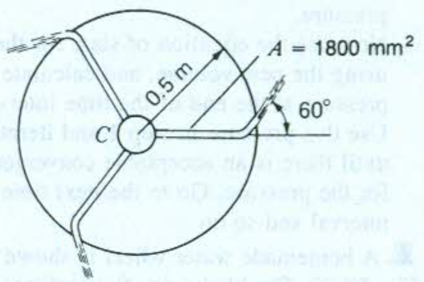


Figure P4.80

4.81 A rotor in Fig. P4.81 with four channels is held stationary and, with exits at B blocked, is filled with water through inlet at C. Now at $t = 0$, the outlets are opened at B, the rotor is released, and a flow q is started at the inlet such that q varies as $q = 0.05t$ m^3/s, with t in seconds. What is the differential equation for ω of the rotor if there is no resistance to rotation about the axis of the rotor at C? The area of each of the channels is 1500 mm^2. Use a stationary control volume. The moment of inertia I_{zz} of rotor and water ($\iiint \bar{r}^2 \rho \, dv$) is 10 kg · m^2.

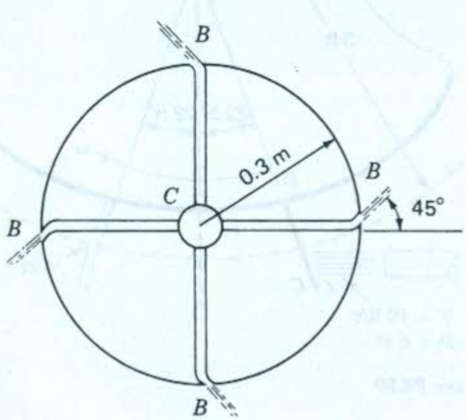


Figure P4.81

4.82 A horizontal disc A in Fig. P4.82 has a torque T applied about axis C that causes a constant angular acceleration $\dot{\omega} = \kappa$. A chain of length l and weight per unit length w lies on a frictionless horizontal surface. At time $t = 0$, $\omega = 0$, and at this instant the chain is connected to the disc at D. What is the torque T needed? The moment of inertia about the axis C of the disc is I. Use a stationary control volume that includes the entire chain and disc. *Hint:* For the disc, $\iiint (rV_\theta)\rho \, dv = \iiint r(r\omega) \, dm = \omega \iiint r^2 \, dm = I\omega$. Is the torque T that you have computed valid after H touches the disc?

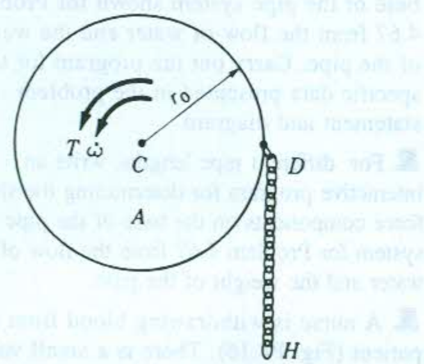


Figure P4.82

4.83 Do Prob. 4.82 for the case where the chain that has not come into contact with the disc rests on a horizontal surface having with the chain a dynamic coefficient of friction of μ_d. Compute the torque for the time interval before H comes into contact with the disc. The solution to the preceding problem is $T_s = r_0^2 \kappa(w/gl) + I\kappa$.

4.84 Do Computer Example 4.1, this time with the system in a vertical plane. Be sure to include the weight of 100 N of water in your control volume. Neglect the effect of friction and gravity on the speed of the water. Plot vertical and horizontal force components. Use applicable programming of Computer Example 4.1.

4.85 Write an interactive computer program for steady flow through a double-exit elbow (see Fig. 4.9) for determining the exit velocity from a free jet having a diameter D_3 and flowing in the xy plane at an angle α_3 with the horizontal x axis. The entering velocity through entrance 1 is V_1 at a pressure of p_1 psi gage and a diameter D_1 and is oriented vertically in the minus y direction. The second exit is a free jet with velocity V_2 in the negative x direction with a diameter D_2. The inside volume is V. Determine the vertical and horizontal force components on this elbow. Run your program for the data specified in Example 4.6.

4.86 Write an interactive computer program to determine the bending moment about the base of the pipe system shown for Problem 4.67 from the flow of water and the weight of the pipe. Carry out the program for the specific data presented in the problem statement and diagram.

4.87 For different pipe lengths, write an interactive program for determining the shear force components on the base of the pipe system for Problem 4.67 from the flow of water and the weight of the pipe.

4.88 A nurse is withdrawing blood from a patient (Fig. P4.16). There is a small volume of air ahead of the piston of 0.002 in^3, having an absolute pressure of 10 psi just as the piston starts to be withdrawn. The piston allows air to come through on its periphery as the piston moves so as to increase the volume of air just ahead of the blood. This volume is governed by the perfect gas equation of state for adiabatic expansion, namely pV^k = const. The amount of air bypassing the piston depends approximately on the difference of pressure between atmospheric pressure of 14.7 psi abs and the initial pressure in the volume of air ahead of the piston face. The following formula is presumed to be valid for this flow rate.

$$V = 0.001(14.7 - p_{av})^{0.2} \text{ ft}^3/\text{s}$$

Under these conditions, estimate the volume of blood extracted for the piston moving a distance of 3 in.

Suggested procedure.

1. Choose a pressure a little smaller than the initial pressure for the initial volume of entrained air.
2. Compute the amount of air coming through during a small time interval. Calculate the volume of the air at the end of this first time interval at the chosen pressure.
3. Now use the equation of state for the air, using the new volume, and calculate a pressure at the end of the time interval. Use this pressure in step 1 and iterate until there is an acceptable convergence for the pressure. Go to the next time interval and so on.

4.89 A homemade water wheel is shown in Fig. P4.89. The blades are flat surfaces. One such blade is shown rotating over a range of 45°. The constant angular speed ω of the wheel is 5 rad/min. What is the average torque developed by the jet of water having a speed of 10 ft/s on this blade over a

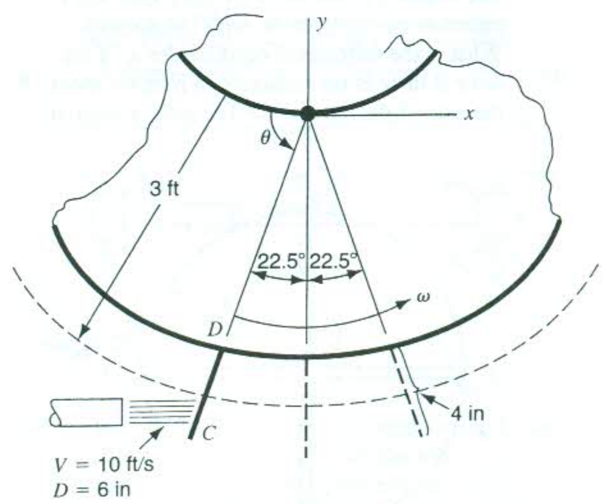
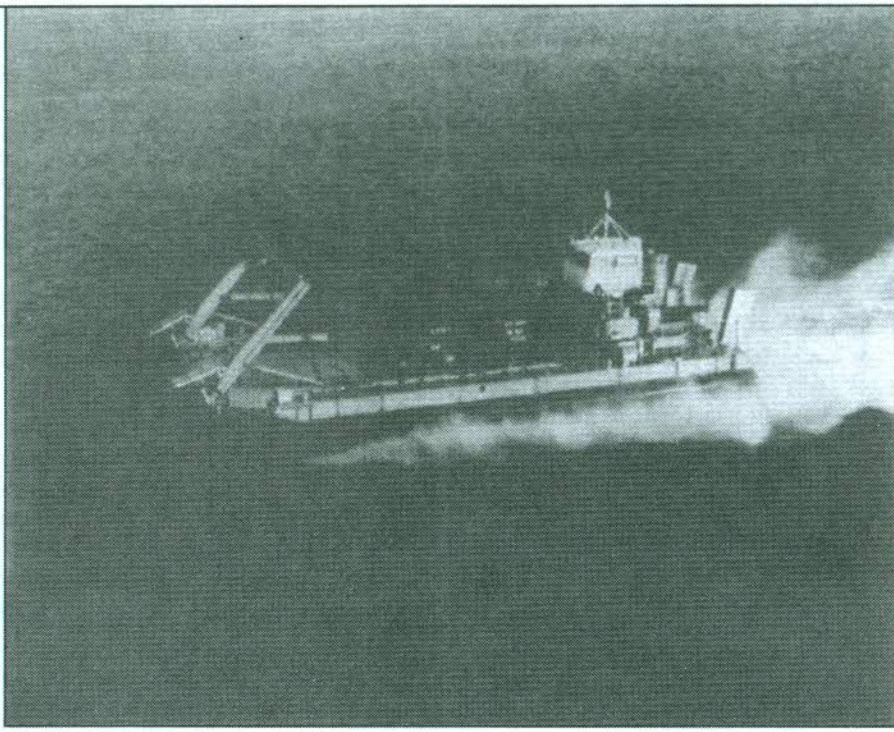


Figure P4.89

rotation of the wheel of 45°? The jet has a diameter of 6 in. Do a quasi-steady analysis with the blade at 3° intervals, wherein the blade is assumed to be *translating* at the appropriate constant speed for the position considered at the *beginning* of each 3° interval. Neglect the effects of friction and gravity. Thus, you will have a torque for each 3° interval which you will finally average over the entire 45° interval. Comment on the accuracy of your computation compared to the same operation over the same interval of a real wheel having the same construction and operating conditions. Use the linear momentum equation.



Air cushion vehicle. (*Courtesy Bell Aerospace Textron.*)

Bell Aerospace Textron's LACV-30 (Lighter, Amphibian Air Cushion Vehicle - 30 ton payload) is the result of the first production contract for an air cushion vehicle awarded to an American company.

The LACV-30 is a high-speed amphibious cargo carrier that can run at speeds up to 56 miles per hour carrying 60 tons of gross weight over water, land, snow, ice, even marshes and swamps. It can haul a wide variety of wheeled and tracked equipment and containerized cargo, and it needs no dock or berth facilities. The bow has a crane for unloading.

CHAPTER 5

Basic Laws for Finite Systems and Finite Control Volumes II: Thermodynamics

5.1 INTRODUCTION

We have thus far considered conservation of mass and Newton's law, the latter in the forms of linear momentum and moment of momentum, for both finite systems and finite control volumes. We started with the familiar system approach and then using the Reynolds' transport equation quickly went to the corresponding control volume formulations. In this chapter we shall do the same for the first law of thermodynamics. We shall defer, however, detailed discussion of the second law of thermodynamics until Chap. 10 where its use will be more directly needed.

5.2 PRELIMINARY NOTE

The first law of thermodynamics is a statement of macroscopic experience which stipulates that energy must at all times be conserved. Hence, the first law accounts for energy entering, leaving, and accumulating in either a system or a control volume.

It will be convenient to classify energy under two main categories: *stored energy* and *energy in transition.* Energy primarily associated with a given mass will be considered stored energy. On the other hand, energy going from one system to another is called energy in transition. We will consider only stored energy as extensive from the viewpoint that this energy is directly identified with and "resides in" the matter involved in a discussion. One may list the following types of stored energy of an element of mass:

1. *Kinetic energy* E_K: energy associated with the motion of the mass.
2. *Potential energy* E_P: energy associated with the position of the mass in conservative external fields.

3. *Internal energy U:* molecular and atomic energy associated with the internal fields of the mass.[1]

Two types of energy in transition are listed: heat and work. *Heat* is the energy in transition from one mass to another as a result of a temperature difference. On the other hand, *work*, as we learned in mechanics, is the energy in transition to or from a system which occurs when external forces, acting on the system, move through a distance. In thermodynamics, we further generalize the concept of work to include energy transferred from or to a system by any action such that the total external effect of the given action can be reduced by hypothetical frictionless mechanisms entirely to that of raising a mass in the gravitational field.[2] Thus electric current can be arranged to lift a weight by using an electric motor, and if there is no friction or electric resistance, this can be the only effect of the current. Hence, it represents a flow of energy which we classify as work. Heat, however, even with frictionless machines, cannot raise a weight and have no other effect. There must be rejected heat to a sink.

Considering stored energy again, we note that since the kinetic energy of an infinitesimal particle is equal to $\frac{1}{2}dm\,V^2$, the change in kinetic energy during a process is obviously dependent only on the final and initial *velocity* of the infinitesimal system for the process. Change in the potential energy is defined only for *conservative* force fields and is by definition equal to minus the work done *by* these conservative force fields on the infinitesimal system during a process. As you will recall from your studies in mechanics and electrostatics, this work depends only on the final and initial *positions* of the infinitesimal system for the process. Finally, the internal atomic and molecular energy of a fluid stems from force fields which are approximately conservative. Hence, it should be noted that *stored energy* is a *point function;* i.e., all changes during a process are expressible in terms of values at the end points. On the other hand, *energy in transition* is a *path function;* i.e., changes are dependent not only on the end points but also on the actual path between the end points.

5.3 SYSTEM ANALYSIS

An arbitrary system, (shown in Fig. 5.1) by definition, may move and deform without restriction but may not transfer mass across its boundary. The net heat *added* to the system and the net work *done* by the system on the surroundings during the time interval Δt are designated as Q and W_K, respectively.

If E is used to represent the total stored energy of a system at any time t and its property as a point function is employed, conservation of energy demands that for a process occurring during interval t_1 to t_2,

$$Q - W_K = \Delta E = E_2 - E_1 = (E_K + E_P + U)_2 - (E_K + E_P + U)_1 \quad [5.1]$$

[1]By considering stored energy as the sum of these quantities, we impose the restrictions of classical mechanics on the resulting equations. This means that the control volumes later used for the first law of thermodynamics will of necessity be inertial control volumes.

[2]The reader is referred to current textbooks on thermodynamics for a more detailed discussion of work and heat.

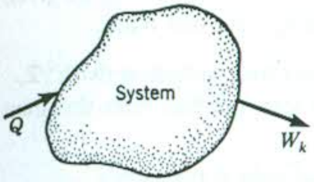


Figure 5.1
Heat and work on system.

The differential form of Eq. 5.1 may be written in the following manner:[3]

$$dE = dQ - dW_K \tag{5.2}$$

We have now listed the usual form of the first law of thermodynamics applied to systems. Since Q and W_K are not point functions, they are then representable as explicit functions of time. Accordingly, we can employ the usual derivative notation dQ/dt and dW_K/dt for time derivatives. However, E is a point function and expressible in terms of spatial variables and time. To indicate that we are following the system, we use the substantial derivative. Thus we have for the time variations of stored energy and energy in transition for a system

$$\boxed{\frac{DE}{Dt} = \frac{dQ}{dt} - \frac{dW_K}{dt}} \tag{5.3}$$

5.4 CONTROL-VOLUME ANALYSIS

To develop the control-volume approach, we will consider E to be the extensive property to be used in the Reynolds' transport equation. The term e will then represent stored energy per unit mass. We can then say using the Reynolds' transport equation

$$\frac{DE}{Dt} = \underset{CS}{\oint\!\!\oint} (e)(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \underset{CV}{\iiint} (e)(\rho\, dv) \tag{5.4}$$

Using Eq. 5.3 in the left side of Eq. 5.4, we get

$$\boxed{\frac{dQ}{dt} - \frac{dW_K}{dt} = \underset{CS}{\oint\!\!\oint} (e)(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \underset{CV}{\iiint} (e)(\rho\, dv)} \tag{5.5}$$

Equation 5.5 then states that the net rate of energy transferred into the control volume by heat and work[4] equals the rate of efflux of stored energy from the control volume plus the rate of increase of stored energy inside the control volume.

---

[3]In thermodynamics texts the differentials of Q and W are frequently denoted as δQ and δW_K or as $\text{đ}Q$ and $\text{đ}W_K$, respectively, to remind the reader that since both Q and W depend on the path these are not perfect differentials.

[4]In the present development, dW_K/dt does *not* include work done by gravitational body forces, since this effect is contained in e as potential energy.

According to the discussion of Sec. 5.2, it is seen that the term e may be given as the sum of the following specific types of stored energy per unit mass:

1. *Kinetic energy* e_K. The kinetic energy of an infinitesimal particle is $dm\,V^2/2$, where dm is in units of slugs or kilograms in this text. Per unit mass this then becomes $V^2/2$.
2. *Potential energy* e_P. Assuming that the only external field is the earth's gravitational field, we have for the potential energy of an infinitesimal particle at an elevation z above some datum, the quantity $\int_0^z dm\, g\, dz$. Considering g as constant, we then have as the potential energy per unit mass the quantity gz.
3. *Internal energy* u. If certain properties of a fluid are known, the internal energy per unit mass relative to some datum state may usually be evaluated or found in experimentally contrived tables.

Hence we give e as

$$e = \frac{V^2}{2} + gz + u \tag{5.6}$$

Next let us discuss the term dW_K/dt in Eq. 5.5. It will be convenient to form three classifications of W_K. They are:

1. Net work done *on the surroundings* as a result of *tractions* on that part of the control surface across which there is a *flow of fluid.* We call this work *flow work.*
2. Any other work transferred at the rest of the control surface *to the surroundings* by direct contact between inside and outside nonfluid elements. For example, the work transferred through a control surface by shafts or by electric currents would fall into this category. We call this work *shaft work* and denote it as W_S.
3. Classifications 1 and 2 take care of the total work transferred at the control surface by direct contact. Inside the control surface, we may have work on the surroundings resulting from the reactions to *body forces.* Note carefully, in this regard, that the effects of gravity have already been taken into account as the potential energy (in e), so the body force **B** must *not* include gravity; it may include, for instance, contributions from magnetic and electric force distributions.

Let us examine flow work carefully (see Fig. 5.2). First, note that **T** by definition is the traction force from the *surroundings* acting *on* the control surface. Hence, $\mathbf{T} \cdot \mathbf{V}$ is the time rate of work (power) done *by* the surroundings at the control surface per unit area of the control surface. It thus represents power per unit area *entering* the control volume. Hence, the time rate of work *leaving* the control volume—the total rate of *flow work*—is given as

$$\text{Total rate of flow work} = -\oiint_{CS} \mathbf{T} \cdot \mathbf{V}\, dA \tag{5.7}$$

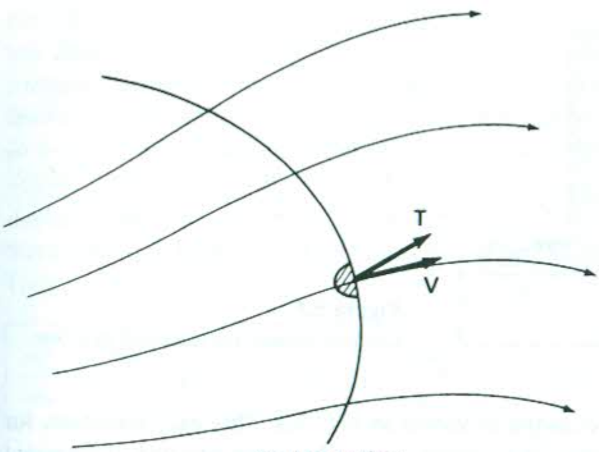


Figure 5.2
Flow work at control surface.

Similarly, the body force **B** represents a force distribution per unit mass on the material *in* the control volume *from* the surroundings not requiring direct contact. Hence $-\mathbf{B} \cdot \mathbf{V}$ is the power *leaving* the control volume per unit mass of material inside the control volume. We can thus give the *total rate of body force work leaving* the control volume as

$$\text{Total rate of body force work} = -\iiint_{CV} \mathbf{B} \cdot \mathbf{V} \rho \, dv \qquad [5.8]$$

A general form of the first law can now be given as

$$\boxed{\begin{aligned} &\frac{dQ}{dt} - \frac{dW_S}{dt} + \oiint_{CS} \mathbf{T} \cdot \mathbf{V} \, dA + \iiint_{CV} \mathbf{B} \cdot \mathbf{V} \rho \, dv \\ &= \oiint_{CS} \left(\frac{V^2}{2} + gz + u \right)(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \iiint_{CV} \left(\frac{V^2}{2} + gz + u \right)(\rho \, dv) \end{aligned}} \qquad [5.9]$$

A very important simplification commonly encountered is the case of[5] a steady flow where inlet and outlet flows to and from a device, respectively, are considered

[5]For the interested reader, we have presented in Appendix I a somewhat different form of the integral form of the first law of thermodynamics that is derived from the general formulation given by Eq. 5.9. The sole difference will be that we will *not* restrict the stationary control volume to be that which entails ingoing and exiting pipes for entering and exiting fluids (a step which admittably makes the resulting equation very useful for most problems). The control volume instead will be of a *general* fixed shape. The interested reader will then attain a better understanding of the concept of *flow work*. This gain in understanding could conceivably be useful in certain problems. Finally, from the equation reached in the appendix, it is a simple matter to move directly to the more practical equations that we have presented in this section.

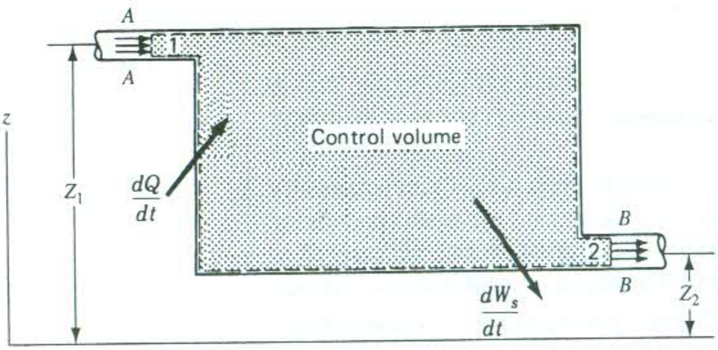


Figure 5.3
Control volume for idealized machine.

one dimensional. Such an example is shown in Fig. 5.3. This may represent, for instance, a steam turbine where the control volume has been selected to represent the inside volume of the turbine casing, and sections AA and BB of the control surface have been established in the inlet and outlet pipes of the turbine.

Since all properties are taken constant over cross sections AA and BB, and since the inlet and outlet velocities are normal to the control surface, one may carry out the surface integrations of Eq. 5.9 with ease. Furthermore, because this is steady flow, the stored energy inside the control volume remains constant with time and the last term on the right side of Eq. 5.9 is zero. Proceeding then, the traction force power occurs at sections AA and BB and is given as $+p_1A_1V_1$ and $-p_2A_2V_2$ respectively. Furthermore, $\rho\mathbf{V}\cdot\mathbf{dA}$ at these sections become $-\rho_1V_1A_1$ and $+\rho_2A_2V_2$, respectively. The equation with these changes then becomes, when we carry out the integrations,[6]

$$\frac{dQ}{dt}-\frac{dW_S}{dt}+p_1A_1V_1-p_2A_2V_2=-\left(\frac{V_1^2}{2}+gz_1+u_1\right)\rho_1V_1A_1 \\ +\left(\frac{V_2^2}{2}+gz_2+u_2\right)\rho_2V_2A_2$$

We now introduce the product $\rho_1 v_1$ and $\rho_2 v_2$, both of which equal unity, into the expressions $+p_1A_1V_1$ and $-p_2A_2V_2$, respectively. We get from these expressions $+(p_1v_1)(\rho_1V_1A_1)$ and $-(p_2v_2)(\rho_2V_2A_2)$. Now rearranging the above equation, we get a form of the first law that is much used in fluid mechanics and thermodynamics. Thus

$$\boxed{\begin{aligned}&\frac{dQ}{dt}+\left(\frac{V_1^2}{2}+gz_1+u_1+p_1v_1\right)(\rho_1V_1A_1)\\&\quad=\frac{dW_S}{dt}\left(\frac{V_2^2}{2}+gz_2+u_2+p_2v_2\right)(\rho_2V_2A_2)\end{aligned}} \qquad [5.10]$$

[6]Note that when we use v in the differential dv, it simply represents volume, but when v is isolated as in Eq. 5.11, it is meant to represent the specific volume. If this is understood, there should be no confusion about the meaning of v in our discussions.

Note that with positive Q and W_S, the left side of Eq. 5.10 represents energies per unit time entering the control volume of Fig. 5.3, while the right side represents energies leaving this control volume per unit time. Noting also that $\rho_1 V_1 A_1 = \rho_2 V_2 A_2$ from continuity, we shall next replace both expressions by dm/dt, the rate of mass flow. Now we divide by this expression in the above equation and note that we can say $\dfrac{dQ/dt}{dm/dt} = \dfrac{dQ}{dm}$ and that $\dfrac{dW_S/dt}{dm/dt} = \dfrac{dW_S}{dm}$. We then may form the first law equation for this simple case based on per unit mass flow rather than based on per unit time flow. Thus we have

$$\boxed{\frac{dQ}{dm} + \left(\frac{V_1^2}{2} + gz_1 + u_1 + p_1 v_1\right) = \frac{dW_S}{dm} + \left(\frac{V_2^2}{2} + gz_2 + u_2 + p_2 v_2\right)} \qquad [5.11]$$

We note next that the *enthalpy, h*, is defined as $(u + pv)$. It is much used in thermodynamics and often appears in the preceding formulations of the first law. A case that bears mentioning is the first law involving enthalpy h expressed in British thermal units (Btu) per *pound mass* flow while mass flow $\dot{m}$ is given as pounds mass per second. For such a situation we would write the first law with Q in Btu as

$$778\frac{dQ}{dt} + \left(\frac{V_1^2}{2g} + z_1 + 778h_1\right)\dot{m} = \left(\frac{V_2^2}{2g} + z_2 + 778h_2\right)\dot{m} + \frac{dW_S}{dt} \qquad [5.12]$$

Note we have divided $V^2/2$ and gz by g. We explain this step in the following way. The terms $V^2/2$ and gz come from Newton's law and so are based on the slug. Thus they are energy per slug of flow. But $\dot{m}$ is based on pounds mass. Hence on dividing by g, the kinetic energy and potential energy terms then are given per unit pound mass of flow as is h in this case. We then get an equation involving energy per unit time. Please be alert in the homework problems for the nomenclature and units described in the preceding paragraphs.

5.5 PROBLEMS INVOLVING THE FIRST LAW OF THERMODYNAMICS

To illustrate the use of some of the preceding equations, several examples will now be given. In the computations, it will be helpful if we remember that $v = 1/\rho$. For water at standard conditions, $\rho = 62.4/\mathrm{g} = 1.938\ \text{slugs/ft}^3 \equiv 1000\ \text{kg/m}^3$.

We now present some information that will be useful in working problems involving pumps and turbines for which the fluid is a liquid and hence, for calculations here, incompressible. For such cases the term *head* is often used in conjunction with the performance of the pump or turbine. Using the notation H_D, head is defined as

$$H_D = \left(\frac{V^2}{2g} + z + \frac{p}{\gamma}\right) \quad \text{(dimension is length)} \qquad [5.13]$$

and may be considered to be the "mechanical energy" per unit *weight* of flow. The *change* in head ΔH_D for a pump or turbine is then

$$\Delta H_D = \Delta\left(\frac{V^2}{2g}\right) + \Delta(z) + \Delta\left(\frac{p}{\gamma}\right) \qquad [5.14]$$

where Δ indicates change between outlet and inlet conditions of the device.

How does ΔH_D relate to dW_S/dm? Since dW_S/dm is energy per unit mass we will now consider $g\ \Delta H_D$ which now also has these dimensions. For a *pump*, $g\ \Delta H_D$ is a *positive* number because of the increase of mechanical energy of the fluid, while clearly for a *turbine*, $g\ \Delta H_D$ will be *negative*. But dW_S/dm for a pump by definition is negative (energy going into the control volume), while for a turbine it will be positive (energy leaving the control volume). Thus we see that

$$\frac{dW_S}{dm} = -g\ \Delta H_D \qquad [5.15]$$

Hence Eq. 5.11 can be written for constant value of u as

$$\frac{dQ}{dm} + \left(\frac{V_1^2}{2} + gz_1 + \frac{p_1}{\rho}\right) = \left(\frac{V_2^2}{2} + gz_2 + \frac{p_2}{\rho}\right) - g\ \Delta H_D \qquad [5.16]$$

with the proper sign applied to ΔH_D (positive for pumps and negative for turbines).

We wish to point out that for cases involving liquid, we may use the gage pressure on both sides of the simplified first law equation since the portion p_{atm}/γ of the absolute pressures will cancel, leaving only the gage pressures.

Before you go to problems we point out that if your stated assumptions warrant, you are encouraged to go directly to the simplified forms of the first law (Eqs. 5.10 and 5.11) as we have here. This is the same procedure proposed for conservation of mass, and is in contrast to the linear and moment-of-momentum equations wherein, because of the complexity of these equations, we urged you to start with the general equations first and then carefully to fashion your working equations. We note furthermore that very often the expression pv is replaced by the expression p/ρ. This will depend on the problem or the discussion. Also, we shall often formally refer to either Eqs. 5.10 or 5.11 in this text as the *simplified first law of thermodynamics* to differentiate them from the earlier, more general forms. We shall use these simpler, more restricted forms of the first law most of the time.

EXAMPLE 5.1

■ Problem Statement

Shown in Fig. 5.4 is a large-diameter pipe through which water flows. The inlet and outlet conditions are specified in the diagram. What must be the power absorbed by the turbine if we neglect friction?

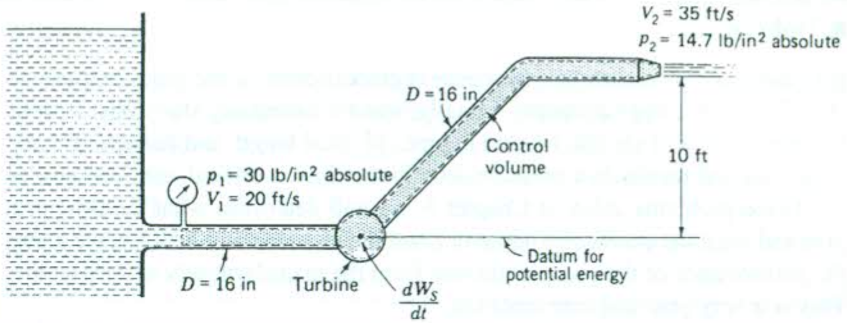


Figure 5.4
Water flows through pipe system.

■ Strategy

We will choose a control which will include the turbine and the entrance and exit of the pipe where known information is present. We will make the following assumptions, so we can use the simplified form of the first law. Thus,

1. Steady flow.
2. Incompressible flow.
3. 1-D flow in and out of the pipe.
4. Neglect friction (short large-diameter pipes).
5. Take $dQ/dt = 0$ as a result of assumption 4.
6. Consider u to be constant because of assumptions 2 and 4.

■ Execution

We choose the interior of the pipe starting at the pressure gage as the control volume, including the interior of the turbine. Noting assumptions 5 and 6, we can write the *simplified first law,*

$$\frac{V_1^2}{2} + g(z_c)_1 + p_1 v_1 = \frac{V_2^2}{2} + g(z_c)_2 + p_2 v_2 + \frac{dW_S}{dm} \qquad \text{[a]}$$

Substituting numerical values, we get

$$\frac{20^2}{2} + 0 + \frac{(30)(144)}{1.938} = \frac{35^2}{2} + 10g + \frac{(14.7)(144)}{1.938} + \frac{dW_S}{dm}$$

Solving for the work per slug of flow gives us $dW_S/dm = 402$ ft·lb/slug. To find the power, we multiply dW_S/dm by dm/dt. Thus noting that $dm/dt = \rho VA$, we have

$$\text{Power} = 402\left[1.938\frac{\pi(16^2/4)}{144}20\right] = 21{,}773 \text{ ft·lb/s}$$

Dividing by 550 produces the desired result—39.6 hp.

■ Debriefing

It is important to realize that we could neglect friction in the pipes because of short length and large diameter. Actually, when we carefully study pipe flow in Chapter 8, we will see that friction in pipes of usual length and diameter is very important and results in a pressure drop in the flow, a critical consideration in most pipe problems. Also, in Chapter 8, we will determine what the flow in a pipe and what the power of a pump or a turbine in a pipe will be when we know the performance of the pump or turbine from the manufacturer's specifications. This is a very practical consideration.

EXAMPLE 5.2

■ Problem Statement

A steam turbine (Fig. 5.5) uses 4500 kg/h of steam while delivering 770 kW of power at the turbine shaft. The inlet and outlet velocities of the steam are 65 m/s and 290 m/s, respectively. Measurements indicate that the inlet and outlet enthalpies of the steam are 2800 kJ/kg and 2000 kJ/kg, respectively. Calculate the rate at which heat is lost from the turbine casing and bearings.

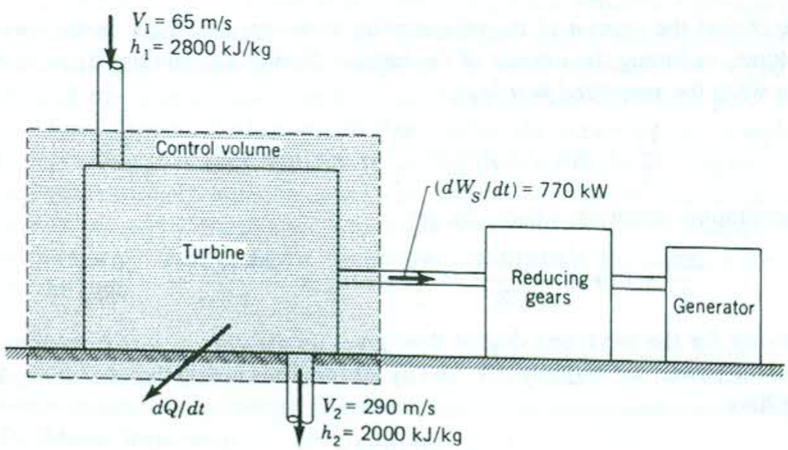


Figure 5.5
Power plant.

■ Strategy

We will need a control volume that completely surrounds the turbine casing and bearings. We will make the following assumptions so that we can use the *simplified first law*.

1. Steady flow.
2. Neglect the change in potential energy between the inlet and exit.
3. 1-D flow in and out.

■ Execution

We show the control volume in Fig. 5.5. The simplified first law is expressed next.

$$\left[\frac{V_1^2}{2} + gz_1 + u_1 + \frac{p_1}{\rho_1}\right]\rho_1 V_1 A_1 + \frac{dQ}{dt}$$

$$= \left[\frac{\mathbf{V}_2^2}{2} + gz_2 + u_2 + \frac{p_2}{\rho_2}\right]\rho_2 V_2 A_2 + dW_K/dt.$$

Introducing $h = u + p/\rho$ and replacing ρVA by dm/dt, we get

$$\left(\frac{V_1^2}{2} + h_1\right)\left(\frac{dm}{dt}\right) + \frac{dQ}{dt} = \left(\frac{V_2^2}{2} + h_2\right)\left(\frac{dm}{dt}\right) + \frac{dW_S}{dt}$$

$$\therefore \left(\frac{65^2}{2} + 2800 \times 10^3\right)\frac{4500}{3600} + \frac{dQ}{dt}$$

$$= \left(\frac{290^2}{2} + 2000 \times 10^3\right)\frac{4500}{3600} + 770 \times 10^3$$

Solving for dQ/dt, we get -180.1 kJ/s, the negative sign indicating that the heat is leaving the control volume.

■ Debriefing

It will be left for the student to demonstrate that the inclusion of the potential energies would not appreciably affect the results for changes in height of 3 m between the inlet and the outlet pipe sections of the control volume.

EXAMPLE 5.3

■ Problem Statement

Air at a pressure of 500 kPa absolute and at a temperature of 35°C enters a highly insulated air motor and leaves as a free jet into the atmosphere at a temperature of -5°C. The average inlet velocity is 25 m/s and the average exit velocity is 70 m/s. Three kg of air per minute enter. If we take the internal energy, u, as $c_v T$ where c_v is a constant specific heat at constant volume, what must be the power being applied to the motor? Take the value of c_v to be 4.08×10^{-5} N · m/kg · K.

■ Strategy

We will choose a control volume encompassing the inside of the motor and the pipe. We will use the *simplified first law*.

$$\left(\frac{V_1^2}{2} + gz_1 + u_1 + \frac{p_1}{\rho_1}\right) + \frac{dQ}{dm} = \left(\frac{V_2^2}{2} + gz_2 + u_2 + \frac{p_2}{\rho_2}\right) + \frac{dW_S}{dm}$$

We will make the following assumptions in working with this equation:

1. 1-D flow into the motor and into the atmosphere.
2. Treat air as a perfect gas with $R = 278\ \mathrm{N \cdot m/kg \cdot K}$.
3. With an insulated motor and short pipes, neglect heat transfer.

We will need densities, so we will use the equation of state for a perfect gas to solve the problem.

■ Execution

With the preceding assumptions, we insert numerical values into the first law. Thus, using $u = c_v T$, we get

$$\frac{25^2}{2} + 0 + (4.08 \times 10^{-5})(35 + 273) + \frac{500 \times 10^3}{\rho_1}$$
$$= \frac{70^2}{2} + 0 + (4.08 \times 10^{-5})(-5 + 273) + \frac{101{,}325}{\rho_2} + \frac{dW_S}{dm}$$

Evaluating terms, we get

$$\frac{dW_S}{dm} + \frac{101{,}325}{\rho_2} - \frac{500 \times 10^3}{\rho_1} + 2.14 \times 10^3 = 0 \qquad \text{[a]}$$

We now determine the entering and exiting densities using the *equation of state* for a perfect gas.

$$p_1 = \rho_1 R T_1 \qquad \therefore\ 500{,}000 = (\rho_1)(287)(273 + 35)$$
$$\rho_1 = 5.66\ \mathrm{kg/m^3}$$

$$p_2 = \rho_2 R T_2 \qquad 101{,}325 = (\rho_2)(287)(273 - 5)$$
$$\rho_2 = 1.317\ \mathrm{kg/m^3}$$

Going back to Eq. a we get, using the above values for ρ.

$$\frac{dW_S}{dm} = -\frac{101{,}325}{1.317} + \frac{500 \times 10^3}{5.66} - 2.14 \times 10^3 = 9263\ \mathrm{W/kg}$$

Hence,

$$\frac{dW_S}{dt} = \frac{dW_S}{dm}\frac{dm}{dt} = (9263)\left(\frac{3}{60}\right) = 463\ \mathrm{W} = 0.463\ \mathrm{kW}$$

■ Debriefing

Please note that we used absolute pressures in the first law since the density was not constant, thus not allowing us to cancel p_{atm}/ρ terms. And, in the equation of state, we must use absolute values of both pressure and temperature.

EXAMPLE 5.4

■ Problem Statement

A small water turbine is shown in a water tunnel test section (Fig. 5.6) absorbing 7.70 kW of power from the flow of water. What is the horizontal force on the tunnel test section from the flow of water inside and the atmospheric pressure outside?

■ Strategy

We will choose a control volume that includes the entire interior space of the water tunnel test section as shown in Fig. 5.6 by the dashed line. You will note that the control surface cuts the supports of the turbine at the wall of the water tunnel test section. The desired force is the reaction to the force from the walls of the test section onto the turbine as transmitted through the cut supports, plus the reaction to the force from the walls onto the water at the wetted area. We shall start with the *linear momentum equation* for this control volume to involve the reaction to the desired force. For simplifying the momentum equation, we will make the following assumptions:

1. Steady flow.
2. Incompressible flow.
3. 1-D flow in and out of the test section.

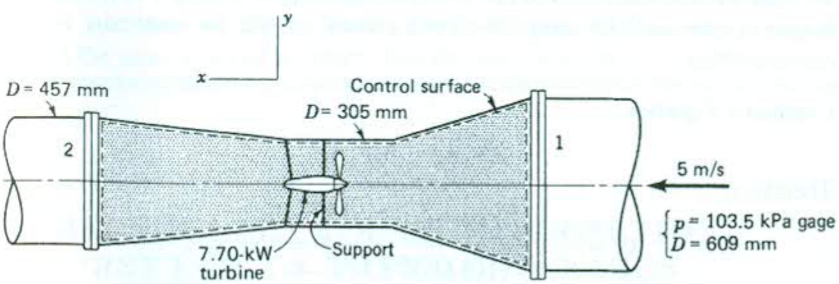


Figure 5.6
Water turbine.

We will find that more information is needed for a solution. Hence, we will go to the *simplified continuity equation* using the above assumptions. Again, we will need additional information, and so we will go to the *simplified first law* for the chosen control volume. To simplify the equation further, we will make the following two additional assumptions over those already in place:

4. Internal energy is constant.
5. Negligible heat transfer.

We will see that we now have enough equations for a solution.

■ Execution

We start by writing the general form of the **linear momentum equation.**

$$\oint\!\!\oint_{CS} T_x\, dA + \iiint_{CV} B_x \rho\, dv = \oint\!\!\oint_{CS} V_x(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \iiint_{CV} V_x(\rho\, dv)$$

The body force integral vanishes for the horizontal direction, and the last expression is zero because of assumption 1. Observing assumptions 2 and 3 we then have as the working equation

Momentum Equation in Horizontal Direction:

$$p_1 A_1 - p_2 A_2 + R_x = \rho_2 V_2^2 A_2 - \rho_1 V_1^2 A_1$$

Putting in numerical values, we get, using gage pressures,

$$(103.5 \times 10^3)\frac{(\pi)(0.609)^2}{4} - (p_2)(\pi)\frac{0.457^2}{4} + R_x$$
$$= (1000)(V_2^2)\frac{(\pi)(0.457^2)}{4} - (1000)(5^2)\frac{(\pi)(0.609^2)}{4}$$

Carrying out the algebraic operations and collecting terms, we get

$$-0.1640 p_2 + R_x = 164.0 V_2^2 - 3.743 \times 10^4 \qquad \text{[a]}$$

We have here as additional unknowns the quantities V_2 and p_2. The quantity V_2 is easily determined by using the chosen control volume for continuity considerations. Thus

Continuity Equation:

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

Therefore,

$$V_2 = \frac{A_1}{A_2} V_1 = \frac{(\pi)(0.609^2)/4}{(\pi)(0.457^2)/4} 5 = 8.88 \text{ m/s} \qquad \text{[b]}$$

We can now go directly to the *simplified first law.* Canceling u_1 and u_2 and deleting dQ/dm we then have in accordance with our assumed flow conditions

First Law of Thermodynamics:

$$\frac{V_1^2}{2} + p_1 v_1 = \frac{dW_S}{dm} + \frac{V_2^2}{2} + p_2 v_2 \qquad [c]$$

Noting that $\dot{m} = \rho V_1 A_1 = (1000)(5)(\pi)(0.609^2)/4 = 1.456 \times 10^3$ kg/s, we have, using gage pressures,

$$\left(\frac{5^2}{2}\right) + (103.5 \times 10^3)\left(\frac{1}{1000}\right) = \frac{7.70 \times 10^3}{1.456 \times 10^3} + \left(\frac{8.88^2}{2}\right) + p_2\left(\frac{1}{1000}\right)$$

Solving for p_2, we get

$$p_2 = 71.28 \text{ kPa gage} \qquad [d]$$

Substituting the solved value of p_2 and V_2 into Eq. a and solving for R_x, we get

$$R_x = -12.81 \text{ kN}$$

Hence,

$$K_x = 12.81 \text{ kN}$$

Because we used gage pressures in the momentum equation K_x is the total horizontal force from water inside and air outside. We could use gage pressure in the first law equation because ρ is constant in this problem.

■ Debriefing

We remind you that by using gage pressures in the linear momentum for this problem, we are involving the internal forces from the water on the control surface of the test section, plus the forces from the atmospheric pressure on the outside of the test section. This will include all forces acting on the test section walls and thus is a good step to take. But you must remember that generally you must use *absolute pressures* in the first law equation. We were able to use gage pressures here in our first law because water, being incompressible (assumption 2), yields the term p_{atm}/ρ on both sides of the equation. Hence, this term cancels out, leaving in the equation only gage pressures. Also, we point out that for a device of this kind, the flow should be reasonably close to adiabatic, and since we are considering the flow incompressible, we can consider, with the same order of accuracy, that the internal energy is constant.

5.6 BERNOULLI'S EQUATION FROM THE FIRST LAW OF THERMODYNAMICS

Let us consider a portion of a streamtube in a *steady, incompressible, nonviscous* flow, as shown in Fig. 5.7, as our control volume. In applying the first law of thermodynamics for this control volume, we note that Eq. 5.11 is valid. There is

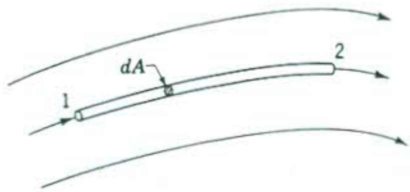


Figure 5.7
Streamtube in a steady, incompressible, nonviscous flow.

obviously no work other than flow work, and hence the term dW_S/dm is zero. We then obtain after some rearrangement

$$\left(\frac{V_1^2}{2} + p_1 v + gz_1\right) = \left(\frac{V_2^2}{2} + p_2 v + gz_2\right) + \left[(u_2 - u_1) - \frac{dQ}{dm}\right]$$

For frictionless flow involving mechanical energy only, i.e., no heat transfer or change in internal energy,[7] the last bracketed expression in the equation above vanishes, and we obtain

$$\boxed{\frac{V_1^2}{2} + p_1 v + gz_1 = \frac{V_2^2}{2} + p_2 v + gz_2} \qquad [5.17]$$

This equation is called *Bernoulli's equation*. Shrinking the streamtube cross-section without limit, Bernoulli then states that along a streamline the *mechanical energy* per unit mass is conserved. Or, along any one streamline:

$$\boxed{\frac{V^2}{2} + pv + gz = \text{const}} \qquad [5.18]$$

The constant may have a different value for each streamline. However, in many problems one can deduce that somewhere in the flow the streamlines have the same mechanical energy per unit mass, so the mechanical energy per unit mass is constant *everywhere* in the flow. We can present Bernoulli's equation in a different form by dividing Eqs. (5.17) and (5.19) by g and replacing v/g by $1/\gamma$. We then get

$$\boxed{\frac{V_1^2}{2g} + \frac{p_1}{\gamma} + z_1 = \frac{V_2^2}{2g} + \frac{p_2}{\gamma} + z_2} \qquad [5.19]$$

Or, one can say that

$$\boxed{\frac{V^2}{2g} + \frac{p}{\gamma} + z = \text{const}} \qquad [5.20]$$

---

[7]We wish to emphasize that the presence of friction in the streamtube will cause the temperature to rise with the concurrent effects on the internal energy u and heat transfer. When we study pipe flow with friction present the expression $\left[(u_2 - u_1) - \frac{dQ}{dm}\right]$ is called the *head loss* and its effect is to decrease the pressure in the pipe.

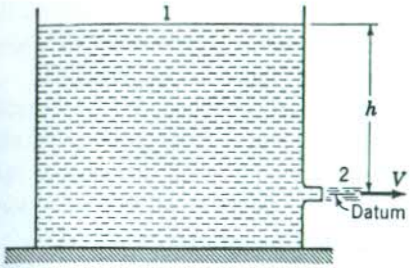


Figure 5.8
Efflux from a large tank through a well-rounded opening.

Note that the dimension for each expression is L (length). The terms are accordingly called *heads*. Recall that we introduced the head H_D in conjunction with pumps and turbines when the fluid was a liquid. We can say for Eq. (5.20) that the sum of the velocity head, the pressure head, and the elevation head is constant along a streamline.

We now illustrate the use of Bernoulli's equation. Note that we often will use gage pressures in Bernoulli's equation since the $p_{\text{atm}}v$ or p_{atm}/γ part of the absolute pressure terms will cancel out.

5.7 APPLICATIONS OF BERNOULLI'S EQUATION

We will now look at the use of Bernoulli's equation for two interesting applications.

Case 1. Orifices A well-rounded, small, simple nozzle acting as an orifice for the tank is shown in Fig. 5.8. A free jet of water is exiting from the tank. It will be of interest to determine the velocity of the water leaving the tank through this orifice.

This is not strictly steady flow, since the elevation of the water surface h is decreasing. However, since h changes slowly, no serious error will be incurred if it is assumed at time t that the corresponding height h is constant in computing the jet velocity. The flow may thus be considered *quasi-steady.* It can be assumed, furthermore, that density is constant and friction can be neglected. However, corrections may later be made to account for the latter. Under these conditions, and in light of the fact *that all streamlines have the same total energy per unit mass at the free surface,* we may use Bernoulli's equation at all positions in the flow.

By equating mechanical heads between point 1, at the free surface, and point 2, at the free jet, known quantities are related to the desired velocity. A position datum is established at the level of the jet. Hence, neglecting the kinetic energy at the free surface and taking the *pressure in the free jet to be atmospheric pressure,* we have[8]

$$h + \frac{p_{\text{atm}}}{\gamma} = \frac{V^2}{2g} + \frac{p_{\text{atm}}}{\gamma}$$

$$\therefore\ V = \sqrt{2gh}$$

[8]Note for the simple conditions we have assumed, we get the same velocity here as that of a freely falling particle in a gravity field when we neglect friction. Thus, thermodynamics and Newton's law give identical results here as was the case in the mechanics of particles and rigid bodies studied in your sophomore classes.

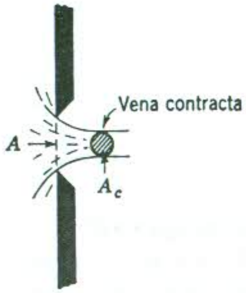


Figure 5.9
Sharp-edged opening.

For more accurate results one may account for friction by utilizing an experimentally determined coefficient called the *velocity coefficient* c_v to multiply $\sqrt{2gh}$. This coefficient depends on the size and shape of the opening as well as the elevation h of the free surface. The value of c_v usually is no smaller than 0.98 for well-rounded openings.

For other than a well-rounded opening there will be a contraction of the jet stream as it leaves the container. The smallest section of the jet is called the *vena contracta* (Fig. 5.9), and the area at this section is determined experimentally. The *coefficient of contraction* c_c, defined by the expression $A_c = c_c A$, is used for this purpose. This coefficient depends on the shape and size of the opening as well as the elevation of the free surface above the jet. Contraction coefficients usually run from 0.60, for a sharp-edged outlet, on up to 1, for the well-rounded outlet.

Therefore, to determine the rate of efflux of fluid, q, we have

$$q = c_v \sqrt{2gh}\, c_c A = c_d \sqrt{2gh}\, A$$

where $c_d = c_v c_c$ is called the *coefficient of discharge.* Tables and charts of the various coefficients can be found in hydraulic handbooks.

Case 2. The D'Alembert Paradox The cross section of an infinitely long cylinder is shown in Fig. 5.10 oriented at right angles to a steady flow which may be assumed to be uniform far from the cylinder. Also, this flow may be considered incompressible.

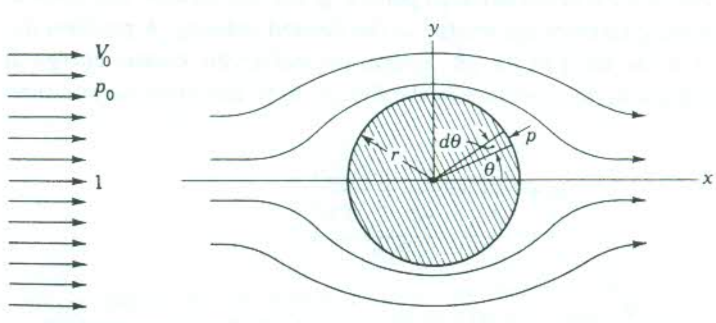


Figure 5.10
Flow around a cylinder.

If friction is to be neglected throughout the entire flow, what are the forces coming onto the cylinder as a result of the flow?

In aerodynamic work, engineers are concerned with two components of the force from the fluid flow. The component parallel to the free-stream velocity is called the *drag* component, while the component normal to the free stream is termed the *lift* component. Since this is a two-dimensional flow, evaluations will be made per unit length of the cylinder.

The free-stream velocity and pressure are given as V_0 and p_0, respectively. One may evaluate the fluid velocity at the boundary from two-dimensional incompressible nonviscous theory to be

$$V = 2V_0 \sin\theta \qquad \text{[a]}$$

In the absence of shear stresses the force on the strip $r\,d\theta$ equals $pr\,d\theta$ and is normal to the boundary of the cylinder. The component parallel to the free stream is then $-pr\,d\theta\cos\theta$. Integrating over the surface gives us the drag per unit length

$$D = -\int_0^{2\pi} pr\cos\theta\,d\theta \qquad \text{[b]}$$

We must now find the pressure at each point on the cylinder surface in terms of the known free-stream characteristics and the variable θ. This can be done by using *Bernoulli's equation* between the free-stream conditions far from the cylinder and the points on the cylinder. If the fluid is of small density, such as air, we may neglect the potential-energy terms in Bernoulli's equation in comparison with the other terms. In this case,

$$\frac{V_0^2}{2g} + \frac{p_0}{\gamma} = \frac{(2V_0\sin\theta)^2}{2g} + \frac{p}{\gamma} \qquad \text{[c]}$$

Solving for p and replacing γ by ρg we get

$$p = p_0 + \frac{\rho V_0^2}{2}(1 - 4\sin^2\theta) \qquad \text{[d]}$$

Substituting into Eq. b, and integrating,

$$D = -\int_0^{2\pi}\left[p_0 + \frac{\rho V_0^2}{2}(1 - 4\sin^2\theta)\right] r\cos\theta\,d\theta = 0 \qquad \text{[e]}$$

Thus, in the complete absence of friction the drag force is zero. It will be left for you to demonstrate that the lift is also zero for this case. Actually, the drag about any streamlined body in a stream will be zero when friction is completely ignored throughout the flow. However, we will learn in Chap. 11 that this is not so for lift. The absence of drag on the cylinder as demonstrated by Bernoulli's equation puzzled early investigators since it contradicted everyday experience. For this reason it began to be called the *D'Alembert paradox.* What was initially lacking was a knowledge of boundary layers, a vital subject that we will study in Chapter 11 on potential flow.

EXAMPLE 5.5

■ Problem Statement

One end of a U-tube is oriented directly into the flow (Fig. 5.11) so that the velocity of the flow of the stream is zero at this point. The pressure at a point where the flow has been halted is called the *stagnation pressure.* At the other end of the U-tube, the flow is undisturbed. The pressure at the pipe section there, if one neglects hydrostatic pressure, is a constant pressure over the section. If one neglects friction, what is the volume of flow in the pipe?

■ Strategy

We shall make the following assumptions for the flow.

1. Incompressible, steady, frictionless flow.
2. Neglect the hydrostatic pressure in a cross section of the pipe flow.
3. The opening from the right end of the U-tube into the pipe does not disturb the flow there (no burrs of the tube there).
4. 1-D flow in the pipe away from point A.

These conditions allow us to use *Bernoulli* between point A and point B, both along the centerline of the pipe. This will involve the 1-D velocity of the pipe. Then, we will use manometry to give us enough information to determine the 1-D velocity and hence the volume flow.

■ Execution

Bernoulli's equation between points A and B with the velocity at A zero is as follows:

$$p_A/\rho = p_B/\rho + \frac{V_B^2}{2} \qquad \text{[a]}$$

$$\frac{V_B^2}{2} = \frac{p_A - p_B}{\rho}$$

Using *manometry,* for the U-tube and noting that the pressure at B is the same as the pressure at the U-tube entrance into the pipe since we are not including gravity induced pressure in the pipe sections, we have

$$p_A - p_B = (\gamma_{\text{Hg}} - \gamma_{\text{H}_2\text{O}})\left(\frac{2}{12}\right) = (12.6)(62.4)\left(\frac{2}{12}\right) = 131.0 \text{ psf}$$

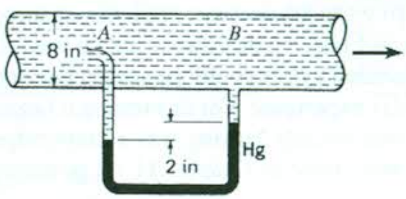


Figure 5.11
U-tube

Now, going to equation a we get the volume flow Q.

$$Q = (11.62)\left(\frac{(\pi)(8)^2}{(4)(144)}\right) = 4.06 \text{ cfs}$$

■ Debriefing

Via this example we have introduced the concept of the stagnation pressure. This pressure is an important measure in fluid flow theory and computations. We will see it often as we go ahead in the text. In problems 5.21, 5.22 and 5.24 we have presented other measuring devices. In Chap. 10, on compressible flow, we have other problems involving flow measurement. We point out finally that for good experimental results the capillary tube at the stagnation end must be carefully aligned so that its centerline is very close to the direction of flow.

EXAMPLE 5.6

■ Problem Statement

Water flows from a very large reservoir and drives a turbine, as shown in Fig. 5.12. Neglecting friction through the pipes (they are short with large diameters), determine the horsepower developed by the flow on the turbine for the data given in the diagram.

■ Strategy

We shall form a control volume encompassing the turbine, and we will use the *simplified first law* for it and deal with the pipes and reservoir system separately.

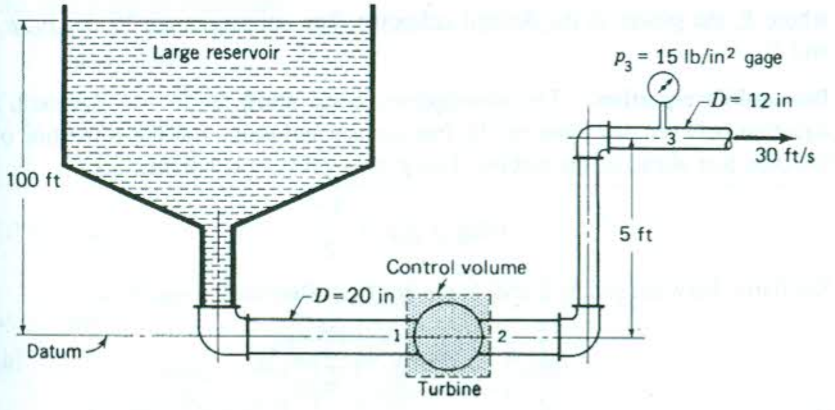


Figure 5.12
Flow from reservoir drives turbine.

For the turbine, we make the following assumptions:

1. Steady flow.
2. No heat transfer.
3. Incompressible flow.
4. 1-D flow into and out of the control volume.
5. Neglect change in specific internal energy.

For the pipe and reservoir system, we make additional assumptions over and above those for the turbine.

6. Neglect friction in the reservoir and the pipes.
7. Neglect the kinetic energy at the free surface of the reservoir.

We will then be able to use *Bernoulli* in the reservoir and the pipes and the first law of thermodynamics for the turbine. Finally, using *continuity* equations, we can find the power of the turbine output.

■ Execution

Simplified first law. We start with the following equation using the assumptions presented for the control volume,

$$\frac{V_1^2}{2} + p_1 v = \frac{V_2^2}{2} + p_2 v + \frac{dW_S}{dm} \qquad \text{[a]}$$

The term dW_S/dm may be expressed as

$$\frac{dW_S}{dm} = \frac{P}{\rho V_1 A_1} \qquad \text{[b]}$$

where P, the power, is the desired unknown. The unknowns are V_1, V_2, p_1, p_2, and P.

Bernoulli's equation. The assumptions made allow us to use Bernoulli's equation between any point on the free surface and point 1 at the centerline of the pipe just ahead of the turbine. Using gage pressures, we then get

$$100g = p_1 v + \frac{V_1^2}{2} \qquad \text{[c]}$$

Similarly, between points 2 and 3, we get from Bernoulli's equation

$$p_2 v + \frac{V_2^2}{2} = p_3 v + \frac{V_3^2}{2} + 5g \qquad \text{[d]}$$

Continuity equations. We get for the control volume about the turbine

$$V_1 = V_2 \qquad \text{[e]}$$

For a control volume not shown consisting of the interior of the pipe from point 2 to point 3 we get

$$V_2 = \left(\frac{12}{20}\right)^2 V_3 = 0.36V_3 \qquad \text{[f]}$$

We now have enough equations for all the unknowns, and we may solve for P in units of horsepower. Thus

$$P = 124.1 \text{ hp}$$

■ Debriefing

One must resist the temptation to use Bernoulli for streamlines going through the control volume enclosing the turbine. There is a work term for the streamtubes comprising the internal turbine flow and a degree of unsteadiness at the streamtube level that precludes the use of Bernoulli, even though at the power output level we can consider steady behavior of the turbine.

EXAMPLE 5.7

■ Problem Statement

If 10 ft^3/s of water is to flow through the pipe Fig. 5.13, what must be the horsepower of the pump? Neglect friction in the pipes.

■ Strategy

We will isolate the pump from the pipe by drawing a control volume between the inlet and outlet of the pump, as shown in Fig. 5.13. We shall first use the simplified first law for this control volume using gage pressures and making the following idealizations:

1. Steady flow.
2. Incompressible flow.
3. u is constant.
4. No heat transfer.

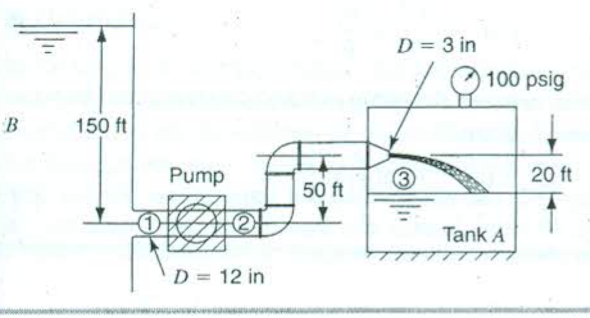


Figure 5.13
Pumping system.

This will bring into play unknown pressures p_1 and p_2 as well as the work per unit mass, all unknowns. Now we will make additional assumptions that will enable us to use Bernoulli between the free surface in tank B and the pump inlet.

5. Neglect kinetic energy at the free surface of tank B.
6. Neglect friction in tank B and take the flow there as quasi-static.

Using the centerline of the lowest pipe as a datum, we will add one more equation for the three unknowns. We will make one more assumption for the flow between the pump outlet and the free jet in tank A.

7. Neglect the friction in the nozzle in tank A and the elbows in the pipe.

We will use Bernoulli between the pump outlet and the free jet. Finally we will need equations of continuity for the inlet and outlet of the pump as well as the velocity in the free jet. This will give us enough independent equations for the desired information.

■ Execution

The *simplified first law* for the pump control volume for no heat transfer is as follows after canceling the internal energy terms u as per assumption 3,

$$\frac{V_1^2}{2} + \frac{p_1}{\rho} = \frac{V_2^2}{\rho} + \frac{p_2}{\rho} + \frac{dW_S}{dm}$$

Hence noting that $V_1 = V_2$

$$\frac{p_2 - p_1}{\rho} = \frac{dW_S}{dm} \qquad [1]$$

Now using *Bernoulli* between the free surface of tank A and the inlet of the pump, we get, using assumptions 5 and 6 and now assumption 3 applied to this flow,

$$150g = \frac{V_1^2}{2} + \frac{p_1}{\rho} \qquad [2]$$

A second Bernoulli equation between the pump exit and the free jet can be written by virtue of assumption 7. Thus,

$$\frac{V_2^2}{2} + \frac{p_2}{\rho} = \frac{V_3^2}{2} + \frac{(100)(144)}{\rho} + 50g \qquad [3]$$

Finally, *continuity* gives us the following simple results:
For the nozzle,

$$V_3 = \frac{12^2}{3^2} V_2 = 16 V_2 \qquad [4]$$

For the pump,

$$V_1 = V_2 = \frac{10}{\frac{\pi 1^2}{4}} = 12.73 \text{ ft/s}$$

Hence, from Eq. [4],

$$V_3 = 203.7 \text{ ft/s}$$

From Eq. 2,

$$p_1 = \left(150g - \frac{12.73^2}{2}\right)(1.938) = 9204 \text{ psf gage}$$

From Eq. 3,

$$\frac{12.73^2}{2} + \frac{p_2}{\rho} = \frac{203.7^2}{2} + \frac{14{,}400}{\rho} + 50g$$

$$p_2 = 57{,}570 \text{ psf gage}$$

From Eq. 1,

$$\frac{9{,}204 - 57{,}570}{1.938} = \frac{dW_S}{dm}$$

$$\frac{dW_S}{dm} = -2.496 \times 10^4 \frac{\text{ft-lb}}{\text{slug}}$$

Hence,

$$\frac{dW_S}{dt} = \frac{(2.496 \times 10^4)(10)(1.938)}{550} = 879.4 \text{ hp}$$

■ Debriefing

In looking back at this solution, the exceedingly large velocity of the water emerging from the nozzle at over 200 ft/s seems unreasonable. This prompts a careful check on the solution of the problem and certainly a consideration of the assumptions made during its solution. In this case, the calculations are correct, and the assumptions are quite reasonable. Should such a situation occur in your work, then you should give careful attention to the data used.

EXAMPLE 5.8

■ Problem Statement

We base this problem on a system for collecting oysters from the bottom of the ocean that was attempted in the past.[9] A pump on board a ship (see Fig. 5.14) passes 1500 gal/min of sea water into a 6-in pipe to a so-called ejector nozzle, which is housed in a second larger nozzle *E* open at end *A* and connected to a 10-in pipe. The jet of water coming out at *B* entrains water into the larger nozzle and draws 250 gal/min of water and oysters into the larger nozzle at *A*. The combined specific gravity of water and oysters *entering* at *A* is 1.3. If we take the pressure at *A* to be that of nearby hydrostatic pressure, what is the horsepower needed by the pump?

■ Strategy

We will use as the control volume the entire interior of the system of smaller and larger pipes as well as the pump. Thus, there is one entrance of water as well as one entrance and one exit of the water and oyster mixtures. We will use the following assumptions using a hypothetical continuum for the mixture.

1. Steady 1-D flow.
2. Incompressible flow.
3. Isothermal flow (no heat transfer).
4. Hydrostatic pressure at the entrance *A*.

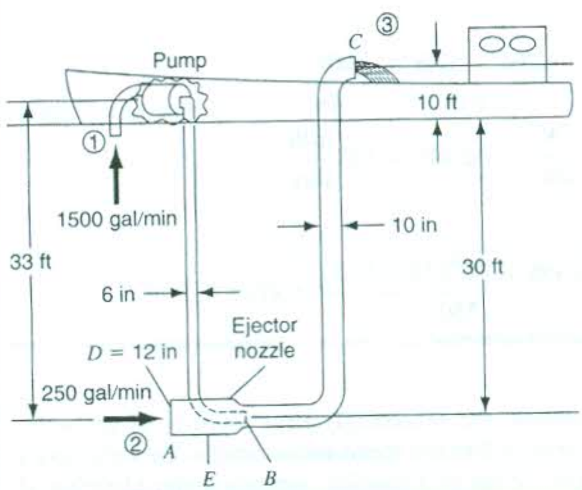


Figure 5.14
A system for collecting oysters.

[9] This system was developed by the author's older brother, the former Albert A. Shames, mechanical engineer and CEO of Alden Engineering Company, which at one time was situated in Foxboro, Massachusetts.

The *first law* that we will use for this control volume is

$$\left(\frac{V_1^2}{2} + gz_1 + p_1v_1 + u_1\right)\left(\frac{dm_1}{dt}\right) + \left(\frac{V_2^2}{2} + gz_2 + p_2v_2 + u_2\right)\left(\frac{dm_2}{dt}\right) = \left(\frac{V_3^2}{2} + gz_3 + p_3v_3 + u_3\right)\left(\frac{dm_3}{dt}\right) + \frac{dW_S}{dt}$$

where dm_1/dt is the mass flow rate of sea water entering at the top, dm_2/dt is the mass flow rate of the mixture of sea water and oysters entering the ejector nozzle at the bottom, and, finally, dm_3/dt is the mass flow rate of sea water and oysters being discharged into the boat. Also, u_1 is the specific internal energy of sea water entering at the top, u_2 is specific internal energy of mixture ① of sea water and oysters coming into the ejector, and u_3 is the specific internal energy of mixture ③ discharging into the boat. Since we have isothermal conditions, there will be *conservation of specific internal energy* in the flow since specific internal energy u will be constant for sea water and for oysters, and hence the amount of specific internal energy for the flow cannot change. Finally, we will use *hydrostatics* for the pressure at A. This will give us the necessary equations for solving for the horsepower of the pump.

■ Execution

Continuity and the isothermal assumption permits us to account for the flow rate of specific internal energy as follows:
(specific internal energy rate into the CV) = (specific internal energy rate out of the CV)

$$u_1\frac{dm_1}{dt} + u_2\frac{dm_2}{dt} = u_3\frac{dm_3}{dt}$$

Going back to the first law, we can drop the u terms as a result of the above equation. Next we evaluate the velocity of the incoming sea water at the top and the velocities of the two mixtures of sea water and oysters, one coming into the ejector nozzle and one discharging into the boat. Thus

$$V_1 = \frac{Q}{A} = \frac{(1500)(0.002228)}{\dfrac{\pi\left(\dfrac{1}{2}\right)^2}{4}} = 17.02 \text{ ft/s}$$

$$V_2 = \frac{(250)(0.002228)}{\dfrac{\pi}{4}(1)^2} = 0.709 \text{ ft/s}$$

$$V_3 = \frac{(1750)(0.0002228)}{\left(\dfrac{\pi}{4}\right)\left(\dfrac{10}{12}\right)^2} = 7.15 \text{ ft/s}$$

The pressures are

$$p_1 = p_{\text{atm}} = 2116 \text{ psf} \quad p_2 = p_{\text{atm}} + (62.4)(30) = 3989 \text{ psf} \quad p_3 = 2116 \text{ psf}$$

Note further that

$$dm_1/dt = (1.938)(1{,}500)(0.002228) = 6.48 \text{ slugs/s}$$
$$dm_2/dt = (1.938)(1.3)(250)(0.0022238) = 1.403 \text{ slugs/s}$$
$$dm_3/dt = 6.48 + 1.405 = 7.88 \text{ slugs/s}$$

We are now ready to go to the first law to substitute values.

$$\left[\frac{17.02^2}{2} + (g)(30) + 2116\left(\frac{1}{1.938}\right)\right]6.48 + \left[\frac{0.709^2}{2} + 0\right.$$
$$\left. + (3989)\frac{1}{(1.938)(1.3)}\right]1.403 = \left[\frac{7.15^2}{2} + (g)(40) + (2116)\frac{1}{\rho_3}\right]7.88 - \frac{dW_S}{dt}$$

We need ρ_3. Hence,

$$\frac{dm_3}{dt} = 7.88 = \rho_3 V_3 A_3 = (\rho_3)(7.15)\left(\frac{\pi}{4}\right)\left(\frac{10}{12}\right)^2$$
$$\rho_3 = 2.02 \text{ slugs/ft}^3$$

Finally, we can solve for dW_S/dt.

$$\frac{dW_S}{dt} = 2108 \text{ ft-lb/s} = 3.83 \text{ hp}$$

■ Debriefing

This problem is of interest from a mechanics point of view in that two species of materials were involved, sea water and oysters. Notice that we made a continuum of the mixture of sea water and oysters first on coming into the ejector nozzle and then another continuum in the flow vertically to the boat. This device is all right as long as there is a reasonably high density of oysters (i.e., a good catch). The problem shows how one can use ingenuity to model a complicated flow process for approximate results.

5.8 A NOTE ON THE SECOND LAW OF THERMODYNAMICS

It will be recalled from Chap. 1 that a property is a measurable characteristic of a material. The *state* of a substance is determined when enough properties are specified to establish uniquely the thermodynamic condition of the substance. A change of state takes place during a *process*. In a process, some or all of the properties will then have changing values.

If a system is involved in a process, there will usually be an exchange of energy between it and its surroundings, as well as a possible change in form of the stored energy. The first law of thermodynamics stipulates that during any and all processes all the energies are to be accounted for. The second law of thermodynamics now places restrictions on the *direction of energy transfer* as well as *the direction in which a real process may proceed.* For instance, heat must always proceed from a higher temperature to a lower temperature if no external influence is exerted on the process. Also, it is known that friction will always tend to retard the relative motion between two solid bodies in contact.

In frictionless incompressible flow, the second law of thermodynamics will be intrinsically satisfied as a result of the basic simplicity of the flow. In such flows there will only be exchanges of kinetic and potential energies between fluid elements, with the complete absence of friction and heat transfer; and under such circumstances there will be no restrictions on the manner of interchange of these energies. If friction without heat transfer is now assumed to be present, only correct directional effects need be prescribed to satisfy the second law. However, the inclusion of heat transfer dictates a more careful procedure, and it is when we reach this part of our study that we actively employ the second law of thermodynamics.

HIGHLIGHTS

We have developed in Chapter 4 integral forms of three basic laws, reaching very general equations. You will recall that they were all developed in much the same way using the Reynolds transport equation, going from the familiar system approach to the control volume approach. We have done the very same thing in this chapter for the first law of thermodynamics.

We started with the first law for a system in the form

$$\frac{DE}{Dt} = \frac{dQ}{dt} - \frac{dW_K}{dt}$$

where E is the stored energy, namely the kinetic energy, plus the potential energy, plus the internal energy, as learned in physics. We used E for N in the Reynolds transport equation, and for the corresponding η the following was used:

$$\eta = \frac{V}{2} + gz + u$$

Now going to the Reynolds transport equation, we used the system first law to replace DE/Dt and, as just indicated, we used the above formulation for η. Also, we noted that dW_K/dt was composed of three categories: shaft work W_S, traction-force work, and body-force work. We then got a very general integral form of the first law which we stated as

$$\frac{dQ}{dt} - \frac{dW_S}{dt} + \oiint_{CS} \mathbf{T} \cdot \mathbf{V}\, dA + \iiint_{CV} \mathbf{B} \cdot \mathbf{V} \rho\, dv$$
$$= \oiint_{CS} \left(\frac{V^2}{2} + gz + u \right)(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t} \iiint_{CV} \left(\frac{V^2}{2} + gz + u \right)(\rho\, dv)$$

What we are saying in this equation is simply this: We add the rate of heat flow through the control surface, plus the rate of work from fluids passing through the control surface (the so-called flow work), plus the equivalent rate of work from electric current in wires passing through the control surface, plus the rate of work transmitted by shafts or any other devices passing through the control surface, plus, finally, the rate of body-force work on the material inside the control volume, all at time t. This then is equated with the rate of flow of mechanical plus internal energy through the control surface, plus the rate of change of mechanical plus internal energy inside the control volume, all at time t. Notice how similar the first law is to the other laws we have studied where, for instance, instead of mechanical plus internal energy being driven, here it is the linear momentum that is driven for Newton's law, and where the driver is for the first law is heat flow and rate of work, for Newton's law it is the traction force and the body force.

We then focused on a simpler setup involving a 1-D flow into the control volume and a 1-D flow out of the control volume, and we called the resulting first law equation the *simplified first law*. Thus we got for this very useful, much-used case

$$\boxed{\begin{aligned} &\left[\frac{V_1^2}{2} + gz_1 + u_1 + \frac{p_1}{\rho_1} \right] \rho_1 V_1 A_1 + \frac{dQ}{dt} \\ &= \left[\frac{V_2^2}{2} + gz_2 + u_2 + \frac{p_2}{\rho_2} \right] \rho_2 V_2 A_2 + \frac{dW_S}{dt} \end{aligned}}$$

We also used another slightly altered form of the above equation wherein we replaced $u + pv$ by h, the enthalpy, and ρVA by dm/dt, the mass flow. Dividing through by dm/dt we have an alternate form of the simplified first law.

$$\boxed{\left[\frac{V_1^2}{2} + gz_1 + h_1 \right] + \frac{dQ}{dm} = \left[\frac{V_2^2}{2} + gz_2 + h_2 \right] + \frac{dW_S}{dm}}$$

From this simplified first law, we were able quite simply to present Bernoulli's equation involving only the mechanical energy conservation. Recall this equation as presented is valid for *steady, incompressible, frictionless* flows.

We will use the basic laws that we developed and Bernoulli's equation throughout the text. And later we will include a more detailed study of the second law of thermodynamics.

5.9 CLOSURE

We have developed the basic laws for systems and control volumes in this chapter and Chap. 4 and have solved a number of problems. You will note that we used essentially the integral forms of these basic laws and solved for certain flow parameters at a certain part of a carefully chosen control surface. Little had to be known of the details of the flow inside the control volume, particularly for steady flow, and we invariably made certain simplifying assumptions concerning flow crossing the control surface (very often the one-dimensional-flow assumptions). In these problems, we required little information, and we reached limited results in the form of resultant forces, average velocities, and so on. In other words, we did not determine the velocity field or the stress distribution throughout a region of flow by the use we made of the integral forms of the basic laws. To do this, and thus to be able to make more detailed deductions, we must use the *differential forms* of the basic laws at a point and integrate them in such a way that the given boundary conditions of the problem are satisfied. This is a very difficult undertaking—so difficult that we cannot present general integration procedures applicable for any fluid under any circumstance. Instead, we must present idealizations for certain classifications of flow, and introducing concepts useful in discussing these flows, we can then consider the basic laws in *differential* form tailored for these cases. Sometimes empirical and experimental results have to be included to reach meaningful results. In Parts II and III of this text, we will examine, in this manner, a number of important flow classifications.

In Chap. 6, we will consider certain key differential equations that will be of much use to us as we proceed through Parts II and III of the text.

5.10 COMPUTER EXAMPLES

COMPUTER EXAMPLE 5.1

■ Computer Problem Statement

An open cylindrical tank (see Fig. C5.1) contains water having a height of 20 ft above the base. At the base two nozzles each have an internal diameter of 1.5 in. The nozzles can be rotated an angle θ simultaneously. For different settings of θ, going from 0° to 45° in 50 steps, how much time does it take for the free surface of the water to go from $h = 20$ ft to $h = 10$ ft for each setting of the nozzles? Take the coefficient of discharge of the nozzles C_D to be dependent on θ measured in radians and given as $C_D = 0.524[1 - 0.01\theta^2]$. The diameter of the tank is 8 ft.

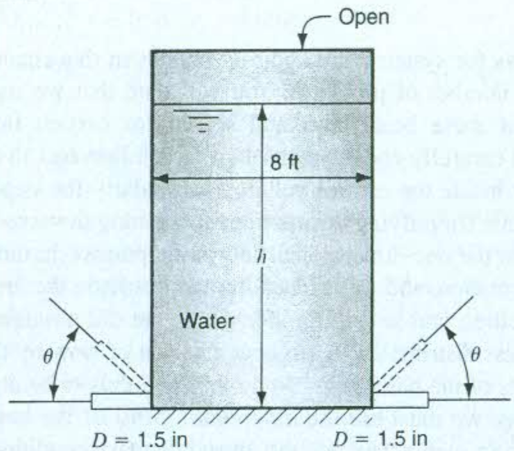


Figure C5.1

■ Strategy

We shall use a quasi-static approach whereby the free surface is held stationary while the associated discharge rate is used to determine the time for a volume increment of water to drain from the tank for the chosen fixed orientation of the exit nozzles. The free surface will then be moved down a distance corresponding to the amount of efflux from the previous volume increment. The process will then be repeated. We will use the given discharge coefficient each time to modify the ideal result from Bernoulli to take into account friction and turbulence at the exit nozzles. We will compute the time for the discharge of each volume increment for each level of a stationary free surface. Summing then will give us the desired time interval for the setting of the nozzles that we are using. Then we will go on to the next nozzle setting for a repeat of the previous steps as part of a loop.

■ Execution

```
clear all;
%Putting this at the beginning of the program ensures
%values and figures don't overlap from previous
%programs.

theta=linspace(0,45,50);
%We are going to vary the nozzles from 0 degrees to
%45 degrees(0->pi/4 rad) in 50 increments.

c=pi./180;
%When "c" is multiplied by theta it will change the
%values in degrees to values in radians which is what
%we need for the discharge coefficient.
```

```
height=20:-.2:10.2;
%We are going to lower the height in .2 ft.
%increments from 20 ft to 10 ft. 10.2 is used for the
%last value since we are solving for time. Plugging
%10.2 into the following equation will give us the
%time for the level to go from 10.2 ft. to 10 ft.

volumechange=10.0531;
%this is how the volume in the main chamber will
%change each time the height drops .2 ft.

for j=1:50;

coeff_d(j)=.524.*(1-.01.*(theta(j).*c).^2);
%This is the given coefficient of discharge of the
%nozzles. It depends on theta.

for i=1:50;

time(i)=volumechange./(coeff_d(j).*sqrt(height (i)).*
.196963);
%The volume change is constant. Each time height
%drops .2 ft. the volume change is the same. For all
%50 iterations of the inner loop we want the same
%coefficient of discharge value that is why the
%coefficient of discharge (coeff_d) is indexed with a
%"j".

end

t(j)=sum(time);
%This will give us the total time it takes for the
%height to drop 10 ft. with a particular theta value
%plugged in.

end

plot(theta, t);
xlabel('Values of theta (degrees)');
ylabel('Time (seconds)');
grid;
title('Time required for a level drop of 10 ft. vs.
angle(theta) of the nozzles');
%This gives us the graph that we want of flow
%performance.
```

■ Debriefing

The time interval plotted vs. the nozzle setting in Fig. C5.2 shows no surprises. For the discharge coefficient we have proposed, the time interval increases close to parabolically with the angle θ. For more accurate results, one would have to make an experimental study of the discharge coefficient. The program we have set forth would then be of value in testing hypotheses of discharge coefficients, which can easily be inserted in the program for the purpose of comparing with experimental results covering a range of nozzle inclinations.

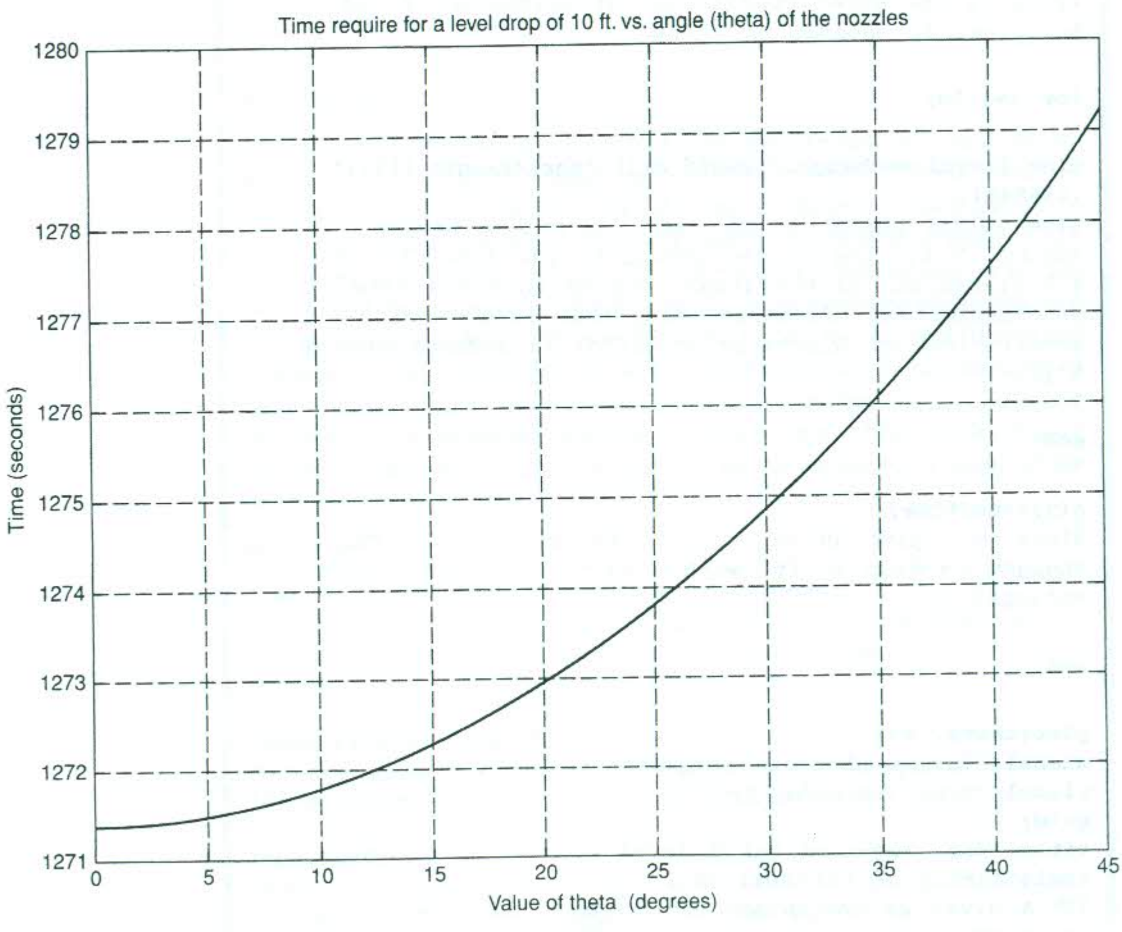


Figure C5.2

COMPUTER EXAMPLE 5.2

■ Computer Problem Statement

A cone is suspended in a tank (see Fig. C5.3) containing water and enclosing air all at a temperature of 60°F. The water initially has a free surface that just touches the tip of the cone. The air above the free surface of the water is at a pressure of 200 psig. At the bottom of the tank are two nozzles out of which water can flow, as shown. If the air behaves isothermally according to the formula pv = const., plot the torques on the tank vs. time. Take the coefficient of discharge $C_D = 0.6$ for the nozzles.

■ Strategy

We will let the free surface descend a small distance. We will compute the volume increment of water that the free surface sweeps out as it moves down the small distance. Using this volume, we will compute the volume of the air as it expands isentropically to accommodate the movement of the free surface. We will compute the pressure of the air at the end of this small movement of the free surface. Note that in using the equation of state for the air, we must use the absolute pressure. Employing a quasi-static approach, we will consider that the free surface does not move in the next calculation and that the expanded pressure of the air has existed during the entire movement of the free surface. Next, we will use Bernoulli for the water, with the absolute pressure of the air at the lower position of the free surface, and also at the nozzle outlets, to compute a constant velocity of the water jets while making use of the coefficient of discharge of 0.6 for the nozzles. We will compute the time interval for this model of the action as well as the value of the moment of the linear momentum from the jets. Using a control volume encompassing the entire interior of the tank and the nozzles, we'll get the torque. We will proceed by moving the

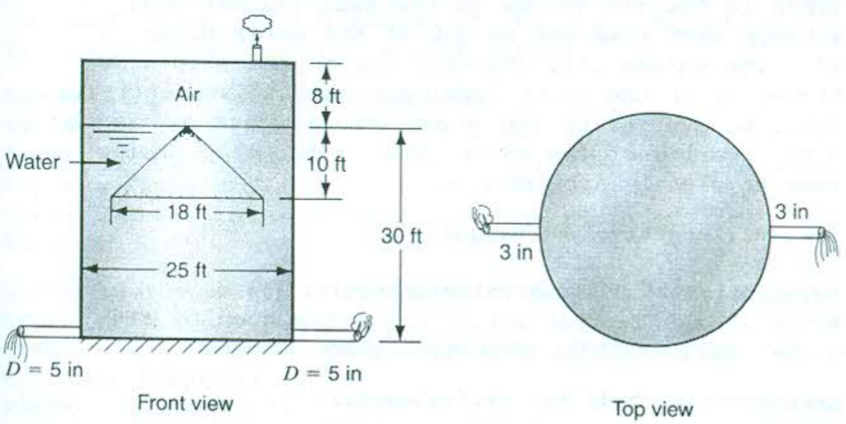


Figure C5.3

free surface downward a step at a time and making time and torque calculations for each movement of the free surface in the quasi-static manner. We will then plot our results for torque vs. time.

■ Execution

```
clear all;
%Putting this at the beginning of the program ensures
%values and figures don't overlap from previous
%programs.

h=.2:.2:10;
%These are the steps in height(ft.) we have chosen to
%use in the volume of the cone calculations.

radius=.9.*h;
%This is how the radius of the cone varies with the
%height of the cone as can be seen by the diagram.

radius1=.9.*(h-.2);
%This is how the radius of the smaller cone whose volume
%is to be subtracted in the next equation.

vol_cone=((1./3).*pi.*radius.^2.*h)-
((1/3).*pi.*radius1.^2.*(h-.2));
%This is the volume of the cone that detracts from
%the volume increase due to the level drop around the
%cone for a .2 ft. step of water level change. In
%this equation we are taking a bigger cone and
%subtracting a cone with a .2 ft. smaller height to
%get a slice of cone for that volume change.

volumechange=98.1748-vol_cone;
%This is how the volume in the main chamber will
%change each time the height of the water drops .2
%ft. The volume will increase due to the step-wise
%lowering of the water level but this volume increase
%will be reduced by the volume of the cone exposed by
%the lowering of the water. The .2 ft. step choice
%was completely arbitrary.

for i=1:length(volumechange);

volume(i)=3926.991+sum(volumechange(1:i));
%This is the initial volume above the apex of the
%cone (3926.991ft^3) plus the volume changes.

pressure(i)=1.2141.*10.^8./volume(i);
%Since p*V=1.2141*10^8, we can figure out how the
%pressure is going to change when the volume changes
```

```
%(using total volume not specific volume). The
%constant was determined using the absolute pressure
%of 200*144+14.696*144 lb/ft^2 and the first volume
%of the air of 3926.991 ft^3.

end

height=29.8:-.2:20;
%We are going to lower the height in .2 ft.
%increments from 30 ft to 20 ft. This height value
%will be used in determining volume flow out of the
%two pipes at the bottom of the tank (using
%Bernoulli's equation) and is the height off the
%bottom of the tank.

velocity=sqrt(1.03199.*(pressure-2116)+64.4.*height);
%This is the velocity of the water out of the nozzles
%at the bottom based on Bernoulli's equation.

area=((pi./4).*(5./12).^2).*2;
%This is the total area where the flow is leaving at
%the bottom.

time1=volumechange./(velocity.*area.*.6);
%This is the time for each .2 ft. level drop in the
%tank using a discharge coefficient of .6. We are
%only using this calculation to get the range of time
%values to be used as the independent variable in our
%plot.

time=linspace(0,sum(time1),50);
%This is time in seconds and is the independent
%variable in our plot. We want to run the time from
%the initiation of draining until the level has
%dropped 10 ft. to the bottom of the cone. "time" is
%an array of 50 linearly spaced values.

torque=24.7095.*area.*velocity.^2;
%This is the torque determined using the angular
%momentum equation. We are not multiplying this value
%by two since the area value we are using is total
%area and not the area of one of the pipes on the
%bottom.

plot(time,torque);
grid;
xlabel('Total elapsed time since the initiation of
draining (seconds)');
ylabel('Torque (ft*lb)');
title('Torque on the tank vs. Total elapsed time');
%This just gives us the plot that we want.
```

■ Debriefing

In this problem, we have presented an unsteady flow using a quasi-steady approach. The torque vs. time is shown in Fig. C5.4. The accuracy will depend on the size of the increments of movement used for the free surface in the program. Also, note that we have in the control volume a compressible and an incompressible fluid having a common free surface. We repeat now that the equation of state for the air requires the use of absolute pressure. For the water in using Bernoulli we could have used gage pressures at the free surface and at the exiting jet. The reason for this is that the expression p_{atm}/γ can be subtracted from both sides of the equation because of the constant value of γ leaving only gage pressures.

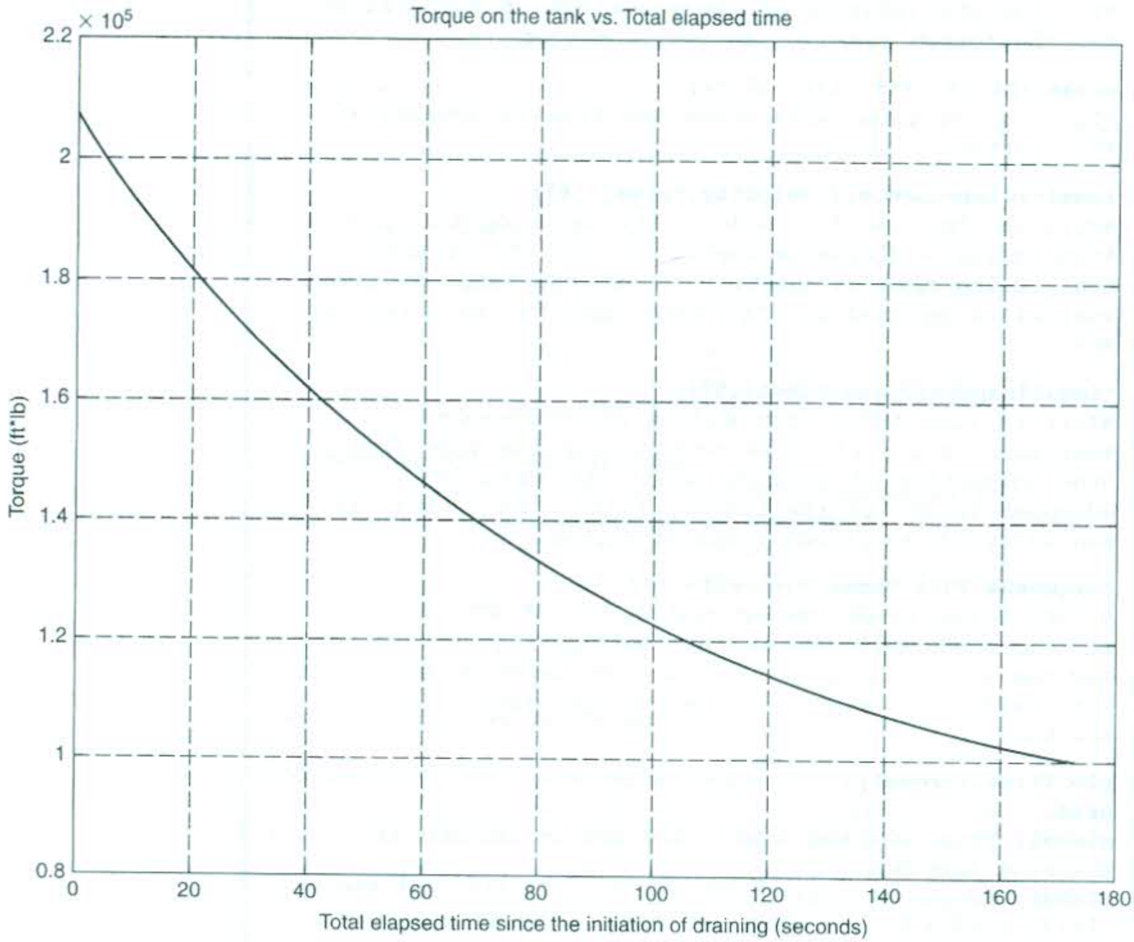


Figure C5.4

PROBLEMS

Problem Categories

First law problems 5.1–5.15
Bernoulli's equation 5.16–5.37
First law plus Bernoulli 5.38–5.48
Computer problems 5.49–5.51

5.1 A sump pump is a sealed pump usually underground that pumps water from its inlet at ① to above ground at ②. The inlet has an inside diameter of 75 mm, and the outlet at ② has a diameter of 50 mm. A current of 10 amps is flowing at a voltage of 220 V to the pump. What is the maximum possible capacity of the pump? Neglect friction in the pipes and heat transfer. Take p_1 as p_{atm}.

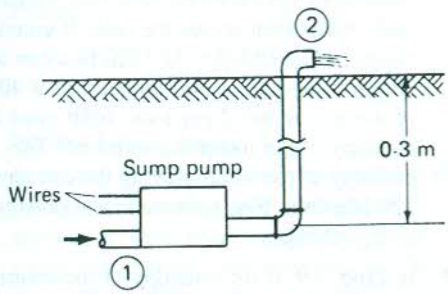


Figure P5.1

5.2 Air at an absolute pressure of 500 kPa and at a temperature of 35°C enters a highly insulated air motor and leaves as a free jet into the atmosphere at a temperature of −5°C. The inlet velocity is 25 m/s and the exit velocity is 70 m/s. If 3 kg of air flows per minute and if we take the internal energy, u, as $c_v T$, with c_v as a constant giving the specific heat at constant volume, what power is developed by the air motor? Take the specific heat as 4.08×10^{-5} N · m/(kg)(K). The atmospheric pressure is 101.4 kPa.

5.3 Steam enters a condenser at the rate of 600 kg/h with an enthalpy h of 2.70×10^6 N · m/(kg). To condense the steam, water at 15°C is brought in at the ratio of 7 kg of water per kilogram of steam. The water enters through a pipe with a 75-mm inside diameter and mixes directly with the steam. The velocity of the entering steam is 120 m/s. What is the temperature of the water leaving the condenser at the same elevation as the water inlet in a pipe having an inside diameter of 100 mm? We may take the enthalpy of a liquid to be $c_p T$ where c_p, the specific heat at constant pressure, is given as 4210 N · m/(kg) (K) for water. Neglect heat transfer from the condenser to the surroundings.

5.4 Water moves steadily through the turbine shown at the rate of Q = 220 l/s. The pressures at 1 and 2 are 170 kPa gage and −20 kPa gage, respectively. If we neglect heat transfer, what is the horsepower delivered to the turbine from the water?

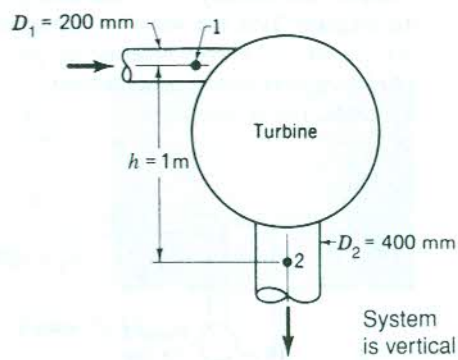


Figure P5.4

5.5 The flow rate through a turbine is 9000 kg/h, and the heat loss through the casing is 100,000 kJ/h. The inlet and exit enthalpies are 2300 kJ/kg and 1800 kJ/kg, respectively, while the inlet and exit velocities are 25 m/s and 115 m/s, respectively. Compute the shaft horsepower of the turbine.

5.6 It takes 50 hp to drive the centrifugal water pump. The pressure of the water at 2 is 30 lb/in^2 gage, and at 1, where the water enters, it is at 10 lb/in^2 gage. How much water is the pump delivering?

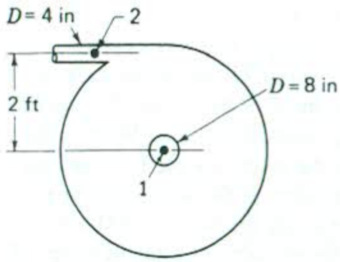


Figure P5.6

5.7 Shown in Fig. P5.7 is a system of highly insulated pipes through which water flows. In the upper pipe, the water leaving the pipe shows an increase of internal energy of 23 kJ/kg over the water entering at A; and the water leaving the lower pipe has an increase in internal energy of 116 kJ/kg (these increases are a result of friction in the flow). Compute the velocity V_3 for the data given in the diagram. Take the water as incompressible with an internal energy entering the pipe of 140 kJ/kg. (Set up two simultaneous equations, but do not solve.)

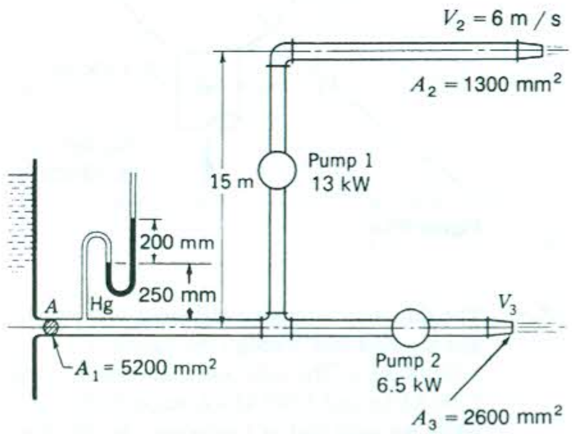


Figure P5.7

5.8 A gas undergoes steady flow through a porous plug in a *well insulated* pipe as

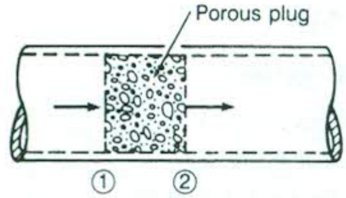


Figure P5.8

shown in Fig. P5.8. Show that if we have no change in kinetic energy and no heat transfer the enthalpy h is conserved on going through the plug. This is an example of what is called a *throttling process* which mimics what occurs as a gas passes through a partially opened valve.

5.9 A jet condenser condenses steam into water by mixing a spray of water with exhaust steam from some device inside of a well-insulated tank. Water then leaves the tank. If entering steam has an enthalpy of 1200 Btu/lbm and enters at the rate of 300 lbm/h, and if 4000 lb of water is injected per hour, what must the enthalpy of the incoming water be? The enthalpy of the water leaving the condenser is 120 Btu/lbm. Neglect kinetic and potential energy changes.

5.10 In Prob. 5.9, if the enthalpy of the entering water is 41 Btu/lbm, what must be the heat loss per hour from the condenser?

5.11 A *heat exchanger* shown in Fig. P5.11 has water entering at A, going through a set of horizontal pipes, and leaving at B. The purpose of this flow is to heat a flow of kerosene entering the heat exchanger at C and leaving at D after passing over the horizontal pipes. Water comes in at A at a temperature of 200°F and leaves at a temperature of 100°F. The kerosene is to be heated from 40°F to 120°F. If we are to heat 3 lbm/s of kerosene, what is the mass flow of water required? Diameters of pipes at A, B, C, and D are equal. The heat exchanger is well insulated. Use the following

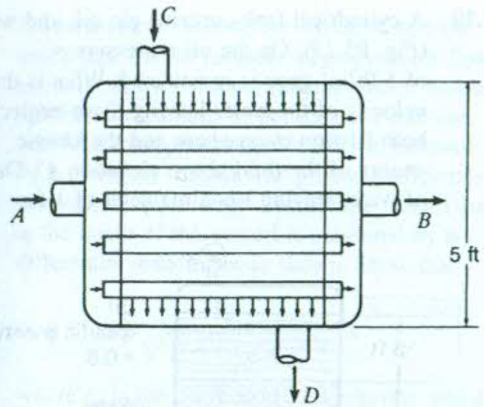


Figure P5.11

formulations for specific enthalpy per pound mass of the fluids (where t is in degrees Fahrenheit):

$$h_{\text{water}} = t - 32 \text{ Btu/lbm}$$

$$h_{\text{kerosene}} = 0.5t + 0.0003t^2 \text{ Btu/lbm}$$

Neglect kinetic energy.

5.12 In Prob. 5.11 what is the heat transfer from the heat exchanger to the atmosphere if the amount of water used is 5000 lbm/h for the conditions given in this problem, and the exit temperature of the kerosene is 100°F?

5.13 A gas turbine is idling at steady state incurring very little heat transfer with the surroundings. Preheated air at a temperature of 400°F enters the *combustion chamber* of the gas turbine at the rate of 40 lbm/s with a velocity of 340 ft/s. Liquid fuel is brought in at the rate of 68 parts by weight of air to fuel. The liquid fuel is at 60°F. The combustion products leave the combustion chamber at a temperature of 1400°F, a velocity of 680 ft/s, and an enthalpy of 360 Btu/lbm. What is the enthalpy of the entering fuel? The enthalpy of the preheated air is given as

$$h = 124.3 + \int_{60}^{T} c_p \, dT \text{ Btu/lbm}$$

where the reference enthalpy is taken at 60°F and T is in degrees Fahrenheit. Also for air we have at low pressure

$$c_p = 0.219 + \frac{0.342T}{10^4} - \frac{0.293T^2}{10^8} \text{ Btu/lbm °R}$$

where T is in degrees Rankine.

5.14 If in Prob. 5.13 the enthalpy of the liquid fuel is 12,000 Btu/lbm, what is the heat loss per second from the combustion chamber?

5.15 Show that for flow into a tank (see Fig. P5.15) the first law can be given as

$$Q = (U_2 - U_1) - \left(\frac{V_p^2}{2} + h_p\right)(m_2 - m_1)$$

where m_1 and m_2 are the masses at time t_1 and t_2 and where U_1 and U_2 are the internal energies in the tank at these times. List the assumptions needed to get the above result. Take V_p and h_p as constant.

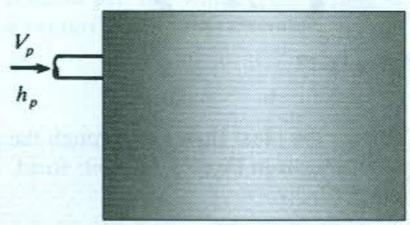


Figure P5.15

5.16 If friction is neglected, what is the velocity of the water issuing from the tank as a free jet? What is the discharge rate?

5.17 One end of a U-tube is oriented directly into the flow so that the velocity of the stream is zero at this point. The pressure at a point in the flow which has been stopped in this way is the *stagnation pressure*. The other end of the U tube measures the "undisturbed" pressure at a section in the flow. Neglecting friction, determine the volume flow of water in the pipe. What is the error by deleting hydrostatic pressure?

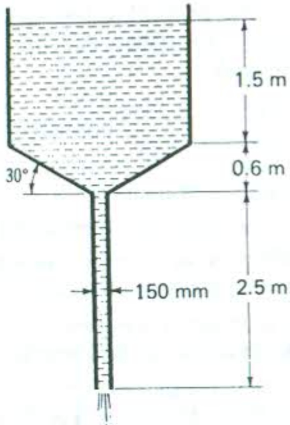


Figure P5.16

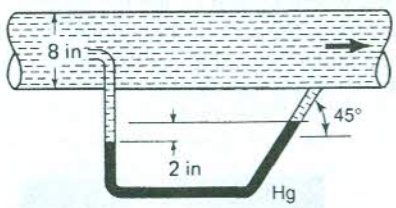


Figure P5.17

5.18 Compute the ideal flow rate through the pipe system shown in Fig. P5.18 *Hint:* Read Prob. 5.17.

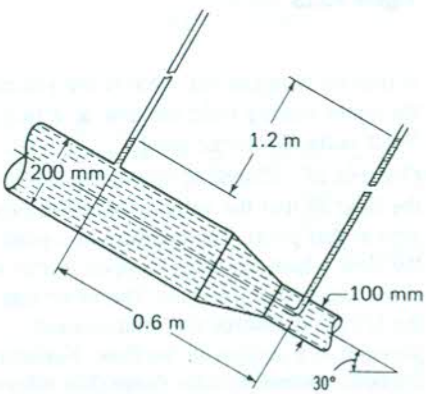


Figure P5.18

5.19 A cylindrical tank contains air, oil, and water (Fig. P5.19). On the oil a pressure p of 5 lb/in^2 gage is maintained. What is the velocity of the water leaving if we neglect both friction everywhere and the kinetic energy of the fluid above elevation A? The jet of water leaving has a diameter of 1 ft.

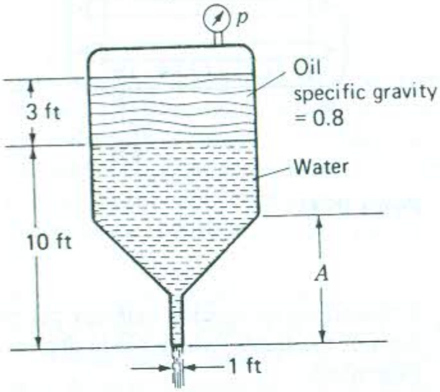


Figure P5.19

5.20 A large tank contains compressed air, gasoline at specific gravity 0.68, light oil at specific gravity 0.80, and water. The pressure p of the air is 150 kPa gage. If we neglect friction, what is the mass flow $\dot{m}$ of oil from a 20-mm diameter jet?

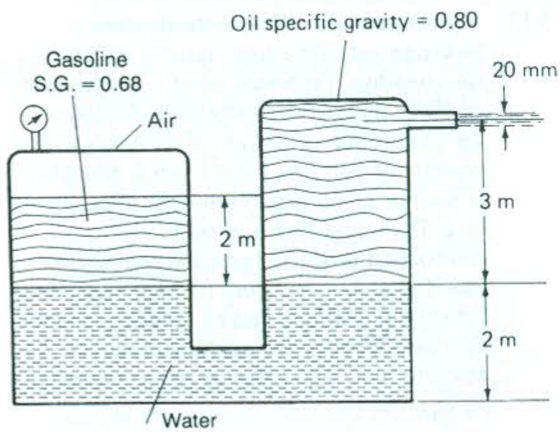


Figure P5.20

5.21 A *venturi meter* is a device which is inserted into a pipe line to measure incompressible flow rates. It consists of a convergent section which reduces the diameter to between one-half and one-fourth the pipe diameter. This is followed by a divergent section. The pressure difference between the position just before the venturi and at the throat of the venturi is measured by a differential manometer as shown. Show that

$$q = c_d\left[\frac{A_2}{\sqrt{1-(A_2/A_1)^2}}\sqrt{2g\frac{p_1-p_2}{\gamma}}\right]$$

where c_d is the *coefficient of discharge,* which takes into account frictional effects and is determined experimentally.

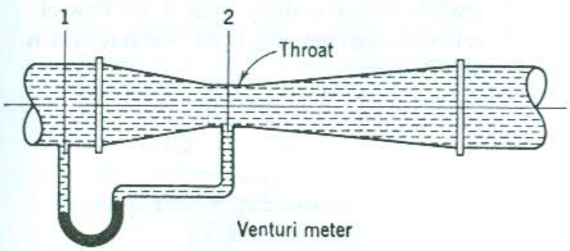


Figure P5.21

5.22 Another way of measuring flow rates is to use the *flow nozzle,* which is a device inserted into the pipe as shown in Fig. P5.22. If A_2 is the exit area of the flow nozzle, show that for incompressible flow we get for q

$$q = c_d\left[\frac{A_2}{\sqrt{1-(A_2/A_1)^2}}\sqrt{2g\frac{p_1-p_2}{\gamma}}\right]$$

where c_d is the *coefficient of discharge,* which takes into account frictional effects and is determined experimentally.

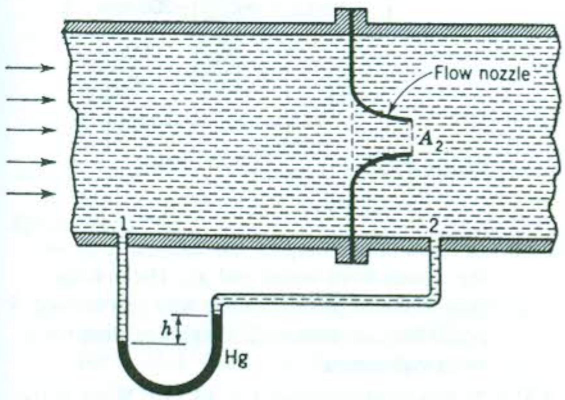


Figure P5.22

5.23 In Prob. 5.22 express q in terms of h, the height of the mercury column, as shown in Fig. P5.22 and the diameters of the pipe and flow nozzle.

5.24 In Probs. 5.21 and 5.22 we considered methods of measuring the flow in a *pipe.* Now we consider the measurement of flow in a rectangular *channel* of uniform width. A hump of height δ is placed on the channel bed over its entire width. The free surface then has a dip d as shown. If we neglect friction we can consider that we have one-dimensional flow. Compute the flow q for the channel per unit width. This system is called a *venturi flume.*

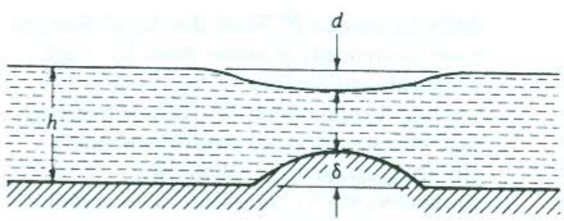


Figure P5.24

5.25 A *siphon* is shown in Fig. P5.25. If we neglect friction entirely, what is the velocity of the water leaving at C as a free jet? What are the pressures of the water in the tube at B and at A?

5.26 If the vapor pressure of water at 15°C is given in the handbook as 0.1799 m of water, how high h above the free surface can point B be before the siphon action breaks down?

5.27 Water flows in a rectangular channel, as shown in Fig. P5.27. The bed of the channel

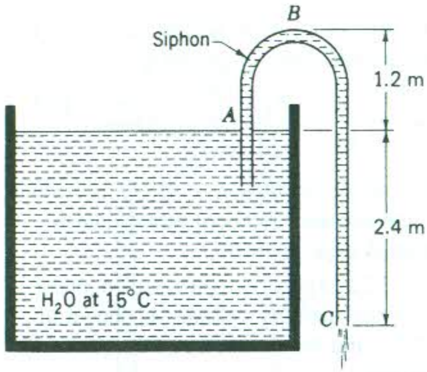


Figure P5.25

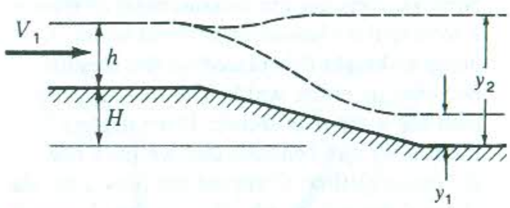


Figure P5.27

drops an amount H. Show that three values of y are theoretically possible. Now, by rough graphical consideration of the function yielding the three roots for y, show that only two roots y_1 and y_2 are positive and hence physically meaningful. The flow corresponding to y_1 is called *shooting flow*, and the flow corresponding to y_2 is called *tranquil flow*, as you will see in Chap. 13. Neglect friction and consider one-dimensional flow upstream and downstream of the drop.

5.28 Water moves with high velocity V_1 in a rectangular channel. There is a rise of height H in the channel bed. Show that to the right of this rise there are three depths given by the fluids calculations. By rough graphical considerations of the equation yielding the three roots, show that only two roots, y_1 and y_2, are meaningful. Neglect friction and consider one-dimensional flow at the sections shown downstream and upstream of the rise.

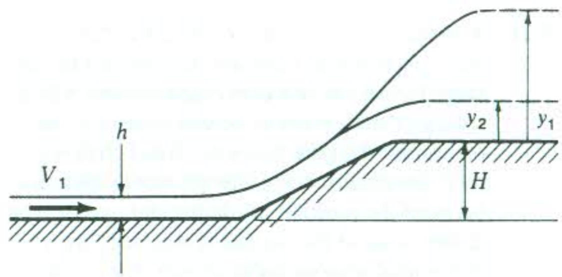


Figure P5.28

5.29 Water flows steadily up the vertical pipe and enters the annular region between the circular plates as shown. It then moves out radially, issuing out as a free sheet of water. If we neglect friction entirely, what is the flow of water through the pipe if the pressure at A is 69 kPa gage?

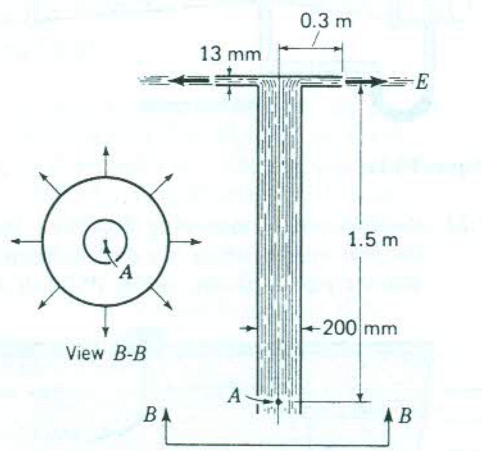


Figure P5.29

5.30 In Prob. 5.29, compute the upward force on the device from water and air. The volume flow is 0.408 m^3/s. Explain why you cannot profitably use Bernoulli's equation here for a force calculation.

5.31 The velocity at point A is 18 m/s. What is the pressure at point B if we neglect friction?

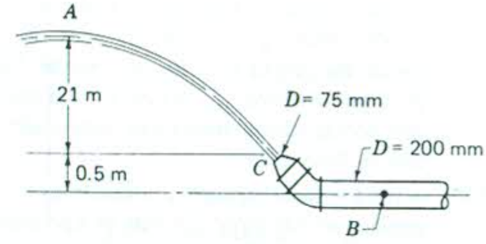


Figure P5.31

5.32 A diver directs a flexible pipe into which is sucked sand and water so as to expose part of a sunken ship. If the pressure at the inlet A is close to the hydrostatic pressure of the surrounding water, what amount of sand will be sucked up per second by a 2-kW pump? The specific gravity of the sand and water mixture picked up is 1.8. The inside diameter of the pipe is 250 mm. Neglect friction losses in the pipe.

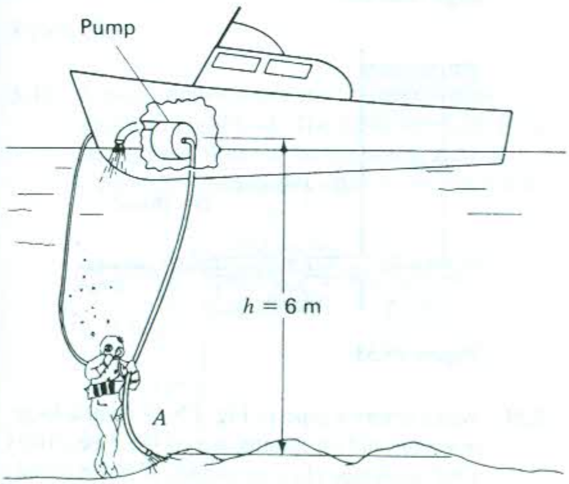


Figure P5.32

5.33 Air is made to flow through a well-insulated pipe by a pump. It is desired that 50 ft^3 of air per second flow by A. The inlet pressure p_A is 10 lb/in^2 absolute, and the temperature is 60°F. What power is required for the blower? Take u equal to c_vT, where c_v is the specific heat at constant volume and T is the absolute temperature. Use the value 0.171 Btu/(lbm)(°R) for c_v. The exit temperature of the air is 90°F.

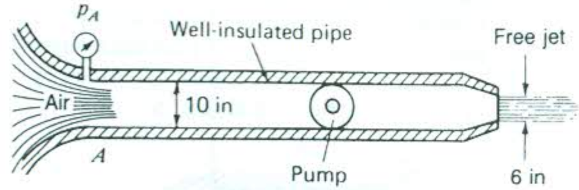


Figure P5.33

5.34 A rocket-powered test sled slides over rails. This test sled is used for experimentation on the ability of human beings to undergo large persistent accelerations. To brake the sled from high speeds, small scoops are lowered to deflect water from a stationary tank of water placed near the end of the run. If the sled is moving at a speed of 100 km/h at the instant of interest, compute h of the deflected stream of water as seen from the sled. Assume no loss in speed of the water relative to the scoop.

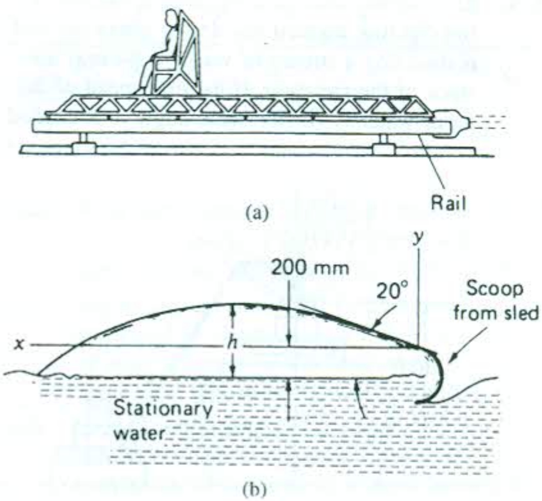


Figure P5.34

5.35 A firefighter is directing water from a hose into the broken window of a burning house. The velocity of the water is 15 m/s as it leaves the hose. What are the angles α needed to do the job? *Hint:* In addition to Bernoulli's equation, you will have to consider components of Newton's law for a water particle. Toward the end of your calculations it will also help if you replace $1/(\cos^2 \alpha)$ by $(1 + \tan^2 \alpha)$ which equals $\sec^2 \alpha$.

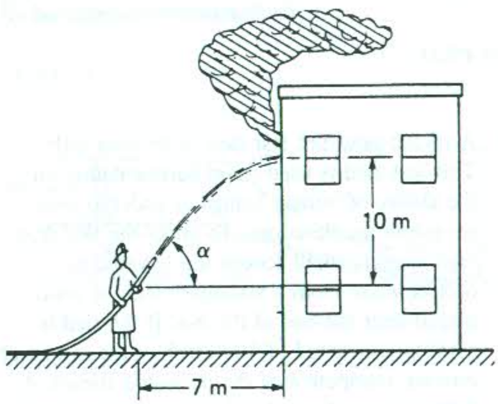


Figure P5.35

5.36 The engine room of a freighter is on fire. A fire-fighting tugboat has drawn alongside and is directing a stream of water to go into the stack of the freighter. If the exit speed of the jet of water is 70 ft/s, what angle α is needed to accomplish the task? *Hint:* Only one α will result in water getting into the stack. To decide the proper α of the two results, locate positions x where y_{max} occurs for the stream and decide which stream can enter stack. See hint in Prob. 6.35.

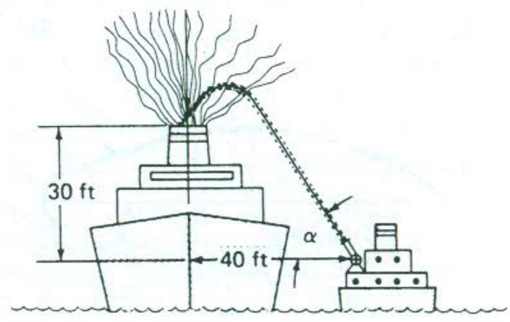


Figure P5.36

5.37 A fluid expands through a nozzle from a pressure of 300 lb/in^2 absolute to a pressure of 5 lb/in^2 absolute. The initial and final enthalpies of the fluid are 1187 Btu/lbm and 1041 Btu/lbm, respectively. Calculate the final velocity by neglecting the inlet velocity (called the approach velocity), gravitational effects, and heat transfer out of the casing and along the fluid flow. If the internal energy u of the fluid is known at the exit conditions to be 800 Btu/lbm and the inlet and outlet areas are 3 in^2 and 2 in^2, respectively, compute the thrust of the nozzle.

5.38 Neglecting friction in the pipe shown in Fig. P5.38, compute the power developed on the turbine from the water coming from a large reservoir.

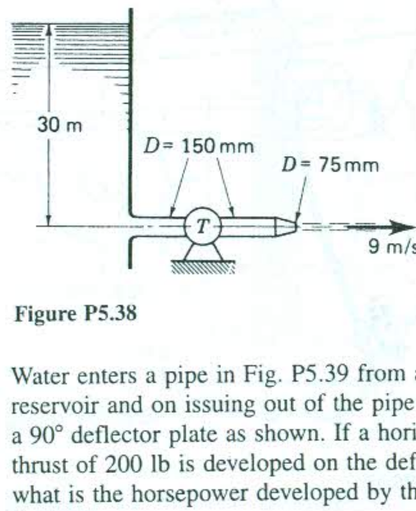


Figure P5.38

5.39 Water enters a pipe in Fig. P5.39 from a large reservoir and on issuing out of the pipe strikes a 90° deflector plate as shown. If a horizontal thrust of 200 lb is developed on the deflector, what is the horsepower developed by the turbine?

5.40 Water in a large tank is under a pressure of 35 kPa gage at the free surface (see Fig. 5.40). It is pumped through a pipe as shown and issues out of a nozzle to form a free jet. For the data given, what is the power required by the pump?

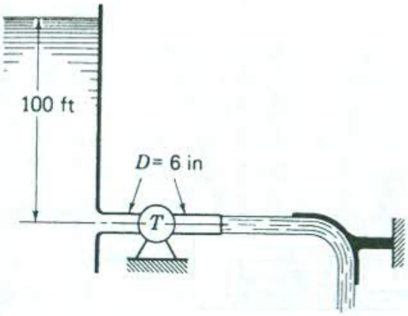


Figure P5.39

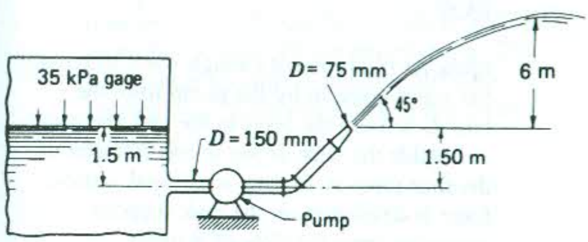


Figure P5.40

5.41 A pump draws water out of a reservoir as shown in Fig. P5.41. The pump develops 10 hp on the flow. What is the horizontal force at support D required as a result of the fluid flow?

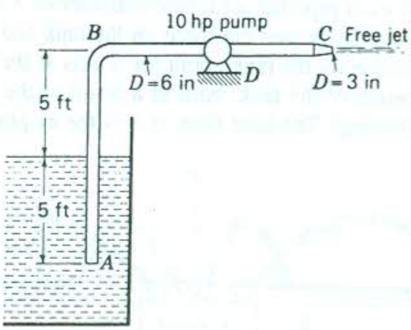


Figure P5.41

5.42 If the pump in Fig. P5.42 develops 3.75 kW on the flow, what is the flow rate? *Hint:* What is the velocity of flow at the right opening of the U-tube?

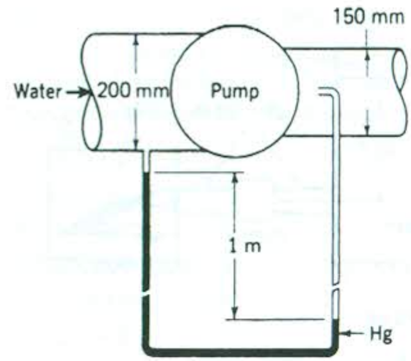


Figure P5.42

5.43 A ground effects ship is moving on the water at a speed of 100 km/h. Each of the two propulsion systems is composed of an intake of area A_1. The water is scooped in and a pump driven by a gas turbine drives the water at high speed out through area A_2. If the total drag of the ship is 25 kN, and if there are two drive systems described above, what is the area A_1 for each inlet? Each pump develops 400 kW.

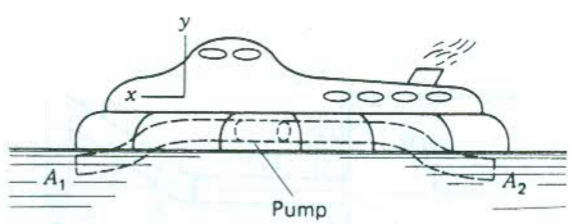


Figure P5.43

5.44 What horsepower is needed to cause 30 ft^3/s of water flow in Fig. P5.44? Neglect friction in pipes. The exit diameter of the nozzle is 10 in.

5.45 Neglecting friction, what is the power developed by the turbine in Fig. P5.45? At B we have a free jet at the hydrostatic pressure in the tank. The mass flow is 500 kg/s.

5.46 The internal diameter of the pipe system in Fig. 5.46 is 6 in. The exit nozzle diameter is 3 in.

a. What is the velocity V_e of flow leaving the nozzle? (Do *not* consider the flow inside the pipe proper to be inviscid.)

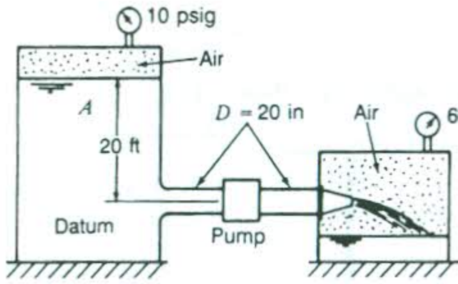


Figure P5.44

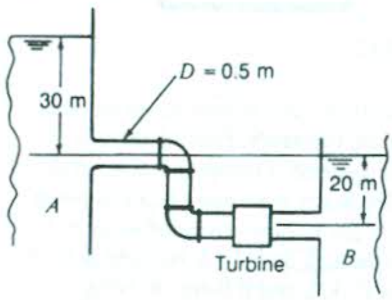


Figure P5.45

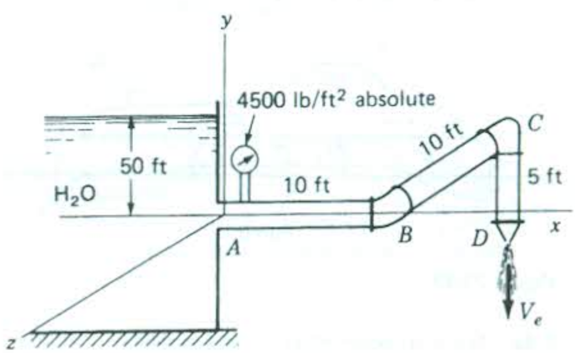


Figure P5.46

b. What is the moment about A coming from the water alone onto the pipe? BC is parallel to the z direction. (Set up moment-of-momentum equation only.) The free surface may be considered at constant height.

5.47 A fountain consists of a tank G containing a water pump feeding four pipes out of which come water streams. The top of the tank is

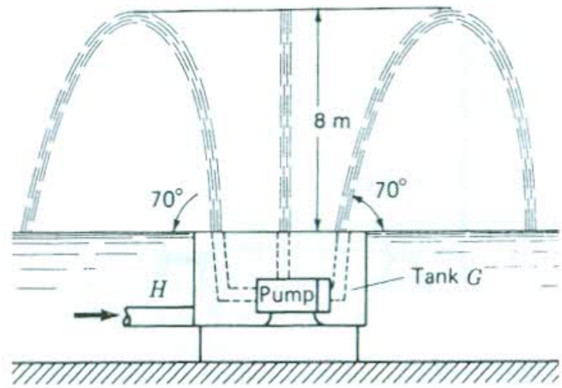


Figure P5.47

open. At H, we inject enough water to replace the water taken in by the pump from the tank G to keep the level in the tank the same as outside the tank. If the inside diameter of the four pipes is 75 mm, what total vertical force is developed on the tank supports stemming from the flow of water?

5.48 A water fountain has four identical spouts of water emerging from a tank inside of which (not shown) is a pump driving the flows in the four spouts. Consider spout A. The unit vector $\boldsymbol{\epsilon}$ at the centerline of pipe A is given as

$$\boldsymbol{\epsilon} = 0.5\mathbf{i} + 0.4\mathbf{j} + 0.768\mathbf{k}$$

If each pipe has an inside diameter of 3 in. what is the vertical force on the tank and the torque on the tank about the y axis at the center of the tank, both as a result of the water flowing? The inlet flow at B is the xy plane.

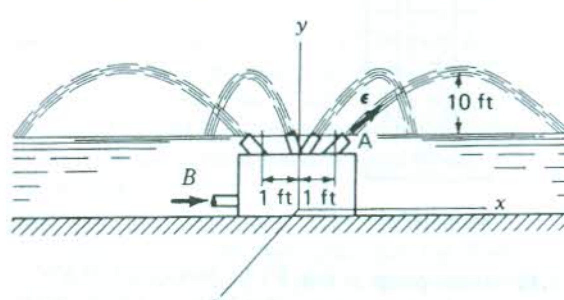


Figure P5.48

5.49 In Prob. 5.19, determine the time it takes for the water above position A to drain from the tank. The value of A is 5 ft. The pressure $p = C_1\sqrt{z} + C_2$ with z the distance that the free surface of the oil descends starting at 5 psi gage and going to 2 psi gage when the free surface of the water reaches the level denoted as A. The diameter of the cylindrical portion of the tank is 4 ft.

5.50 In Prob. 5.4, formulate an interactive program for the horsepower developed by the turbine as a function of the rate of flow Q for inlet pressures p_1, exit pressure p_2, height h, diameter D_1, and diameter D_2. Using the data of Prob. 5.4, evaluate the power output. Plot horsepower vs. rate of flow for different heights h using the other data of the problem statement.

5.51 Develop an interactive computer program for the horsepower output of the turbine in Example 5.1 wherein the following terms are interactively inserted in the program:

Diameter D in inches

Pressure p_1 in psi abs

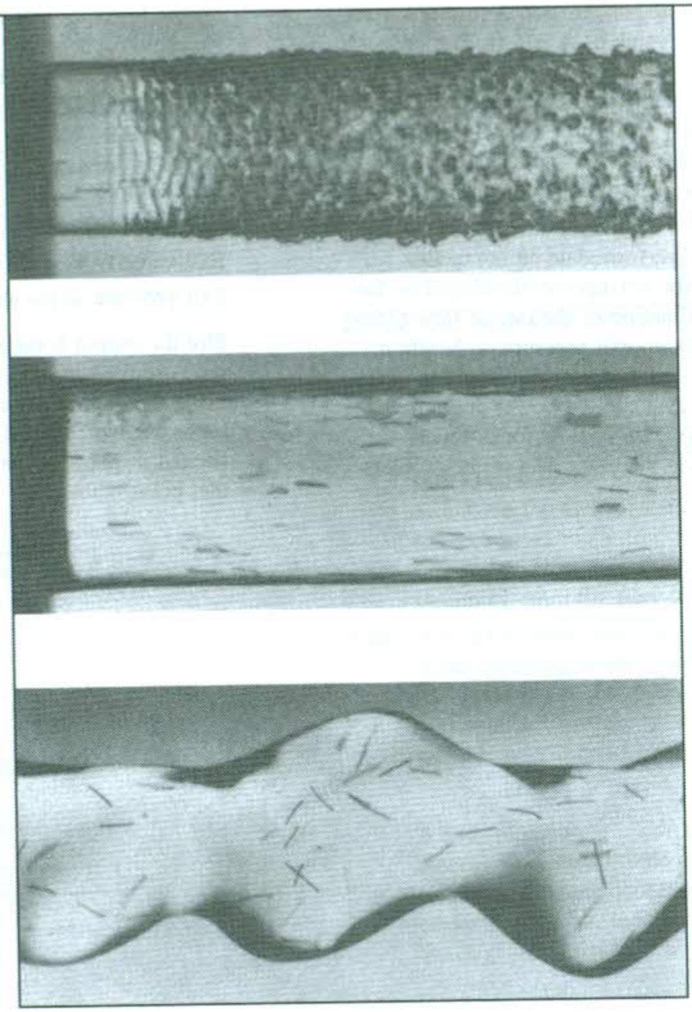
Inlet velocity V_1

Height h in ft (now 10 ft)

Exit velocity V_2 in ft/s

Exit pressure in psi abs

Plot the output horsepower vs. the inlet pressure for varying values of height but using the data in the example for the other terms. At which point in the plot is a pump needed in place of the turbine for the data at that point?



Photographs of free jets showing effects on surface due to turbulence as well as impending breakup of a jet. (*Courtesy J. W. Hoyt and J. J. Taylor, flow visualization using the "floc" technique, Flow Visualization II by Wolfgang Merzkirch, Hemisphere Publishing Corp., 1982, p. 683.*)

A free jet is shown in the top figure at exit, illustrating a transition to turbulence on the jet surface. In the second photo, we have flow from a nozzle designed to delay transition to turbulence on the jet surface. The third photo shows a jet about to break up. Notice the random orientation of the "floc" particles. We have only considered in the book the free jet at immediate exit. Obviously, the free jet in its entirety represents a complex flow about which books have been written.

CHAPTER 6

Differential Forms of the Basic Laws

6.1 INTRODUCTION

In Chaps. 4 and 5 we considered basic laws for finite systems, and using the Reynolds transport equation, we formulated the basic laws for finite control volumes. Particularly for steady flows, we were able to solve for certain resultant force components, average velocities, and so on, by utilizing information concerning the flow at the control surface. We did not need detailed information concerning the characteristics of the flow everywhere inside the control volume.[1] At times this is an advantage in that it affords great simplicity in computing certain quantities. The weakness of this approach is that you can get only average values of quantities or only components of resultant forces but cannot learn about the details of the flow. For this information we employ appropriate *differential equations* valid at a point. These equations must then be integrated to satisfy the boundary conditions of the problem.

In this chapter, we first present differential equations for the conservation of mass by considering an infinitesimal control volume and then for Newton's law by considering an infinitesimal system. To illustrate the use of the differential form of Newton's law, we examine the steady-state configuration of liquids under constant acceleration and under constant rotation about a vertical axis.

It is important to say once again that infinitesimal systems and infinitesimal control volumes yield exactly the same differential equations for any given law. We will use both approaches in the ensuing work.

6.2 CONSERVATION OF MASS

In dealing with the differential forms of the basic laws, we generally use the notation u, v, w to respectively replace V_x, V_y, V_z. We shall do this in this and succeeding chapters.

[1]This is analogous to the blackbox approach of systems theory.

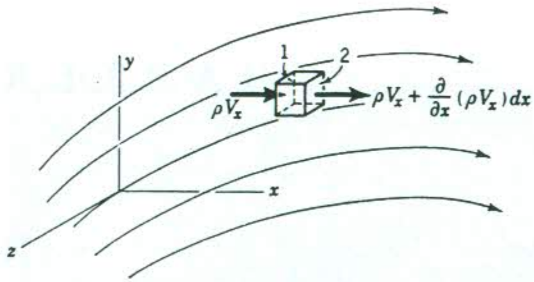


Figure 6.1
Infinitesimal fixed control volume.

We now examine an infinitesimal control volume in the shape of a rectangular parallelepiped (Fig. 6.1) fixed in xyz for some general flow $\mathbf{V}(x, y, z, t)$ measured relative to xyz. In computing the net efflux rate for this control volume, we first consider flow through the surfaces 1 and 2, which are parallel to the yz plane. Note from Fig. 6.1 that the efflux rate through area 1 is $-\rho u$ per unit area and that for a continuum this varies continuously in the x direction, so the efflux rate per unit area through area 2 can be given as $\rho u + [\partial(\rho u)/\partial x]\, dx$.[2] The net efflux rate through these surfaces is then $[\partial(\rho u)/\partial x]\, dx\, dy\, dz$. Performing similar computations for the other pairs of sides and adding the results, we get the net efflux rate:

$$\text{Net efflux rate} = \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dx\, dy\, dz \qquad [6.1]$$

Equating this to the rate of decrease of mass inside the control volume, $-(\partial\rho/\partial t)\, dx\, dy\, dz$, we get the desired equation for conservation of mass after we cancel $dx\, dy\, dz$.

$$\boxed{\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = -\frac{\partial\rho}{\partial t}} \qquad [6.2]$$

We often call this the *differential continuity equation*. For *steady flow* this equation becomes

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad [6.3]$$

and for *incompressible flow* we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad [6.4]$$

even if the flow is unsteady.

By using other suitable infinitesimal control volumes you can develop corresponding differential equations for cylindrical and spherical coordinates. These

[2] We are using a two-term Taylor series expansion here.

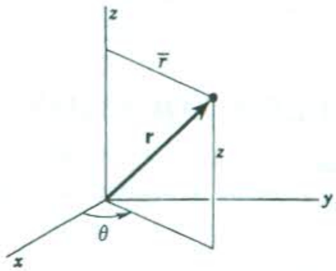


Figure 6.2
Cylindrical coordinates.

equations are all *differential* forms of the continuity equation. In the case of cylindrical coordinates (see Fig. 6.2) we get

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}\rho v_{\bar{r}}) + \frac{1}{\bar{r}}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = -\frac{\partial \rho}{\partial t} \qquad [6.5]$$

As in vector mechanics, we have used $\bar{r}$ as the radial distance from the z axis. However, if the context is clear we may simply use r for the radial coordinate. Although the left sides of Eqs. 6.2 and 6.5 appear quite different, they both convey the same information, namely, the measure of the *rate of efflux of mass per unit volume at a point*. Thus the differences in form are due only to the use of different coordinate systems. To divorce ourselves from the artificially contrived coordinate systems in expressing the efflux rate per unit volume at a point, we introduce the *divergence operator*[3] so that for any and all coordinate systems we can say for the continuity equation that

$$\boxed{\text{div}(\rho \mathbf{V}) = -\frac{\partial \rho}{\partial t}} \quad \text{or} \quad \boxed{\boldsymbol{\nabla} \cdot (\rho \mathbf{V}) = -\frac{\partial \rho}{\partial t}} \qquad [6.6]$$

For different coordinate systems the divergence operator (like the gradient operator) takes on different forms. Thus, for Cartesian coordinates the divergence operating on any vector field **A** becomes

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}} \qquad [6.7]$$

and in cylindrical coordinates we have

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}(\bar{r}A_{\bar{r}}) + \frac{1}{\bar{r}}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}} \qquad [6.8]$$

---

[3]The mathematical definition of the divergence operator is

$$\text{div}\,\mathbf{F} \equiv \boldsymbol{\nabla} \cdot \mathbf{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_S \mathbf{F} \cdot \mathbf{dA}$$

where ΔV is any volume element in space and S is the surface of this volume element. In electricity, div **E** is the flux per unit volume at a point in space from an electric field.

Clearly, for different vector fields **A**, the divergence of **A** takes on a somewhat different meaning from that corresponding to the vector field $\rho\mathbf{V}$.

6.3 NEWTON'S LAW; EULER'S EQUATION

Linear momentum of an element of mass *dm* is a vector quantity defined as *dm* **V**. The fundamental statement of Newton's law for an inertial reference is given in terms of linear momentum as

$$\mathbf{dF} = \frac{D}{Dt}(dm\,\mathbf{V})$$

In the case where the infinitesimal element of mass *dm* is part of a velocity field $\mathbf{V}(x, y, z, t)$ as seen from the inertial reference, this equation may be given as

$$\mathbf{dF} = \frac{D}{Dt}(dm\,\mathbf{V}) = dm\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} + \frac{\partial \mathbf{V}}{\partial t}\right) \qquad [6.9]$$

The extreme right-hand quantity in parentheses is the substantial derivative discussed in Chap. 3. This equation, restricted to the case of *no shear* stress with *only gravity* as a body force is often called *Euler's equation*. The surface force on a fluid element is then due only to pressure p, and in accordance with our discussion in Sec. 1.13 we can express this force in the form $-(\boldsymbol{\nabla}p)\,dv$. We take the negative z direction to correspond to the direction of gravity. We can thus express the gravity force as $-g\rho\,dv\,\mathbf{k}$, which can also be given as $-\mathrm{g}(\boldsymbol{\nabla}z)(\rho\,dv)$. Employing the aforesaid forces in Eq. 6.9 and dividing through by $\rho\,dv(= dm)$, we then have Euler's equation

$$\boxed{-\frac{1}{\rho}\,\boldsymbol{\nabla}p - g\,\boldsymbol{\nabla}z = \left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right) + \frac{\partial \mathbf{V}}{\partial t} = \frac{D\mathbf{V}}{Dt}} \qquad [6.10]$$

Using familiar vector operations, we can next express the acceleration of transport in the following way:

$$u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V}$$

You may readily verify this yourself by carrying out the operations on the right side of the equation in terms of Cartesian components, remembering to evaluate the operator in the parentheses first before operating on the last term **V**. We then may present another form of Euler's equation:

$$\boxed{-\frac{1}{\rho}\,\boldsymbol{\nabla}p - g\,\boldsymbol{\nabla}z = (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} + \frac{\partial \mathbf{V}}{\partial t}} \qquad [6.11]$$

The expanded forms of the above equation in rectangular, cylindrical, and streamline coordinates are shown as follows.

Rectangular Coordinates

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + B_x = \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + \frac{\partial u}{\partial t} \quad [6.12a]$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} + B_y = \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) + \frac{\partial v}{\partial t} \quad [6.12b]$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + B_z = \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) + \frac{\partial w}{\partial t} \quad [6.12c]$$

Cylindrical Coordinates

$$-\frac{1}{\rho}\frac{\partial p}{\partial \bar{r}} + B_{\bar{r}} = \left(v_{\bar{r}}\frac{\partial v_{\bar{r}}}{\partial \bar{r}} + \frac{v_\theta}{\bar{r}}\frac{\partial v_{\bar{r}}}{\partial \theta} - \frac{v_\theta^2}{\bar{r}} + v_z\frac{\partial v_{\bar{r}}}{\partial z}\right) + \frac{\partial v_{\bar{r}}}{\partial t} \quad [6.12d]$$

$$-\frac{1}{\rho}\frac{\partial p}{\bar{r}\partial \theta} + B_\theta = \left(v_{\bar{r}}\frac{\partial v_\theta}{\partial \bar{r}} + \frac{v_\theta}{\bar{r}}\frac{\partial v_\theta}{\partial \theta} + \frac{v_{\bar{r}}v_\theta}{\bar{r}} + v_z\frac{\partial v_\theta}{\partial z}\right) + \frac{\partial v_\theta}{\partial t} \quad [6.12e]$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + B_z = \left(v_{\bar{r}}\frac{\partial v_z}{\partial \bar{r}} + \frac{v_\theta}{\bar{r}}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) + \frac{\partial v_z}{\partial t} \quad [6.12f]$$

Streamline Coordinates in the Osculating Plane[4]

$$-\frac{1}{\rho}\frac{\partial p}{\partial s} + B_s = V\frac{\partial V}{\partial s} + \frac{\partial V}{\partial t} \quad [6.12g]$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial n} + B_n = \frac{V^2}{R} + \frac{\partial V_n}{\partial t} \quad [6.12h]$$

We consider next two applications of Euler's equation.

*6.4 LIQUIDS UNDER CONSTANT RECTILINEAR ACCELERATION OR UNDER CONSTANT ANGULAR SPEED

Case 1. Uniformly Accelerating Liquid Consider a liquid in a container undergoing constant acceleration **a** in inertial space. Suppose that we choose the z axis of the inertial reference to be vertical to the earth's surface and with the y axis to form coordinate plane yz parallel to the acceleration and velocity vectors of the container (see Fig. 6.3). Thus $a_x = V_x = 0$. After a period of time, the liquid will reach a fixed orientation relative to the container.[5] Accordingly, all fluid elements will then have

---

[4]The osculating plane at a position on a curve is the limiting plane of three different points on the curve as they come together at the position of interest on the curve.

[5]This is analogous to a linear mass-spring problem of sophomore mechanics, where a sudden constant force is applied to the mass. The mass will oscillate as a result of the disturbance, but this motion will die out as a result of friction, leaving in due course a fixed deflection of the spring. The motion that dies out is the so-called *transient* leaving the *steady-state* fixed deflection. We are here concerned with the steady-state orientation of the liquid.

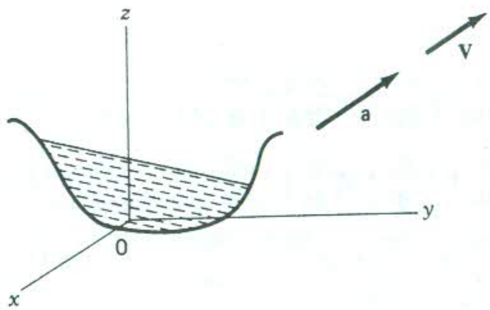


Figure 6.3
Container of liquid accelerates uniformly along a straight line parallel to yz plane.

the same velocity at any time t. This means that $\partial \mathbf{V}/\partial x = \partial \mathbf{V}/\partial y = \partial \mathbf{V}/\partial z = 0$, and $\partial \mathbf{V}/\partial t = \mathbf{a}$ throughout the liquid. From Newton's viscosity law, furthermore, with $\partial \mathbf{V}/\partial n = 0$ in any direction $\mathbf{n}$, we conclude that there is no shear stress present. In that case, the stress field degenerates to a pressure field. We can accordingly apply Euler's equation for this case. Thus, using p_g as the gage pressure we have

$$-\frac{1}{\rho}\,\nabla(p_g + p_{\text{atm}}) - g\,\nabla z = \mathbf{a}$$

$$\therefore\ -\frac{1}{\rho}\,\nabla p_g - g\,\nabla z = \mathbf{a} \tag{6.13}$$

Let us first consider the z component of this equation. We get

$$-\frac{1}{\rho}\frac{\partial p_g}{\partial z} - g = a_z$$

$$\therefore\ \boxed{\frac{\partial p_g}{\partial z} = -\rho(g + a_z)} \tag{6.14}$$

The gage pressure p_g thus varies in the z direction like a *hydrostatic* pressure variation, except that we add to the acceleration of gravity g the acceleration component a_z. With $a_z = 0$ the pressure variation is *exactly hydrostatic,* as studied in Chap. 2.

What is the shape of the free surface? To determine this, take the dot product of Eq. 6.13 with the infinitesimal position vector $\mathbf{dr}$ in *any* direction. Thus dropping the subscript g on gage pressure p for convenience,

$$-\frac{1}{\rho}\,\nabla p \cdot \mathbf{dr} - g\,\nabla z \cdot \mathbf{dr} = \mathbf{a} \cdot \mathbf{dr}$$

$$\therefore\ -\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right) - g\,dz = a_z\,dz + a_y\,dy$$

Noting that the expression in parentheses is the total differential dp, we get

$$-\frac{dp}{\rho} - g\,dz = a_z\,dz + a_y\,dy \tag{6.15}$$

Remember that the term dp is the change in pressure in *any* direction. We now integrate Eq. 6.15. Thus

$$
\begin{aligned}
-\frac{p}{\rho} - gz &= a_z z + a_y y + C' \\
\therefore p &= -\rho a_z z - \rho a_y y - \rho g z + C
\end{aligned}
\qquad [6.16]
$$

We determine the constant of integration C from knowledge of the pressure at some known location. We may rewrite Eq. 6.16 on noting that $\rho = \gamma/g$ to have the following form:

$$
\boxed{p = C - \gamma y \frac{a_y}{g} - \gamma z\left(1 + \frac{a_z}{g}\right)} \qquad [6.17]
$$

At the *free surface*, $p = 0$ gage, so we have from Eq. 6.17 for this surface

$$
\left(\gamma \frac{a_y}{g}\right) y + \gamma\left(1 + \frac{a_z}{g}\right) z = C \qquad [6.18]
$$

This is clearly the equation of a *plane surface*—a not unexpected result. The slope of this surface is

$$
\boxed{\left(\frac{dz}{dy}\right)_{\text{fs}} = -\frac{a_y/g}{1 + a_z/g} = -\frac{a_y}{g + a_z}} \qquad [6.19]
$$

We now consider an example to illustrate the use of the formulations above.

EXAMPLE 6.1

■ Problem Statement

An open, small, rectangular container of water is placed on a conveyor belt (Fig. 6.4) which is accelerating at the rate of 5 m/s^2. Will the water spill from the container during the steady-state configuration of the water?

■ Strategy

We will directly focus on the free surface of the water (Fig. 6.5) during steady-state geometry. We will start with the slope equation, Eq. 6.19. Integrating this equation, we will get the angle β that the free surface forms with the y axis in Fig. 6.5. The angle α is then easily determined by inspection. Using simple trigonometry, we will then get an equation involving distances d_1 and d_2 which are the heights on the sides of the container wetted by the water. Next, using conservation of mass, we will get a second equation for these distances. Solving simultaneously, we will be able directly to make the decision we are seeking as to spillage of the water.

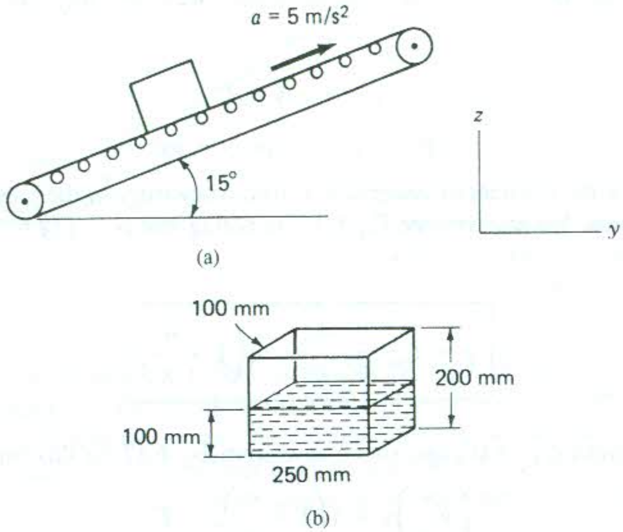


Figure 6.4
Tank of water on an accelerating conveyor belt.

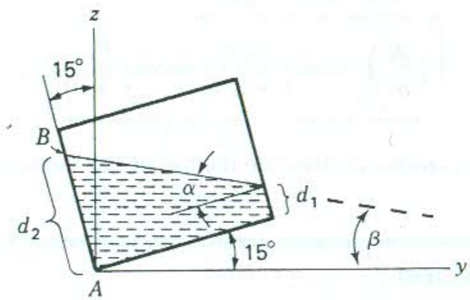


Figure 6.5
Water in steady-state configuration.

■ Execution

We go directly to Eq. 6.19 for the slope of the free surface. Thus,

$$\left(\frac{dz}{dy}\right)_{fs} = -\frac{a_y}{g + a_z} = -\frac{5\cos 15°}{9.81 + 5\sin 15°} = -0.435 \qquad [a]$$

From Fig. 6.6, we can then say for the angle of inclination β (see right end) of the free surface

$$\tan^{-1}\left(\frac{dz}{dy}\right)_{fs} = \tan^{-1}(-0.435) = 180° - \beta$$

$$\therefore \beta = 23.5°$$

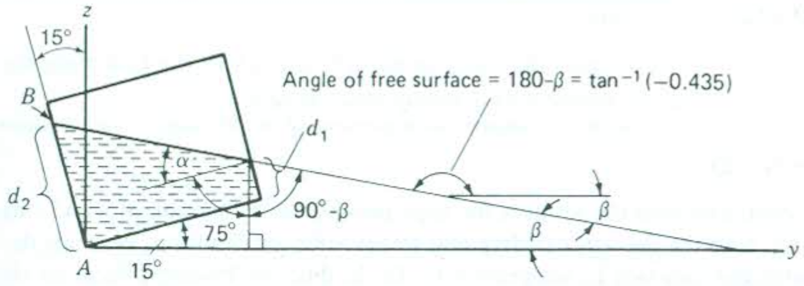


Figure 6.6
Orientation of the free surface.

Next consider Fig. 6.6 again. It may be clear from this diagram that $\alpha = 15° + \beta$. If not, note from Fig. 6.6 that

$$\alpha + 75° + (90° - \beta) = 180°$$
$$\therefore \alpha = 15° + \beta \qquad [b]$$
$$= 15° + 23.5° = 38.5°$$

Now, considering Fig. 6.5 again, we can relate d_1 and d_2 for the free surface as follows:

$$\frac{d_2 - d_1}{250} = \tan\alpha \qquad [c]$$

Hence, from Eq. c,

$$d_2 = 250\tan\alpha + d_1 = 198.9 + d_1 \qquad [d]$$

We now consider *conservation of mass* of the water. Thus considering Figs. 6.4b and 6.5, we compute the mass using each figure.

$$\rho[(250)(100)(100)] = \rho\left[(250)(100)\left(\frac{d_1 + d_2}{2}\right)\right] \qquad [e]$$
$$\therefore d_1 + d_2 = 200$$

Solving Eqs. d and e simultaneously, we get

$$d_2 = 199.5 \text{ mm}$$

On consulting Figs. 6.4b and 6.5, we can conclude that the water does *not* spill during steady-state operation.

■ Debriefing

We were able to use the integrated forms of the famous Euler equations for solving this problem. Thus, we went back to rock-bottom fundamentals. Another approach that is quite prevalent is to consider the concept of *relative equilibrium* for working problems of this kind. To do this, we use the D'Alembert principle from dynamics by considering the inertia term in the direction of acceleration as a fictitious force and treating this component of Newton's law as one of the component equations of equilibrium.

EXAMPLE 6.2

■ Problem Statement

In Example 6.1, determine the force on the left side AB of the tank from the water inside and air outside during steady-state operation.

■ Strategy

We shall start with the formula for gage pressure as a function of y and z as per Eq. 6.16 for the water undergoing steady state acceleration. We must determine the constant of integration C. To do this, we choose a location on the free surface and substitute coordinates of this point as well as the zero gage pressure into the aforementioned equation thus permitting the calculation of C. Now we use the pressure variation along AB to determine via calculus the desired force.

■ Execution

The gage pressure in the tank is given as follows in accordance with Eq. 6.17:

$$\begin{aligned} p &= -\rho a_z z - \rho a_y y - \rho g z + C \\ &= -(1000)(5\sin 15^\circ)z - (1000)(5\cos 15^\circ)y - (1000)(9.81)z + C \qquad \text{[a]} \\ &= -4830y - 11{,}100z + C \end{aligned}$$

where C is the constant of integration. In Fig. 6.6, note that $p = 0$ at position B where

$$\begin{aligned} y &= -d_2 \sin 15^\circ = -0.1995 \sin 15^\circ = -0.0516 \text{ m} \\ z &= d_2 \cos 15^\circ = 0.1995 \cos 15^\circ = 0.1927 \text{ m} \end{aligned}$$

Going back to Eq. a, we have at B

$$\begin{aligned} 0 &= -(4830)(-0.0516) - (11{,}100)(0.1927) + C \\ \therefore C &= 1890 \text{ N/m}^2 \end{aligned} \qquad \text{[b]}$$

Now consider the side BA in Fig. 6.7. The force df on segment $d\eta$ is found using Eq. a as:

$$\begin{aligned} df &= p\,dA = p(0.100)\,d\eta \\ &= [-4830(-\eta \sin 15^\circ) - (11{,}100)(\eta \cos 15^\circ) + 1890](0.100)\,d\eta \qquad \text{[c]} \\ &= [-947\eta + 189.0]\,d\eta \quad \text{N} \end{aligned}$$

Integrating over the entire side AB, we have

$$F = \int_0^{0.1995} (-947\eta + 189.0)\,d\eta = 18.86 \text{ N}$$

Can you find the center of pressure?

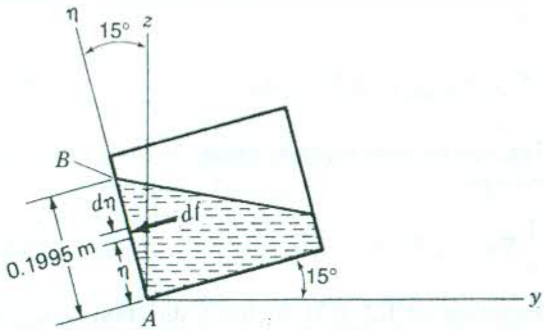


Figure 6.7
Force df on wall from pressure.

■ Debriefing

An interesting question involves determining whether the tank starts to tip for the given steady-state acceleration. We will leave this as a homework problem, and as a small project using MATLAB we suggest you plot steady-state acceleration for tipping vs. the angle of incline of the conveyor.

Case 2. Uniform Rotation about a Vertical Axis We now consider a cylindrical container of liquid (see Fig. 6.8) which is maintained at a uniform angular rotation for a long enough period of time to have the liquid reach a steady-state orientation relative to the container. We would like the shape of the free surface as well as the pressure distribution below the free surface for this steady-state configuration.

Clearly, for steady-state operation, the liquid rotates as if it were a solid (i.e., with no relative deformational motion between particles), and hence there is no shearing action between elements in the flow. Thus, there is no shear stress on the

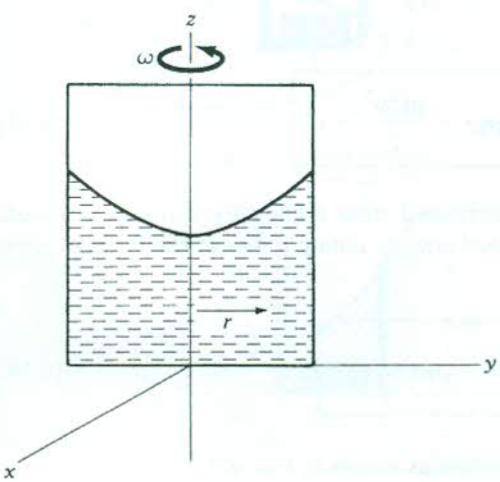


Figure 6.8
Cylinder rotating at constant speed. Liquid has reached steady-state configuration.

elements, and the stress field once more degenerates to a pressure distribution. We can again use Eq. 6.13 as follows:

$$-\frac{1}{\rho}\nabla(p + p_{\text{atm}}) - g\,\nabla z = \mathbf{a} \tag{6.20}$$

where p is the gage pressure. The acceleration vector is toward the axis of rotation and is given as $-\bar{r}\omega^2\boldsymbol{\epsilon}_{\bar{r}}$. We then have

$$-\frac{1}{\rho}\nabla p - g\,\nabla z = -\bar{r}\omega^2\boldsymbol{\epsilon}_{\bar{r}} \tag{6.21}$$

First, we consider the component of Eq. 6.21 in the z direction—i.e., the direction of gravity. We get

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = 0$$

$$\therefore \quad \boxed{\frac{\partial p}{\partial z} = -\gamma} \tag{6.22}$$

We see that the pressure p varies *hydrostatically* in the z direction. We next take the dot product of Eq. 6.21 with the arbitrary infinitesimal position vector $\mathbf{dr} = d\bar{r}\,\boldsymbol{\epsilon}_{\bar{r}} + \bar{r}\,d\theta\,\boldsymbol{\epsilon}_\theta + dz\,\boldsymbol{\epsilon}_z$, where we have used cylindrical coordinates. Recalling from case 1 of Sec. 6.4 that $\nabla p \cdot \mathbf{dr}$ results in the change in pressure dp, we get

$$-\frac{dp}{\rho} - g\,dz = -\bar{r}\omega^2\,d\bar{r}$$

We now integrate the equation to get

$$-\frac{p}{\rho} - gz = -\frac{\bar{r}^2\omega^2}{2} + C'$$

Rearranging the equation, we have

$$\boxed{p = -\rho g z + \frac{\rho\bar{r}^2\omega^2}{2} + C} \tag{6.23}$$

The constant of integration C is determined from a known pressure at a known location. The *free surface* is determined now by setting $p = 0$ in Eq. 6.23. We then get for the free surface

$$\boxed{\frac{\rho\omega^2}{2}\bar{r}^2 - \rho g z = -C} \tag{6.24}$$

The free surface is a *paraboloidal* surface as shown in Fig. 6.8.

EXAMPLE 6.3

■ Problem Statement

Two tubes of length 6 in and internal diameter of 0.1 in are connected to a small tank (see Fig. 6.9). The tubes and tank contain water as shown. The system is attached to a platform. At what angular speed must the platform rotate so that the steady-state configuration of the water will cause water to spill out from one of the tubes? Disregard capillary effects.

■ Strategy

The key idea to use is to insert a *hypothetical cylinder* of water having at rest the same height of water as the tubes. We will note that the heights reached by the water in the tubes will be the same as those of the free surface of water in the hypothetical cylindrical tank at the centerline, if the latter is imagined to rotate with the same speed as the *tank* and if we consider the same corresponding radial distances from the vertical centerline for the tubes and the cylinder. This must be true since the pressures will be the same for the aforementioned points of the two systems. This is shown in Fig. 6.10.

■ Execution

We focus now on the free surface of the hypothetical, rotating cylinder. We state Eq. 6.24 as follows:

$$\frac{\rho\omega^2}{2}\bar{r}^2 - \rho g z = -C \qquad [a]$$

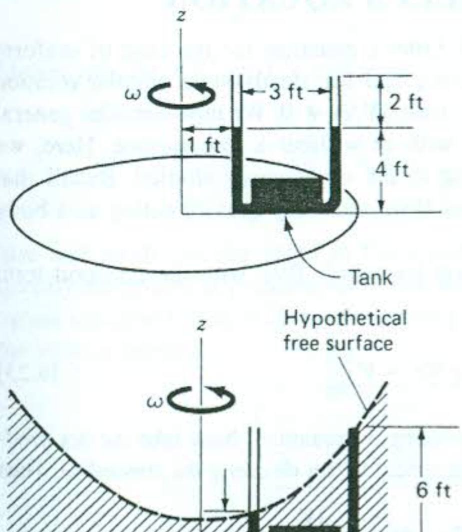


Figure 6.9
Initial configuration for $\omega = 0$.

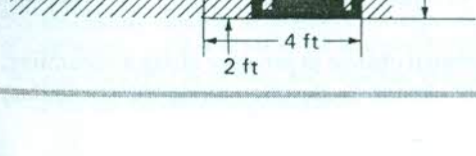


Figure 6.10
Desired steady-state configuration.

We have two unknowns—ω and C. However, we know for spillage (see Fig. 6.10) that when $\bar{r} = 4$ ft, then $z = 6$ ft, and from conservation of mass in the tube-tank system, that when $\bar{r} = 1$ ft, then $z = 2$ ft. Applying these conditions to Eq. a, we have

$$\frac{1.938\omega^2}{2}4^2 - (1.938)(32.2)(6) = -C$$

$$\frac{1.938\omega^2}{2}1^2 - (1.938)(32.2)(2) = -C$$

Solving simultaneously, we get

$$\omega = 4.14 \text{ rad/s}$$

■ Debriefing

You will note that the pressure in the hypothetical cylinder of water under steady-state rotation is hydrostatic. Question for discussion: How would you determine stresses in the wall of a real tank under conditions outlined for the theoretical case of this problem?

6.5 INTEGRATION OF THE STEADY-STATE EULER EQUATION; BERNOULLI'S EQUATION

In the previous section, we integrated Euler's equation for the case of uniform acceleration of liquid with a free surface and for steady-state angular rotation of a liquid with a free surface. In both cases, $\partial \mathbf{V}/\partial t \neq \mathbf{0}$. We now consider general steady-state flow of an inviscid fluid with or without a free surface. Here, we will require that $\partial \mathbf{V}/\partial t = \mathbf{0}$ in contrast to the other cases studied. Recall that Euler's equation is valid for frictionless flow with only gravity acting as a body force.

Let us then express Euler's equation for *steady flow* with the transport term given for streamline coordinates. Thus

$$-\frac{\nabla p}{\rho} - g\,\nabla z = V\frac{\partial \mathbf{V}}{\partial s} \qquad [6.25]$$

where s, you will recall, is the coordinate along a streamline. Now take the dot product of each term in Eq. 25, using the displacement vector **ds** *along the streamline.* Thus

$$-\frac{\nabla p \cdot \mathbf{ds}}{\rho} - g\,\nabla z \cdot \mathbf{ds} = V\frac{\partial \mathbf{V}}{\partial s} \cdot \mathbf{ds}$$

The term $\nabla p \cdot \mathbf{ds}$ becomes dp, the differential change in pressure along a streamline; and $\nabla z \cdot \mathbf{ds}$ becomes dz, the differential change in elevation along the streamline.

The right side of the equation simplifies to $V\,dV$, where dV is the velocity change along a streamline. Thus[6]

$$-\frac{dp}{\rho} - g\,dz = V\,dV = d\left(\frac{V^2}{2}\right)$$

Taking g as constant and integrating along a streamline, we get

$$\boxed{\int_0^p \frac{dp}{\rho} + gz + \frac{V^2}{2} = \text{const}} \qquad [6.26]$$

This equation is often called the *compressible* form of Bernoulli's equation. If ρ is expressible as a function of p only, that is, $\rho = \rho(p)$, the first expression is integrable. Flows having this characteristic are called *barotropic flows.* If, on the other hand, the flow is *incompressible,* we then get[7]

$$\boxed{\frac{p}{\rho} + gz + \frac{V^2}{2} = \text{const}} \qquad [6.27]$$

This is Bernoulli's equation, derived in Chap. 5 from the first law of thermodynamics. Between any two points along a streamline, we have

$$\boxed{\frac{p_1}{\rho} + gz_1 + \frac{V_1^2}{2} = \frac{p_2}{\rho} + gz_2 + \frac{V_2^2}{2}} \qquad [6.28]$$

Multiplying Eq. 6.27 by $1/g$ and replacing $g\rho$ by γ, we get

$$\boxed{\frac{p}{\gamma} + z + \frac{V^2}{2g} = \text{const}} \qquad [6.29]$$

You will recall that the terms in this equation are in units of length and are frequently designated as pressure, elevation, and velocity *heads,* respectively. The analogous equation to Eq. 6.28 between two points in the flow can then be given for the various heads as

$$\boxed{\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}} \qquad [6.30]$$

[6]We have had to restrict **ds** to be along a streamline to get the right side of the equation into the form of a differential and thus to permit the integration to give the simple result $V^2/2$.

[7]We may use p/ρ or pv in Bernoulli's equation. In thermodynamics, the tendency is to use pv. We will generally use p/ρ in fluid mechanics.

We see that for frictionless, isothermal flow the first law of thermodynamics is equivalent to Newton's law. This is the same situation that prevailed in particle and rigid-body mechanics, where Newton's law, linear momentum methods, and energy methods were equivalent to each other. That is, one could solve a problem considering any of these three methods. (To be sure, certain problems were more easily solved by certain ones of the three methods.) We point out next that if the flow is compressible or if friction causes changes in the properties of the fluid by causing temperature changes, among other things, the first law of thermodynamics and Newton's law (in the form of momentum equations) are *independent* equations and must be separately satisfied.

6.6 BERNOULLI'S EQUATION APPLIED TO IRROTATIONAL FLOW

The Bernoulli equation developed in Sec. 6.5 may be applied between points along any one streamline when there is frictionless, incompressible, steady flow. If we stipulate further that the flow be *irrotational*, we can show that Bernoulli's equation is valid between *any two* points in a flow.

Thus consider Euler's equation for steady flow in the form given by Eq. 6.13:

$$-\frac{\boldsymbol{\nabla} p}{\rho} - g\,\boldsymbol{\nabla} z = (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V}$$

You will recall that this equation is valid for frictionless flow with gravity as the only body force. The term $(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V}$ in this equation may be replaced in the following manner:

$$(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} = \boldsymbol{\nabla}\left(\frac{V^2}{2}\right) - \mathbf{V}\times\mathbf{curl}\,\mathbf{V} \qquad [6.31]$$

a step which you may readily verify by carrying out the operations on both sides of the equation, using Cartesian components of the vectors and operators. Euler's equation can then be written as

$$-\frac{\boldsymbol{\nabla} p}{\rho} - g\,\boldsymbol{\nabla} z = \boldsymbol{\nabla}\left(\frac{V^2}{2}\right) - \mathbf{V}\times\mathbf{curl}\,\mathbf{V} \qquad [6.32]$$

If the flow is *irrotational*, $\mathbf{curl}\,\mathbf{V} = \mathbf{0}$ and Eq. 6.32 simplifies to

$$\frac{\boldsymbol{\nabla} p}{\rho} + g\,\boldsymbol{\nabla} z + \boldsymbol{\nabla}\left(\frac{V^2}{2}\right) = \mathbf{0} \qquad [6.33]$$

We next take the dot product of the terms in the equation with the *arbitrary* infinitesimal displacement vector $\mathbf{dr}$. Noting that $\boldsymbol{\nabla} p\cdot\mathbf{dr} = dp$, $\boldsymbol{\nabla} z\cdot\mathbf{dr} = dz$, and $\boldsymbol{\nabla}(V^2/2)\cdot\mathbf{dr} = d(V^2/2)$, where the differentials represent infinitesimal changes of the quantities in the *arbitrary* direction of **dr**, we get

$$\frac{dp}{\rho} + g\,dz + d\left(\frac{V^2}{2}\right) = 0$$

Taking g as a constant, and integrating, we get

$$\int_0^p \frac{dp}{\rho} + gz + \frac{V^2}{2} = \text{const} \qquad [6.34]$$

By limiting ourselves to incompressible flow we then get

$$\frac{p}{\rho} + gz + \frac{V^2}{2} = \text{const} \qquad [6.35]$$

Multiplying through by $1/g$,

$$\frac{p}{\gamma} + z + \frac{V^2}{2g} = \text{const} \qquad [6.36]$$

Since **dr** was arbitrary in direction, there was no directional restriction on the differentials formed, with the result that the integrated formulations are valid *everywhere* in the flow. Thus, by including the restriction of irrotationality, we can throw out the earlier requirement of remaining on a particular streamline for a given constant in Bernoulli's equation. Hence, between *any* two points 1 and 2 in such a flow we have

$$\boxed{\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g}} \qquad [6.37]$$

We now consider Newton's law for a general flow.

*6.7 NEWTON'S LAW FOR GENERAL FLOWS

In Sec. 6.3, we considered the differential form of Newton's law for inviscid flows in a gravitational field and developed the Euler equations of motion for a fluid. Now we consider a general flow of fluid of any kind under any circumstance. Accordingly, consider an infinitesimal element of fluid which at time t is a rectangular parallelepiped (Fig. 6.11). We show only stresses on the faces of the element which give force increments in the x direction. Note that we have assumed the stresses to vary continuously and have used Taylor series expansions limited to two terms. Thus, on face 1, we have shown τ_{xx} and on face 2, a distance dx apart from face 1, we have used $\tau_{xx} + (\partial\tau_{xx}/\partial x)\,dx$, where τ_{xx} in this expression corresponds to the stress on face 1 and the derivative is taken at a position corresponding to face 1. Now we

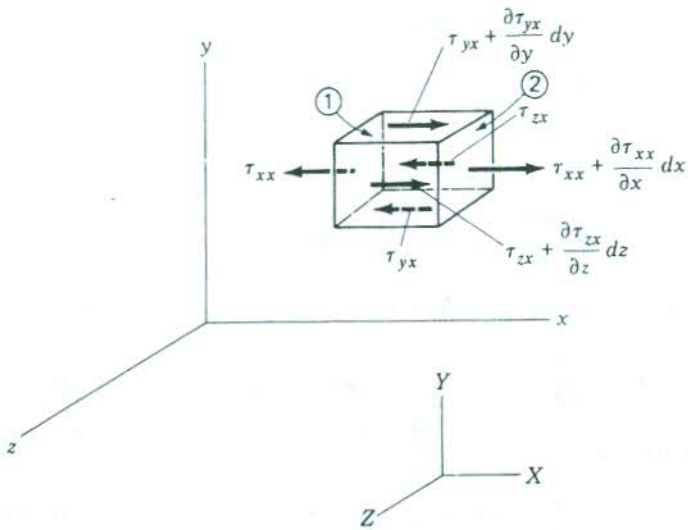


Figure 6.11
Stresses contributing force in x direction.

use Newton's law in the x direction relative to inertial reference XYZ. With body-force component B_x, we have

$$\left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x} dx\right) dy\, dz - \tau_{xx}\, dy\, dz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx\, dz - \tau_{yx}\, dx\, dz$$
$$+ \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) dx\, dy - \tau_{zx}\, dx\, dy + B_x \rho\, dx\, dy\, dz = \rho(dx\, dy\, dz) a_x \qquad [6.38]$$

where a_x is the acceleration component in the x direction. Canceling terms and dividing through by $dx\, dy\, dz$, we get

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho B_x = \rho a_x$$

Now using the complementary property of shear ($\tau_{ij} = \tau_{ji}$) and formulating Newton's law in the y and z directions, we get the desired form of Newton's law as follows:

$$\boxed{\begin{aligned} \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho B_x &= \rho \frac{Du}{Dt} \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho B_y &= \rho \frac{Dv}{Dt} \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho B_z &= \rho \frac{Dw}{Dt} \end{aligned}} \qquad [6.39]$$

Note that the equations have a simple format if you simply start off with the matrix array of stresses. These equations may be familiar to some readers who have seen them in strength of materials with the right side of the equation set equal to zero for equilibrium.

If we consider that the stresses are the superposition of a dilatation deformation associated with a scalar pressure field p plus a distortional deformation associated with a stress field τ'_{ij} called the *deviatoric* stress field,[8] we can give Eq. 6.39 as follows:

$$-\frac{\partial p}{\partial x} + \frac{\partial \tau'_{xx}}{\partial x} + \frac{\partial \tau'_{xy}}{\partial y} + \frac{\partial \tau'_{xz}}{\partial z} + \rho B_x = \rho \frac{Du}{Dt} \qquad [6.40a]$$

$$-\frac{\partial p}{\partial y} + \frac{\partial \tau'_{yx}}{\partial x} + \frac{\partial \tau'_{yy}}{\partial y} + \frac{\partial \tau'_{yz}}{\partial z} + \rho B_y = \rho \frac{Dv}{Dt} \qquad [6.40b]$$

$$-\frac{\partial p}{\partial z} + \frac{\partial \tau'_{zx}}{\partial x} + \frac{\partial \tau'_{zy}}{\partial y} + \frac{\partial \tau'_{zz}}{\partial z} + \rho B_z = \rho \frac{Dw}{Dt} \qquad [6.40c]$$

If we have an inviscid flow ($\therefore\ \tau_{xx} = \tau_{yy} = \tau_{zz} = -p$) and with only gravity as the body force, Eq. 6.40 becomes

$$\begin{aligned} -\frac{\partial p}{\partial x} &= \rho \frac{Du}{Dt} \\ -\frac{\partial p}{\partial y} &= \rho \frac{Dv}{Dt} \\ -\frac{\partial p}{\partial z} - \rho g &= \rho \frac{Dw}{Dt} \end{aligned} \qquad [6.41]$$

In vector form, we have

$$-\nabla p - \rho g\ \nabla z = \rho \frac{D\mathbf{V}}{Dt} = \rho\left[(\mathbf{V} \cdot \nabla)\mathbf{V} + \frac{\partial \mathbf{V}}{\partial t}\right] \qquad [6.42]$$

which you will recognize as Euler's equation.

---

[8] In the study of continua in general, the *deviatoric stress tensor* τ'_{ij} is given as

$$\begin{aligned} \tau'_{xx} &= \tau_{xx} - \tfrac{1}{3}(\tau_{xx} + \tau_{yy} + \tau_{zz}) \\ \tau'_{yy} &= \tau_{yy} - \tfrac{1}{3}(\tau_{xx} + \tau_{yy} + \tau_{zz}) \\ \tau'_{zz} &= \tau_{zz} - \tfrac{1}{3}(\tau_{xx} + \tau_{yy} + \tau_{zz}) \\ \tau'_{xy} &= \tau_{xy} \qquad \tau'_{xz} = \tau_{xz} \qquad \tau'_{yz} = \tau_{yz} \end{aligned}$$

We have pointed out in Sec. 1.12 that $\frac{1}{3}(\tau_{xx} + \tau_{yy} + \tau_{zz})$ for most fluids is $-p$. Equation 6.40 is thus Newton's law for such fluids involving the deviatoric stress tensor. Hence, substituting for τ'_{ij} in Eq. 6.40 and replacing p, we then get back to Eq. 6.39.

6.8 PROBLEMS INVOLVING LAMINAR[9] PARALLEL FLOWS

In Part II of the text we shall be concerned with integrating differential forms of basic laws. At this time as a preview we shall integrate certain differential equations of previous sections.[10] This will allow us to use these equations immediately while their development is still fresh. We consider here two *parallel, steady, incompressible isothermal* flows of a *Newtonian* fluid.

Before we begin, we wish to show from the continuity equation that for incompressible parallel flow the velocity profile *cannot change* in the direction of flow. Thus for such a flow in the x direction, we have $v = w = 0$ and from *continuity* we have at a point

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\therefore \frac{\partial u}{\partial x} + 0 + 0 = 0$$

The velocity V which equals u thus cannot change in the direction of flow with the result that the *velocity profile must remain intact.*

We now examine two cases, namely, flows between infinite parallel plates and flows through pipes.

Case 1. Flow between Two Infinite Parallel Plates We now consider steady, laminar, incompressible flow between two infinite parallel plates shown in Fig. 6.12. We shall apply the *linear momentum equation.* Clearly the only nonzero stress from τ'_{ij} is τ'_{xy}. Furthermore, this stress can only be a function of y since the velocity profile does not change in the direction of flow x and also since the problem is two dimensional thus eliminating variations with z. With this in mind, Eq. 6.40 reduces to

$$-\frac{\partial p}{\partial x} + \frac{\partial \tau'_{xy}}{\partial y} = 0 \qquad [6.43a]$$

$$-\frac{\partial p}{\partial y} - \rho g = 0 \qquad [6.43b]$$

$$-\frac{\partial p}{\partial z} = 0 \qquad [6.43c]$$

Note we have used the fact that $v = w = 0$ and $u = u(y)$, as well as the condition of steady flow to render all substantial derivatives equal to zero. Note also that the

[9]You will recall from Chap. 1 that a laminar flow is a well-ordered flow in contrast to a turbulent flow which has a random small velocity variation superposed over what otherwise would be well ordered. We shall begin studies of turbulent flows in Chap. 8.

[10]The reader may now opt to go to Chap. 9, Secs. 9.1, 9.3, and 9.6, to first develop the *Navier-Stokes equations* and then to return here once the differential equations of the parallel plate and pipe flows have been reached. The reader need then only consider here the solutions to these differential equations.

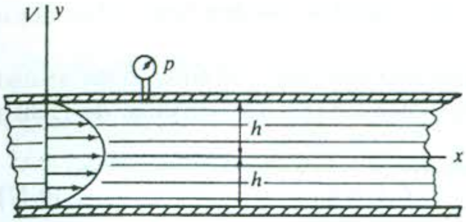


Figure 6.12
Flow between two infinite parallel plates of a Newtonian fluid.

pressure *varies hydrostatically* in the y direction (Eq. 6.43b)[11] and will be a function additionally only of x (Eqs. 6.43a and 6.43c). Hence, at any section x, the gage pressure at the upper plate will be that recorded by a pressure gage mounted on the plate at that position (see Fig. 6.12) and will increase hydrostatically as one descends into the flow at that position x.

Now we apply the *Newton viscosity law* and we focus in on Eq. 6.43a. Thus we have

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \qquad [6.44]$$

If we delete the hydrostatic increase in pressure in the y direction (since it is constant at any level y and clearly has no influence on the velocity field), then p, which is the measurement at the top, will be constant at a section x. Also, $\partial p/\partial x$ is constant at a section x. Hence $\partial p/\partial x$ can only vary with x.

On the other hand, the velocity u has been shown to be independent of x, and in the two-dimensional problem can be a function of only the coordinate y. Thus each side of the equation is a function of a separate distinct variable. If an equality is to be maintained for all values of the independent variables x and y, it is necessary that each side always equal the same constant. Otherwise, one could vary one variable, say, x, and alter the value of the left side of the equation independently of the right side and so invalidate the equality. Hence, it may be said that

$$\frac{\partial p}{\partial x} = -\beta \qquad [6.45a]$$

$$\mu \frac{\partial^2 u}{\partial y^2} = -\beta \qquad [6.45b]$$

where $-\beta$ we take as the aforementioned constant. Since Eq. 6.45a involves only the independent variable x and Eq. 6.45b involves only the independent variable y, it is clear that the preceding equations are in effect two independent ordinary differential equations. They may then be written as

$$\frac{dp}{dx} = -\beta \qquad [6.46a]$$

$$\mu \frac{d^2 u}{dy^2} = -\beta \qquad [6.46b]$$

[11]This will also be shown more generally in Sec. 9.3.

In essence, a second-order *partial* differential equation has then been replaced by a couple of *ordinary* differential equations.

It is now a simple matter to integrate both equations and arrive at the desired results of a velocity profile and an expression for pressure variation. Integrating Eq. 6.46b first, we get

$$u = -\frac{\beta}{\mu}\left(\frac{y^2}{2} - C_1 y + C_2\right) \qquad [6.47]$$

The constants of integration may be determined by employing the boundary conditions, which are

$$u = 0 \quad \text{when } y = \pm h \qquad [6.48]$$

Applying these conditions to the equation leads to the following relations:

$$\begin{aligned} 0 &= -\frac{\beta}{\mu}\left(\frac{h^2}{2} + C_1 h + C_2\right) \\ 0 &= -\frac{\beta}{\mu}\left(\frac{h^2}{2} - C_1 h + C_2\right) \end{aligned} \qquad [6.49]$$

These equations are satisfied if $C_2 = -h^2/2$ and $C_1 = 0$. The velocity profile may then be given as

$$u = \frac{\beta}{2\mu}(h^2 - y^2) \qquad [6.50]$$

As has been indicated in Fig. 6.12, the profile is that of a two-dimensional parabolic surface. The maximum velocity occurs at $y = 0$, so that

$$(u)_{\max} = \frac{\beta h^2}{2\mu} \qquad [6.51]$$

We wish next to express the separation constant β in terms of the flow q. Hence, we have for q per unit width of the flow

$$\begin{aligned} q &= \int_{-h}^{+h} u(1)(dy) \\ &= \int_{-h}^{+h} \frac{\beta}{2\mu}(h^2 - y^2)\,dy \\ &= \frac{\beta}{2\mu}\left[h^2 y - \frac{y^3}{3}\right]\Bigg|_{-h}^{+h} = \frac{\beta}{2\mu}\left[\frac{4}{3}h^3\right] \end{aligned} \qquad [6.52]$$

Solving for β, we get

$$\beta = \tfrac{3}{2}h^{-3}q\mu \qquad [6.53]$$

With β established, the velocity profile from Eq. 6.50 is now fully determined.

Let us now turn our attention to the pressure. Substituting from Eq. 6.53, we may rewrite Eq. 6.46a as

$$\frac{dp}{dx} = -\frac{3}{2}h^{-3}q\mu \qquad [6.54]$$

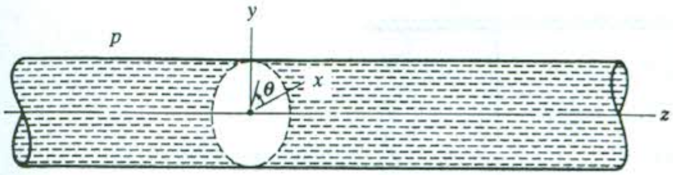


Figure 6.13 Flow in a circular pipe or duct.

Integrating from position 1 to position 2, a distance L apart along the direction of flow, we get

$$(p)_1 - (p)_2 = \tfrac{3}{2}qh^{-3}L\mu \qquad [6.55]$$

The loss in pressure in the direction of flow represented in Eq. 6.55 is attributable only to the action of friction. By dividing both sides of the equation by ρ, there results an expression which we call *head loss*, h_l. This expression represents the loss of pressure due to friction over distance L per unit of mass flowing.[12] Thus

$$\frac{p_1 - p_2}{\rho} \equiv h_l = \frac{3}{2}\frac{qL\mu}{\rho h^3} \qquad [6.56]$$

We now consider axisymmetric laminar incompressible steady flow in a circular duct of pipe.

Case 2. Flow in a Circular Duct Consider the flow in a circular duct or pipe in Fig. 6.13. (Please note the coordinates we are using.) This is another case of parallel flow. As will be shown in Sec. 9.4 the pressure p must vary hydrostatically as will be the case of all parallel incompressible flows. We shall delete hydrostatic contributions and will consider pressure p which is that pressure at the top and which is constant over the section but which will vary with z.

We must now develop an equation for cylindrical coordinates corresponding to Eq. 6.40c. To do this we consider a suitable element in the flow in Fig. 6.14 wherein the shear stresses contributing force in the z direction are shown. For axisymmetric flow Newton's law gives us (noting that the overbars on r are optional)

$$\begin{aligned} -\tau_{zz}\bar{r}\,d\theta\,d\bar{r} &+ \left(\tau_{zz} + \frac{\partial \tau_{zz}}{\partial z}dz\right)\bar{r}\,d\theta\,d\bar{r} - \tau_{\bar{r}z}\bar{r}\,d\theta\,dz \\ &+ \left(\tau_{\bar{r}z} + \frac{\partial \tau_{\bar{r}z}}{\partial \bar{r}}d\bar{r}\right)(\bar{r} + d\bar{r})\,d\theta\,dz + \rho B_z\,d\bar{r}\,\bar{r}\,d\theta\,dz \qquad [6.57] \\ &= \rho\bar{r}\,d\theta\,d\bar{r}\,dz\,\frac{Dv_z}{Dt} \end{aligned}$$

Canceling terms, we get on dividing by $\bar{r}\,d\bar{r}\,d\theta\,dx$,

$$\frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{\bar{r}z}}{\bar{r}} + \frac{\partial \tau_{\bar{r}z}}{\partial \bar{r}} + \frac{\partial \tau_{\bar{r}z}}{\partial \bar{r}}\frac{d\bar{r}}{\bar{r}} + \rho B_z = \rho\frac{Dv_z}{Dt} \qquad [6.58]$$

[12]With no friction clearly there would be zero pressure change. The physical interpretation of h_l is set forth in more detail when we approach it from a thermodynamic viewpoint in Sec. 8.3.

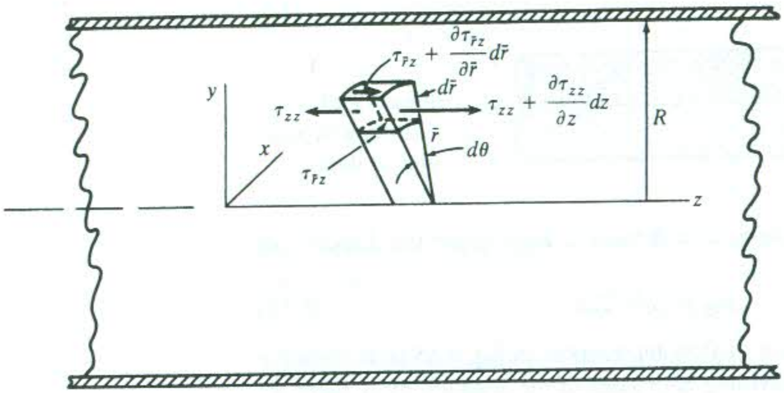


Figure 6.14
Infinitesimal element for cylindrical coordinates with stresses shown only in the z direction.

We can delete the fourth term as negligible. Now we use the Newton viscosity law. We get

$$\frac{\partial \tau_{zz}}{\partial z} + \frac{1}{\bar{r}}\mu\frac{\partial v_z}{\partial \bar{r}} + \mu\frac{\partial^2 v_z}{\partial \bar{r}^2} + \rho B_z = \rho\frac{Dv_z}{Dt}$$

Next, we replace τ_{zz} by $(\tau'_{zz} - p)$ as we have done earlier. We thus have

$$-\frac{\partial p}{\partial z} + \frac{\partial \tau'_{zz}}{\partial z} + \mu\left[\frac{1}{\bar{r}}\frac{\partial v_z}{\partial \bar{r}} + \frac{\partial^2 v_z}{\partial \bar{r}^2}\right] + \rho B_z = \rho\frac{Dv_z}{Dt}$$

For the case at hand, $B_z = Dv_z/Dt = 0$. Also τ'_{zz} cannot vary with z because of the fixed profile in the direction of flow, so that we end up with the following equation to solve:

$$\frac{\partial p}{\partial z} = \mu\left(\frac{\partial^2 v_z}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial v_z}{\partial \bar{r}}\right) \qquad [6.59]$$

Exactly as in the previous analysis one sees that by disregarding hydrostatic pressure variation, the left side of the equation is a function of only the independent variable z. The right side, meanwhile, cannot vary in the direction of flow because of the fixed profile and consequently is a function of only the variable $\bar{r}$. Hence, a separation of the independent variables may be carried out as in Case 1. The resulting ordinary differential equations may then be expressed as

$$\frac{dp}{dz} = -\beta \qquad [6.60a]$$

$$\mu\left(\frac{d^2 v_z}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{dv_z}{d\bar{r}}\right) = -\beta \qquad [6.60b]$$

where $-\beta$ represents a new separation constant to be found later in terms of q, the volume flow per unit time.

Equation (6.60b) may be more easily handled if $dv_z/d\bar{r}$ is replaced by G, thereby forming a first-order differential equation in G. Rearranging terms, we get

$$\frac{dG}{d\bar{r}} + \frac{1}{\bar{r}}G = -\frac{\beta}{\mu} \qquad [6.61]$$

The complementary solution G_c may be found by solving the homogeneous portion of the equation. That is,

$$\frac{dG_c}{d\bar{r}} = -\frac{1}{\bar{r}}G_c$$

Separating variables and integrating,

$$\ln G_c = -\ln \bar{r} + \ln C_1$$

where $\ln C_1$ is the constant of integration. Combining the logarithmic expressions and taking the antilogarithm gives the complementary solution as

$$G_c = \frac{C_1}{\bar{r}}$$

A particular solution may easily be found by inspection to be $G_p = -\beta\bar{r}/2\mu$, so that the general solution for G becomes

$$G = -\frac{1}{2}\frac{\beta}{\mu}\bar{r} + \frac{C_1}{\bar{r}} \qquad [6.62]$$

Replacing G by $dv_z/d\bar{r}$, we may perform a second integration. Thus

$$v_z = -\frac{\beta}{4\mu}\bar{r}^2 + C_1 \ln \bar{r} + C_2 \qquad [6.63]$$

When $\bar{r} = 0$, the term $\ln \bar{r}$ "blows up" and becomes infinite (see Fig. 6.15b), so that the constant C_1 must be zero to render the equation physically meaningful. Also at the pipe wall—that is, $\bar{r} = D/2$—v_z is zero, so that the other constant C_2 must equal $\beta D^2/16\mu$. The final equation for the velocity profile then becomes

$$\boxed{v_z = \frac{\beta}{4\mu}\left(\frac{D^2}{4} - \bar{r}^2\right)} \qquad [6.64]$$

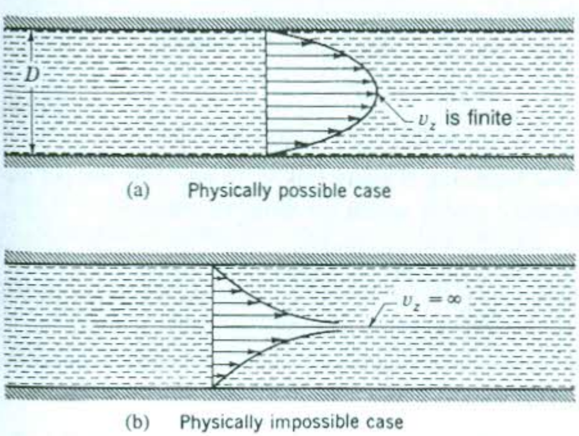


Figure 6.15
Mathematical velocity profile possibilities.

The reader may recognize from this result that the profile for this case is that of a paraboloidal surface of revolution (see Fig. 6.15a).

We wish next to express the separation constant $-\beta$ in terms of the volume flow q. We accordingly have for q

$$
\begin{aligned}
q &= \int_0^{D/2} v_z(2\pi\bar{r})\,d\bar{r} = \int_0^{D/2}\left(\frac{\beta}{4\mu}\right)\left(\frac{D^2}{4} - \bar{r}^2\right)(2\pi\bar{r}\,d\bar{r}) \\
&= \frac{\beta}{4\mu}(2\pi)\left[\frac{D^2}{4}\frac{\bar{r}^2}{2} - \frac{\bar{r}^4}{4}\right]_0^{D/2} \\
&= \frac{\beta\pi D^4}{128\mu}
\end{aligned}
$$

Solving for β,

$$\beta = \frac{128q\mu}{\pi D^4} \tag{6.65}$$

The profile is now fully established in terms of the volume flow.

The differential equation (Eq. 6.60a) will now be examined to give us the second item of information of this section, namely, the pressure drop. This equation may be written as

$$\frac{dp}{dz} = -\beta = -\frac{128q\mu}{\pi D^4}$$

Integrating between sections 1 and 2 a distance L apart, we get

$$(p)_1 - (p)_2 = \frac{128q\mu L}{\pi D^4} \tag{6.66}$$

Since this pressure drop is due only to friction, we can get the *head loss* h_l by dividing through by ρ.[13] Thus

$$\boxed{\frac{\Delta p}{\rho} = h_l = \frac{128q\mu L}{\pi D^4\rho}} \tag{6.67}$$

Solving for q in Eq. 6.66 we may form the following useful formula:

$$q = \frac{\pi(p_1 - p_2)D^4}{128\mu L} \tag{6.68}$$

Also, going back to Eq. 6.64 and replacing β using Eqs. 6.65 and 6.68 we get another useful formula for the velocity profile which we write as follows:

$$\boxed{v_z = \frac{p_1 - p_2}{4\mu L}\left(\frac{D^2}{4} - \bar{r}^2\right)} \tag{6.69}$$

[13]Here again the head loss is the drop in pressure over distance L per unit mass flow (to be discussed more generally in Sec. 8.3).

HIGHLIGHTS

In the previous two chapters, we presented the integral form of the following basic laws:

1. Conservation of mass.
2. The linear momentum equation.
3. The angular momentum equation.
4. The first law of thermodynamics.

We started with a system approach as learned in previous courses and then using the Reynolds transport equation we were able to get the corresponding control volume formulations and thus the integral forms of the above laws. We pointed out in these developments that the system approach and the control volume approach give identical results when applied to infinitesimal systems or infinitesimal control volumes. Thus, to get to differential forms of the basic laws we can use either approach. Now, for the control volume formulation involving the variable N, we want the corresponding differential formulation involving the distribution form of this variable or what we have labeled as η.

The first law that we looked at was the **conservation of mass.** We employed an infinitesimal, rectangular parallelepiped as the control volume. We reached the following result which, using the divergence operator, is valid for any orthogonal coordinate system.

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = -\frac{\partial \rho}{\partial t}$$

Next, we moved on to the **linear momentum equation.** This time we used an infinitesimal system which we took as part of a continuum. Using the substantial derivative to follow this element, we reached the celebrated **Euler** equation as the desired differential form of the linear momentum equation:

$$-\frac{1}{\rho}\mathbf{\nabla}p - g\,\mathbf{\nabla}z = \left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right) + \frac{\partial \mathbf{V}}{\partial t} = \frac{D\mathbf{V}}{Dt}$$

Of course, you will realize that it is valid only for inertial references.

You may have wondered what happened to the **angular momentum equation** in this chapter. If you look at this law for an infinitesimal control volume, one can show that to satisfy this law in the limit as the control volume vanishes, it is necessary that the stress tensor τ_{ij} must be symmetric, i.e., $\tau_{ij} = \tau_{ji}$. Thus we would satisfy the complementary property of shear in generality comparable to what you did under static conditions in your course in solid mechanics.[14]

[14]See I. H. Shames, *Introduction to Solid Mechanics*, 3rd ed., Prentice-Hall.

Finally, we refer you to this book's website (www.mhhe.com/shames) for the development of the **first law of thermodynamics** in terms of the field variables present in the integral form of the first law. We present the equation as follows, summing over index i

$$\rho\frac{Du}{Dt} + p\frac{\partial V_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) + \Phi + \dot{\Theta}$$

where Φ is the so-called *dissipation function*.

As an interesting and useful application of Euler's equation, we examined the case of a liquid under steady-state linear acceleration and steady-state angular motion about a fixed vertical axis. We did not use the *relative equilibrium* concept that is so often used. Your author wanted the student to see the Euler equation "in the flesh" since there was no need to skirt around this equation known to every fluid mechanician. Then, to add more kudos to Euler, we derived the Bernoulli equation from it valid along a streamline. It was immediately shown that Bernoulli is valid everywhere in an irrotational flow.

Finally, we presented Newton's law for general flows and made use of these equations for analyzing parallel laminar flows between parallel plates and also for laminar parallel flow in a circular duct. We got the velocity profile (parabolic) in the latter case which will be useful to us when we study pipe flow, and we introduced the concept of **head loss** and its formulation also to be used extensively later. Keep in mind that head loss is the pressure loss due to friction divided by the density.

6.9 CLOSURE

We will make much use of differential forms of the basic and subsidiary laws in Part II of the text, wherein we examine in far greater detail certain key flows that are important in engineering.

In addition to these differential formulations, we have also to make considerable use of *experimental* information and data. Accordingly, in Chap. 7 we examine certain general and vital aspects concerning the use of experimental data as well as the important procedure of model testing. The results of Chap. 7 will be vital for proper understanding of much of what will follow in later chapters. In particular, you will find that model testing, to be meaningful, requires a sound knowledge of the fundamentals of fluid mechanics.

SUMMARY
PART I

INTEGRAL FORMS OF BASIC LAWS

We use the **Reynolds transport equation** for each development.

$$\boxed{\frac{DN}{Dt} = \oiint_{CS} \eta(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t}\iiint_{CV} \eta(\rho\, dv)}$$

1. Conservation of mass

$$\boxed{N = m, \quad \eta = 1} \qquad \boxed{0 = \oiint_{CS} 1(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t}\iiint_{CV} 1(\rho\, dv)}$$

2. Linear momentum

$$\boxed{\mathbf{N} = m\mathbf{V}, \qquad \boldsymbol{\eta} = \mathbf{V}}$$

$$\boxed{\oiint_{CS} \mathbf{T}\, dA + \iiint_{CV} \mathbf{B}\rho\, dv = \oiint_{CS} \mathbf{V}(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t}\iiint_{CV} \mathbf{V}(\rho\, dv)}$$

3. Angular Momentum

$$\boxed{\mathbf{N} = m(\mathbf{r} \times \mathbf{V}), \qquad \boldsymbol{\eta} = (\mathbf{r} \times \mathbf{V})}$$

$$\boxed{\begin{aligned}\oiint_{CS} \mathbf{r} \times \mathbf{T}\, dA = \iiint_{CV} \mathbf{r} \times \mathbf{B}\, \rho\, dv = \oiint_{CS} (\mathbf{r} \times \mathbf{V})(\rho \mathbf{V} \cdot \mathbf{dA}) \\ + \iiint_{CV} (\mathbf{r} \times \mathbf{V})(\rho\, dv)\end{aligned}}$$

4. First Law of Thermodynamics

$$\boxed{N = m\left(\frac{V^2}{2} + gz + \frac{p}{\rho}\right), \quad \eta = \left(\frac{V^2}{2} + gz + \frac{p}{\rho}\right)}$$

$$\boxed{\frac{dQ}{dt} - \frac{dW_K}{dt} = \oiint_{CS} \left(\frac{V^2}{2} + gz + \frac{p}{\rho}\right)(\rho \mathbf{V} \cdot \mathbf{dA}) + \frac{\partial}{\partial t}\iiint_{CV} \left(\frac{V^2}{2} + gz + \frac{p}{\rho}\right)(\rho\, dv)}$$

Comment: Please note the similarity of these formulations of the basic integral forms of the basic laws. Also note that integration symbols almost always are simplified in problems via the insertion of assumptions. And, at the same time, the integrals give the student a "roadmap" for building up numeric values in problems.

DIFFERENTIAL FORMS OF BASIC LAWS

1. Conservation of Mass

$$\boxed{\operatorname{div}(\rho \mathbf{V}) = -\frac{\partial \rho}{\partial t}}$$

2. Newton's Law (Linear Momentum)

$$\boxed{-\frac{1}{\rho}\nabla p - g\nabla z = (\mathbf{V} \cdot \nabla)\mathbf{V} + \frac{\partial \mathbf{V}}{\partial t}}$$

3. Newton's Law (Angular Momentum)

From dynamics course, $\boxed{\mathbf{M}_A = \dot{\mathbf{H}}_A}$

4. First Law of Thermodynamics

$$\boxed{\frac{DE}{Dt} = \frac{dQ}{dt} - \frac{dW_K}{dt}}$$

Comment: In this text, we will only use Equation 1. Equation 2 for steady flow is called **Euler's equation.** It was used for the steady-state movement of liquids undergoing constant acceleration and constant angular rotation. Also, integration of Euler's equation led to the vital **Bernoulli equation** which has been and will continue to be used often. It is stated as follows:

$$\boxed{\frac{p}{\rho} + gz + \frac{V^2}{2} = \text{const}}$$

Bernoulli's equation is valid along a streamline for steady, inviscid, incompressible flow and, in addition, is valid everywhere for irrotational flow. The third equation, if satisfied, assures us that the stress tensor is symmetric. The fourth equation was used in conjunction with the development of the integral form of the first law of thermodynamics.

6.10 COMPUTER EXAMPLE

COMPUTER EXAMPLE 6.1

■ Computer Problem Statement

In Problem 5.9, we are going to give the apparatus the ability to change the depth that the ejector-nozzle system is below the free surface. This will permit operation above the ocean floor to be adjustable (see Fig. C6.1). We shall make the 30 ft height a variable which we denote as h. We will assume that the distribution of oysters on the average is some function of the depth below the free surface for this case. We wish to plot the power of the pump needed for a flow of 1500 gpm of a mix of water and oysters into the boat versus the depth h of the ejector-nozzle centerline below the free surface including an initial value of 30 ft. We will propose here that the specific gravity sg of the water and oyster mix at the entrance to the ejector-nozzle is given as

sg = 1.5 - (2.54204 - 0.08404.*depth).^0.416

This equation is assumed to be valid from the depth of 28 ft above where there are no oysters, and the water is clear to the maximum depth of 30 ft.

■ Strategy

We will form a hypothetical continuum of water and oysters with the above specific gravity. The control volume will be the entire interior of the pipe system and pump. We will use the *first law of thermodynamics*, the *continuity equation*, and we will assume *isothermal, steady, incompressible* flow throughout. Also, we will consider that we have very close to hydrostatic pressure at the inlet to the jet-ejector nozzle. The height h will be the variable.

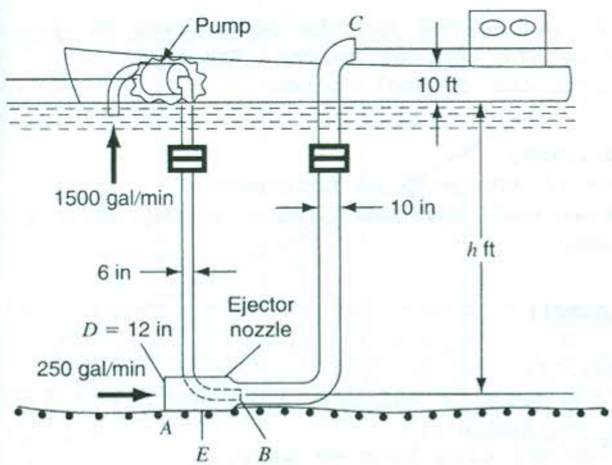


Figure C6.1

■ Execution

```
clear all;
%Putting this at the beginning of the program ensures
%values don't overlap from previous programs.

depth=linspace(28,30,500);
%The depth is an array of 500 values between 28 ft.
%and 30 ft.

sg=1.5-(2.54204-.08404.*depth).^.416;
%This is the specific gravity as a function of depth
%with the constants being found by knowing that at a
%depth of 28 feet the specific gravity equals 1 and
%at a depth of 30 feet the specific gravity equals
%1.3.

mass_flow3=6.48+1.0806.*sg;
%The mass flow out of the end of pipe 3 is the sum of
%the mass flow at 1 and the mass flow at 2. The mass
%flow at 2 depends on the specific gravity which
%depends on depth.

density3=mass_flow3./3.8997;
%The density of the substance exiting at 3 is the
%mass flow exiting at 3 divided by the velocity of
%flow and the cross-sectional area of pipe 3.

dws_dt=-((13718.16+2287.6302.*sg)./density3)-
34.795.*depth.*sg-375.3024.*sg+34.8.*depth
+6945.292;
%This is the first law solved for the pump power on
%the surroundings by the control volume. Remember,
%the pump is part of the control volume.

power_pump=-dws_dt./550;
%This is the power of the pump in horsepower.
%Negative because we want the power value acting on
%the control volume.

plot(depth,power_pump);
grid;
xlabel('Depth (feet)');
ylabel('Power of the pump to maintain 1500 gpm');
title('Pump power vs. Depth');
%This just gives us the plot that we want.
```

■ Debriefing

The specific gravity of the oyster-water mix is just a hypothetical estimation. Of course, for actual use there would have to be experimental verification and adjustment of the proposed specific gravity equation. In Fig. C6.2 notice that in the plot pump power must increase as the height h is increased. This obtains because of the increase in the specific gravity we have used for the incoming mix and also because of the greater height the mix has to be raised.

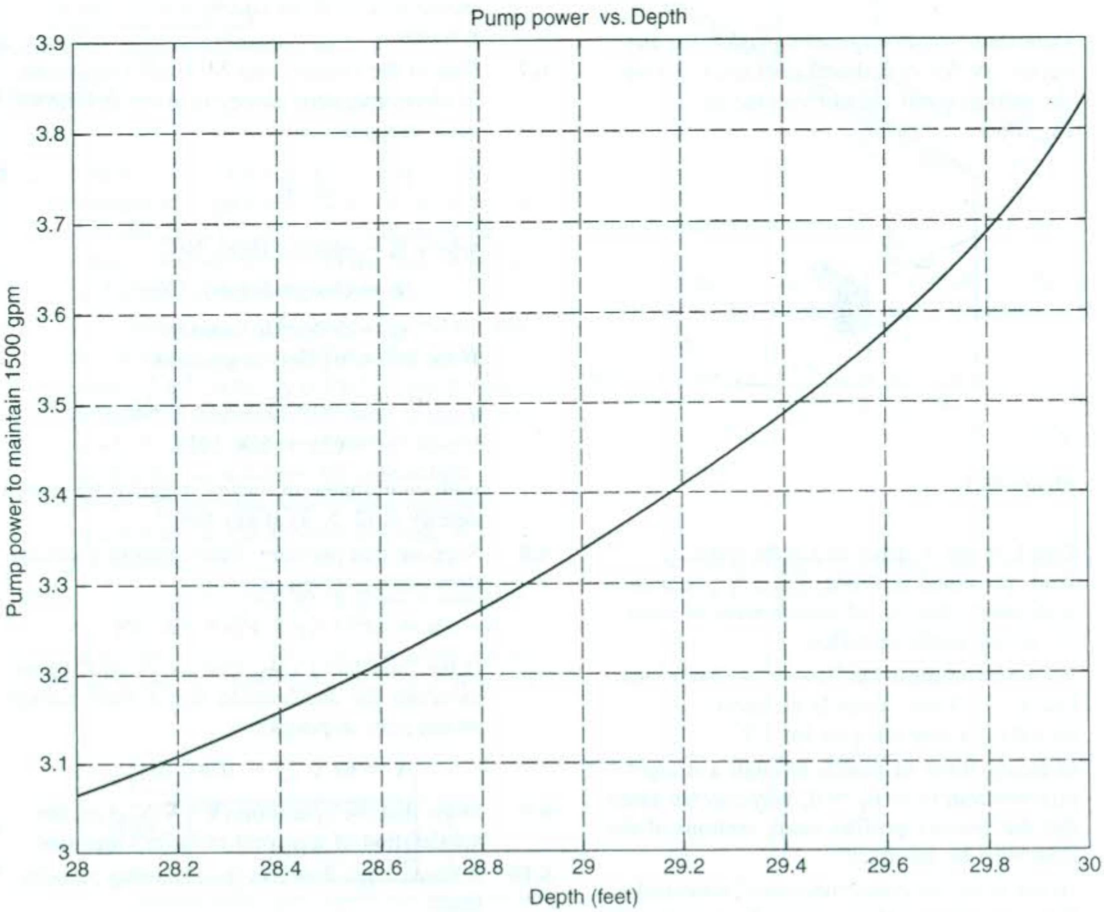


Figure C6.2

PROBLEMS

Problem Categories

Conservation of mass 6.1–6.6
Divergence operator 6.7
Newton's law (Euler's equation) 6.8–6.10
Constant acceleration problems 6.11–6.18
Constant rotation problems 6.19–6.23
Newton's law (general flows) 6.24–6.35
Computer problems 6.36–6.37

Starred Problems

6.18, 6.25

6.1 Determine the divergence operator for the vector $\rho\mathbf{V}$ for cylindrical coordinates. Use the infinitesimal control volume in Fig. P6.1.

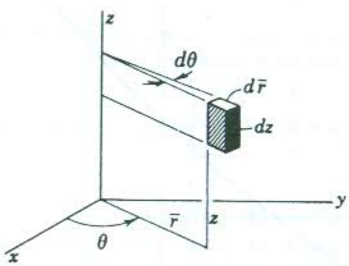


Figure P6.1

6.2 Check to see whether or not the velocity fields presented in Probs. 3.1, 3.5, 3.6, and 3.16 satisfy the law of conservation of mass for an incompressible flow.

6.3 For an incompressible flow, $V_x = 30y^2x^3$ m/s and $V_z = 20$ m/s. What is the most information you can give for V_y?

6.4 In steady flows of liquids through a straight pipe wherein $v_{\bar{r}} = v_\theta = 0$, why can we assert that the velocity profiles along sections of the flow must be invariant?

6.5 In the study of two-dimensional potential flow, we express the velocity $\mathbf{V}$ in terms of a function ψ called the *stream function.* That is,

$$V_x = \frac{\partial \psi}{\partial y}$$
$$V_y = -\frac{\partial \psi}{\partial x}$$

Show that we can satisfy conservation of mass by doing this for incompressible flow.

6.6 Given the following hypothetical velocity field

$$\mathbf{V} = x^2\mathbf{i} + yx\mathbf{j} + t^2\mathbf{k} \quad \text{m/s}$$

and a density distribution given as

$$\rho = \rho_0[1 + x \times 10^{-2}] \quad \text{kg/m}^3$$

what is the time rate of change of ρ at a position (2, 2, 0) m at time $t = 2$ s? ρ_0 is a constant.

6.7 One of the famous four Maxwell's equations in electromagnetic theory is given as follows for a vacuum:

$$\text{div } \mathbf{E} = \frac{\rho}{\epsilon_0}$$

where $\mathbf{E}$ = electric field, N/C
ρ = charge density, C/m^3
ϵ_0 = *dielectric* constant

If the following field is given as

$$\mathbf{E} = (y^2 + x^3)\mathbf{i} + (xy + t^2)\mathbf{j} + (3z + 5)\mathbf{k} \quad \text{N/C}$$

with coordinates in meters, what is the charge density at (2, 5, 3) at any time?

6.8 Suppose that pressure distribution in a steady-flow wave is given as

$$p = 6x^2 + (y + z^2) + 10 \quad \text{Pa}$$

If the fluid has a mass density of 1000 kg/m^3, ascertain the acceleration that a fluid particle would have at position

$$\mathbf{r} = 6\mathbf{i} + 2\mathbf{j} + 10\mathbf{k} \quad \text{m}$$

6.9 Show that the operation $(\mathbf{V} \cdot \nabla)\mathbf{V}$ gives the acceleration of transport in Euler's equation.

6.10 A nonviscous flow has the following velocity field:

$$\mathbf{V} = (x^2 + y^2)\mathbf{i} + 3xy^2\mathbf{j} + (16t^2 + z)\mathbf{k} \quad \text{m/s}$$

The density ρ is to be considered constant. What is the rate of change of pressure in the x direction at a position (1, 1, 0)? Is the pressure variation in any coordinate direction changing with time? What is this time variation at (1, 0, 2) m?

6.11 A tank weighs 80 N and contains 0.25 m^3 of water. A force of 100 N acts on the tank. What is θ when the free surface of the water assumes a fixed orientation relative to the tank?

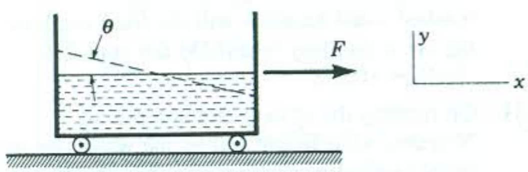


Figure p6.11

6.12 A tank of water in Fig. P6.12 is given a constant acceleration a_y. If the water is not to spill out when a fixed configuration of the water is reached relative to the tank, what is the largest acceleration permissible?

6.13 To construct a simple device for measuring acceleration, take a capillary tube in the shape of a U-tube (Fig. P6.13) and put in oil to the level shown of 300 mm. If the vehicle to which this U-tube is attached accelerates so that the oil assumes the orientation shown, what is the acceleration that you would mark on the acceleration scale at position A?

6.14 A rectangular container is given a constant acceleration a of 0.4 g's. What is the force from fluids on the left wall AB when a fixed configuration of the water has been reached

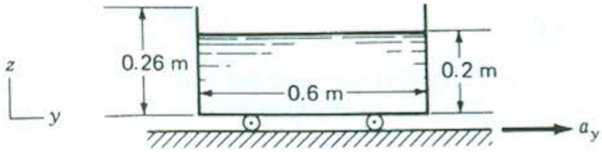


Figure P6.12

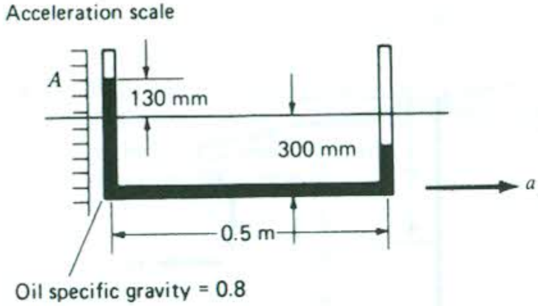


Figure P6.13

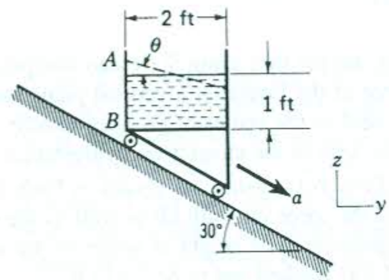


Figure P6.14

relative to the tank? The width of the tank is 1.5 ft. Use integration procedures with Eq. 6.17.

6.15 Using Eq. 6.17, show that for a *vertical* plane submerged surface in a liquid having constant acceleration components a_y and a_z, the gage pressure p can be given as

$$p = \overline{\gamma} d$$

where d is the depth *below* the free surface and $\overline{\gamma}$ is given as

$$\overline{\gamma} = \gamma\left(1 + \frac{a_z}{g}\right)$$

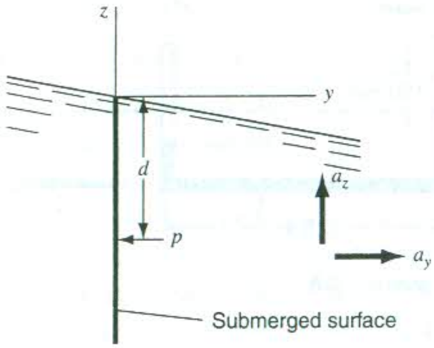


Figure P6.15

This means that using $\bar{\gamma}$ we can compute the force of the liquid on a vertical plane surface as well as the center of pressure exactly as was done in the chapter on hydrostatics.

6.16 In Prob. 6.14, using the results of Prob. 6.15, find the force on wall *AB* as well as the center of pressure. The height of water on the wall *AB* was worked out to be 1.433 ft.

6.17 In Example 6.2, locate the position of the center of pressure from point *A*.

***6.18** A container has a constant width of 500 mm and contains water as shown in Fig. P6.18. It is accelerated uniformly to the right at a rate of 2 m/s^2. What is the total force on side *AB* when the water has assumed a stationary configuration relative to the container?

6.19 Show that the free-surface profile of a rotating liquid once steady state has been achieved is independent of the density ρ.

6.20 The system shown at rest is rotated at a speed of 24 r/min. When steady state has been reached, what height h will the fluid reach in the outer capillary tubes? Do not consider capillary effects.

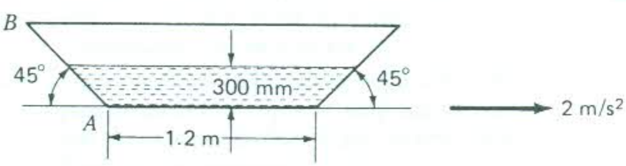


Figure P6.18

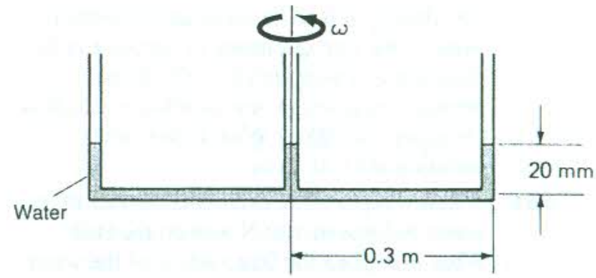


Figure P6.20

6.21 On rotating the system at a speed ω of 30 r/min, what height h does the water rise to in the vertical capillary tubes after steady state has been achieved? Do not consider capillary effects.

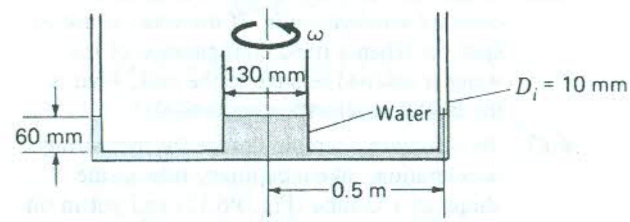


Figure P6.21

6.22 A tank of water is to be spun at an angular speed of ω radians per second. At what speed will the water begin to spill out when it reaches a steady-state configuration relative to the tank?

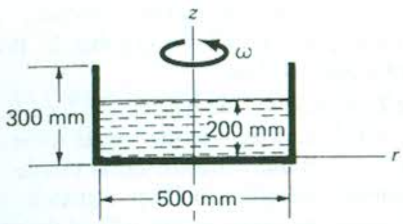


Figure P6.22

6.23 In Prob. 6.22, we found that $\omega = 7.92$ rad/s and the equation of the free surface is

$$z = \frac{1}{\rho g}\left(\frac{\rho\omega^2}{2}r^2 + 2942 - 31.25\omega^2\right) \quad \text{m}$$

Find the center of pressure at the bottom of the tank for a semicircular part of the base area.

6.24 What body-force distribution is needed to maintain the following stress field in equilibrium in a solid?

$$\tau_{ij} = \begin{bmatrix} 500x^3 & 0 & (10z^2 + 580) \\ 0 & -800y^2x^2 & -1000zy^2 \\ (10z^2 + 580) & -1000zy^2 & 0 \end{bmatrix} \quad \text{Pa}$$

***6.25** For cylindrical coordinates, derive the equations of motion using the differential element shown in Fig. P6.25.

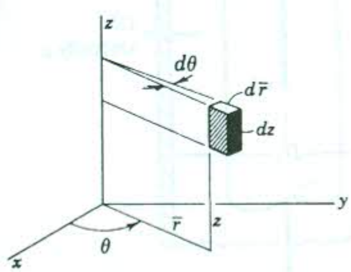


Figure P6.25

6.26 A flow field has the stress field given in Prob. 6.24 with only gravity as a body force in the z direction. What is the convective acceleration at position (1, 2, 0)m? Take ρ as constant.

6.27 What are the equations of motion for two-dimensional flow parallel to the xy plane with no body forces except gravity in the z direction? Show that if

$$\tau_{xx} = \frac{\partial^2\Phi}{\partial y^2}$$

$$\tau_{yy} = \frac{\partial^2\Phi}{\partial x^2}$$

$$\tau_{xy} = -\frac{\partial^2\Phi}{\partial x\,\partial y}$$

where Φ is a scalar function, then there will be zero acceleration everywhere. This is done in solid mechanics to satisfy equilibrium. The function Φ is then called the *Airy function.*

6.28 Consider a flow as shown in Fig. P6.28. Formulate Newton's law in the direction n normal to the streamline using the indicated infinitesimal system. Reach the following result:

$$\frac{V^2}{R} - \frac{1}{\rho}\frac{\partial p}{\partial n} - g\frac{\partial z}{\partial n} = \frac{\partial V_n}{\partial t} \qquad \text{[a]}$$

where V_n is the component of velocity normal to the streamline.

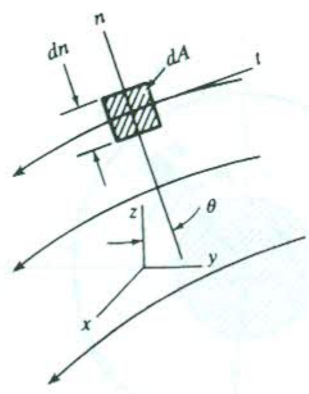


Figure P6.28

6.29 A viscous Newtonian fluid film flows steadily down a tube of radius r_2 (see Fig. P6.29). The film thickness is constant equal to $(r_2 - r_1)$. What is the velocity profile V as a function of r, γ, μ, r_1, and r_2? The ends of the film at top and between are at atmospheric pressure.

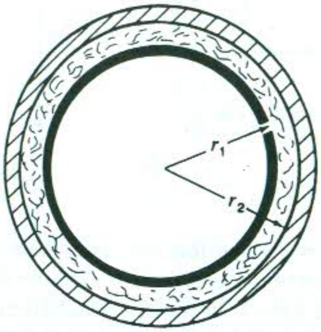


Figure P6.29

6.30 An infinite cylinder of radius a rotates with constant angular speed of ω rad/s inside a stationary journal of radius b as shown in Fig. P6.30. Oil of viscosity μ kg/ms separates the cylinder from the journal. The oil is Newtonian. Find the transverse velocity field v_θ of the oil as a function of r and the pertinent parameters of geometry and fluid properties. Assume steady-state conditions have been reached and that we can use Newton's viscosity law despite the fact that the flow is not a parallel flow.

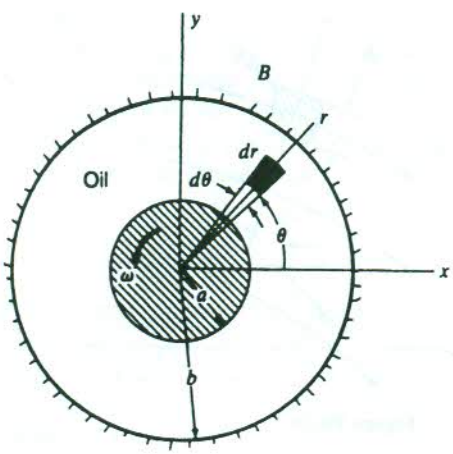


Figure P6.30

6.31 In Prob. 6.30 for the data $\omega = 0.1$ rad/s and $r_b = 0.1$ m, determine the largest value of $(r_b - r_a)$ in order that the linear profile approach presented in Chap. 1 gives a torsional resistance within 10% of the exact resistance for laminar flow.

6.32 In Problem 6.30, develop the differential equation for the pressure as the dependent variable and r as independent variable. Do not try to solve analytically. The equation is nonlinear and must be solved numerically. Use v_θ of Problem 6.30.

6.33 A vertical shaft weighing w per unit length is sliding down concentrically inside a pipe (see Fig. P6.33). Oil separates the two members. Determine the terminal velocity V_T of the shaft without the linear profile

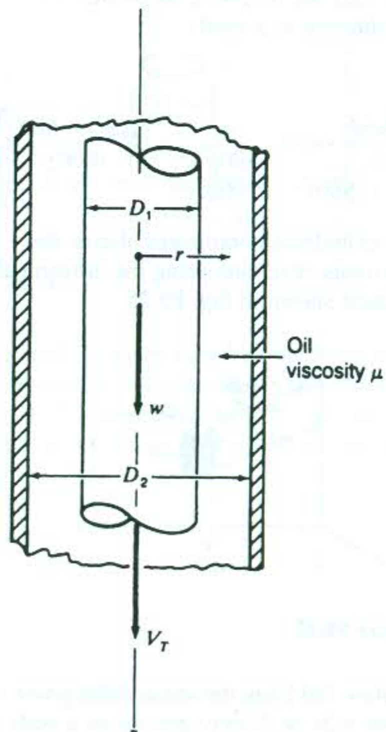


Figure P6.33

assumption made in Chap. 1. Neglect the weight of oil.

6.34 In Prob. 6.33 we got for the terminal velocity the result

$$V_T = \frac{w \ln(D_2/D_1)}{2\pi\mu}$$

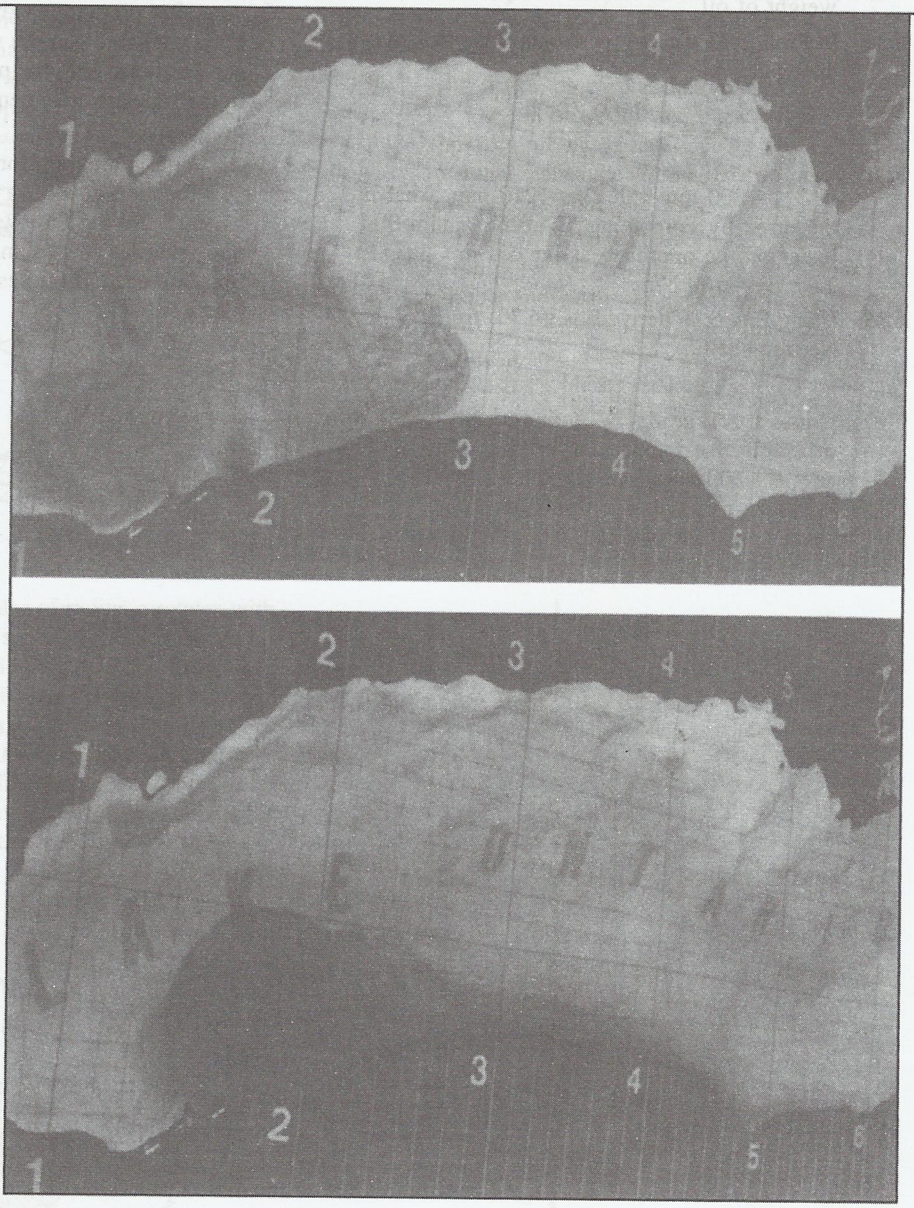
If $D_1 = 200$ mm and $D_2 = 210$ mm, what is the error by computing V_T using a thin-film linear profile approach as we did in Chap. 1. Take $w = 100$ N/m.

6.35 Using Fig. 6.16 derive Eqs. (6.70) from Newton's law.

6.36 In Problem 6.14, develop an iterative computer program to give the angle θ and the horizontal force on the wall for different accelerations. Run the program using the data in the problem statement to check your program.

6.37 Write a program for Problem 6.21 to get h as a function of ω. The dimensions for the general case are L (now 0.5 m), h (now 60 mm), and D (now 30 mm). Run your program for the preceding values of L, h, and D and plot h vs. ω.

Model of Lake Ontario for environmental studies. *(Courtesy Dr. J. Atkinson, State University of New York at Buffalo.)*



A model of Lake Ontario is shown in the photos. The horizontal scale of this model is 1/100,000. The model is about 3 m long and 1 m wide. The model is rotated at 1.71 r/min to include the Coriolis force. Dynamic similarity is achieved via duplicating the Froude and Rossby numbers. The effect of Coriolis is illustrated in the two photos: the top one without rotation and the bottom one with rotation. The dark dye is admitted from the Niagara River at the left. Notice that the Coriolis force caused the dye to move to the right hugging the shoreline. Such studies are valuable for environmental considerations.

CHAPTER 7

Dimensional Analysis and Similitude

7.1 DIMENSIONLESS GROUPS

Before proceeding to specialized discussions of selected types of flow, it will be useful to study the dimensional aspects of fluid flow. This will enable us to understand more clearly the differences between the various flows to be considered in Chapters 8–14. Furthermore, fundamental considerations essential for experimental investigation of fluid phenomena will be set forth in this chapter with the aid of dimensional studies.

In mechanics, a formalism was presented for expressing a dependent dimension in terms of a chosen set of independent basic dimensions. Thus, velocity is given dimensionally by the relation $V \equiv L/T$. To give the most simple dimensional representation of a product of quantities, one need only carry out ordinary algebraic operations on the basic dimensions appearing in the dimensional representation of the quantities. For instance, $Vt \equiv (L/T)T \equiv L$, thus indicating that the product of velocity and time most simply dimensionally is a distance. If a group of quantities has a dimensional representation most simply of unity when multiplied together, the group is called a *dimensionless group*. As an example, the product $\rho VD/\mu$ is a dimensionless group, since

$$\frac{\rho VD}{\mu} \equiv \frac{(M/L^3)(L/T)L}{M/LT} \equiv 1$$

Many of these dimensionless products have been given names, the above group being the well-known *Reynolds* number. In a later section the physical significance of the Reynolds number as well as other dimensionless groups will be discussed.

PART A
DIMENSIONAL ANALYSIS

7.2 NATURE OF DIMENSIONAL ANALYSIS

Recall from mechanics that analytically derived equations are correct for any system of units and consequently each group of terms in the equation must have the same dimensional representation. This is the law of *dimensional homogeneity*. Use was made of this law in establishing the dimensions of quantities such as viscosity.

Another important use may be made of this rule in a situation whereby *the variables involved in a physical phenomenon are known, while the relationship between the variables is not known*. By a procedure, called *dimensional analysis,* the phenomenon may be formulated as a relation between a set of dimensionless groups of the variables, the groups numbering less than the variables. The immediate advantage of this procedure is that considerably less experimentation is required to establish a relationship between the variables over a given range. Furthermore, the nature of the experimentation will often be considerably simplified.

To illustrate this, consider the problem of determining the drag F of a smooth sphere of diameter D moving comparatively slowly with velocity V through a viscous fluid. Other variables involved are ρ and μ, the mass density and viscosity, respectively, of the fluid. The drag F may be stated as some unknown function of these variables. That is,

$$F = f(D, V, \rho, \mu)$$

To determine this relationship experimentally would be a considerable undertaking, since only one variable in the parentheses must be allowed to vary at a time, with the resulting accumulation of many plots. A possible representation of the results of such a procedure is indicated in Fig. 7.1, where F is plotted against D for various values of V. Each plot, however, corresponds to a definite value of ρ and μ, so you can see from the diagram that there will be very many plots required for an effective description of the process. Also, such an approach would mean the use of many spheres of varying diameter and a variety of fluids of varying viscosity and density. Thus, one sees that this could be an extremely time-consuming, as well as expensive, investigation.

Let us now employ dimensional analysis before embarking on an experimental program. As will be shown presently, the drag process above may be formulated as a functional relation between only two dimensionless groups. Each group is called a π (with no relation implied to the number 3.1416...).
Thus

$$\frac{F}{\rho V^2 D^2} = g\left(\frac{\rho VD}{\mu}\right)$$

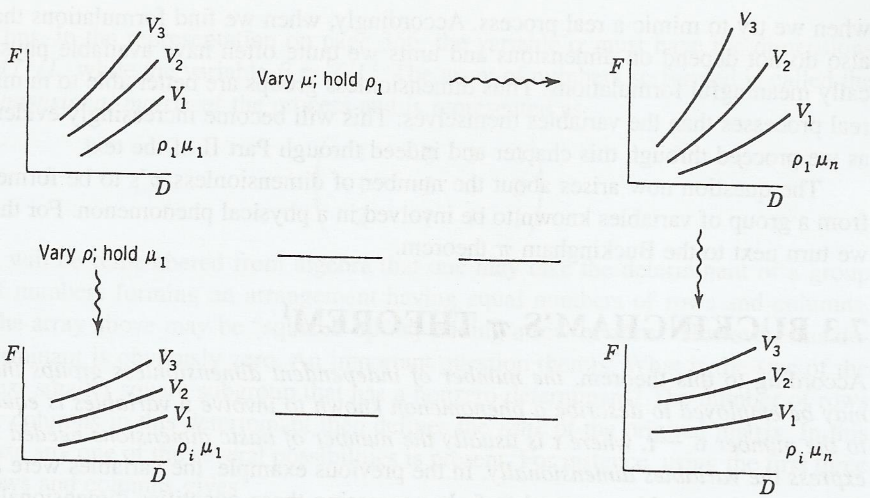


Figure 7.1
Many plots needed to get F versus D, ρ, V, μ.

where the nature of the function g is not known. However, by experiment, a *single* curve may be established relating the π's. This is indicated in Fig. 7.2. Such a simple plot can give as complete quantitative information as hundreds of plots of the type discussed earlier. Suppose that the drag is desired for conditions V_a, D_a, ρ_a, μ_a. The dimensionless group $(\pi_2)_a$ can immediately be evaluated as $\rho_a V_a D_a / \mu_a$. Corresponding to this value of $(\pi_2)_a$, the value of $(\pi_1)_a$ is read off the plot as shown. F_a is then computed as $(\rho_a V_a^2 D_a^2)(\pi_1)_a$.

To establish such a useful curve, one may use a wind or water tunnel where for a given sphere and fluid the value of π_2 may be easily and continuously adjusted by varying the free-stream velocity V. The force on the sphere is measured for each setting of V so that the corresponding values of π_1 may easily be ascertained. Thus, with considerably less time and expense, a curve of the dimensionless groups is established which, as a result of dimensional analysis, is valid for any fluid or any diameter sphere in a flow within the range of the π's tested.

Philosophically one may explain the rationale of what is to ensue as follows. Nature does not bother knowing about the coordinates and dimensions that we use

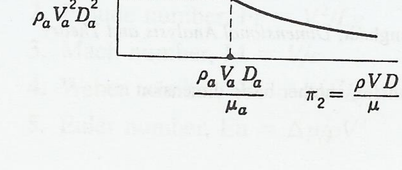


Figure 7.2
Plot of π_1 versus π_2.

when we try to mimic a real process. Accordingly, when we find formulations that also do not depend on dimensions and units we quite often have available physically meaningful formulations. Thus dimensionless groups are better able to mimic real processes than the variables themselves. This will become increasingly evident as we proceed through this chapter and indeed through Part II of the text.

The question now arises about the number of dimensionless π's to be formed from a group of variables known to be involved in a physical phenomenon. For this we turn next to the Buckingham π theorem.

7.3 BUCKINGHAM'S π THEOREM[1]

According to this theorem, *the number of independent dimensionless groups that may be employed to describe a phenomenon known to involve* n *variables is equal to the number* n − r, *where* r *is usually the number of basic dimensions needed to express the variables dimensionally*. In the previous example, the variables were F, V, D, ρ, and μ, making n equal to 5. In expressing these quantities dimensionally three basic dimensions M, L, T or F, L, T must be employed so that $n - r$ becomes equal to 2.[2] Also, it is clear that the dimensionless groups employed are independent, i.e., not related to each other through algebraic operations, since F appears only in one group and μ appears only in the other group. The theorem above states that there can be no additional independent dimensionless group. Hence, any other dimensionless group that the reader may propose will invariably be one that can be developed by algebraic operations on the groups $F/\rho V^2 D^2$ and $\rho VD/\mu$. For example, $F/\mu VD$ is a dimensionless group formed by the product of the preceding groups.

The computation of r in Buckingham's π theorem as the number of basic dimensions needed to express the variables dimensionally is not always correct. For instance, in stress analysis there are problems dealing with force and distance whereby the basic dimensions may number two (F, L) for the FLT system or may number three (M, L, T) if the MLT system is selected. A correct procedure for ascertaining the value of r will now be put forth.

The variables α, β, γ, and so forth are listed along a horizontal axis, and the basic dimensions M, L, T, and so forth are listed along a vertical axis, as shown below. Under each variable a column of numbers is listed representing the powers to which the basic dimension must be raised in the dimensional representation of the particular variable.

	α	β	γ	δ
M	1	0	3	0
L	−1	−2	1	2
T	2	1	1	1

[1]For a complete discussion including proofs, read H. L. Langhaar, *Dimensional Analysis and Theory of Models*, Wiley, NY, 1951.

[2]It is to be pointed out that for situations involving heat transfer, another basic dimension is *temperature* which we shall denote as θ.

Thus, in the representation on page 356, the variable α must have the dimensions MT^2/L, while the variable $\beta \equiv T/L^2$. The array of numbers so formed is called the *dimensional matrix* of the process and is represented as

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ -1 & -2 & 1 & 2 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

It will be remembered from algebra that one may take the determinant of a group of numbers forming an arrangement having equal numbers of rows and columns. The array above may be "squared up" by adding a row of zeros. However, this determinant is obviously zero. An important question then is: What is the size of the next smaller square subgroup that has a nonzero determinant? The number of rows or columns in this determinant then defines the *rank* of the original matrix. In this case any one of the several possibilities is present. For instance, using the first three rows and columns gives

$$\begin{vmatrix} 1 & 0 & 3 \\ -1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 6$$

making the rank of the dimensional matrix equal to 3.

The correct value for r in Buckingham's π theorem may now be stated as the rank of the dimensional matrix.

7.4 IMPORTANT DIMENSIONLESS GROUPS IN FLUID MECHANICS

In most fluid phenomena where heat transfer can be neglected, the following variables may be important:

1. Pressure change, Δp
2. Length, L
3. Viscosity, μ
4. Surface tension, σ
5. Velocity of sound, c
6. Acceleration of gravity, g
7. Mass density, ρ
8. Velocity, V

From these variables the following dimensionless groups can be formed:

1. Reynolds number, $\mathrm{Re} = \rho VD/\mu$
2. Froude number, $\mathrm{Fr} = V^2/Lg$
3. Mach number, $\mathrm{M} = V/c$
4. Weber number, $\mathrm{We} = \rho V^2 L/\sigma$
5. Euler number, $\mathrm{Eu} = \Delta p/\rho V^2$

Note that since three basic dimensions are needed to describe the variables, there are $8 - 3 = 5$ independent dimensionless groups in this listing in accordance with the Buckingham theorem. That these groups are independent is easily seen by noting that the Euler number is the only expression containing the variable Δp; the Weber number is the only one with σ, and so on. Hence, no one of these groups could have been formulated by any manner of algebraic combination or operation of the others.

Fortunately, in many engineering problems, only a few of the variables in the previous listing are simultaneously involved to any appreciable degree. For instance, in aeronautical work, surface tension and gravity are not important enough to warrant consideration so that the Froude number and the Weber number would not be involved. Later, the physical significance of the dimensionless groups listed will be discussed so that the reader will better appreciate when certain groups may be deleted.

7.5 CALCULATION OF THE DIMENSIONLESS GROUPS

Now that the correct number of dimensionless groups in a process has been determined, we turn to the problem of how to form the groupings. We can establish forms of the correct number of independent groups by trial and error. However, when this is not feasible, either of the following procedures will be effective.

Procedure 1

As an illustration of practical value, let us dimensionally investigate the pressure drop in a viscous incompressible flow through a length L of straight pipe. The variables known to be involved in such a process are pressure drop, Δp; average velocity, V; viscosity, μ; inside diameter of pipe, D; length of pipe section, L; density, ρ; and, finally, the roughness of the pipe as represented by the average variation e of inside radius. Functionally, the pressure drop may be expressed as

$$\Delta p = h(\rho, \mu, V, L, D, e) \qquad [7.1]$$

The right side of Eq. 7.1 is replaced by an infinite series[3]

$$\Delta p = (K_1\rho^{a_1}\mu^{b_1}V^{c_1}L^{d_1}D^{f_1}e^{g_1}) + (K_2\rho^{a_2}\mu^{b_2}V^{c_2}L^{d_2}D^{f_2}e^{g_2}) + \cdots \qquad [7.2]$$

where $K_1, K_2, \ldots$ are dimensionless coefficients and $a_1, b_1, \ldots, a_2, b_2, \ldots$ are exponents required by the series. Since each grouping in Eq. 7.2 must have the same dimensions by the law of dimensional homogeneity, we need include in the dimensional representation of Eq. 7.2 only the first expression of the series. Hence,

[3]Here again we are attempting to mimic nature. Thus since natural phenomena proceed such that the variables relate continuously to the variable of interest (here Δp), then we should expect that Δp would be expressible as an infinite series which converges uniformly.

dropping the subscript of the exponents and expressing the equation dimensionally, we get

$$\left[\frac{M}{LT^2}\right] \equiv \left[\frac{M}{L^3}\right]^a\left[\frac{M}{LT}\right]^b\left[\frac{L}{T}\right]^c[L]^d[L]^f[L]^g$$

Now the exponents of the basic dimensions M, L, and T on both sides of the equation may be, respectively, equated according to the law of dimensional homogeneity to form the following set of simultaneous algebraic equations:

For M: $\quad 1 = a + b \quad$ [1]

For L: $\quad -1 = -3a - b + c + d + f + g \quad$ [2]

For T: $\quad -2 = -b - c \quad$ [3]

Since there are six quantities related by only three equations, we may solve any three quantities in terms of the remaining three quantities. We choose as the three *dependent* quantities to be *eliminated* those quantities that are associated with three variables that you would want in one of the dimensionless groups. Suppose that we wish ρ, V, and D to be in one dimensionless group. Then (see Eq. 7.2) we will take a, c, and f to be eliminated—i.e., to be expressed in terms of the remaining quantities b, d, and g. Equation 1 accordingly gives us

$$a = 1 - b$$

while Eq. 3 gives us c,

$$c = 2 - b$$

Finally, substituting these results into Eq. 2 permits the solution of f in terms of the selected independent variables:[4]

$$f = -b - d - g$$

Returning to Eq. 7.2, we restrict the discussion to the first term of the series and replacing a, c, and f by the previous relations, we get

$$\Delta p = K(\rho^{1-b})(\mu^b)(V^{2-b})(L^d)(D^{-b-d-g})(e^g)$$

Upon grouping those terms with the same exponents together and extending the results to the other members of the series, the equation above may be expressed as

$$\frac{\Delta p}{\rho V^2} = K_1\left(\frac{\mu}{\rho VD}\right)^{b_1}\left(\frac{L}{D}\right)^{d_1}\left(\frac{e}{D}\right)^{g_1} + K_2\left(\frac{\mu}{\rho VD}\right)^{b_2}\left(\frac{L}{D}\right)^{d_2}\left(\frac{e}{D}\right)^{g_2} + \cdots$$

Note that any one of the groupings of variables in the equation above is raised to different powers as one goes from expression 1 onward to expression 2, and so on, as is required by the series expansion. That is, the grouping $(\mu/\rho VD)$ is raised to different powers b_1, b_2, and so on. Because of dimensional homogeneity, it follows

[4]If we had n variables and r basic dimensions, we would get any one chosen set of r exponents in terms of the other $n - r$ exponents.

that each of the groupings must perforce be *dimensionless*. Finally, returning to the functional representation of the series, we have

$$\frac{\Delta p}{\rho V^2} = f\left[\left(\frac{\mu}{\rho VD}\right), \left(\frac{L}{D}\right), \left(\frac{e}{D}\right)\right]$$

where f denotes a function. Note that by this procedure the correct number of independent dimensionless groups has been formed. Also, one grouping has ρVD in it as proposed earlier. Since f is an unknown function, the term $\mu/\rho VD$ may be inverted, thus forming the Reynolds number. Also, note the appearance of the Euler number, as well as two geometrical ratios. The pressure loss through a pipe may then be characterized by the equation

$$\mathrm{Eu} = f\left(\mathrm{Re}, \frac{L}{D}, \frac{e}{D}\right)$$

In Chap. 8, we consider this relationship further. Meanwhile, it has served to show how, in a direct manner, a set of dimensionless groups can be formed which are independent and of a number consistent with Buckingham's π theorem.

EXAMPLE 7.1

■ Problem Statement

The end deflection δ of a tip-loaded cantilever beam of length L, as you learned in strength of materials, is given by the formula

$$\delta = \frac{1}{3}\frac{PL^3}{EI} \qquad [a]$$

where E = modulus of elasticity
P = load
I = second moment of area of the cross section of the beam about the centroidal axis.

What does dimensional analysis tell you about the relation between δ and the other variables?

■ Strategy

This problem primarily involves methodology. The procedures will consist of the steps presented just prior to this Example. You are urged to follow them carefully.

■ Execution

We want to decide precisely about the number of independant π's. For this reason, we look at the dimensional matrix using FLT as our basic dimensions. Thus we have

$$\begin{array}{c|ccccc} & P & L & E & I & \delta \\ \hline F & 1 & 0 & 1 & 0 & 0 \\ L & 0 & 1 & -2 & 4 & 1 \\ T & 0 & 0 & 0 & 0 & 0 \end{array}$$

The rank of the matrix clearly is 2, so there will be three independent π's. We now proceed to find such π's.

$$\delta = f(P, L, E, I)$$

Hence,

$$\delta = K_1[(P)^{a_1}(L)^{b_1}(E)^{c_1}(I)^{d_1}] + \cdots$$

Considering the dimensional representation, we can say on dropping the subscript of the exponents:

$$[L] \equiv [F]^a[L]^b\left[\frac{F}{L^2}\right]^c[L^4]^d \qquad \text{[b]}$$

From the law of dimensional homogeneity, we can say on equating exponents for the basic dimensions:

For F: $\quad 0 = a + c$

For L: $\quad 1 = b - 2c + 4d$

For t: $\quad 0 = 0$

Hence, selecting a and b to be solved in terms of c and d (this means that P and L will appear in one dimensionless group), we require that

$$\begin{aligned} a &= -c \\ b &= 1 + 2c - 4d \end{aligned} \qquad \text{[c]}$$

We can then conclude on going back to Eq. b that:

$$\delta = K_1[(P)^{-c}(L)^{1+2c-4d}(E)^c(I)^d] + \cdots \qquad \text{[d]}$$

Grouping the terms with the same powers,

$$\left(\frac{\delta}{L}\right) = K_1\left[\left(\frac{EL^2}{P}\right)^c\left(\frac{I}{L^4}\right)^d\right] + \cdots$$

Going back to the functional form,

$$\left(\frac{\delta}{L}\right) = f\left[\left(\frac{L^2E}{P}\right), \left(\frac{I}{L^4}\right)\right] \qquad \text{[e]}$$

We thus get the expected three dimensionless groups. This is as much as we can get from dimensional analysis. If we multiply the two dimensionless groups in the function we get on inverting the results:

$$\left(\frac{\delta}{L}\right) = g\left[\frac{PL^2}{EI}\right] \qquad \text{[f]}$$

■ **Debriefing**

Going back to Eq. a from strength of materials, we have

$$\left(\frac{\delta}{L}\right) = \frac{1}{3}\left(\frac{PL^2}{EI}\right) \qquad [g]$$

Either the theory of beams or experiments will reveal that the functional relation between δ/L and PL^2/EI in Eq. f is that of direct proportionality where the constant of proportionality is $\frac{1}{3}$.

In general, you will get different dimensionless groups when you choose different sets of powers to be solved for in terms of the remaining powers. However, in accordance with Buckingham's π theorem, there will be only $n - r$ *independent* π's. Hence the various possible sets of π's that can be found can be brought into coincidence with each other by simple algebraic manipulations such as multiplying, dividing, and/or raising the π's to powers. You will have a chance to do this in your homework.

Procedure 2

We now present an *alternative* procedure for establishing dimensionless groups. This procedure has the virtue of being quick to execute. We first choose three variables

Table 7.1 Dimensions

Quantity	$MLT\theta$ system	$FLT\theta$ system
Force	ML/T^2	F
Area	L^2	L^2
Volume	L^3	L^3
Acceleration	L/t^2	L/t^2
Angular velocity	$1/T$	$1/T$
Angular acceleration	$1/T^2$	$1/T^2$
Linear momentum	ML/T	FT
Moment of momentum	ML^2/T	FLT
Energy	ML^2/T^2	FL
Work	ML^2/T^2	FL
Power	ML^2/T^3	FL/T
Pressure and stress	M/T^2L	F/L^2
Moments and products of area	L^4	L^4
Inertia tensor	ML^2	FT^2L
Moment of torque	ML^2/T^2	FL
Heat	ML^2/T^2	FL
Density	M/L^3	FT^2/L^4
Specific weight	M/T^2L^2	F/L^3
Absolute viscosity	M/LT	FT/L^2
Kinematic viscosity	L^2/T	L^2/T
Enthalpy	L^2/T^2	L^2/T^2
Specific heat	$L^2/T^2\theta$	$L^2/T^2\theta$
Surface tension	M/T^2	F/L
Thermal conductivity	$ML/T^3\theta$	$F/T\theta$

which between them involve the basic dimensions M, L, T. For instance, going back to the pressure drop in a pipe Eq. 7.1 which we now rewrite

$$\Delta p = h(\rho, \mu, V, l, D, e)$$

we may choose D having dimension L, V having a dimension T in it, and ρ having a dimension M in it—thus including all three of our basic dimensions. Now using these variables, form the basic dimensions of L, M, and T in the following way:

$$(L) = (D) \qquad [a]$$

$$(T) = (D/V) \qquad [b]$$

$$(M) = (\rho D^3) \qquad [c]$$

Next take each of the four variables not used above namely Δp, μ, l, and e and divide each one by its dimensional representation. We are thus forming dimensionless expressions. Writing these and calling them π_1, π_2, π_3, and π_4 we have:

$$\pi_1 = \frac{\Delta p}{M/LT^2}$$

$$\pi_2 = \frac{\mu}{M/LT}$$

$$\pi_3 = \frac{l}{L}$$

$$\pi_4 = \frac{e}{L}$$

Finally in each of the expressions replace the basic dimensions using results from Eqs. a, b, and c in place of L, T, and M. Thus we get

$$\pi_1 = \frac{\Delta p}{\dfrac{\rho D^3}{(D)(D/V)^2}} = \frac{\Delta p}{\rho V^2}$$

$$\pi_2 = \frac{\mu}{\dfrac{\rho D^3}{(D)(D/V)}} = \frac{\mu}{\rho V D}$$

$$\pi_3 = \frac{l}{D}$$

$$\pi_4 = \frac{e}{D}$$

We have thus produced the dimensionless groups.[5]

[5]As an aid in performing the chapter exercises, we have shown in Table 7.1 a list of commonly used quantities with their dimensional representations for both the $FLT\theta$ and the $MLT\theta$ systems.

We now examine similitude, an important consideration for model testing, and in Sec. 7.7 we will relate similitude with dimensional analysis.

PART B SIMILITUDE

7.6 DYNAMIC SIMILARITY

Similitude in a general sense is the indication of a known relationship between two phenomena. In fluid mechanics this is usually the relation between a full-scale flow and a flow involving smaller but geometrically similar boundaries. However, it must be pointed out that there are similarity laws in common use in fluid mechanics involving flows with dissimilar boundaries. For instance, there is a similarity relation between a subsonic compressible flow (Mach number less than unity) and an incompressible flow about a body distorted in a prescribed manner from that of the compressible flow.[6] Also, in hydrology one uses a model of a river, which is geometrically similar as a plan view, but which is often not similar in depth to the actual river. In this text we restrict the discussion to that of *geometrically similar flows*, i.e., to flows with geometrically similar boundaries.[7]

Two flows consisting of geometrically similar sets of *streamlines* are called *kinematically similar flows*. Since the boundaries will form some of the streamlines, it is clear that kinematically similar flows must also be geometrically similar. However, the converse to this statement is not true, as it is quite easy to arrange kinematically dissimilar flows despite the presence of geometrically similar boundaries. In Fig. 7.3, streamlines are shown about similar double wedges in two-dimensional flows. The one on the left is a low-speed subsonic flow, $M < 1$, while the one on the right is a high-speed supersonic flow, $M > 1$. The lack of similarity between the streamlines is quite apparent.

We define yet a very important third similarity called *dynamic similarity* whereby the force distribution between the two flows is such that, at corresponding points in the flows, identical types of forces (such as shear, pressure, etc.) are parallel and in addition have a ratio which is the same value at all sets of corresponding points between the two flows. Furthermore, this ratio must be common for the various types of forces present. *For dynamically similar flows there will then be this same ratio between corresponding resultant forces on corresponding boundaries.*

What are the conditions for dynamic similarity? It will now be shown that the flows must be *kinematically similar and, furthermore, must have mass distributions such that the ratio of density at corresponding points of the flows are of the same*

[6] Gothert's subsonic similarity rule.

[7] Geometrically similar boundaries can be brought into coincidence by either a uniform dilatation or a uniform shrinkage.

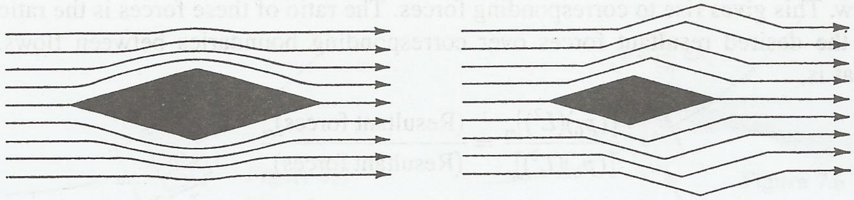


Figure 7.3
Kinematically dissimilar flows with geometrically similar boundaries.

ratio for all sets of corresponding points. Flows satisfying the latter condition are described as flows having *similar mass distributions*. To show that kinematic similarity and mass similarity are *necessary* for dynamic similarity, note first that the condition of *kinematic similarity* means that accelerations

1. Are *parallel* at corresponding points.
2. Have a constant ratio of *magnitude* between all corresponding sets of points.

Item 1 and Newton's law mean that the *resultant* force on each particle must be *parallel* at corresponding points. The condition of *similar mass distributions* and item 2 then mean, also in view of Newton's law, that these *resultant* forces must also have a *constant ratio* of magnitude between all corresponding points in the flow. Since the direction of *each type* of force on a particle is intrinsically tied up with the direction of the streamlines, one may conclude, furthermore, that in kinematically similar flows *identical types of forces* at corresponding points are also *parallel*. Hence we can conclude that since the *resultant* forces on particles have a constant ratio of magnitude between flows, it is necessarily true that *all corresponding components* of the resultant forces (such as shear forces, pressure forces, etc.) have the *same ratio of magnitude* between flows. In short, it is seen that kinematically similar flows with similar mass distributions satisfy all conditions of dynamically similar flows as set forth in our definition at the outset of this paragraph.

Why is the dynamic similarity important in model testing? The reason is very simple and was presented earlier. We now elaborate. If the same ratio exists between corresponding forces at corresponding points, this ratio being the same for the entire flow, then we can say that the *integration of the force distribution giving rise to perhaps lift or drag will also have this ratio between model and prototype flows*. If we do not have flows at least close to having dynamic similarity, then the force ratios between model and prototype flows at different sets of corresponding points will be different, and there is *no simple way* of relating the resultants such as drag and lift between the model and prototype. The testing may be useless. For dynamically similar flows, the ratio between corresponding forces at corresponding points and the accompanying ratio between desired resultant forces for model and prototype flows is not hard to establish. One need only multiply the free-stream pressure times the square of a characteristic length for each flow. This gives rise to corresponding forces. The ratio of these forces is the ratio of the desired resultant forces over corresponding boundaries between flows. That is,

$$\frac{[(p_0)(L^2)]_m}{[(p_0)(L^2)]_p} = \frac{(\text{Resultant forces})_m}{(\text{Resultant forces})_p}$$

7.7 RELATION BETWEEN DIMENSIONAL ANALYSIS AND SIMILITUDE

Let us examine two dynamically similar incompressible, viscous flows about spheres denoted as model and prototype flows in Fig. 7.4. Neglecting body forces, two types of forces may be distinguished acting on each particle, namely, shear and pressure forces. If the inertia term in Newton's law is now written as a D'Alembert force—$m\mathbf{a}$—you will remember from mechanics that for depicting Newton's law we may consider the sum of the two external forces plus the D'Alembert force as equal to zero and thus in "equilibrium." Hence, one may establish a force triangle at each point of the flow. This has been done in Fig. 7.4 for corresponding points A_p and A_m. By rules of dynamic similarity the force triangles for these points are similar, since the sides of the triangles must be parallel. We may form the following equations, which are true for all corresponding points:

$$\frac{(\text{pressure force})_m}{(\text{pressure force})_p} = \frac{(\text{friction force})_m}{(\text{friction force})_p} = \frac{(\text{inertia force})_m}{(\text{inertia force})_p} = \text{const} \qquad [7.3]$$

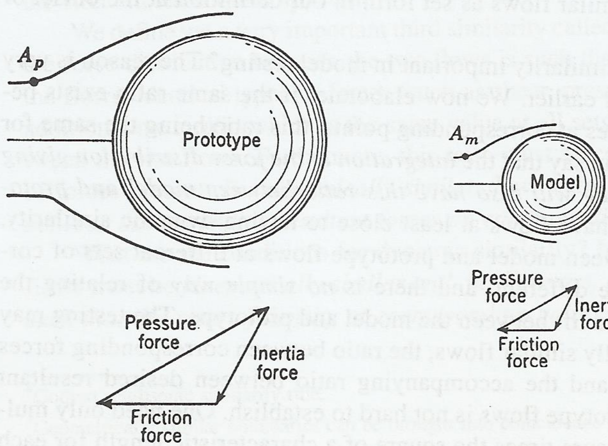


Figure 7.4
Prototype and model flows around sphere.

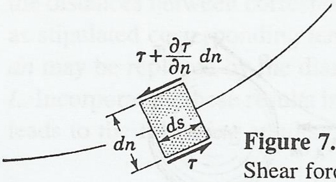


Figure 7.5
Shear forces.

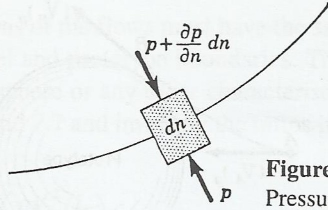


Figure 7.6
Pressure forces.

The following relations may be derived from these equations:

$$\frac{(\text{inertia force})_m}{(\text{friction force})_m} = \frac{(\text{inertia force})_p}{(\text{friction force})_p} = (\text{const})_1 \qquad [7.4]$$

$$\frac{(\text{inertia force})_m}{(\text{pressure force})_m} = \frac{(\text{inertia force})_p}{(\text{pressure force})_p} = (\text{const})_2 \qquad [7.5]$$

It will be informative to evaluate these equations in terms of the flow variables. Let us accordingly examine each of the forces involved:

1. *Viscous or friction force.* An infinitesimal system with rectangular sides is shown in Fig. 7.5 at a position along a streamline. The dimensions are given as ds, dn, and dz, the latter not having been indicated in the diagram. Note from the figure that the net shear force on one pair of sides is $(\partial\tau/\partial n)\,dn\,ds\,dz$. Employing Newton's viscosity law, we may replace τ by $\mu(\partial|\mathbf{V}|/\partial n)$. Using dv as the volume of the system, this shear force is expressed as $\mu(\partial^2|\mathbf{V}|/\partial n^2)\,dv$.
2. *Pressure force.* The net pressure force on the pair of sides shown in Fig. 7.6 is easily evaluated as $(\partial p/\partial n)\,dv$.
3. *Inertia force.* The component selected is along the streamline so that employing Eq. (3.9), we may say for steady flow

$$dm\,a_T = (\rho\,dv)|\mathbf{V}|\frac{\partial|\mathbf{V}|}{\partial s}$$

Although only components of the three types of forces have been developed, it should be clear that the ratios of these components between model and prototype flows will be the same as the ratios of the respective complete forces. Hence, by canceling the term dv, Eqs. 7.4 and 7.5 may be written in the following way:

$$\left[\frac{\rho|\mathbf{V}|\dfrac{\partial|\mathbf{V}|}{\partial s}}{\mu\dfrac{\partial^2|\mathbf{V}|}{\partial n^2}}\right]_m = \left[\frac{\rho|\mathbf{V}|\dfrac{\partial|\mathbf{V}|}{\partial s}}{\mu\dfrac{\partial^2|\mathbf{V}|}{\partial n^2}}\right]_p \qquad [7.6]$$

$$\left[\frac{\rho|\mathbf{V}|\dfrac{\partial|\mathbf{V}|}{\partial s}}{\dfrac{\partial p}{\partial n}}\right]_m = \left[\frac{\rho|\mathbf{V}|\dfrac{\partial|\mathbf{V}|}{\partial s}}{\dfrac{\partial p}{\partial n}}\right]_p \qquad [7.7]$$

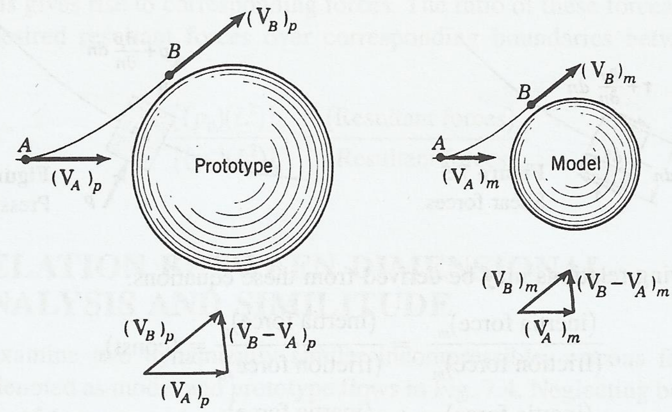


Figure 7.7
Kinematically similar flows.

Now the velocities at all corresponding points in the two flows have the same ratio in magnitude. Hence, one could correctly employ the respective model and prototype *free-stream* velocities for the local velocities in the preceding equations. Also, the difference between the velocity vectors at any two points in one flow and the difference between the velocity vectors at corresponding points of the other flow are vectors which are parallel and of a ratio equal in magnitude to the ratios between velocities themselves at corresponding points. This may be understood by observing Fig. 7.7, where the velocities of points A and B of model and prototype have been indicated. The velocity triangles formed by these velocities and their respective differences form a pair of similar triangles, since two adjacent sides of each triangle, $(\mathbf{V}_A)_m$ and $(\mathbf{V}_B)_m$ of model, and $(\mathbf{V}_A)_p$ and $(\mathbf{V}_B)_p$ of prototype, are parallel and of equal ratio. Thus

$$\frac{|\mathbf{V}_A|_p}{|\mathbf{V}_A|_m} = \frac{|\mathbf{V}_B|_p}{|\mathbf{V}_B|_m} = \frac{|\mathbf{V}_B - \mathbf{V}_A|_p}{|\mathbf{V}_B - \mathbf{V}_A|_m} \tag{7.8}$$

Since the ratio $|\mathbf{V}_A|_p/|\mathbf{V}_A|_m$ holds for all corresponding points, it is proper to replace it by $|\mathbf{V}_0|_p/|\mathbf{V}_0|_m$, the ratio of the free-stream velocities. Furthermore, by taking points A and B infinitesimally close, the equation above may be expressed as

$$\frac{|\mathbf{V}_0|_p}{|\mathbf{V}_0|_m} = \frac{|\mathbf{dV}|_p}{|\mathbf{dV}|_m} \tag{7.9}$$

The same is true for second difference of velocity appearing in the friction forces. From this it is possible to replace all the differential velocities and second differences of velocities of Eqs. 7.6 and 7.7 as well as the local velocities by the free-stream velocities of model and prototype flows. Furthermore, the infinitesimal pressure change can be replaced by a change in pressure Δp between corresponding sets of points between the flows. Finally, one notes from kinematic similarity that the distances between corresponding positions of the flows must have the same ratio as stipulated corresponding lengths of model and prototype boundaries. Thus, term *dn* may be replaced by the diameter of the sphere or any other characteristic length *L*. Incorporating these results into Eqs. 7.6 and 7.7 and inverting the ratios in Eq. 7.7 leads to the following results:

$$\left(\frac{\rho V_0^2/L}{\mu V_0/L^2}\right)_m = \left(\frac{\rho V_0^2/L}{\mu V_0/L^2}\right)_p$$

and therefore on canceling terms

$$\left(\frac{\rho V_0 L}{\mu}\right)_m = \left(\frac{\rho V_0 L}{\mu}\right)_p \qquad [7.10a]$$

Also,

$$\left(\frac{\Delta p/L}{\rho V_0^2/L}\right)_m = \left(\frac{\Delta p/L}{\rho V_0^2/L}\right)_p$$

and therefore

$$\left(\frac{\Delta p}{\rho V_0^2}\right)_m = \left(\frac{\Delta p}{\rho V_0^2}\right)_p \qquad [7.10b]$$

It may be concluded from these results that a *necessary condition for dynamic similarity for the particular flows undertaken is the equality of the Reynolds number and Euler number between the flows, with convenient flow parameters and geometrical measurements being used.* Also a physical interpretation in terms of forces is now available for two of the dimensionless groups introduced in Sec. 7.4. Other groups will be discussed in Sec. 7.8.

In Sec. 7.2, a dimensional analysis was carried out for the drag on a sphere moving in an incompressible viscous flow of precisely the same nature as was investigated in the preceding paragraphs. The following result was given:

$$\left(\frac{F}{\rho V^2 D^2}\right) = g\left(\frac{\rho V D}{\mu}\right) \qquad [7.11]$$

where g is an unknown function. Now the quantity F/D^2 is proportional to the change in pressure Δp between corresponding sets of points between the flows because of the dynamic similarity between the flows. Hence, we may restate the equation above in the form

$$\left(\frac{\Delta p}{\rho V^2}\right) = h\left(\frac{\rho V D}{\mu}\right) \qquad [7.12a]$$

or

$$\text{Eu} = h(\text{Re}) \qquad [7.12b]$$

Thus we see that *dimensional analysis produces the dimensionless groups whose values must be duplicated between geometrically similar flows if dynamic similarity is to be attained between the flows.* Furthermore, we see from Eq. 7.12 that if the Reynolds numbers are duplicated, then the Euler number will be duplicated and therefore, as a result of dimensional analysis, we may relax the requirement for dynamic similarity between the flows under discussion to that of duplicating the Reynolds number only.

Since the shape of the boundary in this discussion was of no consequence beyond the condition of being geometrically similar, we can then say for all viscous, steady, incompressible, geometrically similar flows that conveniently contrived Reynolds numbers must be duplicated between flows to achieve dynamic similarity and so permit data stemming from force distributions on boundaries of models to be meaningful in predicting full-scale results. Experience indicates that this condition with some added corrections is generally sufficient. We will learn more about sufficiency conditions as we study particular flows in more detail in later chapters.

Generalizing once again to *any* flow, we may state that *dimensional analysis will yield the dimensionless groups in a flow for which we must duplicate at least all but one in geometrically similar flows to achieve dynamic similarity.*

7.8 PHYSICAL MEANING OF IMPORTANT DIMENSIONLESS GROUPS OF FLUID MECHANICS

In Sec. 7.7, physical connotations were placed on the Reynolds number and the Euler number. Proceeding in a similar manner, we find that physical interpretations can be developed for the remaining dimensionless groups introduced in Sec. 7.4. These are now listed and discussed in a cursory manner:

1. *Reynolds number* Ratio of the inertia force to the friction force, usually in terms of convenient flow and geometrical parameters.

$$\frac{\rho V^2/L}{\mu V/L^2} = \frac{\rho V L}{\mu}$$

2. *Mach number* Ratio of the square root of the inertia force to the square root of the force stemming from the compressibility of the fluid. This becomes of paramount importance in high-speed flow, where density variations from pressure become significant

$$\sqrt{\frac{\rho V^2/L}{\rho c^2/L}} = \frac{V}{c}$$

3. *Froude number* Ratio of the inertia force to the force of gravity. If there is a free surface, such as in the case of a river, the shape of this surface in the form of waves will be directly affected by the force of gravity, so the Froude number in such problems is significant.

$$\frac{\rho V^2/L}{\gamma} = \frac{V^2}{Lg}$$

4. *Weber number* Ratio of the inertia force to the force of surface tension. This also requires the presence of a free surface, but where large objects are involved, like boats in a fluid such as water, this effect is very small.

$$\frac{\rho V^2/L}{\sigma/L^2} = \frac{\rho V^2 L}{\sigma}$$

5. *Euler number* Ratio of the pressure force to the inertia force. In practical testing work the *pressure coefficient* $\Delta p/(\frac{1}{2}\rho V^2)$ equal to twice the Euler number is ordinarily used.

$$\frac{\Delta p/L}{\rho V^2/L} = \frac{\Delta p}{\rho V^2}$$

With a physical picture of what these dimensionless groups mean, it is much simpler to stipulate which are to be significant and which can be neglected during an investigation. We will continually refer to these throughout the remainder of the text.

EXAMPLE 7.2

■ Problem Statement

The drag on a submarine moving well below the free surface is to be determined by a test on a model scaled to one-twentieth of the prototype. The test is to be carried out in a water tunnel. Determine the ratio of model to prototype drag needed for determining the prototype drag when the speed of the prototype is 5 kn. The kinematic viscosity of seawater is 1.30×10^{-6} m²/s, and the density is 1010 kg/m³ at the depth of the prototype. The water in the tunnel has a temperature of 50°C.

■ Strategy

Since the submarine will be moving well below the free surface, we need not be concerned with wave effects; hence the Froude number will play no role. Because of the slow speed of the submarine, compressibility plays no role, so the Mach number will play no role. Indeed, all we need be concerned about is the Reynolds number and the Euler number for dynamic similarity.

■ Execution

Denoting the length of the submarine as L, we have the following Reynolds number for the prototype flow:

$$[\mathrm{Re}]_p = \frac{V_p L_p}{\nu_p} = \frac{[5\text{ kn}][0.5144(\text{m/s})/\text{kn}][L(m)]_p}{[1.30 \times 10^{-6}\text{ m}^2/\text{s}]} \quad \text{[a]}$$

$$= 1.978 \times 10^6 L_p$$

The Reynolds number for the model flow must then equal the value determined in Eq. a. Using the Table B.1 in the appendix for ν, we have

$$[\text{Re}]_m = \frac{(V_m)(\frac{1}{20}L)_p}{0.556 \times 10^{-6}} \qquad \text{[b]}$$

Equating a and b, we may solve for V_m

$$V_m = 22.0 \text{ m/s} \qquad \text{[c]}$$

This is the free-stream velocity for the water tunnel.

We will measure a drag F_m in the water-tunnel test. We want the drag F_p on the prototype. For **dynamic similarity** all force components have the same ratio at corresponding points. The drag forces for the submarine must have this *same* ratio. What is this ratio? For this information, we can look at the Euler numbers which we know must be duplicated between the flows.

$$\frac{\Delta p_m}{\rho_m V_m^2} = \frac{\Delta p_p}{\rho_p V_p^2}$$

Replacing Δp_m by F_m/L_m^2, which is proportional to Δp_m, and for the same reason, Δp_p by F_p/L_p^2, we get

$$\frac{F_m/L_m^2}{\rho_m V_m^2} = \frac{F_p/L_p^2}{\rho_p V_p^2}$$

$$\therefore F_p = \left(\frac{\rho_p V_p^2}{\rho_m V_m^2}\right)\left(\frac{L_p}{L_m}\right)^2 F_m \qquad \text{[d]}$$

Inserting values, we get for F_p

$$F_p = \frac{(1010)[(5)(0.5144)]^2}{(988)(22^2)}\left[\frac{L_p}{(L_p/20)}\right]^2 F_m$$

$$\therefore F_p = 5.59\, F_m$$

Therefore we multiply a drag measured in the water tunnel by 5.59 to get the proper drag on the prototype.

■ Debriefing

To get the drag for the full-scale submarine, we can multiply the water tunnel drag by 5.59. We are leaving out effects of waves and compressibility. The first will clearly be small if the submarine is well below the free surface. And, certainly, the submarine is nowhere moving near Mach 1 even for the fastest attack submarines, so we can discount compressibility effects.

EXAMPLE 7.3

■ Problem Statement

The power P to drive an axial pump depends on the following variables:

Density of the fluid, ρ
Angular speed of rotor, N
Diameter of rotor, D
Head, ΔH_D
Volumetric flow, Q

A model scaled to one-third the size of the prototype has the following characteristics:

$$N_m = 900 \text{ r/min}$$
$$D_m = 5 \text{ in}$$
$$(\Delta H_D)_m = 10 \text{ ft}$$
$$Q_m = 3 \text{ ft}^3/\text{s}$$
$$P_m = 2 \text{ hp}$$

If the full-size pump is to run at 300 r/min, what is the power required for this pump? What head will the pump maintain? What will the volumetric flow rate Q be?

■ Strategy

We will first choose dimensionless groups to work with. Then we will determine the values of two of these groups from the model flow. Finally, going to the full scale flow, we will maintain these values in the full scale flows. This will permit us to get the full scale power requirements.

■ Execution

We leave to you to show that the pump process can be described by three π's. That is,

$$\underbrace{\left[\frac{P}{\rho D^5 N^3}\right]}_{\pi_1} = f\left[\underbrace{\left(\frac{\Delta H_D}{D}\right)}_{\pi_2}, \underbrace{\frac{Q}{ND^3}}_{\pi_3}\right]$$

For the model flow we have

$$\pi_2 = \left[\frac{\Delta H_D}{D}\right]_m = \frac{10}{(5/12)} = 24$$

$$\pi_3 = \left[\frac{Q}{ND^3}\right]_m = \frac{3}{(900)(2\pi/60)(5/12)^3} = 0.440$$

For **dynamic similarity,** we must maintain these values of the π's for the full scale pump. Hence,

$$\left[\frac{\Delta H_D}{D}\right]_p = 24$$

$$\therefore \frac{(\Delta H_D)_p}{(5/12)(3)} = 24 \qquad (\Delta H_D)_p = 30 \text{ ft}$$

Also,

$$\left[\frac{Q}{ND^3}\right]_p = 0.440$$

$$\therefore \frac{Q_p}{(300)(2\pi/60)[3(5/12)]^3} = 0.440 \qquad Q_p = 27 \text{ ft}^3/\text{s}$$

Finally we require

$$\left[\frac{P}{\rho D^5 N^3}\right]_m = \left[\frac{P}{\rho D^5 N^3}\right]_p$$

$$\frac{2}{\rho(5^5)(900^3)} = \frac{P_p}{\rho[(3)(5)]^5(300^3)}$$

$$\therefore P_p = 18 \text{ hp}$$

■ **Debriefing**

This problem is interesting in that it takes us away from the usual wind-tunnel–towing-tank kind of application that is so often used.

7.9 PRACTICAL USE OF THE DIMENSIONLESS GROUPS

If the variables of a fluid phenomenon are known, we have learned that a dimensional analysis will yield a set of independent dimensionless groups which may usually be put in the form of the various "numbers" discussed previously as well as dimensionless groups in the form of simple geometrical ratios. If at least all but one of the groups are duplicated for geometrically similar flows, we reasoned in Sec. 7.7 that the flows will probably be dynamically similar. This fact introduces the possibility of testing a model of some proposed apparatus in order to study, less expensively, full-scale performance and possible design variations, as we have already indicated. For instance, in the aircraft industry, model testing of some proposed airfoil or entire plane is a very important and significant part of a development program (Fig. 7.8). It must be pointed out that model testing is not inexpensive: the models run into many thousands of dollars and use of test facilities often costs thousands of dollars per hour. In addition to these deterrents, there is

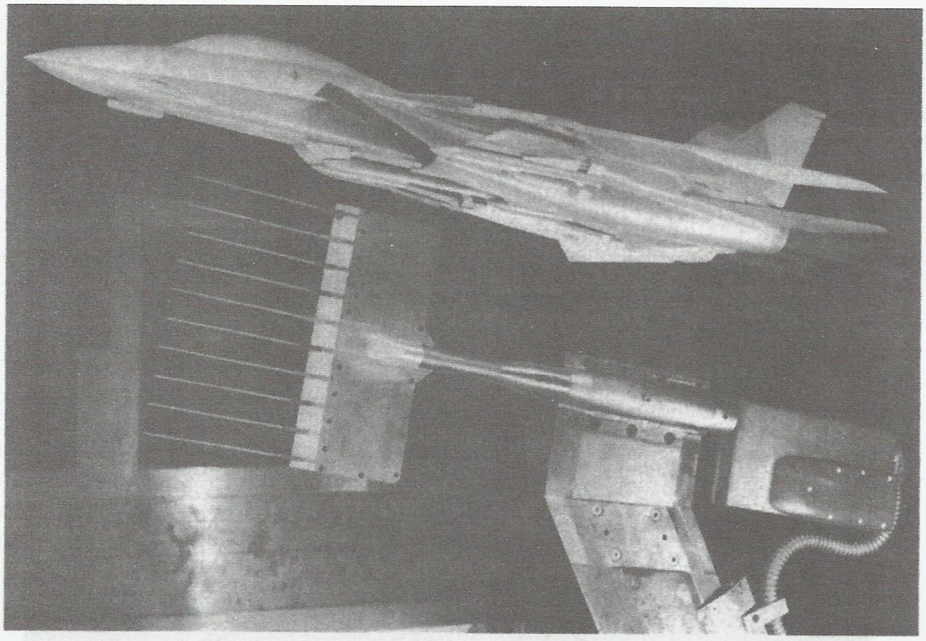


Figure 7.8
Grumman F-14 Tomcat model undergoing flow survey in 7 × 10 ft transonic wind tunnel. (*Aviation and Surface Effects Department, David W. Taylor Naval Ship Research and Development Center, Carderock, Maryland.*)

the important practical question of how much dynamic similarity can be achieved in a test. Of course, this is a very important criterion with respect to the usefulness of the results.

One of the oldest forms of scientific model testing involving considerations of flow similarity is that of the towing tank (see Fig. 7.9), where models of proposed water hulls are moved along a channel of water and drag estimates are made with the aid of certain measurements. It will now be shown that even in this apparently straightforward test, true dynamic similarity cannot be achieved for practical testing purposes. A dimensional analysis will reveal that three groups are involved in determining the drag on the hull—the pressure coefficient, the Reynolds number, and the Froude number. Therefore, for dynamic similarity it is necessary that at least the Reynolds number and Froude number be duplicated for prototype flow and geometrically similar model flow. Suppose that the prototype has a length of 100 ft and a velocity of 10 kn and is to be propelled through fresh water with a viscosity of 2.10×10^{-5} lbf-s/ft^2 and a density of 62.4 lbm/ft^3. Then

$$\mathrm{Re} = \frac{\rho V D}{\mu} = \frac{[(62.4/32.2)\ \text{slugs/ft}^3][(10)(1.689)\ \text{ft/s}][100\ \text{ft}]}{[2.10 \times 10^{-5}\ \text{lbf-s/ft}^2]}$$

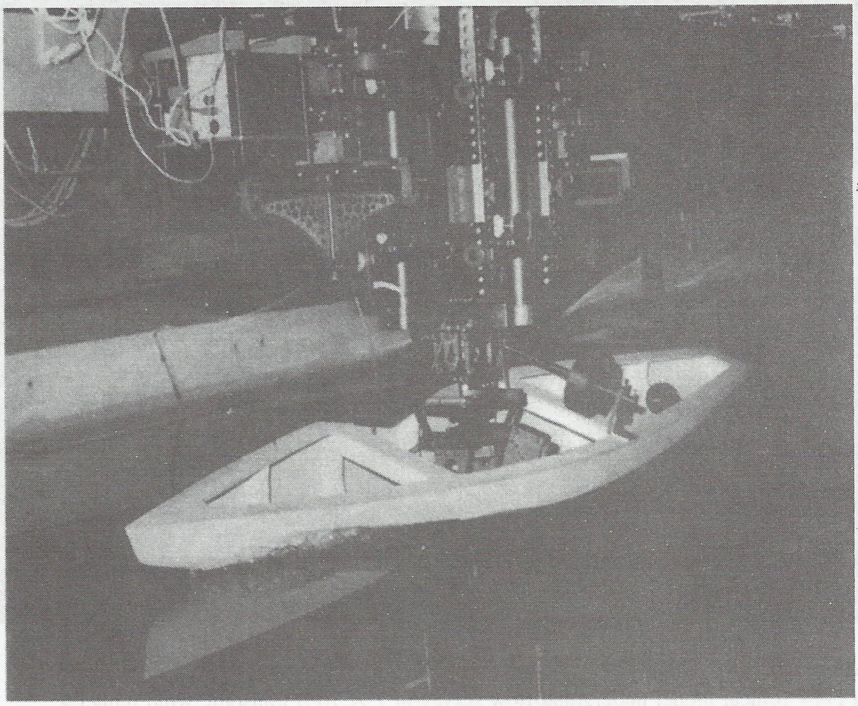


Figure 7.9
Testing a model of America Cup entry *Heart of America* in a towing tank. Unfortunately this sailboat was not a winner. (*Courtesy Davidson Laboratories. Stevens Institute of Technology.*)

Replacing pound-force by (1 slug)(ft/s^2) from Newton's law, we get for Re

$$\mathrm{Re} = 1.559 \times 10^8$$

For the Froude number, we get

$$\mathrm{Fr} = \frac{V^2}{Lg} = \frac{[(10)(1.689)\ \mathrm{ft/s}]^2}{[100\ \mathrm{ft}][32.2\ \mathrm{ft/s^2}]} = 0.0886$$

For a geometrically similar model of a scale one-twentieth of the prototype to duplicate the Froude number, we require a velocity determined as

$$\left(\frac{V^2}{Lg}\right)_m = \frac{V_m^2}{(\frac{1}{20})(100)(32.2)} = 0.0886$$

$$\therefore\ V_m = 2.24\ \mathrm{kn}$$

To duplicate the Reynolds number we require next

$$\left(\frac{\rho VD}{\mu}\right)_m = \left(\frac{VD}{\nu}\right)_m = \left[\frac{[(2.24)(1.689)](\frac{1}{20})(100)}{\nu_m}\right] = 1.559 \times 10^8$$

$$\therefore \nu_m = 1.2113 \times 10^{-7} \text{ ft}^2/\text{s}$$

Here we reach an impasse, since such a fluid cannot be practically formed. Furthermore, for other tests, there would probably be different requirements on the kinematical viscosity of the fluid. Hence, using ordinary water, as is normally the case, dynamic similarity is not achieved in towing-tank tests. However, by duplicating Froude numbers and by additional *theoretical* computations, the errors arising from the dissimilar aspects of the flows can be successfully taken into account. Likewise, in wind-tunnel testing, one sometimes must take wind-tunnel corrections into account so as to render results which are meaningful for full-scale operation.

It must always be kept in mind that model testing, although tremendously cheaper than full-scale testing, is nevertheless usually quite expensive. When theoretical computer computations can be made instead of model testing, this is usually the least-expensive avenue of approach and should be thoroughly explored before embarking on a long, expensive model-testing program.

7.10 SIMILITUDE WHEN THE DIFFERENTIAL EQUATION IS KNOWN

In this chapter, we considered a process where the significant variables were known. This permitted us to ascertain the dimensionless groups characterizing the process. You will recall that we *arbitrarily defined* various similarity relations and then, with the aid of Newton's second law and Newton's viscosity law applied to a simple type of flow, we showed that by duplicating all but one of the dimensionless groups, we could achieve these similitude relations between flows with geometrically similar boundaries. In some other process of another field of study where we do not know the laws involved, how would we establish the effect of duplicating dimensionless groups? In such a case much experience with the process might lead to an insight about the nature of the similarity laws achieved by duplicating dimensionless groups.

If, at the other extreme, the *differential equation* depicting the process is known, we can *deduce* the dimensionless groups *and* the similarity laws resulting from their duplication, even if the differential equation is not solvable. In Chap. 9 we develop the powerful Navier-Stokes equations governing fluid flow. We will then *deduce* at that time (Sec. 9.8) the kinematic and dynamic-similarity laws of the present chapter in a direct and straightforward fashion. Nevertheless, the approach we have followed here, although not so direct, will give us a greater physical feel for the dimensionless groups and will permit us to think of similitude early in our study of fluid mechanics.

HIGHLIGHTS

In this chapter, we exploit the use of the **law of dimensional homogeneity** in two distinct ways. First, we show that if we know the independent dimensionless groups that can be formed from the variables of a particular process (i.e., the π's), then we can express the relations between the variables in a much simpler, vastly more efficient manner. Recall that we showed that a single curve relating two dimensionless groups was as useful as hundreds of curves among the variables themselves. Even more important was the fact that experimentation using the dimensionless groups involved much **less** and much **simpler** experimentation. It should be eminently clear to the student that, knowing the variables involved, it is vital to first start with a dimensional analysis of the variables to benefit from its advantages. We then stated Buckingham's π theorem, whereby the number of variables less the number of independent dimensions yields the number of independent dimensionless groups involved in the process. For processes involving stress, one should check for this number using the rank of the dimensional matrix.

Before going on, we might ask why do the dimensionless groups have the benefits for experimentation? Since nature in no way heeds our dimensional considerations, then dimensionless groups having this same characteristic may then have physical significance. This may explain why **a set of dimensionless groups numbering fewer than the variables** can portray the behavior of a process in a more efficient way than the variables themselves. Also, consider a dimensionless group having, say, four variables. With dimensionless groups characterizing the process, we can vary this group at will by **varying just one** of the variables rather than having to independently vary all four variables as would be the case if we do not use dimensional analysis. This is what accounts for the tremendous gain that can be achieved.

We presented two ways of determining a set of dimensionless groups for this process. The first was longer and made very evident the role of the law of dimensional homogeneity in the process. The second procedure had the virtue of being short but lacked the outright presence of the law of dimensional homogeneity, being more a matter of manipulation. Your author obviously preferred the first approach. Students, however, will find the second approach of particular value.

The last part of the chapter deals with the vital consideration of **dynamic similarity.** To develop the concept of dynamic similarity we presented these definitions:

Geometric similarity. Boundaries between two flows are exact dilations or shrinkages of each other.

Kinematic similarity. Streamlines between flows are exact dilatations or shrinkages of each other.

Dynamic similarity. Corresponding forces at corresponding locations are mutually parallel and have the same ratio of magnitudes. This ratio is the same for all sets of corresponding forces.

We went on to show that the *necessary conditions* for dynamic similarity were the presence of both *geometric* and *kinematic similarity* plus having *similar mass distributions* whereby the ratio of densities at corresponding positions had to be of one value. This brings up two questions.

1. Why is this condition of dynamic similarity important?
2. How do we achieve it?

First, if the ratio of corresponding forces at corresponding points is of one value, then the resultant forces between a model and a prototype will of necessity be the same ratio. This ratio is easily obtained by looking at free-stream conditions between the flows, thus giving the experimentalist the proper value of some force component (like the drag or the lift) of an airfoil prototype from a scaled down model.

Second, the dimensional analysis of the flow yields the dimensionless groups, all but one of which must be duplicated between the flows to attain the valuable necessary condition of dynamic similarity and thus a good grasp of the expected prototype performance.

Thus we see that simply having a model scaled down geometrically to 1/100 the size of the prototype does not mean that something like the drag will be reduced by the same ratio. It is the dimensionless groups that govern such ratios and not the individual variables.

The most gratifying surprise of the chapter occurred when we showed that the dimensionless groups needed for dynamic similarity for a flow consisted of all but one of the dimensionless groups found by the methods of Part A of this chapter.

Actually, for many practical reasons indicated in the chapter, complete dynamic similarity is actually never attained. However, one can often approach dynamic similarity to a point where small theoretical corrections can be inserted to get very good results from model testing.

In Chap. 9, we derive the similarity laws for incompressible, laminar flows using the powerful Navier-Stokes equations derived in that chapter. The dimensionless groups are also derived, and we show mathematically how they govern the ratios of forces between model and prototype flows. This is the most emphatic way of getting to the root of the role of dynamic similarity in testing using models.

7.11 CLOSURE

In this chapter we have examined the procedures and concepts involved in dimensional analysis and similitude and the relation between these concepts. Although the techniques presented and the concepts discussed are important in themselves, there is an even more important lesson to be inferred from this chapter which has not been stressed in the discussion thus far.

It is no doubt clear from the tremendous cost of many devices such as ships, airplanes, and rockets that careful preliminary design and model testing are a requirement before one can "freeze" plans and start on the construction of such a system. Wind-tunnel tests and towing-tank tests on *models* are a major part of a development program. And in these tests dynamic similarity between model and prototype is *almost never* actually completely achieved. There is then usually the necessity of making corrections and adjustments of the model data to make it more accurate. Such steps can be effectively taken only by engineers who are well grounded in the fundamental theory of fluid mechanics. Thus, the "practical" field of testing requires a thorough understanding of fundamentals for other than the most routine full-scale testing.

In retrospect, you can see that the engineer must first get as close to dynamic similarity as is possible. Even then he or she may have to make additional corrections via theoretical procedures. Therefore, this chapter may perhaps motivate you to examine carefully the fundamental concepts of the various types of flow to be examined in Parts II and III of the text and not to become despondent when for certain situations we cannot employ theory to produce answers to given problems by straightforward analytical methods. In such cases, one must often resort to experiments, particularly with models, and such experiments and tests can be carried out successfully only when the basic laws of fluid mechanics that we have been studying are well understood.

PROBLEMS

Problem Categories

Dimensionless expressions 7.1–7.5, 7.7
Dimensional matrix 7.6, 7.10
Dimensional analysis 7.8, 7.9, 7.11–7.45
Similitude 7.46–7.68

7.1 Show that the Weber number given as $\rho V^2 L/\sigma$ is dimensionless

where ρ = density of fluid
V = velocity
L = length
σ = surface tension given as force per unit length

7.2 A dimensionless group that is used in studies of heat transfer is the *Prandtl number* given as

$$\text{Pr} = \frac{c_p \mu}{k}$$

where c_p = specific heat at constant pressure
μ = coefficient of viscosity
k = thermal conductivity of a fluid

What is the dimensional representation of k in the *MLT* system of basic dimensions and in the *FLT* system of basic dimensions?

7.3 In Chap. 12, we will be introduced to the so-called shear velocity τ_* defined as

$$\tau_* = \sqrt{\tau_w/\rho}$$

where τ_w is the shear stress at the wall. Show that τ_* has the dimensions of a velocity—hence its name.

7.4 In heat transfer the heat *convection coefficient* h is defined as the wall heat flux (energy per unit time per unit area) divided by the difference between the wall temperature and average temperature of the fluid at the wall. The *Stanton* number St is a useful dimensionless group defined as

$$\text{St} = \frac{h}{\rho c_p V}$$

Show that it is dimensionless.

7.5 The *Grashof* number Gr is used in buoyancy induced flows where temperature is nonuniform and is defined as

$$\text{Gr} = \frac{g\beta L^3 t}{\nu^2}$$

where β is the thermal expansion coefficient defined as the change in volume per unit volume per unit temperature and t is temperature. Show that the Grashof number is dimensionless.

7.6 What is the rank of the following dimensional matrix? What are the dimensions of α, β, γ, and δ?

	α	β	γ	δ
M	1	0	1	0
L	2	1	−1	1
T	−1	2	−1	0

7.7 Consider a mass on a weightless, frictionless spring as in Fig. P7.7. The spring constant is K, and the position of the body of mass M is measured by the displacement x from the position of static equilibrium. In ascertaining the amplitude of vibration A on such a system resulting from a harmonic disturbance, we know that the following variables are involved:

A = amplitude of vibration
M = mass of body
K = spring constant
F_0 = amplitude of disturbance
ω = frequency of disturbance

Assume that this problem cannot be handled theoretically but must be done experimentally.

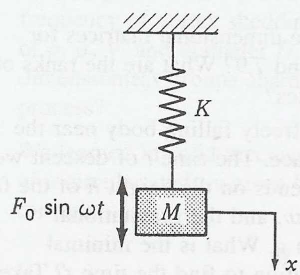


Figure P7.7

a. Explain how you would carry out an experimental program without the use of dimensional analysis.
b. Form two independent dimensionless groups by trial and error, and then explain how you would carry out your experiments.

7.8 The maximum pitching moment that is developed by the water on a flying boat as it lands is noted as $C_{\max}$. The following are the variables that are involved in this action:

α = angle made by flight path of plane with horizontal
β = angle defining attitude of plane
M = mass of plane
L = length of hull
ρ = density of water
g = acceleration of gravity
R = radius of gyration of plane about axis of pitching

According to Buckingham's π theorem, how many independent dimensionless groups should there be which characterize this problem?

7.9 The power required to drive a propeller is known to depend on

D = diameter of propeller
ρ = density of fluid
c = velocity of sound in fluid
ω = angular velocity of propeller
V = free-stream velocity
μ = viscosity of fluid

According to Buckingham's π theorem, how many dimensionless groups characterize this problem?

7.10 What are the dimensional matrices for Probs. 7.8 and 7.9? What are the ranks of these matrices?

7.11 Consider a freely falling body near the earth's surface. The time t of descent we believe depends on the height h of the fall, the weight w, and the gravitational acceleration g. What is the minimal experimentation to find the time t? Take g as a constant.

7.12 The period τ of oscillation for a pendulum is known to depend on l, the length of the pendulum, its mass m, and gravitational acceleration g. How close can you come to the well-known formula

$$\tau = 2\pi\sqrt{l/g}$$

by dimensional analysis?

7.13 In Sec. 7.5, we used the MLT system of basic dimensions for pressure drop in a pipe. Now carry out the development using the FLT system of basic dimensions. If you don't get the same π's as in Sec. 7.5, manipulate π's algebraically until you do.

7.14 By a formal procedure, evaluate a set of dimensionless groups for Prob. 7.7.

7.15 Determine a set of dimensionless groups for Prob. 7.8.

7.16 Determine a set of dimensionless groups for Prob. 7.9. Adjust your π's by algebra to get

$$\frac{P}{D^5\rho\omega^3} = f\left(\text{Re, M, } \frac{D\omega}{c}\right)$$

7.17 In strength of materials, you learned that the shear stress in a rod under torsion is given as

$$\tau = \frac{M_x r}{J} \qquad \text{[a]}$$

where M_x is the torque, and J is the polar moment of area of the cross section. We can give the formula above in dimensionless form as follows:

$$\left(\frac{\tau r^3}{M_x}\right) = \left(\frac{r^4}{J}\right) \qquad \text{[b]}$$

How close to this formula can you get by dimensional analysis? Use the FLT system of basic dimensions.

7.18 Do Example 7.1 using the MLT system of basic dimensions.

7.19 A disc A having a moment of inertia I_{xx} is held by a light rod of length L (see Fig. 7.19). The formula for free torsional oscillation of the disc is given as:

$$\theta = A \sin\sqrt{\frac{K_t}{I_{xx}}}\,t + B \cos\sqrt{\frac{K_t}{I_{xx}}}\,t$$

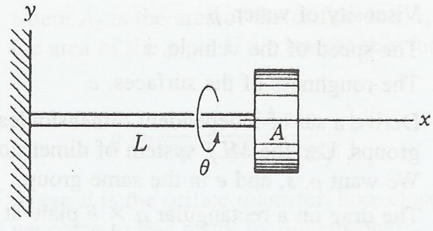


Figure P7.19

where K_t is the equivalent torsional spring constant coming from the rod and has a value given as

$$K_t = \frac{GJ}{L}$$

where G = shear modulus of the rod
J = polar moment of area of the rod
L = length of the rod

How close can you come to these results by using dimensional analysis?

7.20 Experience dictates that the head ΔH_D developed by turbomachines depends on the following variables:

Diameter of rotor, D

Rotational speed, N

Volume flow through the machine, Q

Kinematic viscosity, ν

Gravity, g

Show that

$$\frac{\Delta H_D}{D} = f\left(\frac{Q}{ND^3}, \frac{g}{N^2 D}, \frac{ND^2}{\nu}\right)$$

7.21 The velocity of sound c in a perfect gas is given as $c = \sqrt{kRT}$, where k is the ratio of specific heats and hence dimensionless, and T is the temperature. What can you learn about c by using dimensional analysis only?

7.22 In the laminar flow of a viscous fluid through a capillary tube, the pressure drop over length L is a function of the velocity, diameter, viscosity, and length. Determine the π's involved. How close can you get to the formula $\Delta p = 32(V\mu/L)(L/D)^2$ to be derived in Chap. 8?

7.23 A boat moving along the free surface has a drag D which we know depends on the following variables:

$V \equiv$ velocity
$L \equiv$ length of boat
$\mu \equiv$ viscosity
$g \equiv$ gravity
$\rho \equiv$ density
$B \equiv$ beam or width of boat

Formulate the dimensionless groups involved. If you don't get the Reynolds number, Froude number, and Euler number, manipulate your π's algebraically to get them.

7.24 The pressure drop Δp in a compressible one-dimensional flow in a circular duct is a function of the following variables:

Density, ρ

Velocity of sound, c

Viscosity, μ

Velocity of flow, V

Diameter of duct, D

Length of duct, L

What are the dimensionless groups involved? Manipulate your results until you get an Euler number, a Reynolds number, and a Mach number as three of your π's.

7.25 Formally develop the dimensionless groups given in Sec. 7.4 from the variables presented. If you don't get the particular groups presented in this section, manipulate the π's until you do.

7.26 In Fig. 12.31 we have shown vortices **shedding from flow** past a cylinder. If the frequency of vortex shedding N is a function of ρ, μ, V, and diameter D, what are the dimensionless groups characterizing the process?

7.27 We learned in solid mechanics that the twist of a circular shaft is given by the following formula:

$$\Delta\phi = \frac{M_x L}{GJ} \qquad [1]$$

How close can you come to this result by using dimensional analysis? Proceed as follows:

1. Using *FLT* system write dimensional matrix.
2. What is the rank r of this matrix?
3. Now get as close as possible to Eq. 1 using dimensional analysis.

7.28 A jet of liquid (1) enters liquid (2). The length L from discharge to complete disintegration is to be studied. If the variables known to be involved are the densities and viscosities of the fluids and the jet velocity V_j as well as the jet diameter D_j, determine the dimensionless groups involved.

7.29 Stoke's law for a small sphere of radius R has a drag F for steady creeping flow around the sphere given as

$$F = 6\pi\mu VR$$

How would you get this formula experimentally with a minimum of experimentation knowing the variables involved?

7.30 In strength of materials you learned that the buckling load of a pin ended column is given as

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

How close can you come to this formulation via dimensional analysis?

7.31 The drag D of a diving bell (see Fig. P7.31) depends on the following variables:

Volume of vehicle, V

Density of water, ρ

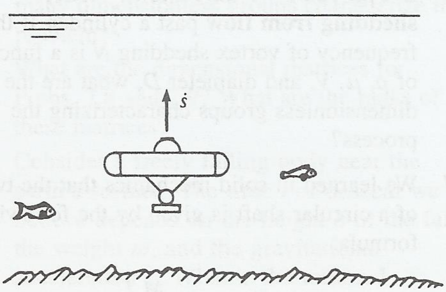


Figure P7.31

Viscosity of water, μ

The speed of the vehicle, $\dot{s}$

The roughness of the surfaces, e

Derive a set of independent dimensionless groups. Use the *MLT* system of dimensions. We want ρ, $\dot{s}$, and e in the same group.

7.32 The drag on a rectangular $a \times b$ plate at an angle α relative to a wind of velocity V is desired. The drag depends on a, b, α, V, μ, and ρ. What dimensionless groups characterize the process?

7.33 Fourier's law of heat conduction in a solid is known to be

$$q = \frac{kA}{L}(t_2 - t_1)$$

where $q \equiv$ energy flow per unit time
$k \equiv$ is the thermal conductivity and is energy times thickness per unit area, per unit time, per unit temperature
$t \equiv$ temperature

How close can you come to Fourier's conduction law by dimensional analysis?

7.34 In the chapter on boundary layers (Chap. 12, see Fig. 12.1), you will learn that the boundary layer thickness δ depends for a smooth plate on the following items:

μ, viscosity of the fluid

ρ, mass density of the fluid

V_0, the free stream velocity

x, the distance from the leading edge of the plate Theory indicates for a laminar boundary layer that

$$\frac{\delta}{x} = 4.96 \Bigg/ \sqrt{\frac{\rho V_0 x}{\mu}}$$

How close can you come by dimensional analysis? Notice that $\rho V_0 x/\mu$ is a form of Reynolds number with x as the length dimension.

7.35 The flow through a square-edged circular orifice is given as follows for an inviscid liquid:

$$q = A_2 \left\{ \frac{2(p_1 - p_2)/\rho}{[1 - (A_2/A_1)^2]} \right\}^{1/2}$$

where A_2 is the area of the orifice and A_1 is the area of the pipe. If we rewrite this formula as

$$q = \frac{\pi d^2}{4}\left\{\frac{2\Delta p/\rho}{1-(d/D)^2}\right\}^{1/2} \qquad \text{(a)}$$

where d is the orifice diameter, how close can we come to this result by dimensional analysis alone?

7.36 The following variables are known to be involved in a flow:

ρ, mass density
L, characteristic length
c, velocity of sound
μ, viscosity
g, acceleration of gravity
V, average velocity
Δp, pressure change

What are the π's involved? Form the Reynolds number, Froude number, Mach number, and Euler number from your results.

7.37 The rise in a tube due to capillary action is a function of D, θ, σ, g, and ρ.

$$\therefore h = f(D, \theta, \sigma, g, \rho)$$

where σ is the surface tension (F/L). Rewrite this relation in terms of dimensionless groups. Have σ, ρ, and g be in the *same* π. Use F, L, and T as basic dimensions. What is r as determined by the dimensional matrix? By algebraic manipulation of π's reach the following result:

$$\left(\frac{h}{D}\right) = G\left[\left(\frac{\sigma}{D^2\rho g}\right), \theta\right]$$

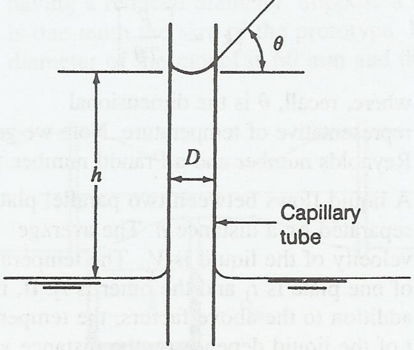


Figure P7.37

7.38 The thrust from an airplane propeller is a function of the following variables:

V_0 = speed of airplane
D = diameter of propeller
ρ = density of air
μ = viscosity of the air
c = speed of sound
ω = ang. speed of propeller

Hence,

$$T = f(V_0, D, \rho, \mu, c, \omega)$$

Find the dimensionless groups that characterize the process. Manipulate so you get:

$$\frac{T}{\rho\omega^2 D^4} = g\left(\frac{\rho V_0 D}{\mu}, \frac{V_0}{c}, \frac{V_0}{\omega D}\right)$$

7.39 A capillary tube (Fig. P7.39) can be used for measuring viscosity. It is known that for this device the viscosity μ is a function of the following variables:

D, diameter of tube
ρ, mass density of fluid
g, acceleration of gravity
L, length of tube from capillary to exit
h, height of capillary fluid
q, volume flow of fluid

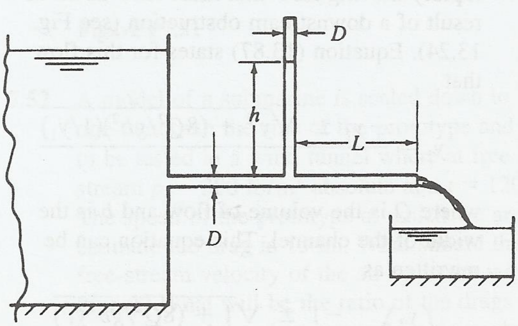


Figure P7.39

How close can you get to the following solution:

$$\left(\frac{\mu}{\rho g^{1/5} q^{2/5}}\right) = \frac{\pi}{128}\left(\frac{D^4 g^{4/5}}{q^{8/5}}\right)\left(\frac{h}{L}\right)$$

7.40 The viscosity μ in a viscosimeter depends on the following variables:

T_q, torque on the spring

ω, angular speed or container

r_1, radius of drum

h, height of drum

ϵ, distance between drum base and container

$\alpha = (r_2 - r_1)$, distance between container and drum bottom

Evaluate the dimensionless groups getting μ, ω, and T in one group. How close can you come to the following analytic solution?

$$\left(\frac{\mu \omega r_1^3}{T_q}\right) = \frac{1}{2\pi}\left[\frac{1}{(h/\alpha) + (1/4)(r_1/\epsilon)}\right]$$

7.41 The viscosity of a fluid is found by observing the terminal speed V_T of a small sphere of radius R and mass density ρ_s in the viscous fluid whose density is ρ_L. We got the following result

$$\mu = \frac{2}{9}\frac{gR^2}{V_T}[\rho_s - \rho_L] \qquad [a]$$

How close can you come using dimensional analysis?

7.42 In Chap. 13 on free surface flow you will learn that a *hydraulic jump* can occur in a rapidly moving flow that must slow down as a result of a downstream obstruction (see Fig. 13.24). Equation (13.87) states for this flow that

$$y_2 = \frac{-y_1 \pm \sqrt{y_1^2 + (8Q^2/gb^2)(1/y_1)}}{2}$$

where Q is the volume of flow and b is the width of the channel. This equation can be rewritten as

$$\left(\frac{y_2}{y_1}\right) = \frac{-1 \pm \sqrt{1 + (8Q^2/gb^2y_1^3)}}{2}$$

a. How close can you come to this result by dimensional analysis?

b. Show that $Q^2/gb^2y_1^3$ is a Froude number with y_1 as the length dimension.

7.43 In the study of turbomachines there are generally six variables involved. They are:

Size of machine diameter, D

Rotational speed, N

Volume flow through the machine, Q

Kinematic viscosity, ν

Gravity, g

Change in total head, ΔH_D

Show that the following describes the performance of turbomachines:

$$f\left(\frac{Q}{ND^3}, \frac{\Delta H_D}{D}, \frac{g}{N^2D}, \frac{ND^2}{\nu}\right) = 0$$

7.44 Consider a flow of fluid past a cylinder involving heat transfer. The heat transfer coefficient h is known for certain conditions to depend on the following variables:

Free-stream velocity, V

Fluid density, ρ

Fluid viscosity, μ

Coefficient of thermal conductivity, k

Diameter of cylinder, D

Specific heat, c_p

What are a set of dimensionless groups for this process? The dimensions for h and k are

$$[h] = \left[\frac{L}{\theta T^3}\right]$$

$$[k] = \left[\frac{F}{T\theta}\right]$$

where, recall, θ is the dimensional representative of temperature. Note we get a Reynolds number and a Prandtl number, $c_p\mu/k$.

7.45 A liquid flows between two parallel plates separated by a distance h. The average velocity of the liquid is V_0. The temperature of one plate is t_1 and the other is t_2. If, in addition to the above factors, the temperature t of the liquid depends on the distance y above the bottom plate, the viscosity of the liquid, the specific heat c_p of the liquid, and thermal conductivity k, what are the dimensionless groups involved to get this temperature? Show that

$$t/t_1 = f[y/h, \mu c_p/k, c_p(t_1 - t_2)/V_0^2]$$

7.46 Explain why dynamic similitude between a model flow and a prototype flow about an airfoil is desirable in a wind-tunnel test.

7.47 The drag of a two-man submarine hull is desired when it is moving far below the free surface of water. A model scaled down one-tenth from the prototype is to be tested. What dimensionless group should be duplicated between model and prototype flows? If the drag of the prototype at 1 kn is desired, at what speed should the model be moved to give the drag to be expected by the prototype?

7.48 Oil having a kinematic viscosity of 6.05×10^{-5} ft²/s is flowing through a 10-in pipe. At what velocity would water at 60°F have to flow through the pipe for dynamically similar flow? What is the ratio of drags for corresponding lengths of pipe from the flows? The specific gravity of the oil is 0.8.

7.49 The wave resistance of an ocean liner scaled down $\frac{1}{100}$ is to be measured. If the drag of the prototype at 20 kn is desired, what must the speed of the model be? Ascertain the ratio of the drag forces between model and prototype.

7.50 A *Venturi meter* is a device for measuring flow in a pipe. It is merely a section of pipe having a reduced diameter. Suppose a model is one-tenth the size of the prototype. If the diameter of the model is 60 mm and the approach velocity is 5 m/s, what is the discharge in liters per second in the prototype for dynamic similarity? The kinematic viscosity in the model fluid is 0.9 times the kinematic viscosity of the prototype fluid.

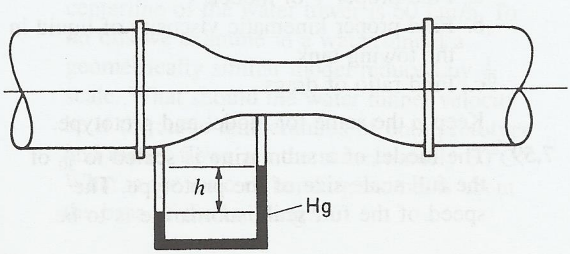


Figure P7.50

7.51 A V-notch *weir* is a vertical plate with a V notch through which fluid in a channel flows. If the shape of the free surface is vital in governing the flow, what π should be duplicated between the flows of a model and a prototype? If a model is one-fiftieth the size of the prototype and the free-stream velocity upstream is 10 ft/s for the prototype, what should the free-stream velocity be for the model? What is the ratio of force on the weir between model and prototype?

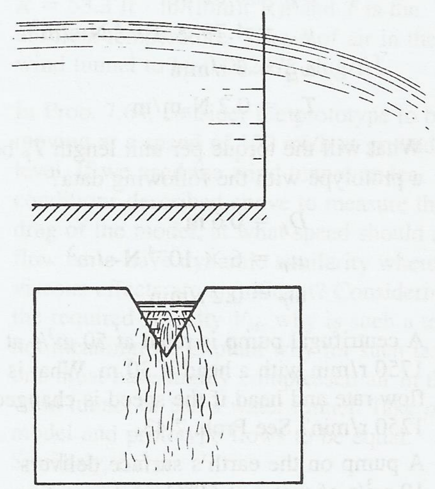


Figure P7.51

7.52 A model of a submarine is scaled down to one-twentieth the size of the prototype and is to be tested in a wind tunnel where at free stream p = 300 lb/in² absolute and t = 120°F. The speed of the prototype at which we are to estimate the drag is 15 kn. What should the free-stream velocity of the air be in the wind tunnel? What will be the ratio of the drags between model and prototype? Explain why, despite the high pressure in the wind tunnel, we can consider the *flow* to be incompressible. The following is given:

$$\left(\frac{\mu}{\rho}\right) \text{for seawater} = 1.121 \times 10^{-5}\ \text{ft}^2/\text{s}$$

Explain why you would not have dynamic similarity if the submarine prototype moved near the free surface.

7.53 A long cylinder is immersed in a large tank of liquid. The diameter of the cylinder is D and the viscosity of the liquid is μ. If the cylinder is spun slowly about its centerline at a speed ω rad/s, what dimensionless group or groups represent the torque per unit length T from viscous action? Suppose the data for a model of this system is known to be

$$D_M = 0.02\ \text{m}$$
$$\mu_M = 4.79 \times 10^{-4}\ \text{N-s/m}^2$$
$$\omega_M = 3\ \text{r/min}$$
$$T_M = 0.2\ \text{N-m/m}$$

What will the torque per unit length T_P be for a prototype with the following data?

$$D_P = 0.6\ \text{m}$$
$$\mu_P = 6 \times 10^{-4}\ \text{N-s/m}^2$$
$$\omega_P = 0.2\ \text{r/min}$$

7.54 A centrifugal pump is rated at 50 m³/s at 1750 r/min with a head if 30 m. What is the flow rate and head if the speed is changed to 1250 r/min? See Prob. 7.43.

7.55 A pump on the earth's surface delivers 10 m³/s of water at 60°C while rotating at 1750 r/min. It has a head of 20 m and the diameter of the impeller is 0.4 m. On a space vehicle a geometrically similar pump $\frac{3}{4}$ the size pumps oil of kinematic viscosity 3×10^{-6} m²/s at a rotational speed of 1450 r/min. At what distance d from the earth's surface will there be possible dynamic similarity between space and earth pump flows? Determine the volume flow and head for the space pump. (The radius of the earth is 6372 km.) See Prob. 7.43.

7.56 A barge (Fig. P7.56) is towed at a speed V of 3 m/s. The width is 20 m. In a towing tank a model scaled down $\frac{1}{30}$ is being tested. What should the ratio of drags be if we duplicate wave drag with the idea that we will correct for skin friction drag later? Take ρ of water in both cases as 1000 kg/m³ and $\nu = 0.0113 \times 10^{-4}$ m²/s for prototype and model.

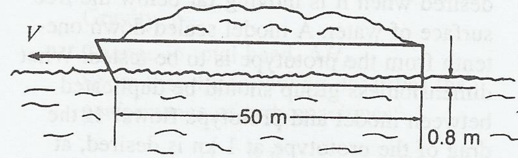


Figure P7.56

7.57 A transport plane is expected to fly at 550 mi/h at an elevation of 30,000 ft standard atmosphere. A model of this plane scaled down to $\frac{1}{15}$ of the prototype is to be tested in a wind tunnel at a temperature of 70°F. To duplicate both *Reynolds* and *Mach* numbers, what is the tunnel *velocity* and the tunnel *pressure* absolute? Take μ_{air} for prototype as 2×10^{-7} lb · s/ft². Take μ_{air} for model as 4.2×10^{-7} lb · s/ft². The speed of sound for a perfect gas is $\sqrt{kRT}$.

7.58 A prototype of a boat of length 100 ft is to move at a speed of 10 kn in fresh water where $\mu = 2.10 \times 10^{-5}$ lbf-s/ft² and $\rho = 62.4$ lbm/ft³. A model scaled down $\frac{1}{20}$ of the prototype is to be tested in a towing tank. For dynamic similarity what three dimensionless groups must be duplicated?

a. Find proper V of model.
b. Find proper kinematic viscosity of liquid in the towing tank.
c. Find ratio of drags.

Keep ρ the same for model and prototype.

7.59 The model of a submarine is scaled to $\frac{1}{30}$ of the full scale size of the prototype. The speed of the full scale submarine is to be

20 kn while at the free surface of sea water where

$$(\nu)_{\text{sea water}} = 1.210 \times 10^{-5}\ \text{ft}^2/\text{s}$$
$$(\rho)_{\text{sea water}} = 1.940\ \text{lbm/ft}^3$$

In the towing tank, where $\nu = 1.217 \times 10^{-5}$ ft^2/s and $\rho = 1.938$ lbm/ft^3, what should be the free-stream velocity for movement at the free surface? What is the ratio of drags? Next, the submarine is considered to move much below the free surface at a speed of 0.5 kn. In a water tunnel, what should be the speed and the ratio of drags? Take $\nu = 1.217 \times 10^{-5}$ ft^2/s again.

7.60 Wave motion along a section of coast is to be studied experimentally in the laboratory using a geometrically similar shape reduced by a factor of 20. The density of ocean water is 1030 kg/m^3, and the laboratory fresh water is 1000 kg/m^3. If we neglect surface tension and friction, what is the wave velocity in the model if the wave velocity in the prototype is 0.15 m/s? What is the ratio of force between prototype and model for these flows?

7.61 A set of blades is used to mix crude oil in a large tank well below the free surface at a temperature of 20°C at an angular speed ω of 0.2 rad/s. A geometrically similar model of this device is reduced by a scale factor of $\frac{1}{5}$. This model is run at a speed ω_M required for dynamic similarity in the large tank of water. It is placed well below the free surface and is at a temperature of 60°C. If the model requires a torque of 0.4 N-m, what are the torque and power for the prototype?

7.62 We wish to determine the wind force on a water tower when a wind normal to the centerline of the water tower is 60 km/h. To do this we examine in a water tunnel a geometrically similar model reduced by $\frac{1}{20}$ scale. What should the water tunnel velocity be if the static temperatures of both prototype and model flows are the same, namely 60°C? What is the ratio of bending moments about the base of the building?

7.63 We wish to model an irrigation canal reduced by $\frac{1}{20}$ scale. Water is flowing in the canal at a speed of 1 m/s at a temperature of 30°C. If we are to duplicate *both* Reynolds and Froude numbers, what must be the kinematic viscosity of the model flow?

7.64 The model of an airfoil reduced to one-twentieth of the prototype is to be tested in a wind tunnel where the temperature is 70°F and the pressure is atmospheric. If the prototype is to fly at 500 mi/h at 5000 ft in the standard atmosphere, what should the velocity be in the wind tunnel for dynamic similarity considering only compressibility? What should the ratio of drags be for model to prototype? The velocity of sound in a perfect gas is $\sqrt{kRT}$, where for air $k = 1.4$, $R = 53.3$ ft · lb/(lbm)(°R), and T is the absolute temperature. Take ρ of air in the wind tunnel to be 0.002378 slug/ft^3.

7.65 In Prob. 7.64, consider the prototype to be moving at a speed of 150 mi/h at ground level. If we used the wind tunnel under conditions described above to measure the drag of the model, at what speed should the flow be to have dynamic similarity where viscous effects are significant? Considering the required velocity V_M, why is such a test not meaningful? Explain why for such tests one must have highly compressed air in the wind tunnel or use a water tunnel. Take ρ for model and prototype flows to be equal. Similarly, for μ.

7.66 We wish to use a model of an airfoil which is one-tenth the size of the prototype. The prototype is at a speed of 150 km/h in the process of landing where T of the air is 25°C. Because viscous effects are significant here, we will test the model in a water tunnel. What speed should we have if for the water the temperature is 50°C and the pressure is atmospheric at free stream? What is the ratio of lifts for the prototype to the model? At larger angles of attack, what must you be concerned with in this kind of a test?

7.67 Suppose in Probs. 7.9 and 7.16 that we exclude viscosity from the variables determining the power required to drive a propeller. A model of a propeller which is 2 ft in length is scaled down to one-fifth of the full-scale propeller. If the model requires 5 hp, what is the power needed for the full-scale propeller rotating at a speed of 150 r/min? The full-size propeller is to operate at 30,000 ft in the standard atmosphere at a free-stream speed of 300 mi/h. What free-stream speed should we use for the model test? What is the angular speed for the model? Take $T_m = 59°\text{F}$.

7.68 In Example 7.3, determine the indicated dimensionless groups presented in the example. Consider the following characteristics of the model.

$$P_m = 5 \text{ kW}$$
$$Q_m = 5 \text{ L/s}$$
$$\Delta H_m = 2 \text{ m}$$
$$N_m = 900 \text{ r/min}$$
$$D_m = 800 \text{ mm}$$

If the full-scale pump is to deliver 50 kW of power at a speed or 400 r/min, what should the scale factor for the full-scale pump be? What are the head and the volumetric flow for the full-scale pump?