The thrust for this XR-5M15 prototype engine is produced by gas particles being ejected at a high velocity. The determination of the forces on the test stand is based on the analysis of the motion of a *variable system of particles,* i.e., the motion of a

number of air particles considered together rather than separately.

CHAPTER 14

Systems of Particles

Chapter 14 Systems of Particles

14.1 INTRODUCTION

In this chapter you will study the motion of *systems of particles,* i.e., the motion of a large number of particles considered together. The first part of the chapter is devoted to systems consisting of well-defined particles; the second part considers the motion of variable systems, i.e., systems which are continually gaining or losing particles, or doing both at the same time.

In Sec. 14.2, Newton's second law will first be applied to each particle of the system. Defining the *effective force* of a particle as the product $m_i\mathbf{a}_i$ of its mass m_i and its acceleration $\mathbf{a}_i$, we will show that the *external forces* acting on the various particles form a system equipollent to the system of the effective forces, i.e., both systems have the same resultant and the same moment resultant about any given point. In Sec. 14.3, it will be further shown that the resultant and moment resultant of the external forces are equal, respectively, to the rate of change of the total linear momentum and of the total angular momentum of the particles of the system.

In Sec. 14.4, the *mass center* of a system of particles is defined and the motion of that point is described, while in Sec. 14.5 the motion of the particles about their mass center is analyzed. The conditions under which the linear momentum and the angular momentum of a system of particles are conserved are discussed in Sec. 14.6, and the results obtained in that section are applied to the solution of various problems.

[illegible] h the application of the work-[illegible] icles, and Sec. 14.9 with the [illegible] n principle. These sections also contain a number of problems of practical interest.

It should be noted that while the derivations given in the first part of this chapter are carried out for a system of independent particles, they remain valid when the particles of the system are rigidly connected, i.e., when they form a rigid body. In fact, the results obtained here will form the foundation of our discussion of the kinetics of rigid bodies in Chaps. 16 through 18.

The second part of this chapter is devoted to the study of variable systems of particles. In Sec. 14.11 you will consider steady streams of particles, such as a stream of water diverted by a fixed vane, or the flow of air through a jet engine, and learn to determine the force exerted by the stream on the vane and the thrust developed by the engine. Finally, in Sec. 14.12, you will learn how to analyze systems which gain mass by continually absorbing particles or lose mass by continually expelling particles. Among the various practical applications of this analysis will be the determination of the thrust developed by a rocket engine.

14.2 APPLICATION OF NEWTON'S LAWS TO THE MOTION OF A SYSTEM OF PARTICLES. EFFECTIVE FORCES

In order to derive the equations of motion for a system of n particles, let us begin by writing Newton's second law for each individual particle of the system. Consider the particle P_i, where $1 \le i \le n$. Let

m_i be the mass of P_i and $\mathbf{a}_i$ its acceleration with respect to the newtonian frame of reference $Oxyz$. The force exerted on P_i by another particle P_j of the system (Fig. 14.1), called an *internal force,* will be denoted by $\mathbf{f}_{ij}$. The resultant of the internal forces exerted on P_i by all the other particles of the system is thus $\sum_{j=1}^{n} \mathbf{f}_{ij}$ (where $\mathbf{f}_{ii}$ has no meaning and is assumed to be equal to zero). Denoting, on the other hand, by $\mathbf{F}_i$ the resultant of all the *external forces* acting on P_i, we write Newton's second law for the particle P_i as follows:

$$\mathbf{F}_i + \sum_{j=1}^{n} \mathbf{f}_{ij} = m_i \mathbf{a}_i \tag{14.1}$$

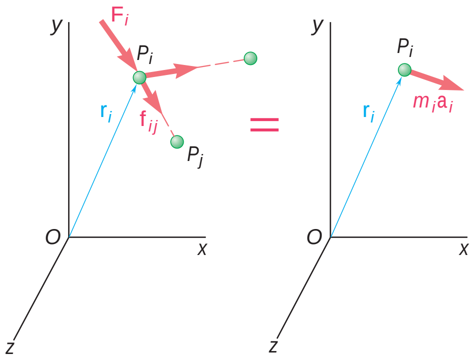


Fig. 14.1

Denoting by $\mathbf{r}_i$ the position vector of P_i and taking the moments about O of the various terms in Eq. (14.1), we also write

$$\mathbf{r}_i \times \mathbf{F}_i + \sum_{j=1}^{n} (\mathbf{r}_i \times \mathbf{f}_{ij}) = \mathbf{r}_i \times m_i \mathbf{a}_i \tag{14.2}$$

Repeating this procedure for each particle P_i of the system, we obtain n equations of the type (14.1) and n equations of the type (14.2), where i takes successively the values 1, 2, . . . , n. The vectors $m_i\mathbf{a}_i$ are referred to as the *effective forces* of the particles.† Thus the equations obtained express the fact that the external forces $\mathbf{F}_i$ and the internal forces $\mathbf{f}_{ij}$ acting on the various particles form a system equivalent to the system of the effective forces $m_i\mathbf{a}_i$ (i.e., one system may be replaced by the other)

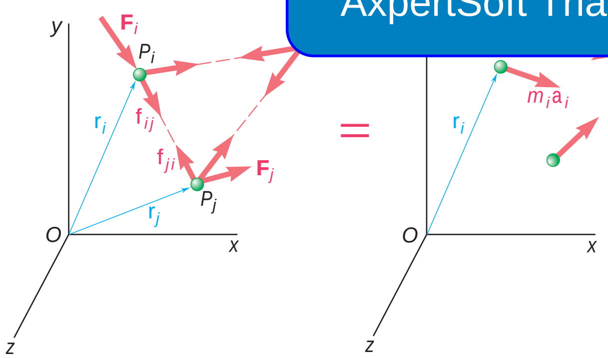


Fig. 14.2

Before proceeding further with our derivation, let us examine the internal forces $\mathbf{f}_{ij}$. We note that these forces occur in pairs $\mathbf{f}_{ij}$, $\mathbf{f}_{ji}$, where $\mathbf{f}_{ij}$ represents the force exerted by the particle P_j on the particle P_i and $\mathbf{f}_{ji}$ represents the force exerted by P_i on P_j (Fig. 14.2). Now, according to Newton's third law (Sec. 6.1), as extended by Newton's law of gravitation to particles acting at a distance (Sec. 12.10), the forces $\mathbf{f}_{ij}$ and $\mathbf{f}_{ji}$ are equal and opposite and have the same line of action. Their sum is therefore $\mathbf{f}_{ij} + \mathbf{f}_{ji} = 0$, and the sum of their moments about O is

$$\mathbf{r}_i \times \mathbf{f}_{ij} + \mathbf{r}_j \times \mathbf{f}_{ji} = \mathbf{r}_i \times (\mathbf{f}_{ij} + \mathbf{f}_{ji}) + (\mathbf{r}_j - \mathbf{r}_i) \times \mathbf{f}_{ji} = 0$$

†Since these vectors represent the resultants of the forces acting on the various particles of the system, they can truly be considered as forces.

since the vectors $\mathbf{r}_j - \mathbf{r}_i$ and $\mathbf{f}_{ji}$ in the last term are collinear. Adding all the internal forces of the system and summing their moments about O, we obtain the equations

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{f}_{ij} = 0 \qquad \sum_{i=1}^{n}\sum_{j=1}^{n}(\mathbf{r}_i \times \mathbf{f}_{ij}) = 0 \tag{14.3}$$

which express the fact that the resultant and the moment resultant of the internal forces of the system are zero.

Returning now to the n equations (14.1), where $i = 1, 2, \ldots, n$, we sum their left-hand members and sum their right-hand members. Taking into account the first of Eqs. (14.3), we obtain

$$\boxed{\sum_{i=1}^{n}\mathbf{F}_i = \sum_{i=1}^{n} m_i\mathbf{a}_i} \tag{14.4}$$

Proceeding similarly with Eq. (14.2) and taking into account the second of Eqs. (14.3), we have

$$\boxed{\sum_{i=1}^{n}(\mathbf{r}_i \times \mathbf{F}_i) = \sum_{i=1}^{n}(\mathbf{r}_i \times m_i\mathbf{a}_i)} \tag{14.5}$$

[illegible] ress the fact that the system of [illegible] f the effective forces $m_i\mathbf{a}_i$ have [illegible] ent resultant. Referring to the definition given in Sec. 3.19 for two equipollent systems of vectors, we can therefore state that *the system of the external forces acting on the particles and the system of the effective forces of the particles are equipollent*† (Fig. 14.3).

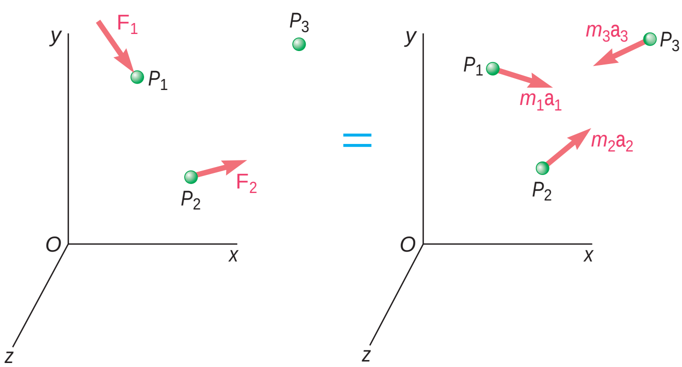


Fig. 14.3

†The result just obtained is often referred to as *d'Alembert's principle*, after the French mathematician Jean le Rond d'Alembert (1717–1783). However, d'Alembert's original statement refers to the motion of a system of connected bodies, with $\mathbf{f}_{ij}$ representing constraint forces which if applied by themselves will not cause the system to move. Since, as it will now be shown, this is in general not the case for the internal forces acting on a system of free particles, the consideration of d'Alembert's principle will be postponed until the motion of rigid bodies is considered (Chap. 16).

Equations (14.3) express the fact that the system of the internal forces $\mathbf{f}_{ij}$ is equipollent to zero. Note, however, that it does *not* follow that the internal forces have no effect on the particles under consideration. Indeed, the gravitational forces that the sun and the planets exert on one another are internal to the solar system and equipollent to zero. Yet these forces are alone responsible for the motion of the planets about the sun.

Similarly, it does not follow from Eqs. (14.4) and (14.5) that two systems of external forces which have the same resultant and the same moment resultant will have the same effect on a given system of particles. Clearly, the systems shown in Figs. 14.4*a* and 14.4*b* have

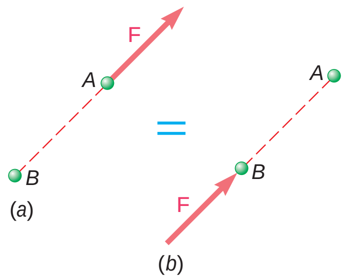


Fig. 14.4

the same resultant and the same moment resultant; yet the first system accelerates particle A and leaves particle B unaffected, while the second accelerates B and does [illegible] that when we stated in Sec. [illegible] forces acting on a rigid body [illegible] noted that this property could *not* be extended to a system of forces acting on a set of independent particles such as those considered in this chapter.

In order to avoid any confusion, blue equals signs are used to connect equipollent systems of vectors, such as those shown in Figs. 14.3 and 14.4. These signs indicate that the two systems of vectors have the same resultant and the same moment resultant. Red equals signs will continue to be used to indicate that two systems of vectors are equivalent, i.e., that one system can actually be replaced by the other (Fig. 14.2).

14.3 LINEAR AND ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES

Equations (14.4) and (14.5), obtained in the preceding section for the motion of a system of particles, can be expressed in a more condensed form if we introduce the linear and the angular momentum of the system of particles. Defining the linear momentum $\mathbf{L}$ of the system of particles as the sum of the linear momenta of the various particles of the system (Sec. 12.3), we write

$$\mathbf{L} = \sum_{i=1}^{n} m_i \mathbf{v}_i \tag{14.6}$$

Defining the angular momentum $\mathbf{H}_O$ about O of the system of particles in a similar way (Sec. 12.7), we have

$$\boxed{\mathbf{H}_O = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i\mathbf{v}_i)} \tag{14.7}$$

Differentiating both members of Eqs. (14.6) and (14.7) with respect to t, we write

$$\dot{\mathbf{L}} = \sum_{i=1}^{n} m_i\dot{\mathbf{v}}_i = \sum_{i=1}^{n} m_i\mathbf{a}_i \tag{14.8}$$

and

$$\begin{aligned} \dot{\mathbf{H}}_O &= \sum_{i=1}^{n} (\dot{\mathbf{r}}_i \times m_i\mathbf{v}_i) + \sum_{i=1}^{n} (\mathbf{r}_i \times m_i\dot{\mathbf{v}}_i) \\ &= \sum_{i=1}^{n} (\mathbf{v}_i \times m_i\mathbf{v}_i) + \sum_{i=1}^{n} (\mathbf{r}_i \times m_i\mathbf{a}_i) \end{aligned}$$

which reduces to

$$\dot{\mathbf{H}}_O = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i\mathbf{a}_i) \tag{14.9}$$

[illegible]near.
[illegible] members of Eqs. (14.8) and [illegible] the right-hand members of Eqs. (14.4) and (14.5). It follows that the left-hand members of these equations are respectively equal. Recalling that the left-hand member of Eq. (14.5) represents the sum of the moments $\mathbf{M}_O$ about O of the external forces acting on the particles of the system, and omitting the subscript i from the sums, we write

$$\boxed{\begin{aligned} \Sigma\mathbf{F} &= \dot{\mathbf{L}} \qquad (14.10) \\ \Sigma\mathbf{M}_O &= \dot{\mathbf{H}}_O \qquad (14.11) \end{aligned}}$$

These equations express that *the resultant and the moment resultant about the fixed point O of the external forces are respectively equal to the rates of change of the linear momentum and of the angular momentum about O of the system of particles*.

14.4 MOTION OF THE MASS CENTER OF A SYSTEM OF PARTICLES

Equation (14.10) may be written in an alternative form if the *mass center* of the system of particles is considered. The mass center of the system is the point G defined by the position vector $\bar{\mathbf{r}}$, which satisfies the relation

$$m\bar{\mathbf{r}} = \sum_{i=1}^{n} m_i \mathbf{r}_i \qquad (14.12)$$

where m represents the total mass $\sum_{i=1}^{n} m_i$ of the particles. Resolving the position vectors $\bar{\mathbf{r}}$ and $\mathbf{r}_i$ into rectangular components, we obtain the following three scalar equations, which can be used to determine the coordinates $\bar{x}, \bar{y}, \bar{z}$ of the mass center:

$$m\bar{x} = \sum_{i=1}^{n} m_i x_i \qquad m\bar{y} = \sum_{i=1}^{n} m_i y_i \qquad m\bar{z} = \sum_{i=1}^{n} m_i z_i \qquad (14.12')$$

Since $m_i g$ represents the weight of the particle P_i, and mg the total weight of the particles, G is also the center of gravity of the system of particles. However, in order to avoid any confusion, G will be referred to as the *mass center* of the system of particles when properties associated with the *mass* of the particles are being discussed, and as the *center of gravity* of the system when properties associated with the *weight* of the particles are being considered. Particles located outside the gravit
have a mass but no weight. We
center, but obviously not to th

Differentiating both members of Eq. (14.12) with respect to t, we write

$$m\dot{\bar{\mathbf{r}}} = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i$$

or

$$m\bar{\mathbf{v}} = \sum_{i=1}^{n} m_i \mathbf{v}_i \qquad (14.13)$$

where $\bar{\mathbf{v}}$ represents the velocity of the mass center G of the system of particles. But the right-hand member of Eq. (14.13) is, by definition, the linear momentum $\mathbf{L}$ of the system (Sec. 14.3). We therefore have

$$\mathbf{L} = m\bar{\mathbf{v}} \qquad (14.14)$$

and, differentiating both members with respect to t,

$$\dot{\mathbf{L}} = m\bar{\mathbf{a}} \qquad (14.15)$$

†It may also be pointed out that the mass center and the center of gravity of a system of particles do not exactly coincide, since the weights of the particles are directed toward the center of the earth and thus do not truly form a system of parallel forces.

where $\bar{\mathbf{a}}$ represents the acceleration of the mass center G. Substituting for $\dot{\mathbf{L}}$ from (14.15) into (14.10), we write the equation

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \tag{14.16}$$

which defines the motion of the mass center G of the system of particles.

We note that Eq. (14.16) is identical with the equation we would obtain for a particle of mass m equal to the total mass of the particles of the system, acted upon by all the external forces. We therefore state that *the mass center of a system of particles moves as if the entire mass of the system and all the external forces were concentrated at that point*.

This principle is best illustrated by the motion of an exploding shell. We know that if air resistance is neglected, it can be assumed that a shell will travel along a parabolic path. After the shell has exploded, the mass center G of the fragments of shell will continue to travel along the same path. Indeed, point G must move as if the mass and the weight of all fragments were concentrated at G; it must, therefore, move as if the shell had not exploded.

It should be noted that the preceding derivation does not involve the moments of the external forces. Therefore, *it would be wrong to assume* that the external forces are equipollent to a vector $m\bar{\mathbf{a}}$ attached at the mass center G. This is not in general the case [illegible] tion, the sum of the moments [illegible] in general equal to zero.

14.5 ANGULAR MOMENTUM OF A SYSTEM OF PARTICLES ABOUT ITS MASS CENTER

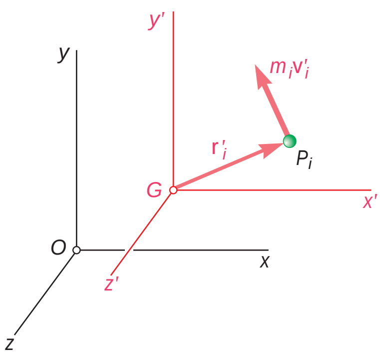


Fig. 14.5

In some applications (for example, in the analysis of the motion of a rigid body) it is convenient to consider the motion of the particles of the system with respect to a centroidal frame of reference $Gx'y'z'$ which translates with respect to the newtonian frame of reference $Oxyz$ (Fig. 14.5). Although a centroidal frame is not, in general, a newtonian frame of reference, it will be seen that the fundamental relation (14.11) holds when the frame $Oxyz$ is replaced by $Gx'y'z'$.

Denoting, respectively, by $\mathbf{r}'_i$ and $\mathbf{v}'_i$ the position vector and the velocity of the particle P_i relative to the moving frame of reference $Gx'y'z'$, we define the *angular momentum* $\mathbf{H}'_G$ of the system of particles *about the mass center* G as follows:

$$\mathbf{H}'_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{v}'_i) \tag{14.17}$$

We now differentiate both members of Eq. (14.17) with respect to t. This operation is similar to that performed in Sec. 14.3 on Eq. (14.7), and so we write immediately

$$\dot{\mathbf{H}}'_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{a}'_i) \tag{14.18}$$

where $\mathbf{a}'_i$ denotes the acceleration of P_i relative to the moving frame of reference. Referring to Sec. 11.12, we write

$$\mathbf{a}_i = \overline{\mathbf{a}} + \mathbf{a}'_i$$

where $\mathbf{a}_i$ and $\overline{\mathbf{a}}$ denote, respectively, the accelerations of P_i and G relative to the frame $Oxyz$. Solving for $\mathbf{a}'_i$ and substituting into (14.18), we have

$$\dot{\mathbf{H}}'_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{a}_i) - \left(\sum_{i=1}^{n} m_i\mathbf{r}'_i\right) \times \overline{\mathbf{a}} \tag{14.19}$$

But, by (14.12), the second sum in Eq. (14.19) is equal to $m\overline{\mathbf{r}}'$ and thus to zero, since the position vector $\overline{\mathbf{r}}'$ of G relative to the frame $Gx'y'z'$ is clearly zero. On the other hand, since $\mathbf{a}_i$ represents the acceleration of P_i relative to a newtonian frame, we can use Eq. (14.1) and replace $m_i\mathbf{a}_i$ by the sum of the internal forces $\mathbf{f}_{ij}$ and of the resultant $\mathbf{F}_i$ of the external forces acting on P_i. But a reasoning similar to that used in Sec. 14.2 shows that the moment resultant about G of the internal forces $\mathbf{f}_{ij}$ of the entire system is zero. The first sum in Eq. (14.19) therefore reduces to the moment resultant about G of the external forces acting on the particles of the system, and we write

$$\Sigma\mathbf{M}_G = \dot{\mathbf{H}}'_G \tag{14.20}$$

which expresses that *the mom*[illegible] *forces is equal to the rate of ch*[illegible] *G of the system of particles.*

It should be noted that in Eq. (14.17) we defined the angular momentum $\mathbf{H}'_G$ as the sum of the moments about G of the momenta of the particles $m_i\mathbf{v}'_i$ *in their motion relative to the centroidal frame of reference* $Gx'y'z'$. We may sometimes want to compute the sum $\mathbf{H}_G$ of the moments about G of the momenta of the particles $m_i\mathbf{v}_i$ *in their absolute motion,* i.e., in their motion as observed from the newtonian frame of reference $Oxyz$ (Fig. 14.6):

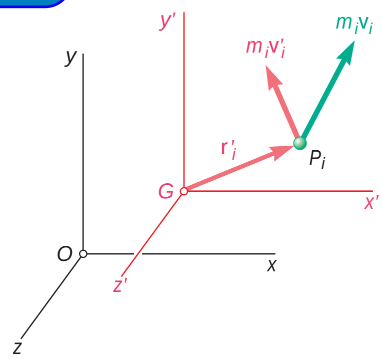


Fig. 14.6

$$\mathbf{H}_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{v}_i) \tag{14.21}$$

Remarkably, the angular momenta $\mathbf{H}'_G$ and $\mathbf{H}_G$ are identically equal. This can be verified by referring to Sec. 11.12 and writing

$$\mathbf{v}_i = \overline{\mathbf{v}} + \mathbf{v}'_i \tag{14.22}$$

Substituting for $\mathbf{v}_i$ from (14.22) into Eq. (14.21), we have

$$\mathbf{H}_G = \left(\sum_{i=1}^{n} m_i\mathbf{r}'_i\right) \times \overline{\mathbf{v}} + \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{v}'_i)$$

But, as observed earlier, the first sum is equal to zero. Thus $\mathbf{H}_G$ reduces to the second sum, which, by definition, is equal to $\mathbf{H}'_G$.†

†Note that this property is peculiar to the centroidal frame $Gx'y'z'$ and does not, in general, hold for other frames of reference (see Prob. 14.29).

Taking advantage of the property we have just established, we simplify our notation by dropping the prime (′) from Eq. (14.20) and writing

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{14.23}$$

where it is understood that the angular momentum $\mathbf{H}_G$ can be computed by forming the moments about G of the momenta of the particles in their motion with respect to either the newtonian frame $Oxyz$ or the centroidal frame $Gx'y'z'$:

$$\mathbf{H}_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i \mathbf{v}_i) = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i \mathbf{v}'_i) \tag{14.24}$$

14.6 CONSERVATION OF MOMENTUM FOR A SYSTEM OF PARTICLES

If no external force acts on the particles of a system, the left-hand members of Eqs. (14.10) and (14.11) are equal to zero and these equations reduce to $\dot{\mathbf{L}} = 0$ and $\dot{\mathbf{H}}_O = 0$. We conclude that

$$\mathbf{L} = \text{constant} \qquad \mathbf{H}_O = \text{constant} \tag{14.25}$$

The equations obtained express that the linear momentum of the system of particles and its angular momentum about the fixed point O are conserved.

In some applications, such as problems involving central forces, the moment about a fixed point O of each of the external forces can be zero without any of the forces being zero. In such cases, the second of Eqs. (14.25) still holds; the angular momentum of the system of particles about O is conserved.

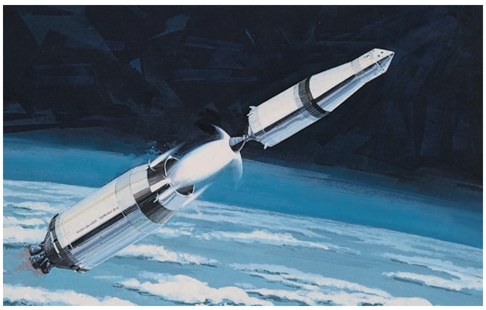


Photo 14.1 If no external forces are acting on the two stages of this rocket, the linear and angular momentum of the system will be conserved.

The concept of conservation of momentum can also be applied to the analysis of the motion of the mass center G of a system of particles and to the analysis of the motion of the system about G. For example, if the sum of the external forces is zero, the first of Eqs. (14.25) applies. Recalling Eq. (14.14), we write

$$\bar{\mathbf{v}} = \text{constant} \tag{14.26}$$

which expresses that the mass center G of the system moves in a straight line and at a constant speed. On the other hand, if the sum of the moments about G of the external forces is zero, it follows from Eq. (14.23) that the angular momentum of the system about its mass center is conserved:

$$\mathbf{H}_G = \text{constant} \tag{14.27}$$

SAMPLE PROBLEM 14.1

A 200-kg space vehicle is observed at $t = 0$ to pass through the origin of a newtonian reference frame $Oxyz$ with velocity $\mathbf{v}_0 = (150 \text{ m/s})\mathbf{i}$ relative to the frame. Following the detonation of explosive charges, the vehicle separates into three parts A, B, and C, of mass 100 kg, 60 kg, and 40 kg, respectively. Knowing that at $t = 2.5$ s the positions of parts A and B are observed to be $A(555, -180, 240)$ and $B(255, 0, -120)$, where the coordinates are expressed in meters, determine the position of part C at that time.

SOLUTION

Since there is no external force, the mass center G of the system moves with the constant velocity $\mathbf{v}_0 = (150 \text{ m/s})\mathbf{i}$. At $t = 2.5$ s, its position is

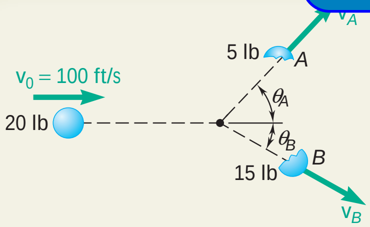
$$\bar{\mathbf{r}} = \mathbf{v}_0 t = (150 \text{ m/s})\mathbf{i}(2.5 \text{ s}) = (375 \text{ m})\mathbf{i}$$

Recalling Eq. (14.12), we write

$$m\bar{\mathbf{r}} = m_A\mathbf{r}_A + m_B\mathbf{r}_B + m_C\mathbf{r}_C$$

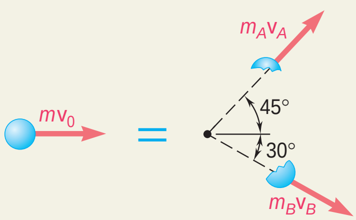
$$(200 \text{ kg})(375 \text{ m})\mathbf{i} = (100 \text{ kg})[(555 \text{ m})\mathbf{i} - (180 \text{ m})\mathbf{j} + (240 \text{ m})\mathbf{k}] + (60 \text{ kg})[(255 \text{ m})\mathbf{i} - (120 \text{ m})\mathbf{k}] + (40 \text{ kg})\mathbf{r}_C$$

$$\mathbf{r}_C = (105 \text{ m})\mathbf{i} + (450 \text{ m})\mathbf{j} - (420 \text{ m})\mathbf{k}$$ ◀



SAMPLE PROBLEM 14.2

A 20-lb projectile is moving with a velocity of 100 ft/s when it explodes into two fragments A and B, weighing 5 lb and 15 lb, respectively. Knowing that immediately after the explosion, fragments A and B travel in directions defined respectively by $u_A = 45°$ and $u_B = 30°$, determine the velocity of each fragment.



SOLUTION

Since there is no external force, the linear momentum of the system is conserved, and we write

$$m_A\mathbf{v}_A + m_B\mathbf{v}_B = m\mathbf{v}_0$$

$$(5/g)\mathbf{v}_A + (15/g)\mathbf{v}_B = (20/g)\mathbf{v}_0$$

$\overset{+}{\text{y}}$ x components: $\quad 5v_A \cos 45° + 15v_B \cos 30° = 20(100)$

+xy components: $\quad 5v_A \sin 45° - 15v_B \sin 30° = 0$

Solving simultaneously the two equations for v_A and v_B, we have

$$v_A = 207 \text{ ft/s} \qquad v_B = 97.6 \text{ ft/s}$$

$\mathbf{v}_A = 207$ ft/s a 45° $\quad \mathbf{v}_B = 97.6$ ft/s c 30° ◀

SOLVING PROBLEMS ON YOUR OWN

This chapter deals with the motion of *systems of particles,* that is, with the motion of a large number of particles considered together, rather than separately. In this first lesson you learned to compute the *linear momentum* and the *angular momentum* of a system of particles. We defined the linear momentum $\mathbf{L}$ of a system of particles as the sum of the linear momenta of the particles and we defined the angular momentum $\mathbf{H}_O$ of the system as the sum of the angular momenta of the particles about O:

$$\mathbf{L} = \sum_{i=1}^{n} m_i \mathbf{v}_i \qquad \mathbf{H}_O = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{v}_i) \qquad (14.6,\ 14.7)$$

In this lesson, you will solve a number of problems of practical interest, either by observing that the linear momentum of a system of particles is conserved or by considering the motion of the mass center of a system of particles.

1. Conservation of the linear momentum of a system of particles. This occurs *when the resultant of the external forces acting on the particles of the system is zero*. You may encounter such a situation in the following types of problems.

a. Problems involving the rectilinear motion of objects such as colliding automobiles and railroad cars. After you have checked that the resultant of the external forces is zero, equate the algebraic sums of the initial momenta and final momenta to obtain an equation which can be solved for one unknown.

b. Problems involving the two-dimensional or three-dimensional motion of objects such as explo[illegible]or billiard balls. After you have check[illegible] zero, add vectorially the initial mo[illegible]l momenta, and equate the two sums to obtain a vector equation expressing that the linear momentum of the system is conserved.

In the case of a two-dimensional motion, this equation can be replaced by two scalar equations which can be solved for two unknowns, while in the case of a three-dimensional motion it can be replaced by three scalar equations which can be solved for three unknowns.

2. Motion of the mass center of a system of particles. You saw in Sec. 14.4 that *the mass center of a system of particles moves as if the entire mass of the system and all of the external forces were concentrated at that point.*

a. In the case of a body exploding while in motion, it follows that the mass center of the resulting fragments moves as the body itself would have moved if the explosion had not occurred. Problems of this type can be solved by writing the equation of motion of the mass center of the system in vectorial form and expressing the position vector of the mass center in terms of the position vectors of the various fragments [Eq. (14.12)]. You can then rewrite the vector equation as two or three scalar equations and solve the equations for an equivalent number of unknowns.

b. In the case of the collision of several moving bodies, it follows that the motion of the mass center of the various bodies is unaffected by the collision. Problems of this type can be solved by writing the equation of motion of the mass center of the system in vectorial form and expressing its position vector before and after the collision in terms of the position vectors of the relevant bodies [Eq. (14.12)]. You can then rewrite the vector equation as two or three scalar equations and solve these equations for an equivalent number of unknowns.

PROBLEMS

14.1 A 30-g bullet is fired with a horizontal velocity of 450 m/s and becomes embedded in block *B* which has a mass of 3 kg. After the impact, block *B* slides on 30-kg carrier *C* until it impacts the end of the carrier. Knowing the impact between *B* and *C* is perfectly plastic and the coefficient of kinetic friction between *B* and *C* is 0.2, determine (*a*) the velocity of the bullet and *B* after the first impact, (*b*) the final velocity of the carrier.

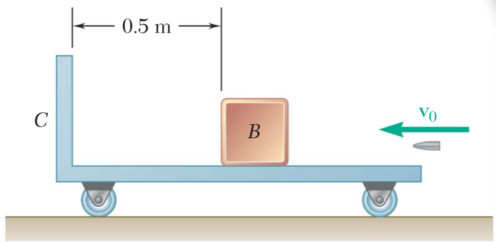


Fig. P14.1

14.2 A 30-g bullet is fired with a horizontal velocity of 450 m/s through 3-kg block *B* and becomes embedded in carrier *C* which has a mass of 30 kg. After the impact, block *B* slides 0.3 m on *C* before coming to rest relative to the carrier. Knowing the coefficient of kinetic friction between *B* and *C* is 0.2, determine (*a*) the velocity of the bullet immediately after passing through *B*, (*b*) the final velocity of the carrier.

14.3 Car *A* weighing 4000 lb and car *B* weighing 3700 lb are at rest on a 22-ton flatcar which is also at rest. Cars *A* and *B* then accelerate and quickly reach constant speeds relative to the flatcar of 7 ft/s and 3.5 ft/s, respectively, before decelerating to a stop at the opposite end of the flatcar. Neglecting friction and rolling resistance, determine the velocity of the flatcar when the cars are moving at constant speeds.

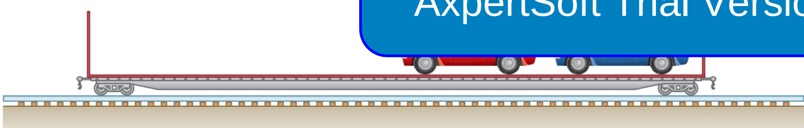


Fig. P14.3

14.4 A bullet is fired with a horizontal velocity of 1500 ft/s through a 6-lb block *A* and becomes embedded in a 4.95-lb block *B*. Knowing that blocks *A* and *B* start moving with velocities of 5 ft/s and 9 ft/s, respectively, determine (*a*) the weight of the bullet, (*b*) its velocity as it travels from block *A* to block *B*.

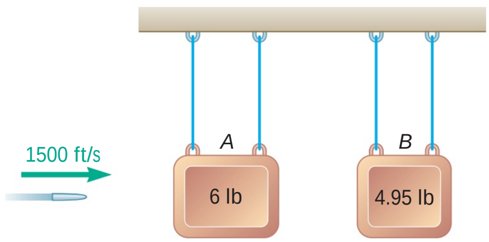


Fig. P14.4

14.5 Two swimmers *A* and *B*, of weight 190 lb and 125 lb, respectively, are at diagonally opposite corners of a floating raft when they realize that the raft has broken away from its anchor. Swimmer *A* immediately starts walking toward *B* at a speed of 2 ft/s relative to the raft. Knowing that the raft weighs 300 lb, determine (*a*) the speed of the raft if *B* does not move, (*b*) the speed with which *B* must walk toward *A* if the raft is not to move.

Fig. *P14.5*

14.6 A 180-lb man and a 120-lb woman stand side by side at the same end of a 300-lb boat, ready to dive, each with a 16-ft/s velocity relative to the boat. Determine the velocity of the boat after they have both dived, if (*a*) the woman dives first, (*b*) the man dives first.

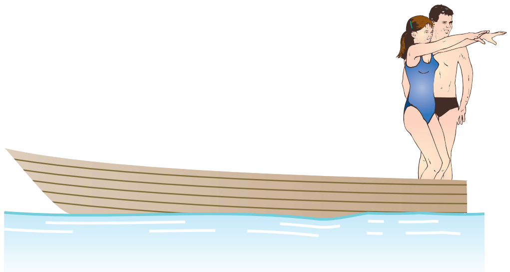


Fig. *P14.6*

14.7 A 40-Mg boxcar A is moving in a railroad switchyard with a velocity of 9 km/h toward cars B and C, which are both at rest with their brakes off at a short distance from each other. Car B is a 25-Mg flatcar supporting a 30-Mg container, and car C is a 35-Mg boxcar. As the cars hit each other they get automatically and tightly coupled. Determine the velocity of car A immediately after each of the two couplings, assuming that the container (*a*) does not slide on the flatcar, (*b*) slides after the first coupling but hits a stop before the second coupling occurs, (*c*) slides and hits the stop only after the second coupling has occurred.

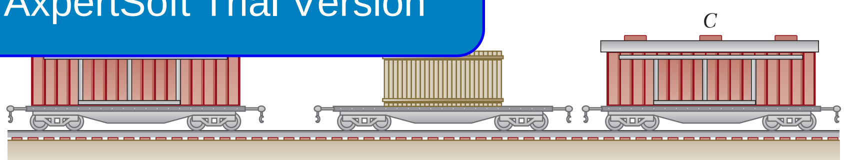


Fig. P14.7

14.8 Packages in an automobile parts supply house are transported to the loading dock by pushing them along on a roller track with very little friction. At the instant shown packages B and C are at rest and package A has a velocity of 2 m/s. Knowing that the coefficient of restitution between the packages is 0.3, determine (*a*) the velocity of package C after A hits B and B hits C, (*b*) the velocity of A after it hits B for the second time.

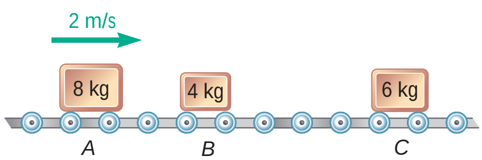


Fig. P14.8

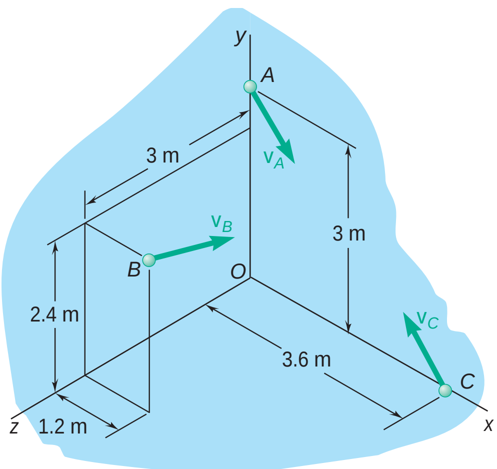


Fig. P14.9

14.9 A system consists of three particles A, B, and C. We know that $m_A = 3$ kg, $m_B = 2$ kg, and $m_C = 4$ kg and that the velocities of the particles expressed in m/s are, respectively, $\mathbf{v}_A = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{v}_B = 4\mathbf{i} + 3\mathbf{j}$, and $\mathbf{v}_C = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. Determine the angular momentum $\mathbf{H}_O$ of the system about O.

14.10 For the system of particles of Prob. 14.9, determine (*a*) the position vector $\bar{\mathbf{r}}$ of the mass center G of the system, (*b*) the linear momentum $m\bar{\mathbf{v}}$ of the system, (*c*) the angular momentum $\mathbf{H}_G$ of the system about G. Also verify that the answers to this problem and to Prob. 14.9 satisfy the equation given in Prob. 14.27.

14.11 A system consists of three particles A, B, and C. We know that $W_A = 5$ lb, $W_B = 4$ lb, and $W_C = 3$ lb and that the velocities of the particles expressed in ft/s are, respectively, $\mathbf{v}_A = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{v}_B = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$, and $\mathbf{v}_C = -3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Determine (*a*) the components v_x and v_y of the velocity of particle B for which the angular momentum $\mathbf{H}_O$ of the system about O is parallel to the x axis, (*b*) the value of $\mathbf{H}_O$.

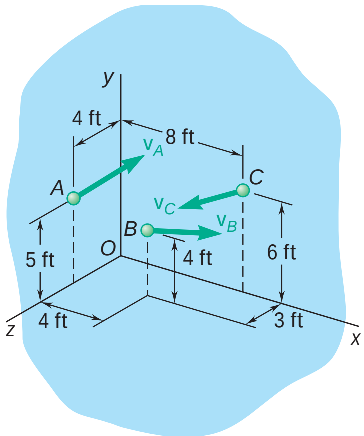


Fig. P14.11

14.12 For the system of particles of Prob. 14.11, determine (*a*) the components v_x and v_z of the velocity of particle B for which the angular momentum $\mathbf{H}_O$ of the system about O is parallel to the z axis, (*b*) the value of $\mathbf{H}_O$.

14.13 A system consists of three particles A, B, and C. We know that $m_A = 3$ kg, $m_B = 4$ kg, and $m_c = 5$ kg and that the velocities of the particles expressed in m/s are, respectively, $\mathbf{v}_A = -4\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$, $\mathbf{v}_B = -6\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$, and $\mathbf{v}_C = 2\mathbf{i} - 6\mathbf{j} - 4\mathbf{k}$. Determine the angular momentum $\mathbf{H}_O$ of the system about O.

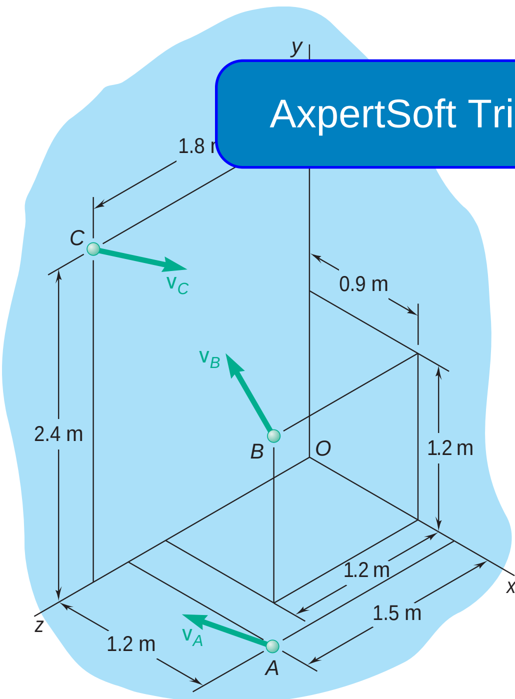


Fig. *P14.13*

14.14 For the system of particles of Prob. 14.13, determine (*a*) the position vector $\bar{\mathbf{r}}$ of the mass center G of the system, (*b*) the linear momentum $m\bar{\mathbf{v}}$ of the system, (*c*) the angular momentum $\mathbf{H}_G$ of the system about G. Also verify that the answers to this problem and to Prob. 14.13 satisfy the equation given in Prob. 14.27.

14.15 A 13-kg projectile is passing through the origin O with a velocity $\mathbf{v}_0 = (35 \text{ m/s})\mathbf{i}$ when it explodes into two fragments A and B, of mass 5 kg and 8 kg, respectively. Knowing that 3 s later the position of fragment A is (90 m, 7 m, −14 m), determine the position of fragment B at the same instant. Assume $a_y = -g = -9.81 \text{ m/s}^2$ and neglect air resistance.

14.16 A 300-kg space vehicle traveling with a velocity $\mathbf{v}_0 = (360 \text{ m/s})\mathbf{i}$ passes through the origin O at $t = 0$. Explosive charges then separate the vehicle into three parts A, B, and C, with mass, respectively, 150 kg, 100 kg, and 50 kg. Knowing that at $t = 4$ s, the positions of parts A and B are observed to be A (1170 m, −290 m, −585 m) and B (1975 m, 365 m, 800 m), determine the corresponding position of part C. Neglect the effect of gravity.

14.17 A 2-kg model rocket is launched vertically and reaches an altitude of 70 m with a speed of 30 m/s at the end of powered flight, time $t = 0$. As the rocket approaches its maximum altitude it explodes into two parts of masses $m_A = 0.7$ kg and $m_B = 1.3$ kg. Part A is observed to strike the ground 80 m west of the launch point at $t = 6$ s. Determine the position of part B at that time.

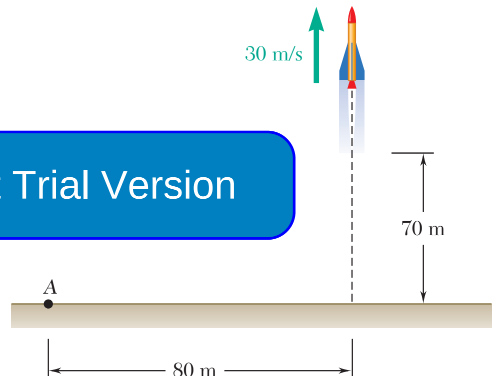


Fig. *P14.17*

14.18 An 18-kg cannonball and a 12-kg cannonball are chained together and fired horizontally with a velocity of 165 m/s from the top of a 15-m wall. The chain breaks during the flight of the cannonballs and the 12-kg cannonball strikes the ground at $t = 1.5$ s, at a distance of 240 m from the foot of the wall, and 7 m to the right of the line of fire. Determine the position of the other cannonball at that instant. Neglect the resistance of the air.

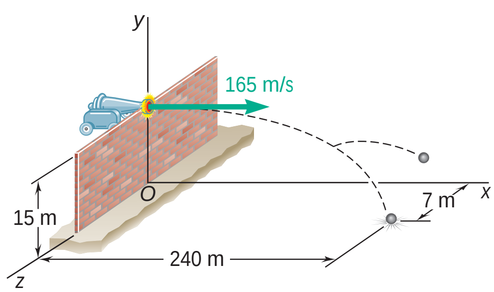


Fig. *P14.18*

14.19 and 14.20 Car A was traveling east at high speed when it collided at point O with car B, which was traveling north at 45 mi/h. Car C, which was traveling west at 60 mi/h, was 32 ft east and 10 ft north of point O at the time of the collision. Because the pavement was wet, the driver of car C could not prevent his car from sliding into the other two cars, and the three cars, stuck together, kept sliding until they hit the utility pole P. Knowing that the weights of cars A, B, and C are, respectively, 3000 lb, 2600 lb, and 2400 lb, and neglecting the forces exerted on the cars by the wet pavement, solve the problems indicated.

14.19 Knowing that the speed of car A was 75 mi/h and that the time elapsed from the first collision to the stop at P was 2.4 s, determine the coordinates of the utility pole P.

14.20 Knowing that the coordinates of the utility pole are $x_p = 46$ ft and $y_p = 59$ ft, determine (*a*) the time elapsed from the first collision to the stop at P, (*b*) the speed of car A.

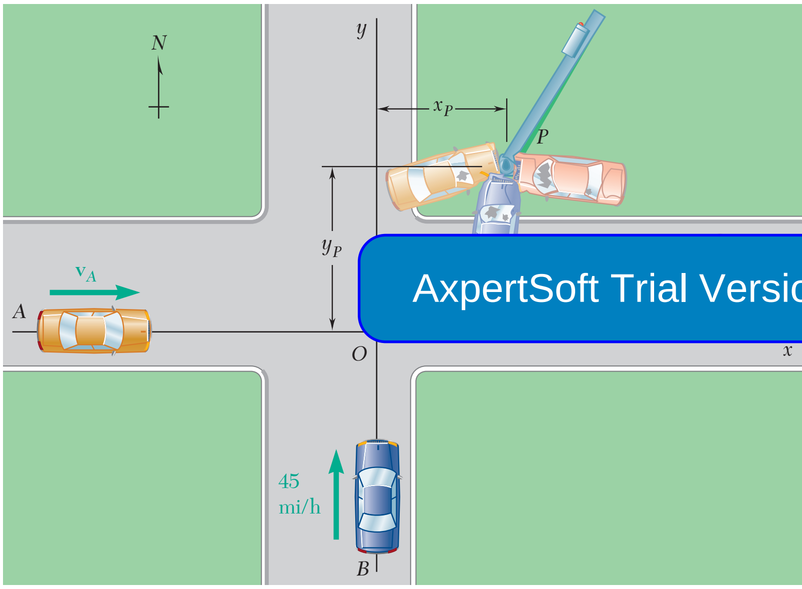


Fig. P14.19 and P14.20

14.21 An expert archer demonstrates his ability by hitting tennis balls thrown by an assistant. A 2-oz tennis ball has a velocity of (32 ft/s)**i** − (7 ft/s)**j** and is 33 ft above the ground when it is hit by a 1.2-oz arrow traveling with a velocity of (165 ft/s)**j** + (230 ft/s)**k** where **j** is directed upwards. Determine the position P where the ball and arrow will hit the ground, relative to point O located directly under the point of impact.

14.22 Two spheres, each of mass m, can slide freely on a frictionless, horizontal surface. Sphere A is moving at a speed $v_0 = 16$ ft/s when it strikes sphere B which is at rest and the impact causes sphere B to break into two pieces, each of mass $m/2$. Knowing that 0.7 s after the collision one piece reaches point C and 0.9 s after the collision the other piece reaches point D, determine (*a*) the velocity of sphere A after the collision, (*b*) the angle u and the speeds of the two pieces after the collision.

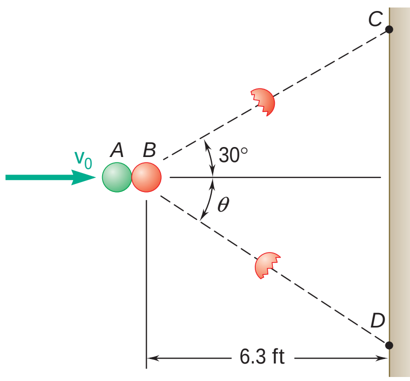


Fig. P14.22

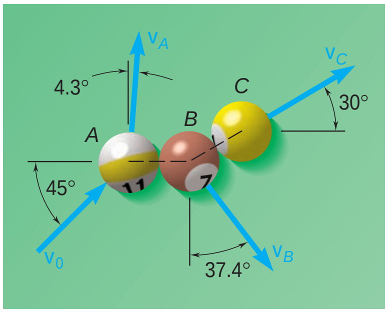


Fig. *P14.23*

14.23 In a game of pool, ball A is moving with a velocity $\mathbf{v}_0$ when it strikes balls B and C which are at rest and aligned as shown. Knowing that after the collision the three balls move in the directions indicated and that $v_0 = 12$ ft/s and $v_C = 6.29$ ft/s, determine the magnitude of the velocity of (*a*) ball A, (*b*) ball B.

14.24 A 6-kg shell moving with a velocity $\mathbf{v}_0 = (12 \text{ m/s})\mathbf{i} - (9 \text{ m/s})\mathbf{j} - (360 \text{ m/s})\mathbf{k}$ explodes at point D into three fragments A, B, and C of mass, respectively, 3 kg, 2 kg, and 1 kg. Knowing that the fragments hit the vertical wall at the points indicated, determine the speed of each fragment immediately after the explosion. Assume that elevation changes due to gravity may be neglected.

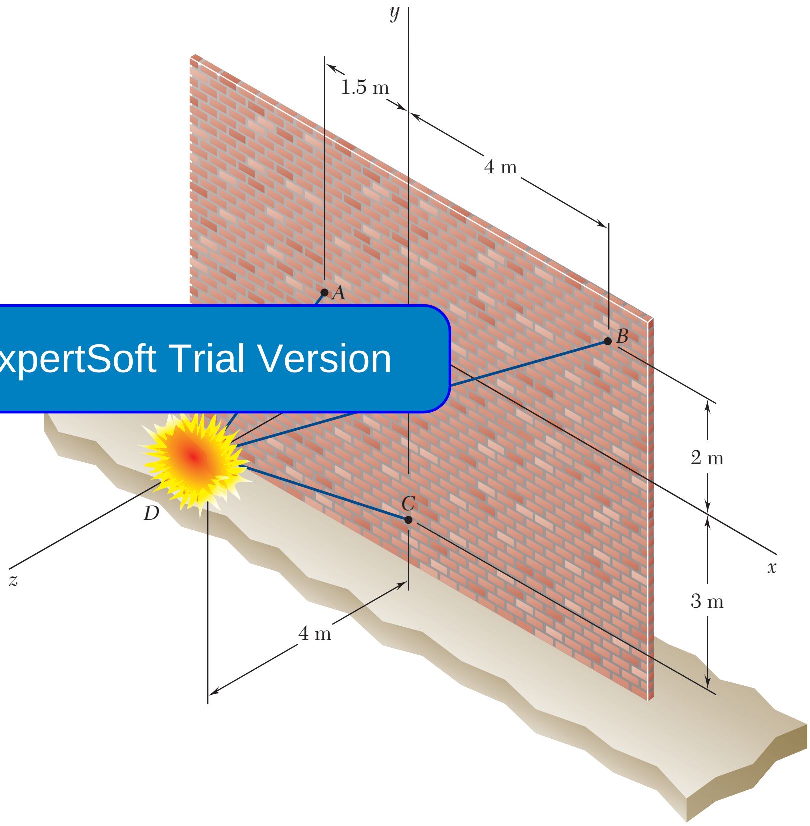


Fig. P14.24 and P14.25

14.25 A 6-kg shell moving with a velocity $\mathbf{v}_0 = (12 \text{ m/s})\mathbf{i} - (9 \text{ m/s})\mathbf{j} - (360 \text{ m/s})\mathbf{k}$ explodes at point D into three fragments A, B, and C of mass, respectively, 2 kg, 1 kg, and 3 kg. Knowing that the fragments hit the vertical wall at the points indicated, determine the speed of each fragment immediately after the explosion. Assume that elevation changes due to gravity may be neglected.

14.26 In a scattering experiment, an alpha particle A is projected with the velocity $\mathbf{u}_0 = -(600\text{ m/s})\mathbf{i} + (750\text{ m/s})\mathbf{j} - (800\text{ m/s})\mathbf{k}$ into a stream of oxygen nuclei moving with a common velocity $\mathbf{v}_0 = (600\text{ m/s})\mathbf{j}$. After colliding successively with the nuclei B and C, particle A is observed to move along the path defined by the points A_1 (280, 240, 120) and A_2 (360, 320, 160), while nuclei B and C are observed to move along paths defined, respectively, by B_1 (147, 220, 130) and B_2 (114, 290, 120), and by C_1 (240, 232, 90) and C_2 (240, 280, 75). All paths are along straight lines and all coordinates are expressed in millimeters. Knowing that the mass of an oxygen nucleus is four times that of an alpha particle, determine the speed of each of the three particles after the collisions.

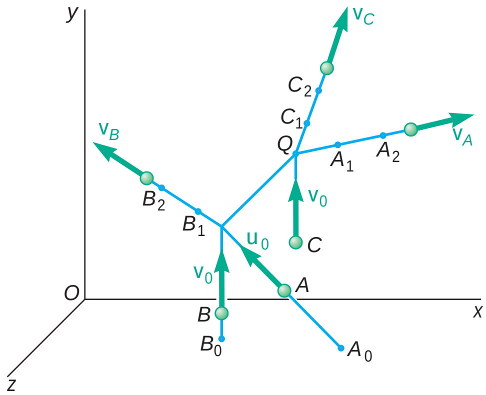


Fig. P14.26

14.27 Derive the relation

$$\mathbf{H}_O = \bar{\mathbf{r}} \times m\bar{\mathbf{v}} + H_G$$

between the angular momenta $\mathbf{H}_O$ and $\mathbf{H}_G$ defined in Eqs. (14.7) and (14.24), respectively. The vectors $\bar{\mathbf{r}}$ and $\bar{\mathbf{v}}$ define, respectively, the position and velocity of the mass center G of the system of particles relative to the newtonian frame of reference $Oxyz$, and m represents the total mass of the system.

14.28 Show that Eq. (14.23) may be derived directly from Eq. (14.11) by substituting for $\mathbf{H}_O$ the expression given in Prob. 14.27.

14.29 Consider the frame of re
to the newtonian frame
momentum $\mathbf{H}'_A$ of a syste

$$\mathbf{H}'_A = \sum_{i=1}^{n} \mathbf{r}'_i \times m_i\mathbf{v}'_i \qquad (1)$$

of the moments about A of the momenta $m_i\mathbf{v}'_i$ of the particles in their motion relative to the frame $Ax'y'z'$. Denoting by $\mathbf{H}_A$ the sum

$$\mathbf{H}_A = \sum_{i=1}^{n} \mathbf{r}'_i \times m_i\mathbf{v}_i$$

of the moments about A of the momenta $m_i\mathbf{v}_i$ of the particles in their motion relative to the newtonian frame $Oxyz$, show that $\mathbf{H}_A = \mathbf{H}'_A$ at a given instant if, and only if, one of the following conditions is satisfied at that instant: (*a*) A has zero velocity with respect to the frame $Oxyz$, (*b*) A coincides with the mass center G of the system, (*c*) the velocity $\mathbf{v}_A$ relative to $Oxyz$ is directed along the line AG.

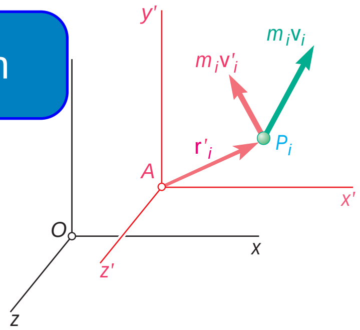


Fig. P14.29

14.30 Show that the relation $\Sigma\mathbf{M}_A = \dot{\mathbf{H}}'_A$, where $\mathbf{H}'_A$ is defined by Eq. (1) of Prob. 14.29 and where $\Sigma\mathbf{M}_A$ represents the sum of the moments about A of the external forces acting on the system of particles, is valid if, and only if, one of the following conditions is satisfied: (*a*) the frame $Ax'y'z'$ is itself a newtonian frame of reference, (*b*) A coincides with the mass center G, (*c*) the acceleration $\mathbf{a}_A$ of A relative to $Oxyz$ is directed along the line AG.

14.7 KINETIC ENERGY OF A SYSTEM OF PARTICLES

The kinetic energy T of a system of particles is defined as the sum of the kinetic energies of the various particles of the system. Referring to Sec. 13.3, we therefore write

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 \tag{14.28}$$

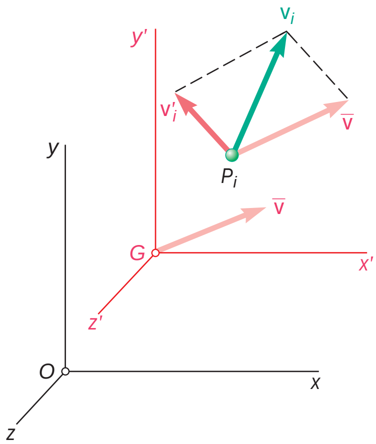


Fig. 14.7

Using a Centroidal Frame of Reference. It is often convenient when computing the kinetic energy of a system comprising a large number of particles (as in the case of a rigid body) to consider separately the motion of the mass center G of the system and the motion of the system relative to a moving frame attached to G.

Let P_i be a particle of the system, $\mathbf{v}_i$ its velocity relative to the newtonian frame of reference $Oxyz$, and $\mathbf{v}'_i$ its velocity relative to the moving frame $Gx'y'z'$ which is in translation with respect to $Oxyz$ (Fig. 14.7). We recall from the preceding section that

$$\mathbf{v}_i = \overline{\mathbf{v}} + \mathbf{v}'_i \tag{14.22}$$

where $\overline{\mathbf{v}}$ denotes the velocity of the mass center G relative to the [illegible]at v_i^2 is equal to the scalar prod-[illegible]rgy T of the system relative to [illegible]

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 = \frac{1}{2} \sum_{i=1}^{n} (m_i \mathbf{v}_i \cdot \mathbf{v}_i)$$

or, substituting for $\mathbf{v}_i$ from (14.22),

$$\begin{aligned} T &= \frac{1}{2} \sum_{i=1}^{n} [m_i(\overline{\mathbf{v}} + \mathbf{v}'_i) \cdot (\overline{\mathbf{v}} + \mathbf{v}'_i)] \\ &= \frac{1}{2}\left(\sum_{i=1}^{n} m_i\right)\overline{v}^2 + \overline{\mathbf{v}} \cdot \sum_{i=1}^{n} m_i \mathbf{v}'_i + \frac{1}{2} \sum_{i=1}^{n} m_i v_i'^2 \end{aligned}$$

The first sum represents the total mass m of the system. Recalling Eq. (14.13), we note that the second sum is equal to $m\overline{\mathbf{v}}'$ and thus to zero, since $\overline{\mathbf{v}}'$, which represents the velocity of G relative to the frame $Gx'y'z'$, is clearly zero. We therefore write

$$T = \tfrac{1}{2} m \overline{v}^2 + \frac{1}{2} \sum_{i=1}^{n} m_i v_i'^2 \tag{14.29}$$

This equation shows that the kinetic energy T of a system of particles can be obtained *by adding the kinetic energy of the mass center G* (assuming the entire mass concentrated at G) *and the kinetic energy of the system in its motion relative to the frame Gx′y′z′.*

14.8 WORK-ENERGY PRINCIPLE. CONSERVATION OF ENERGY FOR A SYSTEM OF PARTICLES

The principle of work and energy can be applied to each particle P_i of a system of particles. We write

$$T_1 + U_{1 y 2} = T_2 \tag{14.30}$$

for each particle P_i, where $U_{1 y 2}$ represents the work done by the internal forces $\mathbf{f}_{ij}$ and the resultant external force $\mathbf{F}_i$ acting on P_i. Adding the kinetic energies of the various particles of the system and considering the work of all the forces involved, we can apply Eq. (14.30) to the entire system. The quantities T_1 and T_2 now represent the kinetic energy of the entire system and can be computed from either Eq. (14.28) or Eq. (14.29). The quantity $U_{1 y 2}$ represents the work of all the forces acting on the particles of the system. Note that while the internal forces $\mathbf{f}_{ij}$ and $\mathbf{f}_{ji}$ are equal and opposite, the work of these forces will not, in general, cancel out, since the particles P_i and P_j on which they act will, in general, undergo different displacements. Therefore, in computing $U_{1 y 2}$, *we must consider the work of the internal forces* $\mathbf{f}_{ij}$ *as well as the work of the external forces* $\mathbf{F}_i$.

If all the forces acting on the particles of the system are conservative, Eq. (14.30) can be replaced by

$$T_1 + V$$

where V represents the potent and external forces acting on the particles of the system. Equation (14.31) expresses the principle of *conservation of energy* for the system of particles.

Photo 14.2 When a golf ball is hit out of a sand trap, some of the momentum of the club is transferred to the golf ball and any sand that is hit.

14.9 PRINCIPLE OF IMPULSE AND MOMENTUM FOR A SYSTEM OF PARTICLES

Integrating Eqs. (14.10) and (14.11) in t from t_1 to t_2, we write

$$\sum \int_{t_1}^{t_2} \mathbf{F}\, dt = \mathbf{L}_2 - \mathbf{L}_1 \tag{14.32}$$

$$\sum \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1 \tag{14.33}$$

Recalling the definition of the linear impulse of a force given in Sec. 13.10, we observe that the integrals in Eq. (14.32) represent the linear impulses of the external forces acting on the particles of the system. We shall refer in a similar way to the integrals in Eq. (14.33) as the *angular impulses* about O of the external forces. Thus, Eq. (14.32) expresses that the sum of the linear impulses of the external forces acting on the system is equal to the change in linear momentum of the system. Similarly, Eq. (14.33) expresses that the sum of the angular impulses about O of the external forces is equal to the change in angular momentum about O of the system.

In order to make clear the physical significance of Eqs. (14.32) and (14.33), we will rearrange the terms in these equations and write

$$\mathbf{L}_1 + \sum \int_{t_1}^{t_2} \mathbf{F}\, dt = \mathbf{L}_2 \tag{14.34}$$

$$(\mathbf{H}_O)_1 + \sum \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2 \tag{14.35}$$

In parts a and c of Fig. 14.8 we have sketched the momenta of the particles of the system at times t_1 and t_2, respectively. In part b we have shown a vector equal to the sum of the linear impulses of the external forces and a couple of moment equal to the sum of the angular impulses about O of the external forces. For simplicity, the particles have been

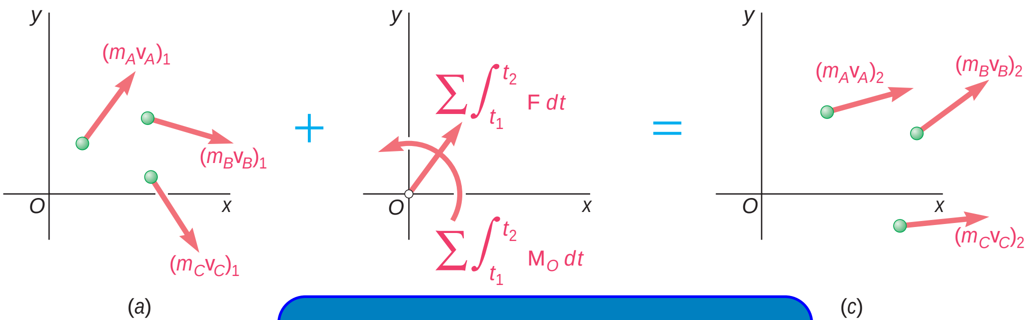


Fig. 14.8

...gure, but the present discussion remains valid in the case of particles moving in space. Recalling from Eq. (14.6) that $\mathbf{L}$, by definition, is the resultant of the momenta $m_i\mathbf{v}_i$, we note that Eq. (14.34) expresses that the resultant of the vectors shown in parts a and b of Fig. 14.8 is equal to the resultant of the vectors shown in part c of the same figure. Recalling from Eq. (14.7) that $\mathbf{H}_O$ is the moment resultant of the momenta $m_i\mathbf{v}_i$, we note that Eq. (14.35) similarly expresses that the moment resultant of the vectors in parts a and b of Fig. 14.8 is equal to the moment resultant of the vectors in part c. Together, Eqs. (14.34) and (14.35) thus express that *the momenta of the particles at time t_1 and the impulses of the external forces from t_1 to t_2 form a system of vectors equipollent to the system of the momenta of the particles at time t_2*. This has been indicated in Fig. 14.8 by the use of blue plus and equals signs.

If no external force acts on the particles of the system, the integrals in Eqs. (14.34) and (14.35) are zero, and these equations yield

$$\mathbf{L}_1 = \mathbf{L}_2 \tag{14.36}$$

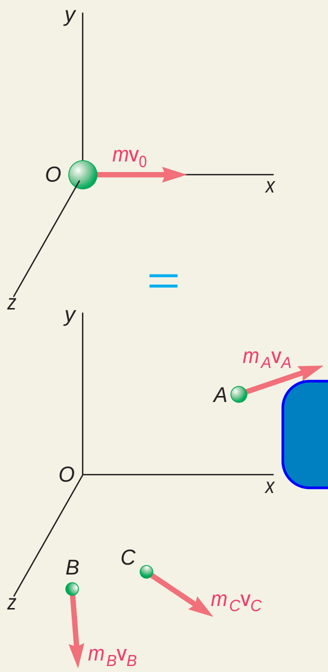
$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2 \tag{14.37}$$

We thus check the result obtained in Sec. 14.6: If no external force acts on the particles of a system, the linear momentum and the angular momentum about O of the system of particles are conserved. The system of the initial momenta is equipollent to the system of the final momenta, and it follows that the angular momentum of the system of particles about *any* fixed point is conserved.

SAMPLE PROBLEM 14.3

For the 200-kg space vehicle of Sample Prob. 14.1, it is known that at $t = 2.5$ s, the velocity of part A is $\mathbf{v}_A = (270\text{ m/s})\mathbf{i} - (120\text{ m/s})\mathbf{j} + (160\text{ m/s})\mathbf{k}$ and the velocity of part B is parallel to the xz plane. Determine the velocity of part C.

SOLUTION



Since there is no external force, the initial momentum $m\mathbf{v}_0$ is equipollent to the system of the final momenta. Equating first the sums of the vectors in both parts of the adjoining sketch, and then the sums of their moments about O, we write

$$\mathbf{L}_1 = \mathbf{L}_2: \qquad m\mathbf{v}_0 = m_A\mathbf{v}_A + m_B\mathbf{v}_B + m_C\mathbf{v}_C \tag{1}$$

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2: \qquad 0 = \mathbf{r}_A \times m_A\mathbf{v}_A + \mathbf{r}_B \times m_B\mathbf{v}_B + \mathbf{r}_C \times m_C\mathbf{v}_C \tag{2}$$

Recalling from Sample Prob. 14.1 that $\mathbf{v}_0 = (150\text{ m/s})\mathbf{i}$,

$$m_A = 100\text{ kg} \qquad m_B = 60\text{ kg} \qquad m_C = 40\text{ kg}$$
$$\mathbf{r}_A = (555\text{ m})\mathbf{i} - (180\text{ m})\mathbf{j} + (240\text{ m})\mathbf{k}$$
$$\mathbf{r}_B = (255\text{ m})\mathbf{i} - (120\text{ m})\mathbf{k}$$
$$\mathbf{r}_C = (105\text{ m})\mathbf{i} + (450\text{ m})\mathbf{j} - (420\text{ m})\mathbf{k}$$

... tatement of this problem, we rewrite

$$200(150\mathbf{i}) = 100(270\mathbf{i} - 120\mathbf{j} + 160\mathbf{k}) + 60[(v_B)_x\mathbf{i} + (v_B)_z\mathbf{k}]$$
$$+ 40[(v_C)_x\mathbf{i} + (v_C)_y\mathbf{j} + (v_C)_z\mathbf{k}] \tag{1'}$$

$$0 = 100\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 555 & -180 & 240 \\ 270 & -120 & 160 \end{vmatrix} + 60\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 255 & 0 & -120 \\ (v_B)_x & 0 & (v_B)_z \end{vmatrix}$$
$$+ 40\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 105 & 450 & -420 \\ (v_C)_x & (v_C)_y & (v_C)_z \end{vmatrix} \tag{2'}$$

Equating to zero the coefficient of $\mathbf{j}$ in (1′) and the coefficients of $\mathbf{i}$ and $\mathbf{k}$ in (2′), we write, after reductions, the three scalar equations

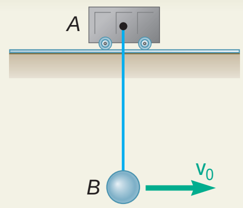
$$(v_C)_y - 300 = 0$$
$$450(v_C)_z + 420(v_C)_y = 0$$
$$105(v_C)_y - 450(v_C)_x - 45\,000 = 0$$

which yield, respectively,

$$(v_C)_y = 300 \qquad (v_C)_z = -280 \qquad (v_C)_x = -30$$

The velocity of part C is thus

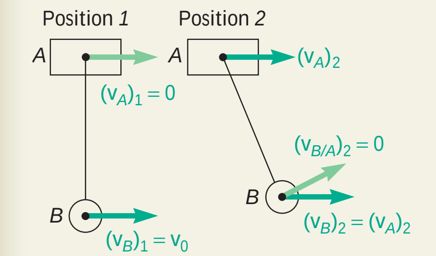
$$\mathbf{v}_C = -(30\text{ m/s})\mathbf{i} + (300\text{ m/s})\mathbf{j} - (280\text{ m/s})\mathbf{k} \quad ◀$$



SAMPLE PROBLEM 14.4

Ball B, of mass m_B, is suspended from a cord of length l attached to cart A, of mass m_A, which can roll freely on a frictionless horizontal track. If the ball is given an initial horizontal velocity $\mathbf{v}_0$ while the cart is at rest, determine (*a*) the velocity of B as it reaches its maximum elevation, (*b*) the maximum vertical distance h through which B will rise. (It is assumed that $v_0^2 < 2gl$.)

SOLUTION

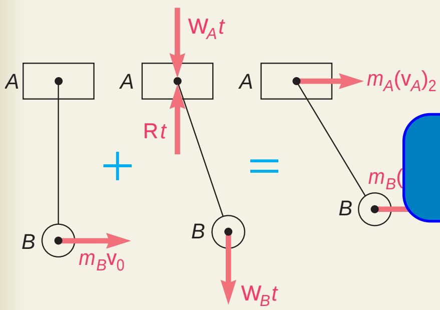


The impulse-momentum principle and the principle of conservation of energy will be applied to the cart-ball system between its initial position *1* and position *2*, when B reaches its maximum elevation.

Velocities ***Position 1:*** $(\mathbf{v}_A)_1 = 0 \qquad (\mathbf{v}_B)_1 = \mathbf{v}_0$ (1)

Position 2: When ball B reaches its maximum elevation, its velocity $(\mathbf{v}_{B/A})_2$ relative to its support A is zero. Thus, at that instant, its absolute velocity is

$$(\mathbf{v}_B)_2 = (\mathbf{v}_A)_2 + (\mathbf{v}_{B/A})_2 = (\mathbf{v}_A)_2 \qquad (2)$$



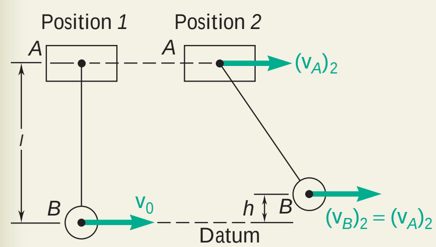
Impulse-Momentum Principle. Noting that the external impulses consist of $\mathbf{W}_A t$, $\mathbf{W}_B t$, and $\mathbf{R}t$, where $\mathbf{R}$ is the reaction of the track on the cart, and recalling (1) and (2), we draw the impulse-momentum diagram and write

$$\Sigma m\mathbf{v}_1 + \Sigma \text{Ext Imp}_{1\to 2} = \Sigma m\mathbf{v}_2$$

$(v_A)_2$

of the system is conserved in the horizontal direction. Solving for $(v_A)_2$:

$$(v_A)_2 = \frac{m_B}{m_A + m_B} v_0 \qquad (\mathbf{v}_B)_2 = (\mathbf{v}_A)_2 = \frac{m_B}{m_A + m_B} v_0 \rightarrow$$



Conservation of Energy

Position 1. ***Potential Energy:*** $V_1 = m_A g l$

Kinetic Energy: $T_1 = \frac{1}{2} m_B v_0^2$

Position 2. ***Potential Energy:*** $V_2 = m_A g l + m_B g h$

Kinetic Energy: $T_2 = \frac{1}{2}(m_A + m_B)(v_A)_2^2$

$$T_1 + V_1 = T_2 + V_2: \qquad \tfrac{1}{2} m_B v_0^2 + m_A g l = \tfrac{1}{2}(m_A + m_B)(v_A)_2^2 + m_A g l + m_B g h$$

Solving for h, we have

$$h = \frac{v_0^2}{2g} - \frac{m_A + m_B}{m_B} \frac{(v_A)_2^2}{2g}$$

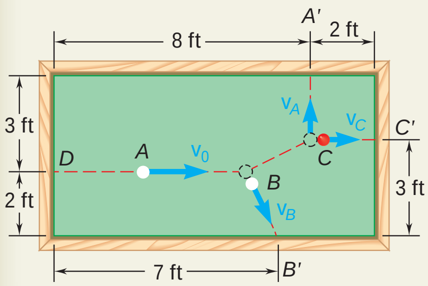
or, substituting for $(v_A)_2$ the expression found above,

$$h = \frac{v_0^2}{2g} - \frac{m_B}{m_A + m_B} \frac{v_0^2}{2g} \qquad h = \frac{m_A}{m_A + m_B} \frac{v_0^2}{2g}$$

Remarks. (1) Recalling that $v_0^2 < 2gl$, it follows from the last equation that $h < l$; we thus check that B stays below A as assumed in our solution.

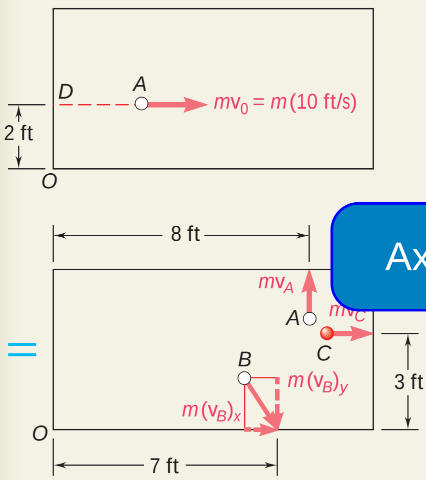
(2) For $m_A \gg m_B$, the answers obtained reduce to $(\mathbf{v}_B)_2 = (\mathbf{v}_A)_2 = 0$ and $h = v_0^2/2g$; B oscillates as a simple pendulum with A fixed. For $m_A \ll m_B$, they reduce to $(\mathbf{v}_B)_2 = (\mathbf{v}_A)_2 = \mathbf{v}_0$ and $h = 0$; A and B move with the same constant velocity $\mathbf{v}_0$.

SAMPLE PROBLEM 14.5



In a game of billiards, ball A is given an initial velocity $\mathbf{v}_0$ of magnitude $v_0 = 10$ ft/s along line DA parallel to the axis of the table. It hits ball B and then ball C, which are both at rest. Knowing that A and C hit the sides of the table squarely at points A' and C', respectively, that B hits the side obliquely at B', and assuming frictionless surfaces and perfectly elastic impacts, determine the velocities $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$ with which the balls hit the sides of the table. (*Remark:* In this sample problem and in several of the problems which follow, the billiard balls are assumed to be particles moving freely in a horizontal plane, rather than the rolling and sliding spheres they actually are.)

SOLUTION



Conservation of Momentum. Since there is no external force, the initial momentum $m\mathbf{v}_0$ is equipollent to the system of momenta after the two collisions (and before any of the balls hits the side of the table). Referring to the adjoining sketch, we write

y+ x components: $\quad m(10 \text{ ft/s}) = m(v_B)_x + mv_C \quad (1)$

+xy components: $\quad 0 = mv_A - m(v_B)_y \quad (2)$

s) $= (8 \text{ ft})mv_A$

$-(7 \text{ ft})m(v_B)_y - (3 \text{ ft})mv_C \quad (3)$

$_B)_x$, and $(v_B)_y$ in terms of v_C,

$$v_A = (v_B)_y = 3v_C - 20 \qquad (v_B)_x = 10 - v_C \tag{4}$$

Conservation of Energy. Since the surfaces are frictionless and the impacts are perfectly elastic, the initial kinetic energy $\frac{1}{2}mv_0^2$ is equal to the final kinetic energy of the system:

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}m_Av_A^2 + \tfrac{1}{2}m_Bv_B^2 + \tfrac{1}{2}m_Cv_C^2$$

$$v_A^2 + (v_B)_x^2 + (v_B)_y^2 + v_C^2 = (10 \text{ ft/s})^2 \tag{5}$$

Substituting for v_A, $(v_B)_x$, and $(v_B)_y$ from (4) into (5), we have

$$2(3v_C - 20)^2 + (10 - v_C)^2 + v_C^2 = 100$$
$$20v_C^2 - 260v_C + 800 = 0$$

Solving for v_C, we find $v_C = 5$ ft/s and $v_C = 8$ ft/s. Since only the second root yields a positive value for v_A after substitution into Eqs. (4), we conclude that $v_C = 8$ ft/s and

$$v_A = (v_B)_y = 3(8) - 20 = 4 \text{ ft/s} \qquad (v_B)_x = 10 - 8 = 2 \text{ ft/s}$$

$\mathbf{v}_A = 4$ ft/sx $\qquad \mathbf{v}_B = 4.47$ ft/s c $63.4°$ $\qquad \mathbf{v}_C = 8$ ft/s y ◀

SOLVING PROBLEMS ON YOUR OWN

In the preceding lesson we defined the linear momentum and the angular momentum of a system of particles. In this lesson we defined the *kinetic energy T* of a system of particles:

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 \qquad (14.28)$$

The solutions of the problems in the preceding lesson were based on the conservation of the linear momentum of a system of particles or on the observation of the motion of the mass center of a system of particles. In this lesson you will solve problems involving the following:

1. Computation of the kinetic energy lost in collisions. The kinetic energy T_1 of the system of particles before the collisions and its kinetic energy T_2 after the collisions are computed from Eq. (14.28) and are subtracted from each other. Keep in mind that, while linear momentum and angular momentum are vector quantities, kinetic energy is a *scalar* quantity.

2. Conservation of linear momentum and conservation of energy. As you saw in the preceding lesson, when the resultant of the external forces acting on a system of particles is z[illegible]s conserved. In problems involving [illegible]e initial linear momentum and the [illegible]uipollent yields two algebraic equations. Equating the initial total energy of the system of particles (including potential energy as well as kinetic energy) to its final total energy yields an additional equation. Thus, you can write three equations which can be solved for three unknowns [Sample Prob. 14.5]. Note that if the resultant of the external forces is not zero but has a fixed direction, the component of the linear momentum in a direction perpendicular to the resultant is still conserved; the number of equations which can be used is then reduced to two [Sample Prob. 14.4].

3. Conservation of linear and angular momentum. When no external forces act on a system of particles, both the linear momentum of the system and its angular momentum about some arbitrary point are conserved. In the case of three-dimensional motion, this will enable you to write as many as six equations, although you may need to solve only some of them to obtain the desired answers [Sample Prob. 14.3]. In the case of two-dimensional motion, you will be able to write three equations which can be solved for three unknowns.

4. Conservation of linear and angular momentum and conservation of energy. In the case of the two-dimensional motion of a system of particles which are not subjected to any external forces, you will obtain two algebraic equations by expressing that the linear momentum of the system is conserved, one equation by writing that the angular momentum of the system about some arbitrary point is conserved, and a fourth equation by expressing that the total energy of the system is conserved. These equations can be solved for four unknowns.

PROBLEMS

14.31 Determine the energy lost due to friction and the impacts for Prob. 14.1.

14.32 In Prob. 14.4, determine the energy lost as the bullet (*a*) passes through block *A*, (*b*) becomes embedded in block *B*.

14.33 In Prob. 14.6, determine the work done by the woman and by the man as each dives from the boat, assuming that the woman dives first.

14.34 Determine the energy lost as a result of the series of collisions described in Prob. 14.8.

14.35 Two automobiles *A* and *B*, of mass m_A and m_B, respectively, are traveling in opposite directions when they collide head on. The impact is assumed perfectly plastic, and it is further assumed that the energy absorbed by each automobile is equal to its loss of kinetic energy with respect to a moving frame of reference attached to the mass center of the two-vehicle system. Denoting by E_A and E_B, respectively, the energy absorbed by automobile *A* and by automobile *B*, (*a*) show that $E_A/E_B = m_B/m_A$, that is, the amount of energy absorbed by each vehicle is inversely proportional to its mass, (*b*) compute E_A and E_B, knowing that $m_A = 1600$ kg and $m_B = 900$ kg and that the speeds of *A* and *B* are, respective

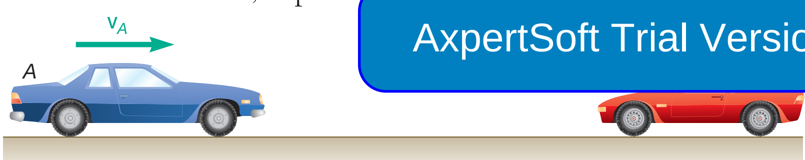


Fig. P14.35

14.36 It is assumed that each of the two automobiles involved in the collision described in Prob. 14.35 had been designed to safely withstand a test in which it crashed into a solid, immovable wall at the speed v_0. The severity of the collision of Prob. 14.35 may then be measured for each vehicle by the ratio of the energy it absorbed in the collision to the energy it absorbed in the test. On that basis, show that the collision described in Prob. 14.35 is $(m_A/m_B)^2$ times more severe for automobile *B* than for automobile *A*.

14.37 Solve Sample Prob. 14.4, assuming that cart *A* is given an initial horizontal velocity $\mathbf{v}_0$ while ball *B* is at rest.

14.38 Two hemispheres are held together by a cord which maintains a spring under compression (the spring is not attached to the hemispheres). The potential energy of the compressed spring is 120 J and the assembly has an initial velocity $\mathbf{v}_0$ of magnitude $v_0 = 8$ m/s. Knowing that the cord is severed when u = 30°, causing the hemispheres to fly apart, determine the resulting velocity of each hemisphere.

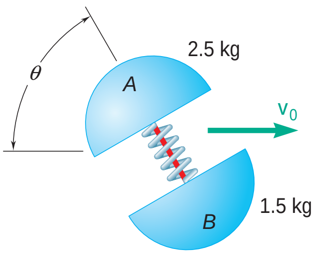


Fig. P14.38

14.39 A 15-lb block *B* starts from rest and slides on the 25-lb wedge *A*, which is supported by a horizontal surface. Neglecting friction, determine (*a*) the velocity of *B* relative to *A* after it has slid 3 ft down the inclined surface of the wedge, (*b*) the corresponding velocity of *A*.

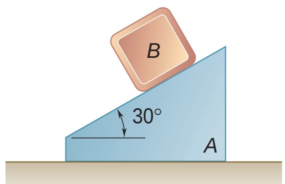


Fig. P14.39

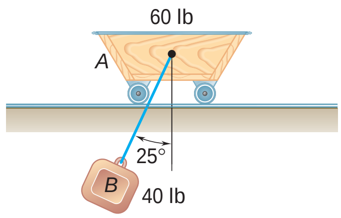


Fig. P14.40

14.40 A 40-lb block B is suspended from a 6-ft cord attached to a 60-lb cart A, which may roll freely on a frictionless, horizontal track. If the system is released from rest in the position shown, determine the velocities of A and B as B passes directly under A.

14.41 and 14.42 In a game of pool, ball A is moving with a velocity $\mathbf{v}_0$ of magnitude $v_0 = 15$ ft/s when it strikes balls B and C, which are at rest and aligned as shown. Knowing that after the collision the three balls move in the directions indicated and assuming frictionless surfaces and perfectly elastic impact (i.e., conservation of energy), determine the magnitudes of the velocities $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$.

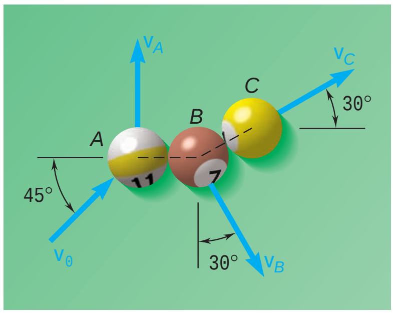


Fig. P14.41

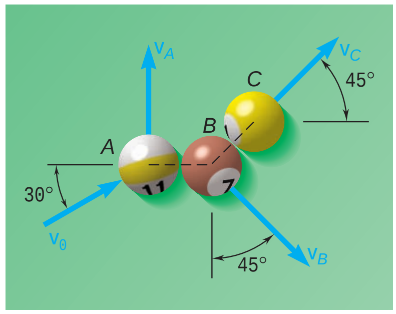


Fig. P14.42

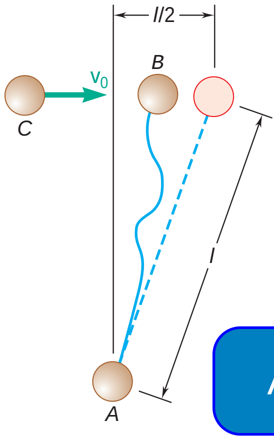


Fig. P14.43

can slide freely on a frictionless, horizontal surface. Spheres A and B are attached to an inextensible, inelastic cord of length l and are at rest in the position shown when sphere B is struck squarely by sphere C which is moving to the right with a velocity $\mathbf{v}_0$. Knowing that the cord is slack when sphere B is struck by sphere C and assuming perfectly elastic impact between B and C, determine (a) the velocity of each sphere immediately after the cord becomes taut, (b) the fraction of the initial kinetic energy of the system which is dissipated when the cord becomes taut.

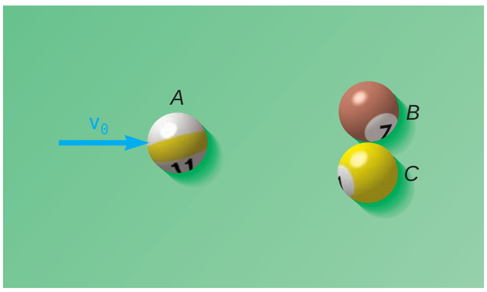


Fig. P14.44

14.44 In a game of pool, ball A is moving with the velocity $\mathbf{v}_0 = v_0\mathbf{i}$ when it strikes balls B and C, which are at rest side by side. Assuming frictionless surfaces and perfectly elastic impact (i.e., conservation of energy), determine the final velocity of each ball, assuming that the path of A is (a) perfectly centered and that A strikes B and C simultaneously, (b) not perfectly centered and that A strikes B slightly before it strikes C.

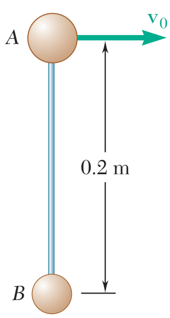


Fig. P14.45

14.45 Two small spheres A and B, of mass 2.5 kg and 1 kg, respectively, are connected by a rigid rod of negligible mass. The two spheres are resting on a horizontal, frictionless surface when A is suddenly given the velocity $\mathbf{v}_0 = (3.5 \text{ m/s})\mathbf{i}$. Determine ($a$) the linear momentum of the system and its angular momentum about its mass center G, (b) the velocities of A and B after the rod AB has rotated through 180°.

14.46 A 900-lb space vehicle traveling with a velocity $\mathbf{v}_0 = (1500 \text{ ft/s})\mathbf{k}$ passes through the origin O. Explosive charges then separate the vehicle into three parts A, B, and C, with masses of 150 lb, 300 lb, and 450 lb, respectively. Knowing that shortly thereafter the positions of the three parts are, respectively, $A(250, 250, 2250)$, $B(600, 1300, 3200)$, and $C(-475, -950, 1900)$, where the coordinates are expressed in feet, that the velocity of B is $\mathbf{v}_B = (500 \text{ ft/s})\mathbf{i} + (1100 \text{ ft/s})\mathbf{j} + (2100 \text{ ft/s})\mathbf{k}$, and that the x component of the velocity of C is -400 ft/s, determine the velocity of part A.

14.47 Four small disks A, B, C, and D can slide freely on a frictionless horizontal surface. Disks B, C, and D are connected by light rods and are at rest in the position shown when disk B is struck squarely by disk A which is moving to the right with a velocity $\mathbf{v}_0 = (38.5 \text{ ft/s})\mathbf{i}$. The weights of the disks are $W_A = W_B = W_C = 15$ lb, and $W_D = 30$ lb. Knowing that the velocities of the disks immediately after the impact are $\mathbf{v}_A = \mathbf{v}_B = (8.25 \text{ ft/s})\mathbf{i}$, $\mathbf{v}_C = v_C\mathbf{i}$, and $\mathbf{v}_D = v_D\mathbf{i}$, determine (*a*) the speeds v_C and v_D, (*b*) the fraction of the initial kinetic energy of the system which is dissipated during the collision.

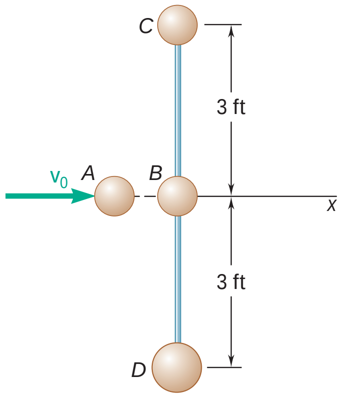


Fig. P14.47

14.48 In the scattering experiment of Prob. 14.26, it is known that the alpha particle is projected from $A_0(300, 0, 300)$ and that it collides with the oxygen nucleus C at $Q(240, 200, 100)$, where all coordinates are expressed in millimeters. Determine the coordinates of point B_0 where the original path of nucleus B intersects the zx plane. (*Hint.* Express that the angular momentum of the three particles about Q is conserved.)

14.49 Three identical small sph[illegible] a horizontal frictionless s[illegible] a light rod and are at res[illegible] struck squarely by sphere A which is moving to the right with a velocity $\mathbf{v}_0 = (8 \text{ ft/s})\mathbf{i}$. Knowing that u = 45° and that the velocities of spheres A and B immediately after the impact are $\mathbf{v}_A = 0$ and $\mathbf{v}_B = (6 \text{ ft/s})\mathbf{i} + (\mathbf{v}_B)_y\mathbf{j}$, determine $(v_B)_y$ and the velocity of C immediately after impact.

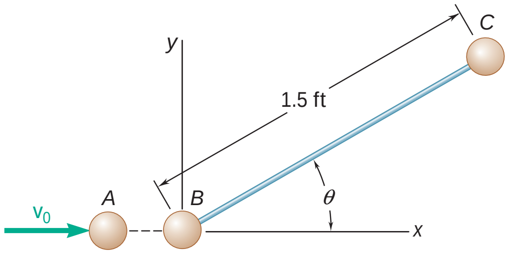


Fig. P14.49

14.50 Three small spheres A, B, and C, each of mass m, are connected to a small ring D of negligible mass by means of three inextensible, inelastic cords of length l. The spheres can slide freely on a frictionless horizontal surface and are rotating initially at a speed v_0 about ring D which is at rest. Suddenly the cord CD breaks. After the other two cords have again become taut, determine (*a*) the speed of ring D, (*b*) the relative speed at which spheres A and B rotate about D, (*c*) the fraction of the original energy of spheres A and B which is dissipated when cords AD and BD again became taut.

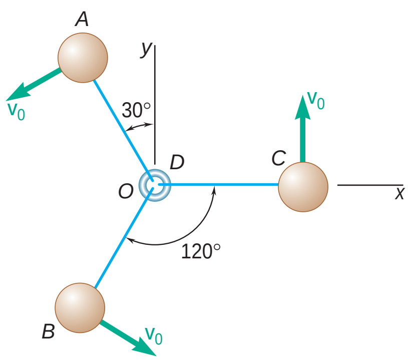


Fig. P14.50

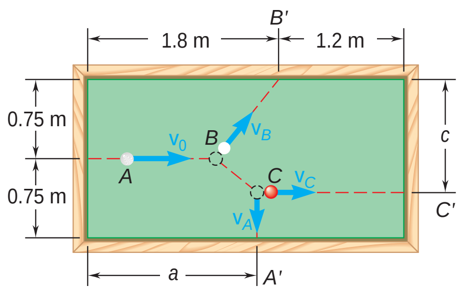


Fig. P14.51

14.51 In a game of billiards, ball A is given an initial velocity $\mathbf{v}_0$ along the longitudinal axis of the table. It hits ball B and then ball C, which are both at rest. Balls A and C are observed to hit the sides of the table squarely at A' and C', respectively, and ball B is observed to hit the side obliquely at B'. Knowing that $v_0 = 4$ m/s, $v_A = 1.92$ m/s, and $a = 1.65$ m, determine (*a*) the velocities $\mathbf{v}_B$ and $\mathbf{v}_C$ of balls B and C, (*b*) the point C' where ball C hits the side of the table. Assume frictionless surfaces and perfectly elastic impacts (i.e., conservation of energy).

14.52 For the game of billiards of Prob. 14.51, it is now assumed that $v_0 = 5$ m/s, $v_C = 3.2$ m/s, and $c = 1.22$ m. Determine (*a*) the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of balls A and B, (*b*) the point A' where ball A hits the side of the table.

14.53 Two small disks A and B, of mass 3 kg and 1.5 kg, respectively, may slide on a horizontal, frictionless surface. They are connected by a cord, 600 mm long, and spin counterclockwise about their mass center G at the rate of 10 rad/s. At $t = 0$, the coordinates of G are $\bar{x}_0 = 0$, $\bar{y}_0 = 2$ m, and its velocity $\bar{\mathbf{v}}_0 = (1.2 \text{ m/s})\mathbf{i} + (0.96 \text{ m/s})\mathbf{j}$. Shortly thereafter the cord breaks; disk A is then observed to move along a path parallel to the y axis and disk B along a path which intersects the x axis at a distance $b = 7.5$ m from O. Determine (*a*) the velocities of A and B after the cord breaks, (*b*) the distance a from the y axis to the path of A.

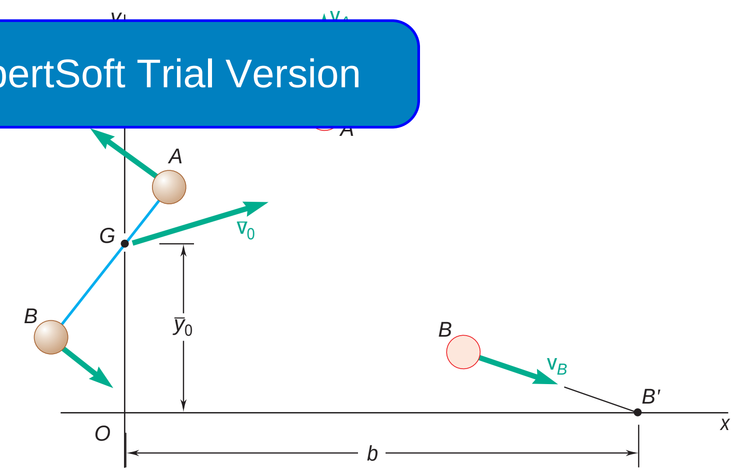


Fig. *P14.53* and *P14.54*

14.54 Two small disks A and B, of mass 2 kg and 1 kg, respectively, may slide on a horizontal and frictionless surface. They are connected by a cord of negligible mass and spin about their mass center G. At $t = 0$, G is moving with the velocity $\bar{\mathbf{v}}_0$ and its coordinates are $\bar{x}_0 = 0$, $\bar{y}_0 = 1.89$ m. Shortly thereafter, the cord breaks and disk A is observed to move with a velocity $\mathbf{v}_A = (5 \text{ m/s})\mathbf{j}$ in a straight line and at a distance $a = 2.56$ m from the y axis, while B moves with a velocity $\mathbf{v}_B = (7.2 \text{ m/s})\mathbf{i} - (4.6 \text{ m/s})\mathbf{j}$ along a path intersecting the x axis at a distance $b = 7.48$ m from the origin O. Determine (*a*) the initial velocity $\bar{\mathbf{v}}_0$ of the mass center G of the two disks, (*b*) the length of the cord initially connecting the two disks, (*c*) the rate in rad/s at which the disks were spinning about G.

14.55 Three small identical spheres A, B, and C, which can slide on a horizontal, frictionless surface, are attached to three 9-in.-long strings, which are tied to a ring G. Initially the spheres rotate clockwise about the ring with a relative velocity of 2.6 ft/s and the ring moves along the x axis with a velocity $\mathbf{v}_0 = (1.3 \text{ ft/s})\mathbf{i}$. Suddenly the ring breaks and the three spheres move freely in the xy plane with A and B following paths parallel to the y axis at a distance $a = 1.0$ ft from each other and C following a path parallel to the x axis. Determine (*a*) the velocity of each sphere, (*b*) the distance d.

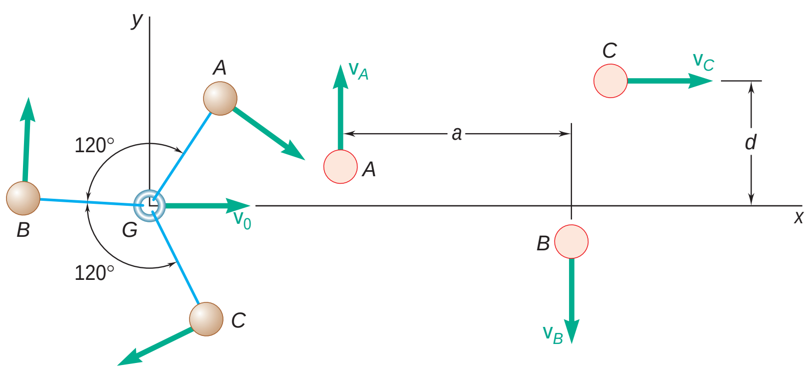


Fig. *P14.55* and *P14.56*

14.56 Three small identical spheres A, B, and C, which can slide on a horizontal, frictionless s[illegible] length l which are tied [illegible] clockwise about the ring [illegible] ity $\mathbf{v}_0$. Suddenly the ring [illegible] in the xy plane. Knowing that $\mathbf{v}_A = (3.5 \text{ ft/s})\mathbf{j}$, $\mathbf{v}_C = (6.0 \text{ ft/s})\mathbf{i}$, $a = 16$ in., and $d = 9$ in., determine (*a*) the initial velocity of the ring, (*b*) the length l of the strings, (*c*) the rate in rad/s at which the spheres were rotating about G.

*14.10 VARIABLE SYSTEMS OF PARTICLES

All the systems of particles considered so far consisted of well-defined particles. These systems did not gain or lose any particles during their motion. In a large number of engineering applications, however, it is necessary to consider *variable systems of particles,* i.e., systems which are continually gaining or losing particles, or doing both at the same time. Consider, for example, a hydraulic turbine. Its analysis involves the determination of the forces exerted by a stream of water on rotating blades, and we note that the particles of water in contact with the blades form an everchanging system which continually acquires and loses particles. Rockets furnish another example of variable systems, since their propulsion depends upon the continual ejection of fuel particles.

We recall that all the kinetics principles established so far were derived for constant systems of particles, which neither gain nor lose particles. We must therefore find a way to reduce the analysis of a variable system of particles to that of an auxiliary constant system. The procedure to follow is indicated in Secs. 14.11 and 14.12 for two broad categories of applications: a steady stream of particles and a system that is gaining or losing mass.

*14.11 STEADY STREAM OF PARTICLES

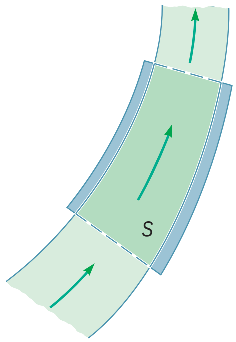


Fig. 14.9

Consider a steady stream of particles, such as a stream of water diverted by a fixed vane or a flow of air through a duct or through a blower. In order to determine the resultant of the forces exerted on the particles in contact with the vane, duct, or blower, we isolate these particles and denote by S the system thus defined (Fig. 14.9). We observe that S is a variable system of particles, since it continually gains particles flowing in and loses an equal number of particles flowing out. Therefore, the kinetics principles that have been established so far cannot be directly applied to S.

However, we can easily define an auxiliary system of particles which does remain constant for a short interval of time Δt. Consider at time t the system S *plus* the particles which will enter S during the interval at time Δt (Fig. 14.10*a*). Next, consider at time $t + \Delta t$ the system S *plus* the particles which have left S during the interval Δt (Fig. 14.10*c*). Clearly, *the same particles are involved in both cases,* and we can apply to those particles the principle of impulse and momentum. Since the total mass m of the system S remains [illegible]stem and those leaving the sys-[illegible]me mass Δm. Denoting by $\mathbf{v}_A$ [illegible] the particles entering S at A and leaving S at B, we represent the momentum of the particles entering S by $(\Delta m)\mathbf{v}_A$ (Fig. 14.10*a*) and the momentum of the particles leaving S by $(\Delta m)\mathbf{v}_B$ (Fig. 14.10*c*). We also represent by appropriate vectors the momenta $m_i\mathbf{v}_i$ of the particles forming S and the impulses of the forces exerted on S and indicate by blue plus and equals signs that the system of the momenta and impulses in parts *a* and *b* of Fig. 14.10 is equipollent to the system of the momenta in part *c* of the same figure.

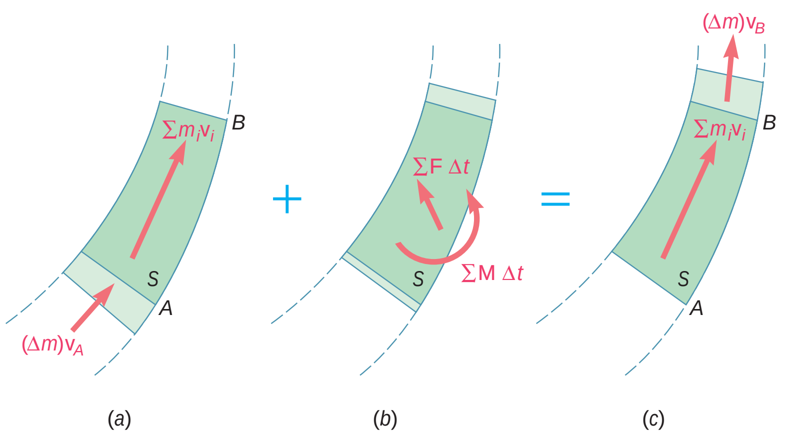


Fig. 14.10

The resultant $\Sigma m_i \mathbf{v}_i$ of the momenta of the particles of S is found on both sides of the equals sign and can thus be omitted. We conclude that *the system formed by the momentum* $(\Delta m)\mathbf{v}_A$ *of the particles entering* S *in the time* Δt *and the impulses of the forces exerted on* S *during that time is equipollent to the momentum* $(\Delta m)\mathbf{v}_B$ *of the particles leaving* S *in the same time* Δt. We can therefore write

$$(\Delta m)\mathbf{v}_A + \Sigma \mathbf{F}\, \Delta t = (\Delta m)\mathbf{v}_B \qquad (14.38)$$

A similar equation can be obtained by taking the moments of the vectors involved (see Sample Prob. 14.5). Dividing all terms of Eq. (14.38) by Δt and letting Δt approach zero, we obtain at the limit

$$\Sigma \mathbf{F} = \frac{dm}{dt}(\mathbf{v}_B - \mathbf{v}_A) \qquad (14.39)$$

where $\mathbf{v}_B - \mathbf{v}_A$ represents the difference between the *vector* $\mathbf{v}_B$ and the *vector* $\mathbf{v}_A$.

If SI units are used, dm/dt is expressed in kg/s and the velocities in m/s; we check that both members of Eq. (14.39) are expressed in the same units (newtons). If U.S. customary units are used, dm/dt must be expressed in slugs/s and the velocities in ft/s; we check again that both members of the equation are expressed in the same units (pounds).†

The principle we have established can be used to analyze a large number of engineering applications. Some of the more common of these applications will [illegible]

Fluid Stream Diverted by [illegible] method of analysis given above can be applied directly to find the force $\mathbf{F}$ exerted by the vane on the stream. We note that $\mathbf{F}$ is the only force which needs to be considered since the pressure in the stream is constant (atmospheric pressure). The force exerted by the stream on the vane will be equal and opposite to $\mathbf{F}$. If the vane moves with a constant velocity, the stream is not steady. However, it will appear steady to an observer moving with the vane. We should therefore choose a system of axes moving with the vane. Since this system of axes is not accelerated, Eq. (14.38) can still be used, but $\mathbf{v}_A$ and $\mathbf{v}_B$ must be replaced by the *relative velocities* of the stream with respect to the vane (see Sample Prob. 14.7).

Fluid Flowing Through a Pipe. The force exerted by the fluid on a pipe transition such as a bend or a contraction can be determined by considering the system of particles S in contact with the transition. Since, in general, the pressure in the flow will vary, the forces exerted on S by the adjoining portions of the fluid should also be considered.

†It is often convenient to express the mass rate of flow dm/dt as the product ρQ, where ρ is the density of the stream (mass per unit volume) and Q its volume rate of flow (volume per unit time). If SI units are used, ρ is expressed in kg/m^3 (for instance, $\rho = 1000$ kg/m^3 for water) and Q in m^3/s. However, if U.S. customary units are used, ρ will generally have to be computed from the corresponding specific weight γ (weight per unit volume), $\rho = \gamma/g$. Since γ is expressed in lb/ft^3 (for instance, $\gamma = 62.4$ lb/ft^3 for water), ρ is obtained in slugs/ft^3. The volume rate of flow Q is expressed in ft^3/s.

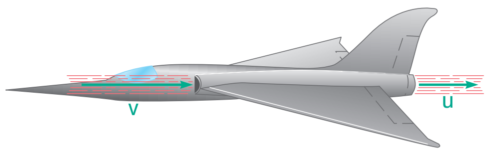


Fig. 14.11

Jet Engine. In a jet engine, air enters with no velocity through the front of the engine and leaves through the rear with a high velocity. The energy required to accelerate the air particles is obtained by burning fuel. The mass of the burned fuel in the exhaust gases will usually be small enough compared with the mass of the air flowing through the engine that it can be neglected. Thus, the analysis of a jet engine reduces to that of an airstream. This stream can be considered as a steady stream if all velocities are measured with respect to the airplane. It will be assumed, therefore, that the airstream enters the engine with a velocity $\mathbf{v}$ of magnitude equal to the speed of the airplane and leaves with a velocity $\mathbf{u}$ equal to the relative velocity of the exhaust gases (Fig. 14.11). Since the intake and exhaust pressures are nearly atmospheric, the only external force which needs to be considered is the force exerted by the engine on the airstream. This force is equal and opposite to the thrust.†

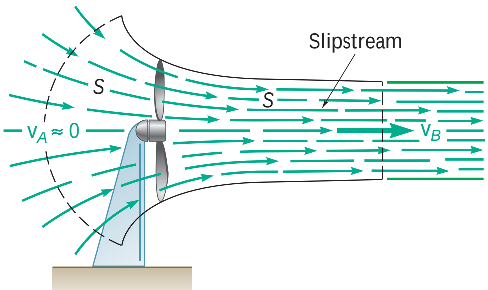


Fig. 14.12

Fan. We consider the system of particles S shown in Fig. 14.12. The velocity $\mathbf{v}_A$ of the particles entering the system is assumed equal to zero, and the velocity $\mathbf{v}_B$ of the particles leaving the system is the velocity of the *slipstream*. The rate of flow can be obtained by multiplying v_B by the cross-sectional area of the slipstream. Since the pressure all around S is atmospheric, the only external force acting on S is the thrust of the fan.

[illegible] the thrust created by the rotat- [illegible] similar to the determination of [illegible] e air particles as they approach [illegible] the rate of flow is obtained by multiplying the magnitude of the velocity $\mathbf{v}_B$ of the slipstream by its cross-sectional area.

*14.12 SYSTEMS GAINING OR LOSING MASS

Let us now analyze a different type of variable system of particles, namely, a system which gains mass by continually absorbing particles or loses mass by continually expelling particles. Consider the system S shown in Fig. 14.13. Its mass, equal to m at the instant t, increases by Δm in the interval of time Δt. In order to apply the principle of impulse and momentum to the analysis of this system, we must consider at time t the system S *plus* the particles of mass Δm which S absorbs during the time interval Δt. The velocity of S at time t is denoted by $\mathbf{v}$, the velocity of S at time $t + \Delta t$ is denoted by $\mathbf{v} + \Delta\mathbf{v}$, and the absolute velocity of the particles absorbed is denoted by $\mathbf{v}_a$. Applying the principle of impulse and momentum, we write

$$m\mathbf{v} + (\Delta m)\mathbf{v}_a + \Sigma\mathbf{F}\,\Delta t = (m + \Delta m)(\mathbf{v} + \Delta\mathbf{v}) \qquad (14.40)$$

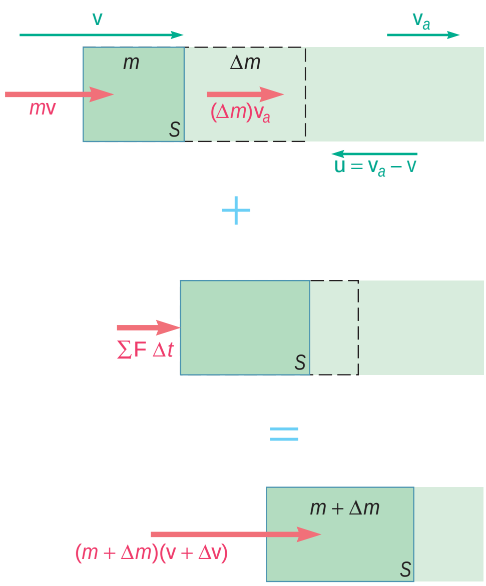


Fig. 14.13

†Note that if the airplane is accelerated, it cannot be used as a newtonian frame of reference. The same result will be obtained for the thrust, however, by using a reference frame at rest with respect to the atmosphere, since the air particles will then be observed to enter the engine with no velocity and to leave it with a velocity of magnitude $u - v$.

Solving for the sum $\Sigma \mathbf{F}\,\Delta t$ of the impulses of the external forces acting on S (excluding the forces exerted by the particles being absorbed), we have

$$\Sigma \mathbf{F}\,\Delta t = m\Delta \mathbf{v} + \Delta m(\mathbf{v} - \mathbf{v}_a) + (\Delta m)(\Delta \mathbf{v}) \tag{14.41}$$

Introducing the *relative velocity* $\mathbf{u}$ with respect to S of the particles which are absorbed, we write $\mathbf{u} = \mathbf{v}_a - \mathbf{v}$ and note, since $v_a < v$, that the relative velocity $\mathbf{u}$ is directed to the left, as shown in Fig. 14.13. Neglecting the last term in Eq. (14.41), which is of the second order, we write

$$\Sigma \mathbf{F}\,\Delta t = m\,\Delta \mathbf{v} - (\Delta m)\mathbf{u}$$

Dividing through by Δt and letting Δt approach zero, we have at the limit†

$$\Sigma \mathbf{F} = m\frac{d\mathbf{v}}{dt} - \frac{dm}{dt}\mathbf{u} \tag{14.42}$$

Rearranging the terms and recalling that $d\mathbf{v}/dt = \mathbf{a}$, where $\mathbf{a}$ is the acceleration of the system S, we write

$$\Sigma \mathbf{F} + \frac{dm}{dt}\mathbf{u} = m\mathbf{a} \tag{14.43}$$

which shows that the action on S of the particles being absorbed is equivalent to a thrust

$\mathbf{P}$

which tends to slow down the motion of S, since the relative velocity $\mathbf{u}$ of the particles is directed to the left. If SI units are used, dm/dt is expressed in kg/s, the relative velocity u in m/s, and the corresponding thrust in newtons. If U.S. customary units are used, dm/dt must be expressed in slugs/s, u in ft/s, and the corresponding thrust in pounds.‡

The equations obtained can also be used to determine the motion of a system S losing mass. In this case, the rate of change of mass is negative, and the action on S of the particles being expelled is equivalent to a thrust in the direction of $-\mathbf{u}$, that is, in the direction opposite to that in which the particles are being expelled. A *rocket* represents a typical case of a system continually losing mass (see Sample Prob. 14.8).

14.3 As the shuttle's booster rockets ed, the gas particles they eject provide the required for liftoff.

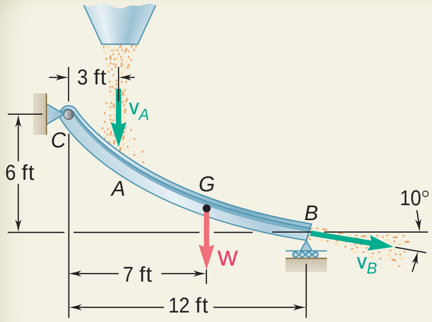
†When the absolute velocity $\mathbf{v}_a$ of the particles absorbed is zero, $\mathbf{u} = -\mathbf{v}$, and formula (14.42) becomes

$$\Sigma \mathbf{F} = \frac{d}{dt}(m\mathbf{v})$$

Comparing the formula obtained to Eq. (12.3) of Sec. 12.3, we observe that Newton's second law can be applied to a system gaining mass, *provided that the particles absorbed are initially at rest*. It may also be applied to a system losing mass, *provided that the velocity of the particles expelled is zero* with respect to the frame of reference selected.

‡See footnote on page 899.

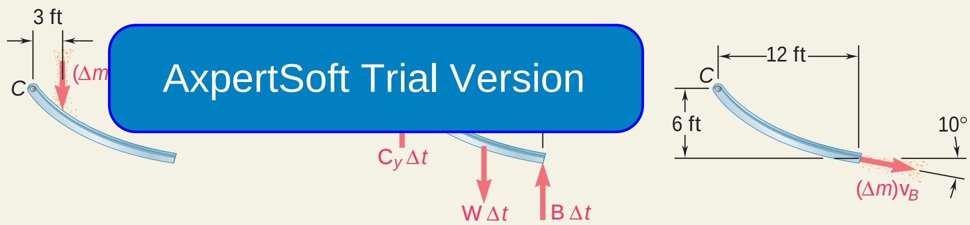
SAMPLE PROBLEM 14.6



Grain falls from a hopper onto a chute CB at the rate of 240 lb/s. It hits the chute at A with a velocity of 20 ft/s and leaves at B with a velocity of 15 ft/s, forming an angle of 10° with the horizontal. Knowing that the combined weight of the chute and of the grain it supports is a force $\mathbf{W}$ of magnitude 600 lb applied at G, determine the reaction at the roller support B and the components of the reaction at the hinge C.

SOLUTION

We apply the principle of impulse and momentum for the time interval Δt to the system consisting of the chute, the grain it supports, and the amount of grain which hits the chute in the interval Δt. Since the chute does not move, it has no momentum. We also note that the sum $\Sigma m_i \mathbf{v}_i$ of the momenta of the particles supported by the chute is the same at t and $t + \Delta t$ and can thus be omitted.



Since the system formed by the momentum $(\Delta m)\mathbf{v}_A$ and the impulses is equipollent to the momentum $(\Delta m)\mathbf{v}_B$, we write

$\overset{+}{y}$ x components: $\quad C_x\,\Delta t = (\Delta m)v_B \cos 10° \qquad (1)$

$+xy$ components: $\quad -(\Delta m)v_A + C_y\,\Delta t - W\,\Delta t + B\,\Delta t = -(\Delta m)v_B \sin 10° \qquad (2)$

$+1$ moments about C: $\quad -3(\Delta m)v_A - 7(W\,\Delta t) + 12(B\,\Delta t) = 6(\Delta m)v_B \cos 10° - 12(\Delta m)v_B \sin 10° \qquad (3)$

Using the given data, $W = 600$ lb, $v_A = 20$ ft/s, $v_B = 15$ ft/s, and $\Delta m/\Delta t = 240/32.2 = 7.45$ slugs/s, and solving Eq. (3) for B and Eq. (1) for C_x,

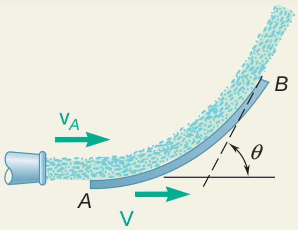
$$12B = 7(600) + 3(7.45)(20) + 6(7.45)(15)(\cos 10° - 2 \sin 10°)$$
$$12B = 5075 \qquad B = 423 \text{ lb} \qquad \mathbf{B} = 423 \text{ lb x} \quad ◀$$

$$C_x = (7.45)(15) \cos 10° = 110.1 \text{ lb} \qquad \mathbf{C}_x = 110.1 \text{ lb y} \quad ◀$$

Substituting for B and solving Eq. (2) for C_y,

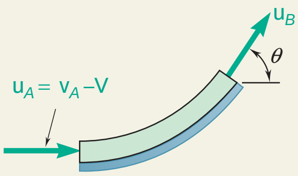
$$C_y = 600 - 423 + (7.45)(20 - 15 \sin 10°) = 307 \text{ lb}$$
$$\mathbf{C}_y = 307 \text{ lb x} \quad ◀$$

SAMPLE PROBLEM 14.7

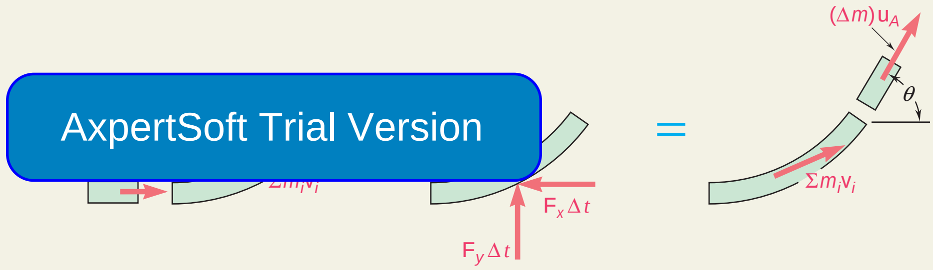


A nozzle discharges a stream of water of cross-sectional area A with a velocity $\mathbf{v}_A$. The stream is deflected by a *single* blade which moves to the right with a constant velocity $\mathbf{V}$. Assuming that the water moves along the blade at constant speed, determine (*a*) the components of the force $\mathbf{F}$ exerted by the blade on the stream, (*b*) the velocity $\mathbf{V}$ for which maximum power is developed.

SOLUTION



a. Components of Force Exerted on Stream. We choose a coordinate system which moves with the blade at a constant velocity $\mathbf{V}$. The particles of water strike the blade with a relative velocity $\mathbf{u}_A = \mathbf{v}_A - \mathbf{V}$ and leave the blade with a relative velocity $\mathbf{u}_B$. Since the particles move along the blade at a constant speed, the relative velocities $\mathbf{u}_A$ and $\mathbf{u}_B$ have the same magnitude u. Denoting the density of water by ρ, the mass of the particles striking the blade during the time interval Δt is $\Delta m = A\rho(v_A - V)\,\Delta t$; an equal mass of particles leaves the blade during Δt. We apply the principle of impulse and momentum to the system formed by the particles in contact with the blade and the particles striking the blade in the time Δt.



Recalling that $\mathbf{u}_A$ and $\mathbf{u}_B$ have the same magnitude u, and omitting the momentum $\Sigma m_i\mathbf{v}_i$ which appears on both sides, we write

$\xrightarrow{+}$ x components: $\quad (\Delta m)u - F_x\,\Delta t = (\Delta m)u\cos\theta$

$+\uparrow y$ components: $\quad +F_y\,\Delta t = (\Delta m)u\sin\theta$

Substituting $\Delta m = A\rho(v_A - V)\,\Delta t$ and $u = v_A - V$, we obtain

$$\mathbf{F}_x = A\rho(v_A - V)^2(1 - \cos\theta) \leftarrow \qquad \mathbf{F}_y = A\rho(v_A - V)^2\sin\theta \uparrow \quad ◀$$

b. Velocity of Blade for Maximum Power. The power is obtained by multiplying the velocity V of the blade by the component F_x of the force exerted by the stream on the blade.

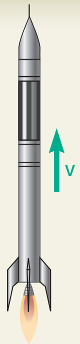
$$\text{Power} = F_xV = A\rho(v_A - V)^2(1 - \cos\theta)V$$

Differentiating the power with respect to V and setting the derivative equal to zero, we obtain

$$\frac{d(\text{power})}{dV} = A\rho(v_A^2 - 4v_AV + 3V^2)(1 - \cos\theta) = 0$$

$$V = v_A \qquad V = \tfrac{1}{3}v_A \qquad \text{For maximum power } \mathbf{V} = \tfrac{1}{3}v_A \rightarrow \quad ◀$$

Note. These results are valid only when a *single* blade deflects the stream. Different results are obtained when a series of blades deflects the stream, as in a Pelton-wheel turbine. (See Prob. 14.81.)

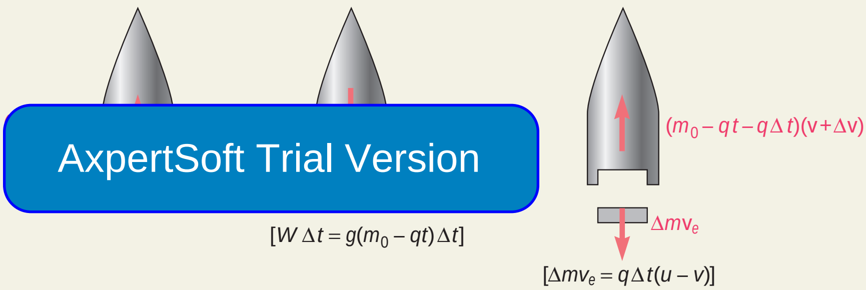


SAMPLE PROBLEM 14.8

A rocket of initial mass m_0 (including shell and fuel) is fired vertically at time $t = 0$. The fuel is consumed at a constant rate $q = dm/dt$ and is expelled at a constant speed u relative to the rocket. Derive an expression for the magnitude of the velocity of the rocket at time t, neglecting the resistance of the air.

SOLUTION

At time t, the mass of the rocket shell and remaining fuel is $m = m_0 - qt$, and the velocity is $\mathbf{v}$. During the time interval Δt, a mass of fuel $\Delta m = q\,\Delta t$ is expelled with a speed u relative to the rocket. Denoting by $\mathbf{v}_e$ the absolute velocity of the expelled fuel, we apply the principle of impulse and momentum between time t and time $t + \Delta t$.



We write

$$(m_0 - qt)v - g(m_0 - qt)\,\Delta t = (m_0 - qt - q\,\Delta t)(v + \Delta v) - q\,\Delta t(u - v)$$

Dividing through by Δt and letting Δt approach zero, we obtain

$$-g(m_0 - qt) = (m_0 - qt)\frac{dv}{dt} - qu$$

Separating variables and integrating from $t = 0$, $v = 0$ to $t = t$, $v = v$,

$$dv = \left(\frac{qu}{m_0 - qt} - g\right)dt \qquad \int_0^v dv = \int_0^t \left(\frac{qu}{m_0 - qt} - g\right)dt$$

$$v = [-u\ln(m_0 - qt) - gt]_0^t \qquad v = u\ln\frac{m_0}{m_0 - qt} - gt \quad \blacktriangleleft$$

Remark. The mass remaining at time t_f, after all the fuel has been expended, is equal to the mass of the rocket shell $m_s = m_0 - qt_f$, and the maximum velocity attained by the rocket is $v_m = u\ln(m_0/m_s) - gt_f$. Assuming that the fuel is expelled in a relatively short period of time, the term gt_f is small and we have $v_m \approx u\ln(m_0/m_s)$. In order to escape the gravitational field of the earth, a rocket must reach a velocity of 11.18 km/s. Assuming $u = 2200$ m/s and $v_m = 11.18$ km/s, we obtain $m_0/m_s = 161$. Thus, to project each kilogram of the rocket shell into space, it is necessary to consume more than 161 kg of fuel if a propellant yielding $u = 2200$ m/s is used.

SOLVING PROBLEMS ON YOUR OWN

This lesson is devoted to the study of the motion of *variable systems of particles*, i.e., systems which are continually *gaining or losing particles* or doing both at the same time. The problems you will be asked to solve will involve (1) *steady streams of particles* and (2) *systems gaining or losing mass*.

1. To solve problems involving a steady stream of particles, you will consider a portion S of the stream and express that the system formed by the momentum of the particles entering S at A in the time Δt and the impulses of the forces exerted on S during that time is equipollent to the momentum of the particles leaving S at B in the same time Δt (Fig. 14.10). Considering only the resultants of the vector systems involved, you can write the vector equation

$$(\Delta m)\mathbf{v}_A + \Sigma\mathbf{F}\,\Delta t = (\Delta m)\mathbf{v}_B \tag{14.38}$$

You may want to consider as well the moments about a given point of the vector systems involved to obtain an additional equation [Sample Prob. 14.6], but many problems can be solved using Eq. (14.38) or the equation obtained by dividing all terms by Δt and letting Δt approach zero,

$$\Sigma\mathbf{F} = \frac{dm}{dt}(\mathbf{v}_B - \mathbf{v}_A) \tag{14.39}$$

where $\mathbf{v}_B - \mathbf{v}_A$ [illegible] he mass rate of flow dm/dt can be e[illegible] of the stream (mass per unit volume) and the volume rate of flow Q (volume per unit time). If U.S. customary units are used, r is expressed as the ratio g/g, where g is the specific weight of the stream and g is the acceleration of gravity.

Typical problems involving a steady stream of particles have been described in Sec. 14.11. You may be asked to determine the following:

a. Thrust caused by a diverted flow. Equation (14.39) is applicable, but you will get a better understanding of the problem if you use a solution based on Eq. (14.38).

b. Reactions at supports of vanes or conveyor belts. First draw a diagram showing on one side of the equals sign the momentum $(\Delta m)\mathbf{v}_A$ of the particles impacting the vane or belt in the time Δt, as well as the impulses of the loads and reactions at the supports during that time, and showing on the other side the momentum $(\Delta m)\mathbf{v}_B$ of the particles leaving the vane or belt in the time Δt [Sample Prob. 14.6]. Equating the x components, y components, and moments of the quantities on both sides of the equals sign will yield three scalar equations which can be solved for three unknowns.

c. Thrust developed by a jet engine, a propeller, or a fan. In most cases, a single unknown is involved, and that unknown can be obtained by solving the scalar equation derived from Eq. (14.38) or Eq. (14.39).

(*continued*)

2. To solve problems involving systems gaining mass, you will consider the system S, which has a mass m and is moving with a velocity $\mathbf{v}$ at time t, and the particles of mass Δm with velocity $\mathbf{v}_a$ that S will absorb in the time interval Δt (Fig. 14.13). You will then express that the total momentum of S and of the particles that will be absorbed, *plus* the impulse of the external forces exerted on S, are equipollent to the momentum of S at time $t + \Delta t$. Noting that the mass of S and its velocity at that time are, respectively, $m + \Delta m$ and $\mathbf{v} + \Delta\mathbf{v}$, you will write the vector equation

$$m\mathbf{v} + (\Delta m)\mathbf{v}_a + \Sigma\mathbf{F}\,\Delta t = (m + \Delta m)(\mathbf{v} + \Delta\mathbf{v}) \qquad (14.40)$$

As was shown in Sec. 14.12, if you introduce the relative velocity $\mathbf{u} = \mathbf{v}_a - \mathbf{v}$ of the particles being absorbed, you obtain the following expression for the resultant of the external forces applied to S:

$$\Sigma\mathbf{F} = m\frac{d\mathbf{v}}{dt} - \frac{dm}{dt}\mathbf{u} \qquad (14.42)$$

Furthermore, it was shown that the action on S of the particles being absorbed is equivalent to a thrust

(14.44)

exerted in the directi … ing absorbed.

Examples of systems gaining mass are conveyor belts and moving railroad cars being loaded with gravel or sand, and chains being pulled out of a pile.

3. To solve problems involving systems losing mass, such as rockets and rocket engines, you can use Eqs. (14.40) through (14.44), provided that you give negative values to the increment of mass Δm and to the rate of change of mass dm/dt. It follows that the thrust defined by Eq. (14.44) will be exerted in a direction opposite to the direction of the relative velocity of the particles being ejected.

PROBLEMS

14.57 A stream of water of cross-section area A_1 and velocity $\mathbf{v}_1$ strikes a circular plate which is held motionless by a force **P**. A hole in the circular plate of area A_2 results in a discharge jet having a velocity $\mathbf{v}_1$. Determine the magnitude of **P**.

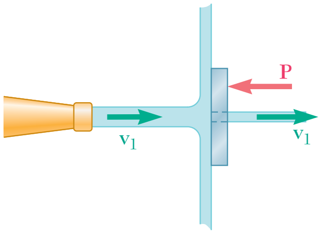


Fig. P14.57

14.58 A jet ski is placed in a channel and is tethered so that it is stationary. Water enters the jet ski with velocity $\mathbf{v}_1$ and exits with velocity $\mathbf{v}_2$. Knowing the inlet area is A_1 and the exit area is A_2, determine the tension in the tether.

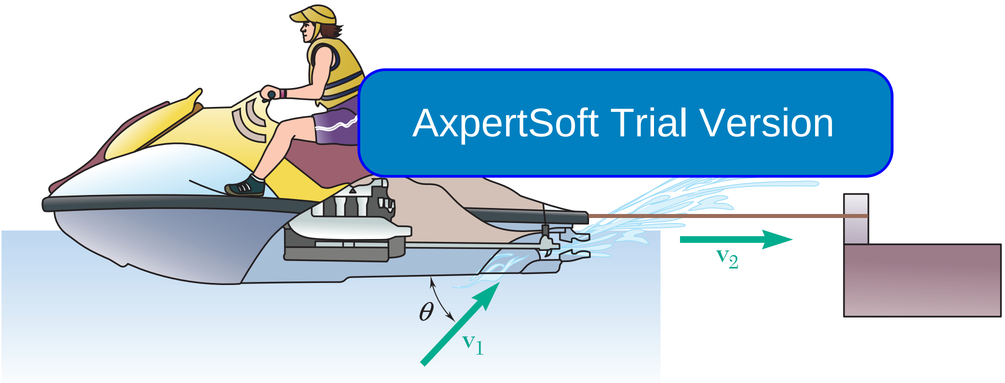


Fig. P14.58

14.59 A stream of water of cross-section area A and velocity $\mathbf{v}_1$ strikes a plate which is held motionless by a force **P**. Determine the magnitude of **P**, knowing that $A = 0.75$ in^2, $v_1 = 80$ ft/s, and $V = 0$.

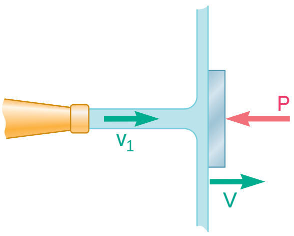


Fig. P14.59 and P14.60

14.60 A stream of water of cross-section area A and velocity $\mathbf{v}_1$ strikes a plate which moves to the right with a velocity **V**. Determine the magnitude of **V**, knowing that $A = 1$ in^2, $v_1 = 100$ ft/s, and $P = 90$ lb.

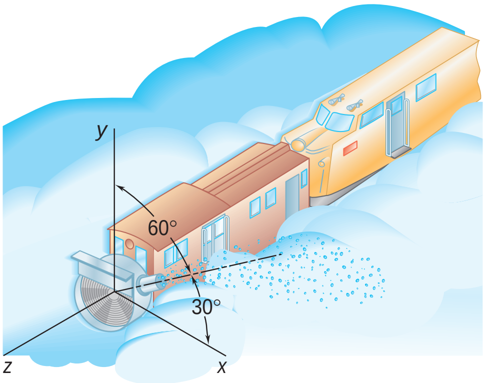
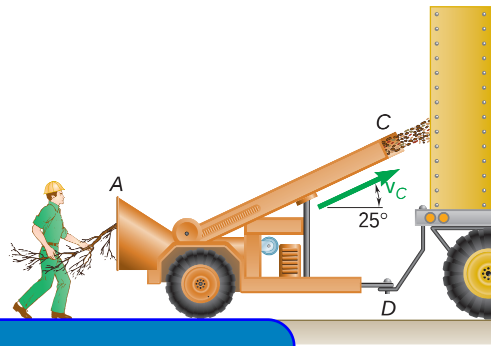


Fig. P14.61

14.61 A rotary power plow is used to remove snow from a level section of railroad track. The plow car is placed ahead of an engine which propels it at a constant speed of 20 km/h. The plow car clears 160 Mg of snow per minute, projecting it in the direction shown with a velocity of 12 m/s relative to the plow car. Neglecting friction, determine (*a*) the force exerted by the engine on the plow car, (*b*) the lateral force exerted by the track on the plow.

14.62 Tree limbs and branches are being fed at *A* at the rate of 5 kg/s into a shredder which spews the resulting wood chips at *C* with a velocity of 20 m/s. Determine the horizontal component of the force exerted by the shredder on the truck hitch at *D*.



14.63 Sand falls from three hoppers onto a conveyor belt at a rate of 90 lb/s for each hopper. The sand hits the belt with a vertical velocity $v_1 = 10$ ft/s and is discharged at *A* with a horizontal velocity $v_2 = 13$ ft/s. Knowing that the combined mass of the beam, belt system, and the sand it supports is 1300 lb with a mass center at *G*, determine the reaction at *E*.

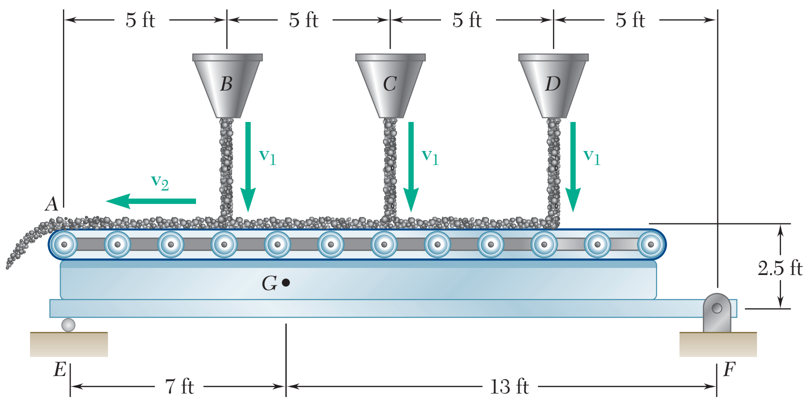


Fig. *P14.63*

14.64 The stream of water shown flows at a rate of 550 L/min and moves with a velocity of magnitude 18 m/s at both A and B. The vane is supported by a pin and bracket at C and by a load cell at D which can exert only a horizontal force. Neglecting the weight of the vane, determine the components of the reactions at C and D.

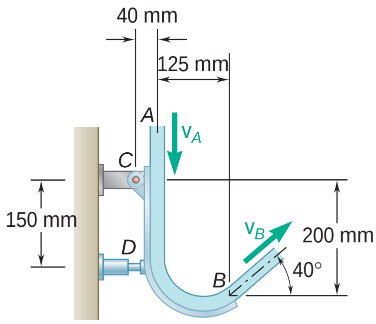


Fig. P14.64

14.65 The nozzle discharges water at the rate of 340 gal/min. Knowing the velocity of the water at both A and B has a magnitude of 65 ft/s and neglecting the weight of the vane, determine the components of the reactions at C and D (1 ft^3 = 7.48 gal).

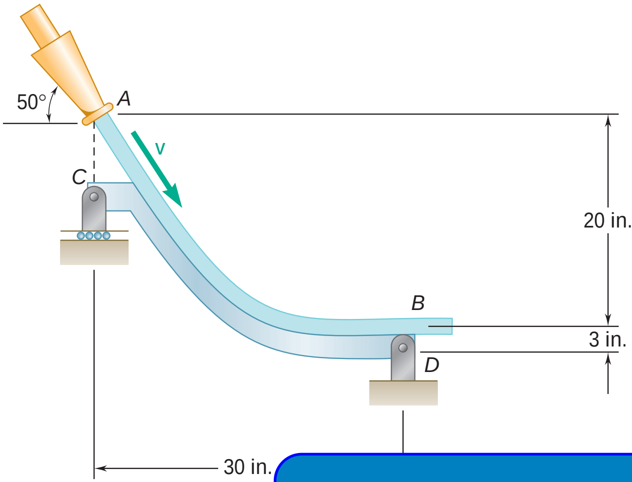


Fig. *P14.65*

14.66 A high-speed jet of air issues from nozzle A with a velocity of $\mathbf{v}_A$ and mass flow rate of 0.36 kg/s. The air impinges on a vane causing it to rotate to the position shown. The vane has a mass of 6 kg. Knowing that the magnitude of the air velocity is equal at A and B, determine (*a*) the magnitude of the velocity at A, (*b*) the components of the reactions at O.

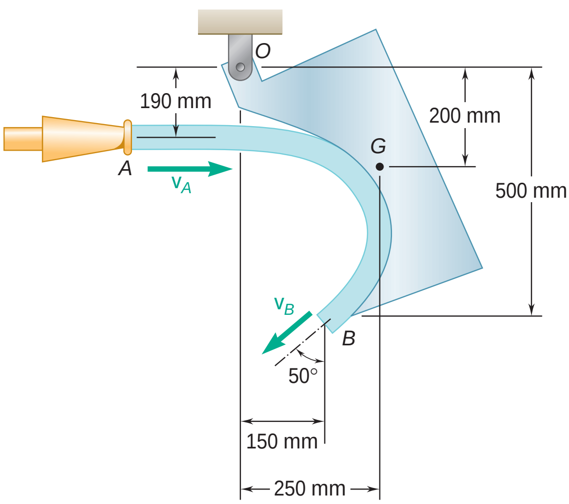


Fig. P14.66

14.67 Coal is being discharged from a first conveyor belt at the rate of 120 kg/s. It is received at A by a second belt which discharges it again at B. Knowing that $v_1 = 3$ m/s and $v_2 = 4.25$ m/s and that the second belt assembly and the coal it supports have a total mass of 472 kg, determine the components of the reactions at C and D.

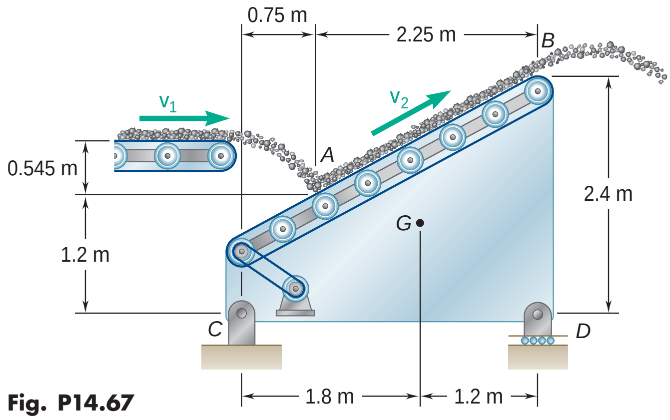


Fig. P14.67

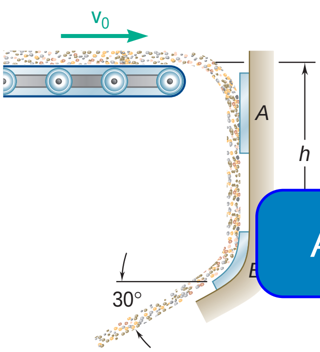


Fig. P14.68

14.68 A mass q of sand is discharged per unit time from a conveyor belt moving with a velocity $\mathbf{v}_0$. The sand is deflected by a plate at A so that it falls in a vertical stream. After falling a distance h the sand is again deflected by a curved plate at B. Neglecting the friction between the sand and the plates, determine the force required to hold in the position shown (*a*) plate A, (*b*) plate B.

a jet airplane traveling at 900 km/h ... st velocity is 600 m/s relative to the ... air which must pass through the engine per second to maintain the speed of 900 km/h in level flight.

14.70 While cruising in level flight at a speed of 600 mi/h, a jet plane scoops in air at the rate of 200 lb/s and discharges it with a velocity of 2100 ft/s relative to the airplane. Determine the total drag due to air friction on the airplane.

14.71 In order to shorten the distance required for landing, a jet airplane is equipped with movable vanes which partially reverse the direction of the air discharged by each of its engines. Each engine scoops in the air at a rate of 120 kg/s and discharges it with a velocity of 600 m/s relative to the engine. At an instant when the speed of the airplane is 270 km/h, determine the reverse thrust provided by each of the engines.

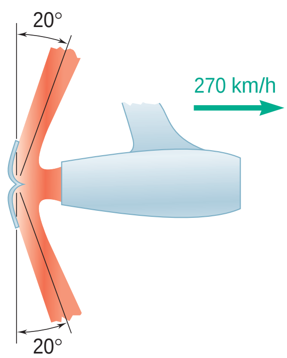


Fig. P14.71

14.72 The helicopter shown can produce a maximum downward air speed of 80 ft/s in a 30-ft-diameter slipstream. Knowing that the weight of the helicopter and its crew is 3500 lb and assuming g = 0.076 lb/ft^3 for air, determine the maximum load that the helicopter can lift while hovering in midair.

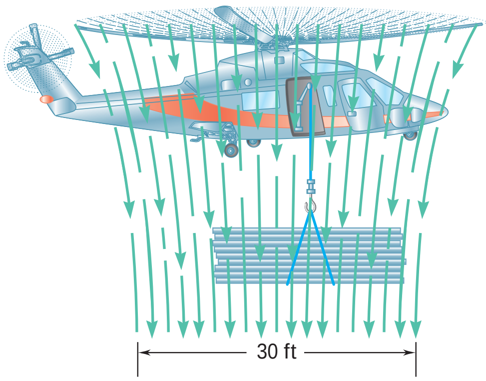


Fig. P14.72

14.73 A floor fan designed to deliver air at a maximum velocity of 6 m/s in a 400-mm-diameter slipstream is supported by a 200-mm-diameter circular base plate. Knowing that the total weight of the assembly is 60 N and that its center of gravity is located directly above the center of th[illegible] height h at which the fa[illegible] Assume r = 1.21 kg/m^3 [illegible] of the air.

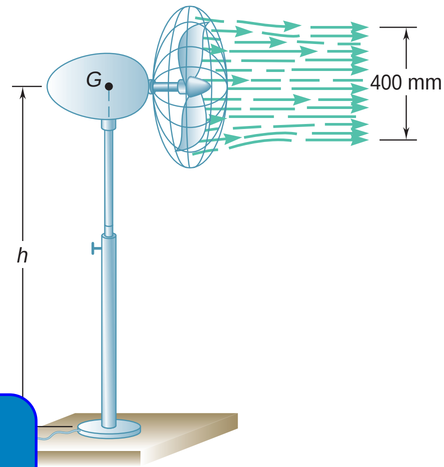


Fig. P14.73

14.74 The jet engine shown scoops in air at A at a rate of 200 lb/s and discharges it at B with a velocity of 2000 ft/s relative to the airplane. Determine the magnitude and line of action of the propulsive thrust developed by the engine when the speed of the airplane is (*a*) 300 mi/h, (*b*) 600 mi/h.

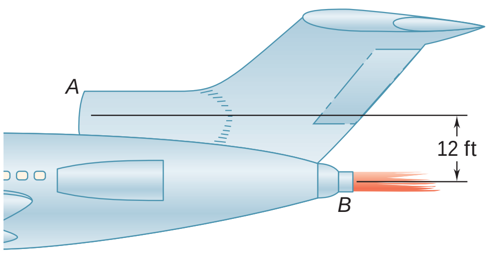


Fig. P14.74

14.75 A jet airliner is cruising at a speed of 900 km/h with each of its three engines discharging air with a velocity of 800 m/s relative to the plane. Determine the speed of the airliner after it has lost the use of (*a*) one of its engines, (*b*) two of its engines. Assume that the drag due to air friction is proportional to the square of the speed and that the remaining engines keep operating at the same rate.

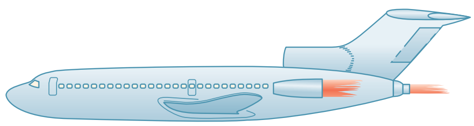


Fig. P14.75

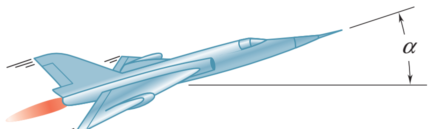


Fig. P14.76

14.76 A 16-Mg jet airplane maintains a constant speed of 774 km/h while climbing at an angle $\alpha = 18°$. The airplane scoops in air at a rate of 300 kg/s and discharges it with a velocity of 665 m/s relative to the airplane. If the pilot changes to a horizontal flight while maintaining the same engine setting, determine (*a*) the initial acceleration of the plane, (*b*) the maximum horizontal speed that will be attained. Assume that the drag due to air friction is proportional to the square of the speed.

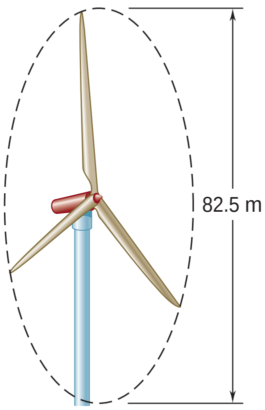


Fig. P14.78 and P14.79

14.77 The propeller of a small airplane has a 2-m-diameter slipstream and produces a thrust of 3600 N when the airplane is at rest on the ground. Assuming $\rho = 1.225$ kg/m^3 for air, determine (*a*) the speed of the air in the slipstream, (*b*) the volume of air passing through the propeller per second, (*c*) the kinetic energy imparted per second to the air in the slipstream.

14.78 The wind turbine–generator shown has an output-power rating of 1.5 MW for a wind speed of 36 km/h. For the given wind speed, determine (*a*) the kinetic energy of the air particles entering the 82.5-m-diameter circle per second, (*b*) the efficiency of this energy conversion system. Assume $\rho = 1.21$ kg/m^3 for air.

14.79 A wind turbine-generator system having a diameter of 82.5 m produces 1.5 MW at a wind speed of 12 m/s. Determine the diameter of blade necessary to produce 10 MW of power assuming the efficiency is the same for both designs and $\rho = 1.21$ kg/m^3 for air.

14.80 While cruising in level flight at a speed of 570 mi/h, a jet airplane … s and discharges it with a velocity … ne. Determine (*a*) the power actu- … (*b*) the total power developed by … efficiency of the airplane.

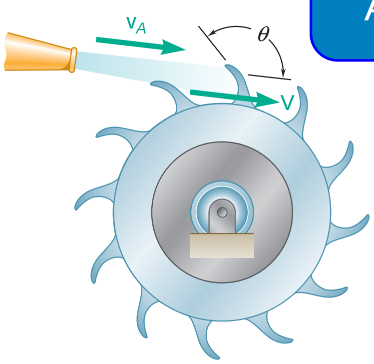


Fig. *P14.81*

14.81 In a Pelton-wheel turbine, a stream of water is deflected by a series of blades so that the rate at which water is deflected by the blades is equal to the rate at which water issues from the nozzle ($\Delta m/\Delta t = A\rho v_A$). Using the same notation as in Sample Prob. 14.7, (*a*) determine the velocity **V** of the blades for which maximum power is developed, (*b*) derive an expression for the maximum power, (*c*) derive an expression for the mechanical efficiency.

14.82 A circular reentrant orifice (also called Borda's mouthpiece) of diameter D is placed at a depth h below the surface of a tank. Knowing that the speed of the issuing stream is $v = \sqrt{2gh}$ and assuming that the speed of approach v_1 is zero, show that the diameter of the stream is $d = D/\sqrt{2}$. (*Hint:* Consider the section of water indicated, and note that P is equal to the pressure at a depth h multiplied by the area of the orifice.)

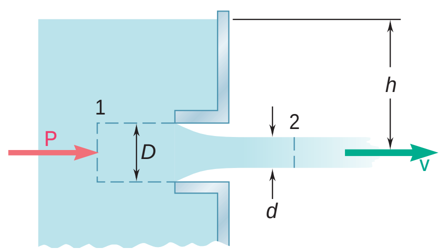


Fig. *P14.82*

14.83 Gravel falls with practically zero velocity onto a conveyor belt at the constant rate $q = dm/dt$. (*a*) Determine the magnitude of the force $\mathbf{P}$ required to maintain a constant belt speed v. (*b*) Show that the kinetic energy acquired by the gravel in a given time interval is equal to half the work done in that interval by the force $\mathbf{P}$. Explain what happens to the other half of the work done by $\mathbf{P}$.

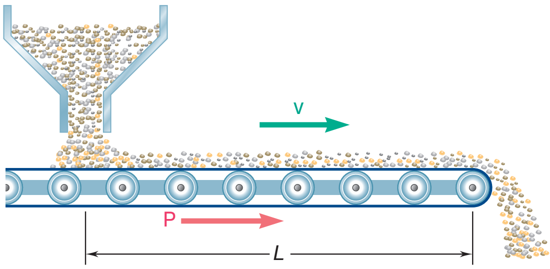


Fig. P14.83

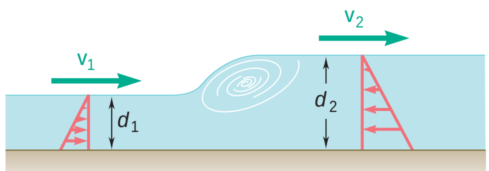


Fig. *P14.84*

***14.84** The depth of water flowing in a rectangular channel of width b at a speed v_1 and a depth d_1 increases to a depth d_2 at a *hydraulic jump*. Express the rate of flow Q in terms of b, d_1, and d_2.

***14.85** Determine the rate of flow in the channel of Prob. 14.84, knowing that $b = 12$ ft, $d_1 = 4$ ft [illegible]

14.86 A chain of length l and [illegible] end A is raised vertically [illegible] of the length y of chain which is off the floor at any given instant (*a*) the magnitude of the force $\mathbf{P}$ applied to A, (*b*) the reaction of the floor.

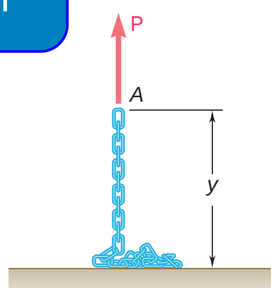


Fig. P14.86

14.87 Solve Prob. 14.86, assuming that the chain is being *lowered* to the floor at a constant speed v.

14.88 The ends of a chain lie in piles at A and C. When released from rest at time $t = 0$, the chain moves over the pulley at B, which has a negligible mass. Denoting by L the length of chain connecting the two piles and neglecting friction, determine the speed v of the chain at time t.

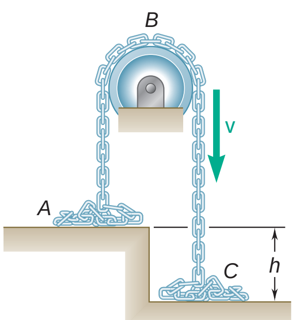


Fig. P14.88

14.89 A toy car is propelled by water that squirts from an internal tank at a constant 6 ft/s relative to the car. The weight of the empty car is 0.4 lb and it holds 2 lb of water. Neglecting other tangential forces, determine the top speed of the car.

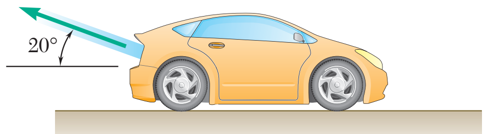


Fig. P14.89 and P14.90

14.90 A toy car is propelled by water that squirts from an internal tank. The weight of the empty car is 0.4 lb and it holds 2 lb of water. Knowing the top speed of the car is 8 ft/s determine the relative velocity of the water that is being ejected.

14.91 The main propulsion system of a space shuttle consists of three identical rocket engines which provide a total thrust of 6 MN. Determine the rate at which the hydrogen-oxygen propellant is burned by each of the three engines, knowing that it is ejected with a relative velocity of 3750 m/s.

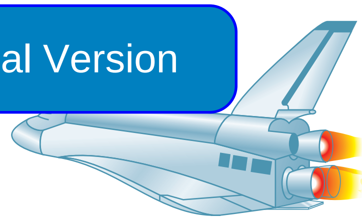


Fig. P14.91 and *P14.92*

14.92 The main propulsion system of a space shuttle consists of three identical rocket engines, each of which burns the hydrogen-oxygen propellant at the rate of 750 lb/s and ejects it with a relative velocity of 12,000 ft/s. Determine the total thrust provided by the three engines.

14.93 A rocket weighs 2600 lb, including 2200 lb of fuel, which is consumed at a rate of 25 lb/s and ejected with a relative velocity of 13,000 ft/s. Knowing that the rocket is fired vertically from the ground, determine its acceleration (*a*) as it is fired, (*b*) as the last particle of fuel is being consumed.

14.94 A space vehicle describing a circular orbit about the earth at a speed of 24×10^3 km/h releases at its front end a capsule which has a gross mass of 600 kg, including 400 kg of fuel. If the fuel is consumed at the rate of 18 kg/s and ejected with a relative velocity of 3000 m/s, determine (*a*) the tangential acceleration of the capsule as its engine is fired, (*b*) the maximum speed attained by the capsule.

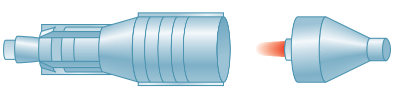


Fig. P14.94

14.95 A 540-kg spacecraft is mounted on top of a rocket with a mass of 19 Mg, including 17.8 Mg of fuel. Knowing that the fuel is consumed at a rate of 225 kg/s and ejected with a relative velocity of 3600 m/s, determine the maximum speed imparted to the spacecraft if the rocket is fired vertically from the ground.

Fig. P14.95

Fig. P14.96

14.96 The rocket used to launch the 540-kg spacecraft of Prob. 14.95 is redesigned to include t [illegible] including 8.9 Mg of fue [illegible] of 225 kg/s and ejecte [illegible] Knowing that when stag [illegible] its casing is released and jettisoned, determine (*a*) the speed of the rocket at that instant, (*b*) the maximum speed imparted to the spacecraft.

Fig. *P14.97*

14.97 A communications satellite weighing 10,000 lb, including fuel, has been ejected from a space shuttle describing a low circular orbit around the earth. After the satellite has slowly drifted to a safe distance from the shuttle, its engine is fired to increase its velocity by 8000 ft/s as a first step to its transfer to a geosynchronous orbit. Knowing that the fuel is ejected with a relative velocity of 13,750 ft/s, determine the weight of fuel consumed in this maneuver.

14.98 Determine the increase in velocity of the communications satellite of Prob. 14.97 after 2500 lb of fuel has been consumed.

14.99 Determine the distance separating the communications satellite of Prob. 14.97 from the space shuttle 60 s after its engine has been fired, knowing that the fuel is consumed at a rate of 37.5 lb/s.

14.100 For the rocket of Prob. 14.93, determine (*a*) the altitude at which all of the fuel has been consumed, (*b*) the velocity of the rocket at this time.

14.101 Determine the altitude reached by the spacecraft of Prob. 14.95 when all the fuel of its launching rocket has been consumed.

14.102 For the spacecraft and the two-stage launching rocket of Prob. 14.96, determine the altitude at which (*a*) stage A of the rocket is released, (*b*) the fuel of both stages has been consumed.

14.103 In a jet airplane, the kinetic energy imparted to the exhaust gases is wasted as far as propelling the airplane is concerned. The useful power is equal to the product of the force available to propel the airplane and the speed of the airplane. If v is the speed of the airplane and u is the relative speed of the expelled gases, show that the mechanical efficiency of the airplane is $\mathrm{h} = 2v/(u + v)$. Explain why $\mathrm{h} = 1$ when $u = v$.

14.104 In a rocket, the kinetic energy imparted to the consumed and ejected fuel is wasted as far as propelling the rocket is concerned. The useful power is equal to the product of the force available to propel the rocket and the speed of the rocket. If v is the speed of the rocket and u is the relative speed of the expelled fuel, show that the mechanical efficiency of the rocket is $\mathrm{h} = 2uv/(u^2 + v^2)$. Explain why $\mathrm{h} = 1$ when $u = v$.

REVIEW AND SUMMARY

In this chapter we analyzed the motion of *systems of particles,* i.e., the motion of a large number of particles considered together. In the first part of the chapter we considered systems consisting of well-defined particles, while in the second part we analyzed systems which are continually gaining or losing particles, or doing both at the same time.

Effective forces

We first defined the *effective force* of a particle P_i of a given system as the product $m_i\mathbf{a}_i$ of its mass m_i and its acceleration $\mathbf{a}_i$ with respect to a newtonian frame of reference centered at O [Sec. 14.2]. We then showed that *the system of the external forces acting on the particles and the system of the effective forces of the particles are equipollent;* i.e., both systems have the *same resultant* and the *same moment resultant* about O:

$$\sum_{i=1}^{n} \mathbf{F}_i = \sum_{i=1}^{n} m_i\mathbf{a}_i \qquad (14.4)$$

$$\sum_{i=1}^{n} (\mathbf{r}_i \times \mathbf{F}$$

Linear and angular momentum of a system of particles

Defining the *linear momentum* $\mathbf{L}$ and the *angular momentum* $\mathbf{H}_O$ *about point* O of the system of particles [Sec. 14.3] as

$$\mathbf{L} = \sum_{i=1}^{n} m_i\mathbf{v}_i \qquad \mathbf{H}_O = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i\mathbf{v}_i) \qquad (14.6,\ 14.7)$$

we showed that Eqs. (14.4) and (14.5) can be replaced by the equations

$$\Sigma\mathbf{F} = \dot{\mathbf{L}} \qquad \Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O \qquad (14.10,\ 14.11)$$

which express that *the resultant and the moment resultant about O of the external forces are, respectively, equal to the rates of change of the linear momentum and of the angular momentum about O of the system of particles*.

Motion of the mass center of a system of particles

In Sec. 14.4, we defined the mass center of a system of particles as the point G whose position vector $\bar{\mathbf{r}}$ satisfies the equation

$$m\bar{\mathbf{r}} = \sum_{i=1}^{n} m_i\mathbf{r}_i \qquad (14.12)$$

where m represents the total mass $\sum_{i=1}^{n} m_i$ of the particles. Differentiating both members of Eq. (14.12) twice with respect to t, we obtained the relations

$$\mathbf{L} = m\overline{\mathbf{v}} \qquad \dot{\mathbf{L}} = m\overline{\mathbf{a}} \qquad (14.14,\ 14.15)$$

where $\overline{\mathbf{v}}$ and $\overline{\mathbf{a}}$ represent, respectively, the velocity and the acceleration of the mass center G. Substituting for $\dot{\mathbf{L}}$ from (14.15) into (14.10), we obtained the equation

$$\Sigma\mathbf{F} = m\overline{\mathbf{a}} \qquad (14.16)$$

from which we concluded that *the mass center of a system of particles moves as if the entire mass of the system and all the external forces were concentrated at that point* [Sample Prob. 14.1].

Angular momentum of a system of particles about its mass center

In Sec. 14.5 we considered the motion of the particles of a system with respect to a centroidal frame $Gx'y'z'$ attached to the mass center G of the system and in translation with respect to the newtonian frame $Oxyz$ (Fig. 14.14). We defined the *angular momentum* of the system *about its mass center* G as the sum of the moments about G of the momenta $m_i\mathbf{v}'_i$ of the particles in their motion relative to the frame $Gx'y'z'$. We also noted that the same result can be obtained by considering the moments about G of the momenta $m_i\mathbf{v}_i$ of the [illegible] therefore wrote

$$[illegible] \sum_{i=1}^{n}(\mathbf{r}'_i \times m_i\mathbf{v}'_i) \qquad (14.24)$$

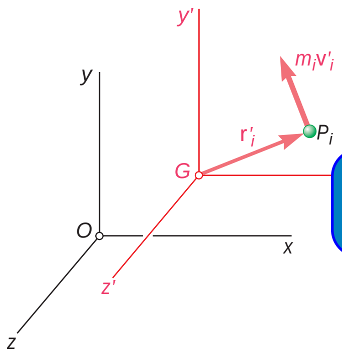


Fig. 14.14

and derived the relation

$$\Sigma\mathbf{M}_G = \dot{\mathbf{H}}_G \qquad (14.23)$$

which expresses that *the moment resultant about G of the external forces is equal to the rate of change of the angular momentum about G of the system of particles*. As will be seen later, this relation is fundamental to the study of the motion of rigid bodies.

Conservation of momentum

When no external force acts on a system of particles [Sec. 14.6], it follows from Eqs. (14.10) and (14.11) that the linear momentum $\mathbf{L}$ and the angular momentum $\mathbf{H}_O$ of the system are conserved [Sample Probs. 14.2 and 14.3]. In problems involving central forces, the angular momentum of the system about the center of force O will also be conserved.

Kinetic energy of a system of particles

The kinetic energy T of a system of particles was defined as the sum of the kinetic energies of the particles [Sec. 14.7]:

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 \qquad (14.28)$$

Using the centroidal frame of reference $Gx'y'z'$ of Fig. 14.14, we noted that the kinetic energy of the system can also be obtained by adding the kinetic energy $\frac{1}{2}m\bar{v}^2$ associated with the motion of the mass center G and the kinetic energy of the system in its motion relative to the frame $Gx'y'z'$:

$$T = \tfrac{1}{2}m\bar{v}^2 + \frac{1}{2}\sum_{i=1}^{n} m_i v_i'^2 \qquad (14.29)$$

Principle of work and energy

The *principle of work and energy* can be applied to a system of particles as well as to individual particles [Sec. 14.8]. We wrote

$$T_1 + U_{1\to 2} = T_2 \qquad (14.30)$$

and noted that $U_{1\to 2}$ represents the work of *all* the forces acting on the particles of the system, internal as well as external.

Conservation of energy

If all the forces acting on the particles of the system are *conservative,* we can determine the potential energy V of the system and write

$$T_1 + V_1 = T_2 + V_2 \qquad (14.31)$$

which expresses the *principle of conservation of energy* for a system of particles.

Principle of impulse and momentum

We saw in Sec. 14.9 that the *principle of impulse and momentum* for a system of particles can be expressed graphically as shown in Fig. 14.15. It states that the mo impulses of the external forces equipollent to the system of the

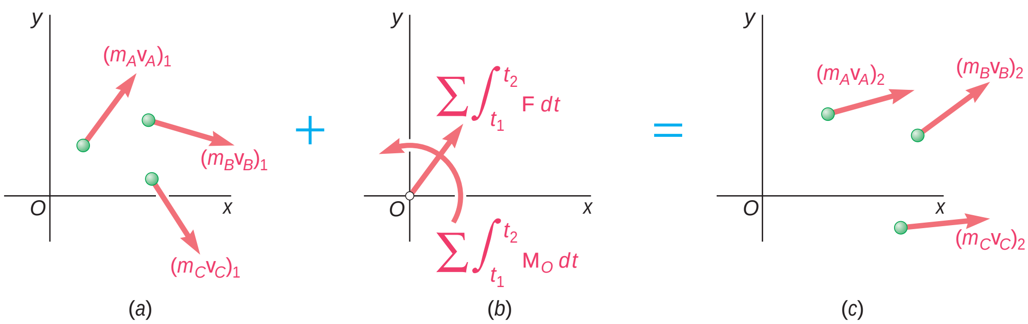


Fig. 14.15

If no external force acts on the particles of the system, the systems of momenta shown in parts *a* and *c* of Fig. 14.15 are equipollent and we have

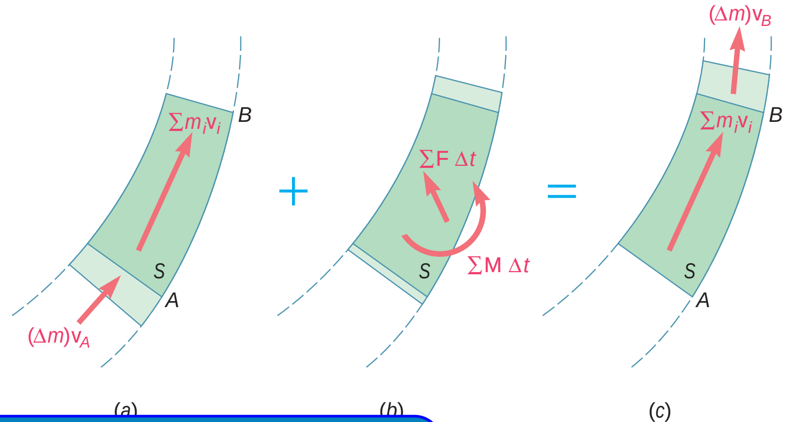
$$\mathbf{L}_1 = \mathbf{L}_2 \qquad (\mathbf{H}_O)_1 = (\mathbf{H}_O)_2 \qquad (14.36,\ 14.37)$$

Use of conservation principles in the solution of problems involving systems of particles

Many problems involving the motion of systems of particles can be solved by applying simultaneously the principle of impulse and momentum and the principle of conservation of energy [Sample Prob. 14.4] or by expressing that the linear momentum, angular momentum, and energy of the system are conserved [Sample Prob. 14.5].

Variable systems of particles
Steady stream of particles

In the second part of the chapter, we considered *variable systems of particles*. First we considered a *steady stream of particles,* such as a stream of water diverted by a fixed vane or the flow of air through a jet engine [Sec. 14.11]. Applying the principle of impulse and momentum to a system S of particles during a time interval Δt, and including the particles which enter the system at A during that time interval and those (of the same mass Δm) which leave the system at B, we concluded that *the system formed by the momentum* $(\Delta m)\mathbf{v}_A$ *of the particles entering* S *in the time* Δt *and the impulses of the forces exerted on* S *during that time is equipollent to the momentum* $(\Delta m)\mathbf{v}_B$ *of the particles leaving* S *in the same time* Δt (Fig. 14.16). Equating



l moments about a fixed point of the vectors involved, we could obtain as many as three equations, which could be solved for the desired unknowns [Sample Probs. 14.6 and 14.7]. From this result, we could also derive the following expression for the resultant $\Sigma\mathbf{F}$ of the forces exerted on S,

$$\Sigma\mathbf{F} = \frac{dm}{dt}(\mathbf{v}_B - \mathbf{v}_A) \qquad (14.39)$$

where $\mathbf{v}_B - \mathbf{v}_A$ represents the difference between the *vectors* $\mathbf{v}_B$ and $\mathbf{v}_A$ and where dm/dt is the mass rate of flow of the stream (see footnote, page 899).

Systems gaining or losing mass

Considering next a system of particles gaining mass by continually absorbing particles or losing mass by continually expelling particles [Sec. 14.12], as in the case of a rocket, we applied the principle of impulse and momentum to the system during a time interval Δt, being careful to include the particles gained or lost during that time interval [Sample Prob. 14.8]. We also noted that the action on a system S of the particles being *absorbed* by S was equivalent to a thrust

$$\mathbf{P} = \frac{dm}{dt}\mathbf{u} \qquad (14.44)$$

where dm/dt is the rate at which mass is being absorbed, and $\mathbf{u}$ is the velocity of the particles *relative to* S. In the case of particles being *expelled* by S, the rate dm/dt is negative and the thrust $\mathbf{P}$ is exerted in a direction opposite to that in which the particles are being expelled.

14.105 Three identical cars are being unloaded from an automobile carrier. Cars B and C have just been unloaded and are at rest with their brakes off when car A leaves the unloading ramp with a velocity of 5.76 ft/s and hits car B, which hits car C. Car A then again hits car B. Knowing that the velocity of car B is 5.04 ft/s after the first collision, 0.630 ft/s after the second collision, and 0.709 ft/s after the third collision, determine (*a*) the final velocities of cars A and C, (*b*) the coefficient of restitution for each of the collisions.

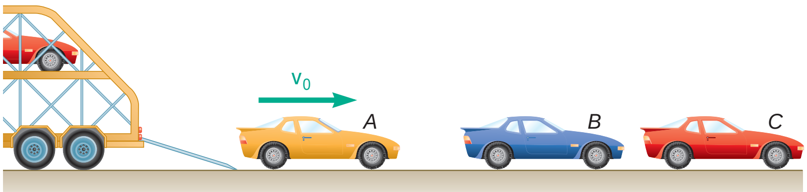


Fig. *P14.105*

14.106 A 30-g bullet is fired with a velocity of 480 m/s into block A, which has a mass of 5 kg. The coefficient of kinetic friction between block A and cart BC is 0.50. Knowing that the cart has a mass of 4 kg and can roll freely, deter[illegible] block, (*b*) the final positi[illegible]

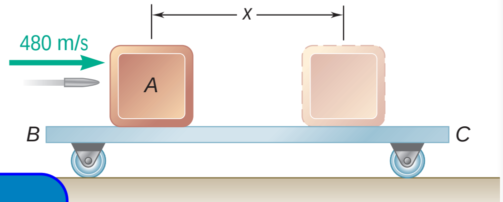


Fig. P14.106

14.107 An 80-Mg railroad engine A coasting at 6.5 km/h strikes a 20-Mg flatcar C carrying a 30-Mg load B which can slide along the floor of the car ($\mu_k = 0.25$). Knowing that the car was at rest with its brakes released and that it automatically coupled with the engine upon impact, determine the velocity of the car (*a*) immediately after impact, (*b*) after the load has slid to a stop relative to the car.

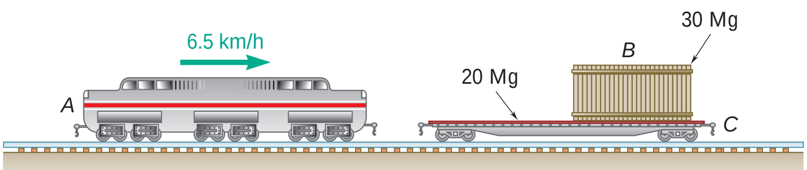


Fig. P14.107

14.108 In a game of pool, ball A is moving with a velocity $\mathbf{v}_0$ when it strikes balls B and C which are at rest and aligned as shown. Knowing that after the collision the three balls move in the directions indicated and that $v_0 = 12$ ft/s and $v_C = 6.29$ ft/s, determine the magnitude of the velocity of (*a*) ball A, (*b*) ball B.

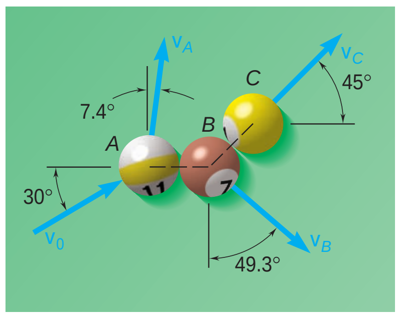


Fig. P14.108

14.109 Mass C, which has a mass of 4 kg, is suspended from a cord attached to cart A, which has a mass of 5 kg and can roll freely on a frictionless horizontal track. A 60-g bullet is fired with a speed $v_0 = 500$ m/s and gets lodged in block C. Determine (*a*) the velocity of C as it reaches its maximum elevation, (*b*) the maximum vertical distance h through which C will rise.

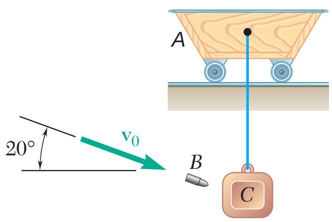


Fig. P14.109

14.110 A 15-lb block B is at rest and a spring of constant $k = 72$ lb/in is held compressed 3 in. by a cord. After 5-lb block A is placed against the end of the spring the cord is cut causing A and B to move. Neglecting friction, determine the velocities of blocks A and B immediately after A leaves B.

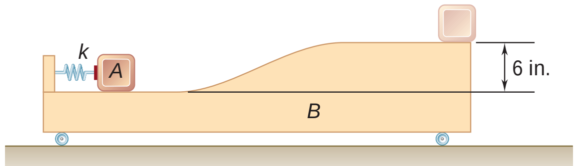


Fig. P14.110

14.111 Car A was at rest 9.28 m south of point O when it was struck in the rear by car B, which was traveling north at a speed v_B. Car C, which was traveling west at a speed v_C, was 40 m east of point O at the time of the collision. Cars A and B stuck together and, because the pavement was covered with ice, they slid into the intersection and were struck by car C which had not changed its speed. Measurements based on a photograph taken from a traffic helicopter shortly after the second collision indicated that the positions of the cars, expressed in meters, were $\mathbf{r}_A = -10.1\mathbf{i} + 16.9\mathbf{j}$, $\mathbf{r}_B = -10.1\mathbf{i} + 20.4\mathbf{j}$, and $\mathbf{r}_C = -19.8\mathbf{i} - 15.2\mathbf{j}$. Knowing that the masses of cars A, B, and C are, respectively, 1400 kg, 1800 kg, and 1600 kg, and that the time elapsed between the first collision and the time the photograph [illegible]he initial speeds of cars B and C.

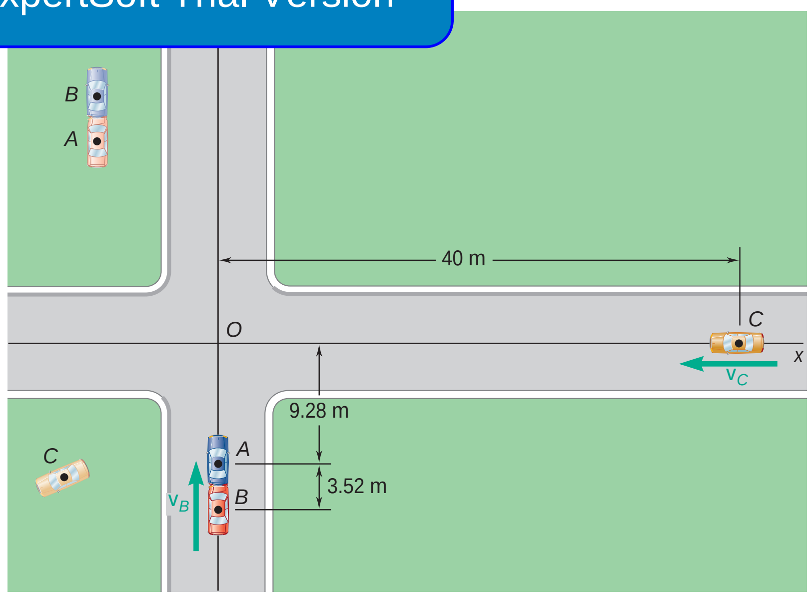


Fig. P14.111

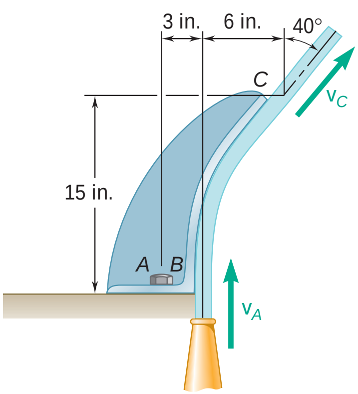


Fig. P14.112

14.112 The nozzle shown discharges water at the rate of 200 gal/min. Knowing that at both B and C the stream of water moves with a velocity of magnitude 100 ft/s, and neglecting the weight of the vane, determine the force-couple system which must be applied at A to hold the vane in place (1 ft^3 = 7.48 gal).

14.113 Prior to takeoff, the pilot of a 6000-lb twin-engine airplane tests the reversible-pitch propellers with the brakes at point B locked. Knowing that the velocity of the air in the two 6.6-ft-diameter slipstreams is 60 ft/s and that point G is the center of gravity of the airplane, determine the reactions at points A and B. Assume g $= 0.075$ lb/ft^3 and neglect the approach velocity of the air.

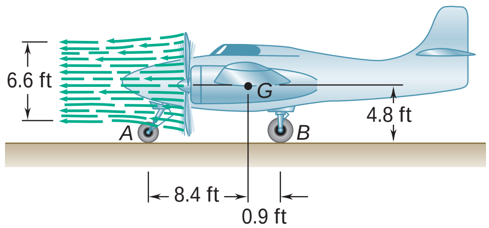


Fig. *P14.113*

14.114 A railroad car of length L and mass m_0 when empty is moving freely on a horizontal track while being loaded with sand from a stationary chute at a rate $dm/dt = q$. Knowing that the car was approaching the chute at a speed v_0, determine (*a*) the mass of the car and its load after the car has cleared the chute, (*b*) the speed of the car at that time.

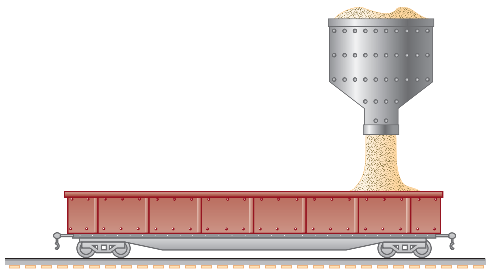


Fig. P14.114

14.115 A garden sprinkler has four rotating arms, each of which consists of two horizontal straight sections of pipe forming an angle of 120° with each other. Each arm discharges water at a rate of 20 L/min with a velocity of 18 m/s relative to the arm. Knowing that the friction between the moving and

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constant rate at which th

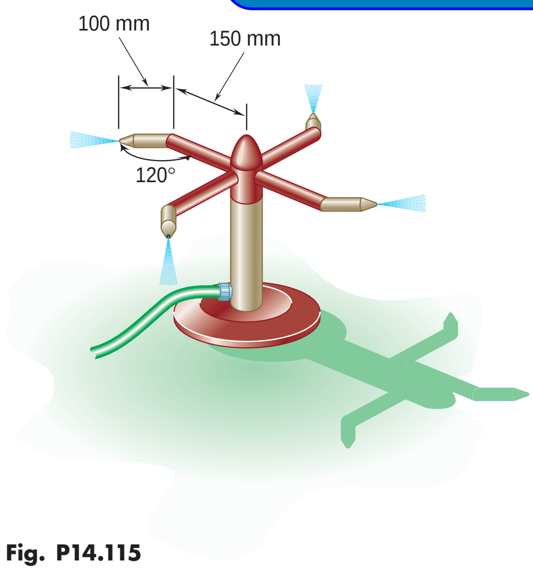


Fig. P14.115

14.116 A chain of length l and mass m falls through a small hole in a plate. Initially, when y is very small, the chain is at rest. In each case shown, determine (*a*) the acceleration of the first link A as a function of y, (*b*) the velocity of the chain as the last link passes through the hole. In case *1* assume that the individual links are at rest until they fall through the hole; in case *2* assume that at any instant all links have the same speed. Ignore the effect of friction.

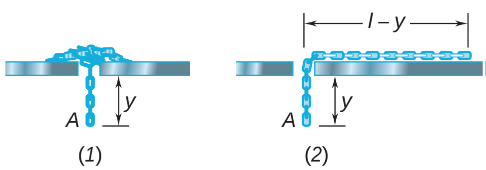


Fig. P14.116

COMPUTER PROBLEMS

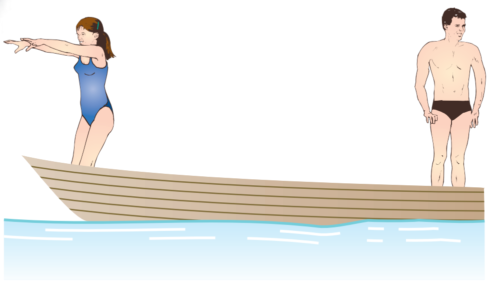


Fig. P14.C1

14.C1 A man and a woman, of weights 180 lb and 120 lb, respectively, stand at opposite ends of a stationary boat of weight 300 lb, ready to dive with velocities v_m and v_w, respectively, relative to the boat. Use computational software to determine the velocity of the boat after both swimmers have dived if (*a*) the woman dives first, (*b*) the man dives first. Solve that problem assuming that the velocities of the woman and the man relative to the boat are, respectively, (i) 14 ft/s and 18 ft/s, (ii) 18 ft/s and 14 ft/s.

14.C2 A system of particles consists of n particles A_i of mass m_i and coordinates x_i, y_i, and z_i, having velocities of components $(v_x)_i$, $(v_y)_i$, and $(v_z)_i$. Derive expressions for the components of the angular momentum of the system about the origin O of the coordinates. Use computational software to solve Probs. 14.11 and 14.13.

14.C3 A shell moving with a velocity of known components v_x, v_y, and v_z explodes into three fragments of weights W_1, W_2, and W_3 at point A_0 at a distance d from a vertical wall. Use computational software to determine the speed of each fragment immediately after the explosion, knowing the coordinates x_i and y_i of the points A_i (i = 1, 2, 3) where the fragments hit the wall. Use this program to solve (*a*) Prob. 14.24, (*b*) Prob. 14.25.

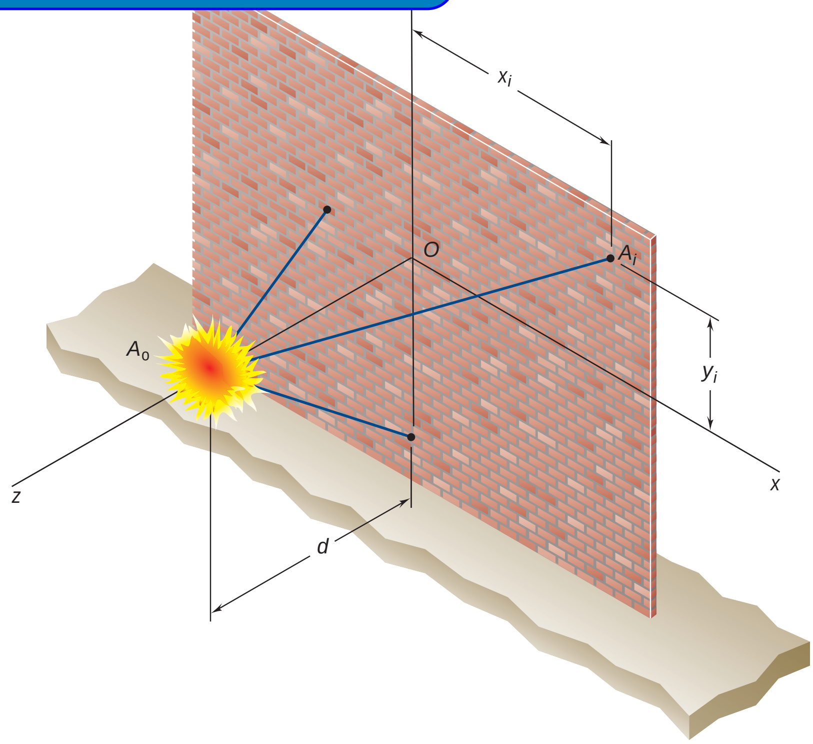


Fig. P14.C3

14.C4 As a 6000-kg training plane lands on an aircraft carrier at a speed of 180 km/h, its tail hooks into the end of an 80-m long chain which lies in a pile below deck. Knowing that the chain has a mass per unit length of 50 kg/m and assuming no other retarding force, use computational software to determine the distance traveled by the plane while the chain is being pulled out and the corresponding values of the time and of the velocity and deceleration of the plane.

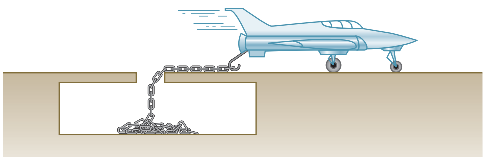


Fig. P14.C4

14.C5 A 16-Mg jet airplane maintains a constant speed of 774 km/h while climbing at an angle $\alpha = 18°$. The airplane scoops in air at a rate of 300 kg/s and discharges it with a velocity of 665 m/s relative to the airplane. Knowing that the pilot changes the angle of climb α while maintaining the same engine setting, use computational software to calculate and plot values of α from 0 to 20° (*a*) the initial acceleration of the plane, (*b*) the maximum speed that will be attained. Assume that the drag due to air friction is proportional to the square of the speed.

14.C6 A rocket has a weight of [illegible] is consumed at the rate of 25 lb [illegible] 12,000 ft/s. Knowing that the ro[illegible] assuming a constant value for the acceleration of gravity, and using 4-s time intervals, use computational software to determine and plot from the time of ignition to the time when the last particle of fuel is being consumed (*a*) the acceleration a of the rocket in ft/s^2, (*b*) its velocity v in ft/s, (*c*) its elevation h above the ground in miles. (*Hint:* Use for v the expression derived in Sample Prob. 14.8, and integrate this expression analytically to obtain h.)

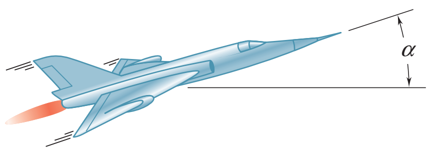


Fig. P14.C5

This huge crank belongs to a Wartsila-Sulzer RTA96-C turbocharged two-stroke diesel engine. In this chapter you will learn to perform the *kinematic* analysis of rigid bodies that undergo *translation, fixed axis rotation, and general plane motion.*

C H A P T E R 15

Kinematics of Rigid Bodies

Chapter 15 Kinematics of Rigid Bodies

15.1 INTRODUCTION

In this chapter, the kinematics of *rigid bodies* will be considered. You will investigate the relations existing between the time, the positions, the velocities, and the accelerations of the various particles forming a rigid body. As you will see, the various types of rigid-body motion can be conveniently grouped as follows:

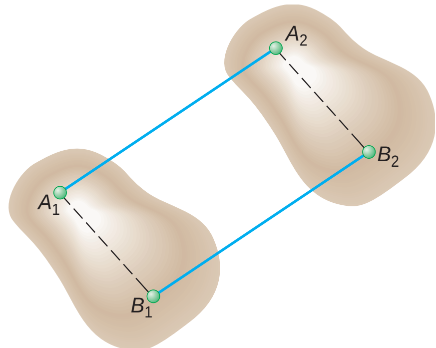


Fig. 15.1

1. *Translation*. A motion is said to be a translation if any straight line inside the body keeps the same direction during the motion. It can also be observed that in a translation all the particles forming the body move along parallel paths. If these paths are [illegible] to be a *rectilinear translation* [illegible]ed lines, the motion is a *curvi-*

2. *Rotation About a Fixed Axis*. In this motion, the particles forming the rigid body move in parallel planes along circles centered on the same fixed axis (Fig. 15.3). If this axis, called the *axis of rotation*, intersects the rigid body, the particles located on the axis have zero velocity and zero acceleration.

Rotation should not be confused with certain types of curvilinear translation. For example, the plate shown in Fig. 15.4*a* is in curvilinear translation, with all its particles moving along *parallel* circles, while the plate shown in Fig. 15.4*b* is in rotation, with all its particles moving along *concentric* circles.

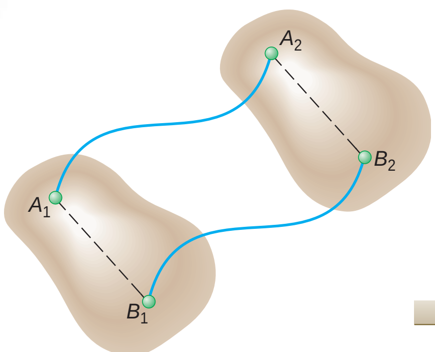


Fig. 15.2

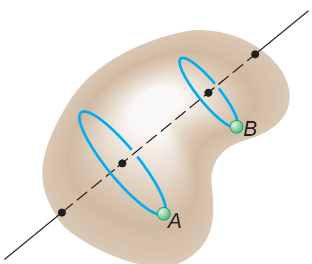
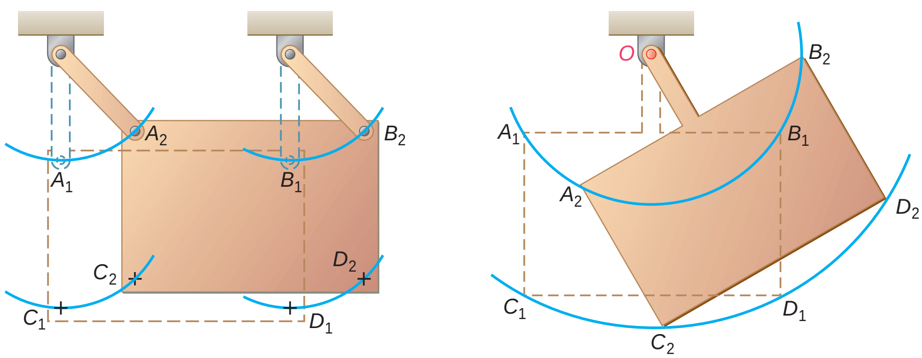


Fig. 15.3



(*a*) Curvilinear translation

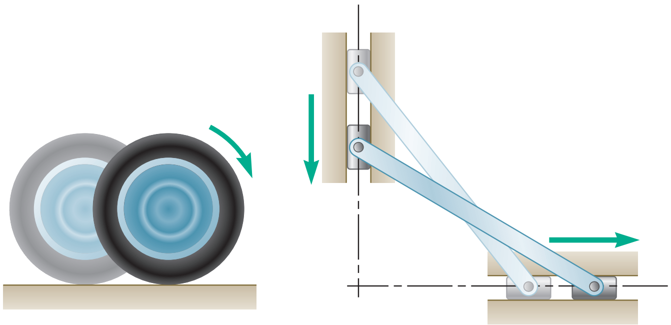
(*b*) Rotation

Fig. 15.4

In the first case, any given straight line drawn on the plate will maintain the same direction, whereas in the second case, point O remains fixed.

Because each particle moves in a given plane, the rotation of a body about a fixed axis is said to be a *plane motion*.

3. *General Plane Motion*. There are many other types of plane motion, i.e., motions in which all the particles of the body move in parallel planes. Any plane motion that is neither a rotation nor a translation is referred to as a general plane motion. Two examples of general plane motion are given in Fig. 15.5.



(*a*) Rolling wheel (*b*) Sliding rod

Fig. 15.5

4. *Motion About a Fixed Po* a rigid body attached at a fixed point O, e.g., the motion of a top on a rough floor (Fig. 15.6), is known as motion about a fixed point.

5. *General Motion*. Any motion of a rigid body that does not fall in any of the categories above is referred to as a general motion.

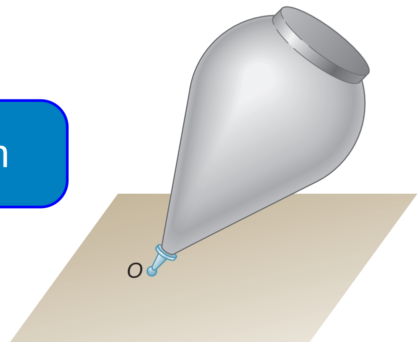


Fig. 15.6

After a brief discussion in Sec. 15.2 of the motion of translation, the rotation of a rigid body about a fixed axis is considered in Sec. 15.3. The *angular velocity* and the *angular acceleration* of a rigid body about a fixed axis will be defined, and you will learn to express the velocity and the acceleration of a given point of the body in terms of its position vector and the angular velocity and angular acceleration of the body.

The following sections are devoted to the study of the general plane motion of a rigid body and to its application to the analysis of mechanisms such as gears, connecting rods, and pin-connected linkages. Resolving the plane motion of a slab into a translation and a rotation (Secs. 15.5 and 15.6), we will then express the velocity of a point B of the slab as the sum of the velocity of a reference point A and of the velocity of B relative to a frame of reference translating with A (i.e., moving with A but not rotating). The same approach is used later in Sec. 15.8 to express the acceleration of B in terms of the acceleration of A and of the acceleration of B relative to a frame translating with A.

An alternative method for the analysis of velocities in plane motion, based on the concept of *instantaneous center of rotation,* is given in Sec. 15.7; and still another method of analysis, based on the use of parametric expressions for the coordinates of a given point, is presented in Sec. 15.9.

The motion of a particle relative to a rotating frame of reference and the concept of *Coriolis acceleration* are discussed in Secs. 15.10 and 15.11, and the results obtained are applied to the analysis of the plane motion of mechanisms containing parts which slide on each other.

The remaining part of the chapter is devoted to the analysis of the three-dimensional motion of a rigid body, namely, the motion of a rigid body with a fixed point and the general motion of a rigid body. In Secs. 15.12 and 15.13, a fixed frame of reference or a frame of reference in translation will be used to carry out this analysis; in Secs. 15.14 and 15.15, the motion of the body relative to a rotating frame or to a frame in general motion will be considered, and the concept of Coriolis acceleration will again be used.

Photo 15.1 This replica of a battering ram at *Château des Baux, France* undergoes curvilinear translation.

15.2 TRANSLATION

Consider a rigid body in translation (either rectilinear or curvilinear ... two of its particles (Fig. 15.7*a*). ... he position vectors of A and B ... erence and by $\mathbf{r}_{B/A}$ the vector ...

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \qquad (15.1)$$

Let us differentiate this relation with respect to t. We note that from the very definition of a translation, the vector $\mathbf{r}_{B/A}$ must maintain a constant direction; its magnitude must also be constant, since A and B

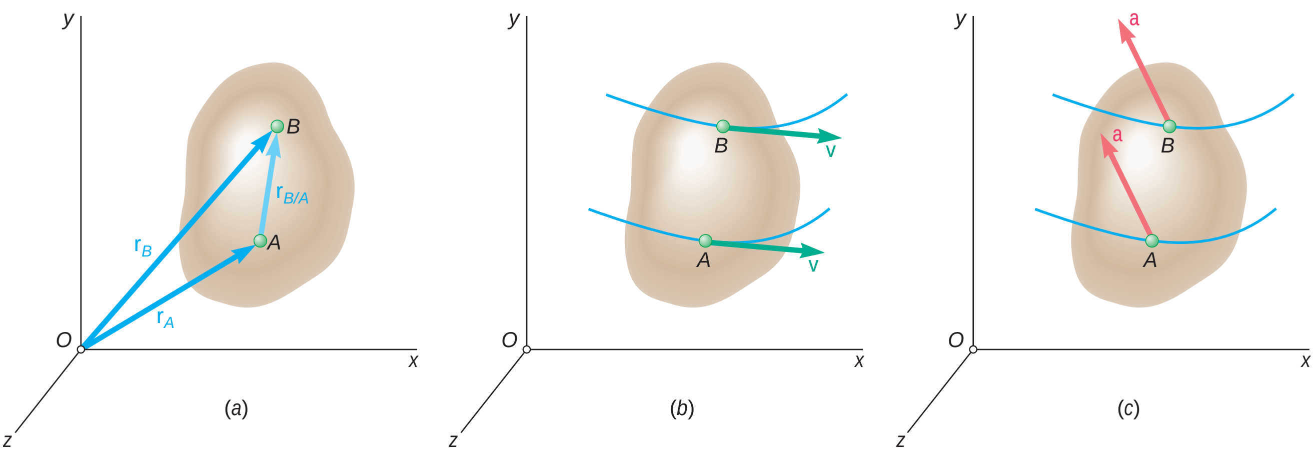


Fig. 15.7

belong to the same rigid body. Thus, the derivative of $\mathbf{r}_{B/A}$ is zero and we have

$$\mathbf{v}_B = \mathbf{v}_A \tag{15.2}$$

Differentiating once more, we write

$$\mathbf{a}_B = \mathbf{a}_A \tag{15.3}$$

Thus, *when a rigid body is in translation, all the points of the body have the same velocity and the same acceleration at any given instant* (Fig. 15.7*b* and *c*). In the case of curvilinear translation, the velocity and acceleration change in direction as well as in magnitude at every instant. In the case of rectilinear translation, all particles of the body move along parallel straight lines, and their velocity and acceleration keep the same direction during the entire motion.

15.3 ROTATION ABOUT A FIXED AXIS

Consider a rigid body which rotates about a fixed axis AA'. Let P be a point of the body and $\mathbf{r}$ its position vector with respect to a fixed frame of reference. For convenience, let us assume that the frame is centered at point O on AA' a [illegible] (Fig. 15.8). Let B be the projec [illegible] at a constant distance from B, i [illegible] of radius $r \sin \phi$, where ϕ denotes the angle formed by $\mathbf{r}$ and AA'.

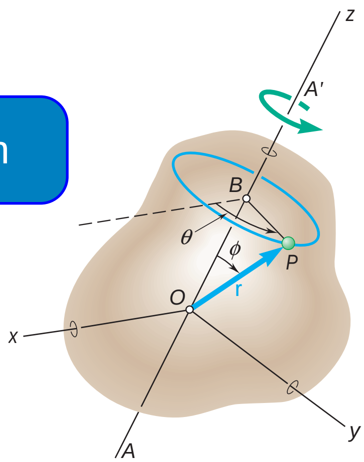


Fig. 15.8

The position of P and of the entire body is completely defined by the angle θ the line BP forms with the zx plane. The angle θ is known as the *angular coordinate* of the body and is defined as positive when viewed as counterclockwise from A'. The angular coordinate will be expressed in radians (rad) or, occasionally, in degrees (°) or revolutions (rev). We recall that

$$1 \text{ rev} = 2\pi \text{ rad} = 360°$$

We recall from Sec. 11.9 that the velocity $\mathbf{v} = d\mathbf{r}/dt$ of a particle P is a vector tangent to the path of P and of magnitude $v = ds/dt$. Observing that the length Δs of the arc described by P when the body rotates through $\Delta\theta$ is

$$\Delta s = (BP)\,\Delta\theta = (r \sin \phi)\,\Delta\theta$$

and dividing both members by Δt, we obtain at the limit, as Δt approaches zero,

$$v = \frac{ds}{dt} = r\dot{\theta} \sin \phi \tag{15.4}$$

where $\dot{\theta}$ denotes the time derivative of θ. (Note that the angle θ depends on the position of P within the body, but the rate of change $\dot{\theta}$ is itself independent of P.) We conclude that the velocity $\mathbf{v}$ of P is a vector perpendicular to the plane containing AA' and $\mathbf{r}$, and of

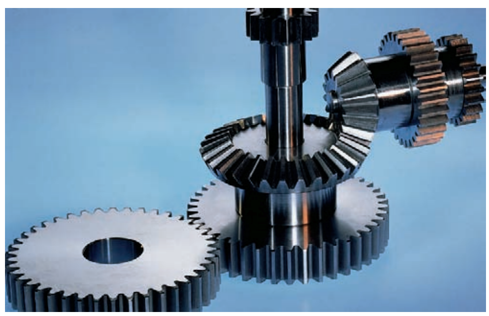


Photo 15.2 For the central gear rotating about a fixed axis, the angular velocity and angular acceleration of that gear are vectors directed along the vertical axis of rotation.

magnitude v defined by (15.4). But this is precisely the result we would obtain if we drew along AA' a vector $\mathsf{V} = \dot{\mathsf{u}}\mathbf{k}$ and formed the vector product $\mathsf{V} \times \mathbf{r}$ (Fig. 15.9). We thus write

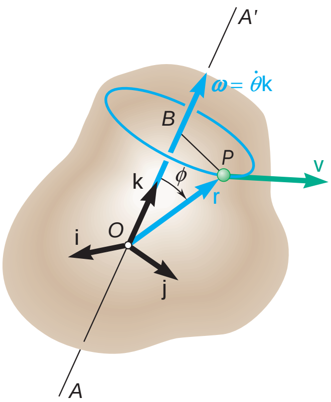


Fig. 15.9

$$\boxed{\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathsf{V} \times \mathbf{r}} \tag{15.5}$$

The vector

$$\mathsf{V} = \mathsf{v}\mathbf{k} = \dot{\mathsf{u}}\mathbf{k} \tag{15.6}$$

which is directed along the axis of rotation, is called the *angular velocity* of the body and is equal in magnitude to the rate of change $\dot{\mathsf{u}}$ of the angular coordinate; its sense may be obtained by the right-hand rule (Sec. 3.6) from the sense of rotation of the body.†

The acceleration $\mathbf{a}$ of the particle P will now be determined. Differentiating (15.5) and recalling the rule for the differentiation of a vector product (Sec. 11.10), we write

$$\begin{aligned} \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathsf{V} \times \mathbf{r}) \\ &= \frac{d\mathsf{V}}{dt} \times \mathbf{r} + \mathsf{V} \times \frac{d\mathbf{r}}{dt} \\ &= \frac{d\mathsf{V}}{dt} \times \mathbf{r} + \mathsf{V} \times \mathbf{v} \end{aligned} \tag{15.7}$$

is called the *angular acceleration* of the body. Substituting also for $\mathbf{v}$ from (15.5), we have

$$\boxed{\mathbf{a} = \mathsf{A} \times \mathbf{r} + \mathsf{V} \times (\mathsf{V} \times \mathbf{r})} \tag{15.8}$$

Differentiating (15.6) and recalling that $\mathbf{k}$ is constant in magnitude and direction, we have

$$\mathsf{A} = \mathsf{a}\mathbf{k} = \dot{\mathsf{v}}\mathbf{k} = \ddot{\mathsf{u}}\mathbf{k} \tag{15.9}$$

Thus, the angular acceleration of a body rotating about a fixed axis is a vector directed along the axis of rotation, and is equal in magnitude to the rate of change $\dot{\mathsf{v}}$ of the angular velocity. Returning to (15.8), we note that the acceleration of P is the sum of two vectors. The first vector is equal to the vector product $\mathsf{A} \times \mathbf{r}$; it is tangent to the circle described by P and therefore represents the tangential component of the acceleration. The second vector is equal to the *vector triple product* $\mathsf{V} \times (\mathsf{V} \times \mathbf{r})$ obtained by forming the vector product of V and $\mathsf{V} \times \mathbf{r}$; since $\mathsf{V} \times \mathbf{r}$ is tangent to the circle described by P, the vector triple product is directed toward the center B of the circle and therefore represents the normal component of the acceleration.

†It will be shown in Sec. 15.12 in the more general case of a rigid body rotating simultaneously about axes having different directions that angular velocities obey the parallelogram law of addition and thus are actually vector quantities.

Rotation of a Representative Slab. The rotation of a rigid body about a fixed axis can be defined by the motion of a representative slab in a reference plane perpendicular to the axis of rotation. Let us choose the xy plane as the reference plane and assume that it coincides with the plane of the figure, with the z axis pointing out of the paper (Fig. 15.10). Recalling from (15.6) that $\mathsf{V} = \mathsf{v}\mathbf{k}$, we

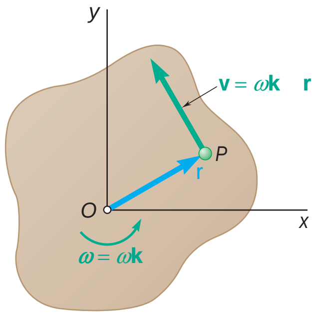


Fig. 15.10

note that a positive value of the scalar v corresponds to a counterclockwise rotation of the representative slab, and a negative value to a clockwise rotation. Substituting $\mathsf{v}\mathbf{k}$ for V into Eq. (15.5), we express the velocity of any given point P of the slab as

$$\mathbf{v} =$$

Since the vectors $\mathbf{k}$ and $\mathbf{r}$ are mutually perpendicular, the magnitude of the velocity $\mathbf{v}$ is

$$v = r\mathsf{v} \qquad (15.10')$$

and its direction can be obtained by rotating $\mathbf{r}$ through 90° in the sense of rotation of the slab.

Substituting $\mathsf{V} = \mathsf{v}\mathbf{k}$ and $\mathsf{A} = \mathsf{a}\mathbf{k}$ into Eq. (15.8), and observing that cross-multiplying $\mathbf{r}$ twice by $\mathbf{k}$ results in a 180° rotation of the vector $\mathbf{r}$, we express the acceleration of point P as

$$\mathbf{a} = \mathsf{a}\mathbf{k} \times \mathbf{r} - \mathsf{v}^2\mathbf{r} \qquad (15.11)$$

Resolving $\mathbf{a}$ into tangential and normal components (Fig. 15.11), we write

$$\begin{aligned} \mathbf{a}_t &= \mathsf{a}\mathbf{k} \times \mathbf{r} \qquad & a_t &= r\mathsf{a} \\ \mathbf{a}_n &= -\mathsf{v}^2\mathbf{r} \qquad & a_n &= r\mathsf{v}^2 \end{aligned} \qquad (15.11')$$

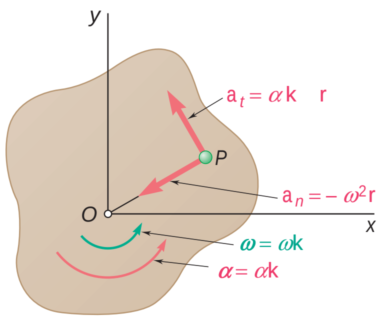


Fig. 15.11

The tangential component $\mathbf{a}_t$ points in the counterclockwise direction if the scalar a is positive, and in the clockwise direction if a is negative. The normal component $\mathbf{a}_n$ always points in the direction opposite to that of $\mathbf{r}$, that is, toward O.

Photo 15.3 If the lower roll has a consta angular velocity, the speed of the paper be wound onto it increases as the radius of the increases.

15.4 EQUATIONS DEFINING THE ROTATION OF A RIGID BODY ABOUT A FIXED AXIS

The motion of a rigid body rotating about a fixed axis AA' is said to be *known* when its angular coordinate θ can be expressed as a known function of t. In practice, however, the rotation of a rigid body is seldom defined by a relation between θ and t. More often, the conditions of motion will be specified by the type of angular acceleration that the body possesses. For example, α may be given as a function of t, as a function of θ, or as a function of ω. Recalling the relations (15.6) and (15.9), we write

$$\omega = \frac{d\theta}{dt} \tag{15.12}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{15.13}$$

or, solving (15.12) for dt and substituting into (15.13),

$$\alpha = \omega\frac{d\omega}{d\theta} \tag{15.14}$$

those obtained in Chap. 11 for the rectilinear motion of a particle, their integration can be performed by following the procedure outlined in Sec. 11.3.

Two particular cases of rotation are frequently encountered:

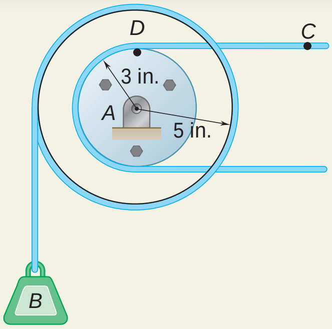
1. *Uniform Rotation.* This case is characterized by the fact that the angular acceleration is zero. The angular velocity is thus constant, and the angular coordinate is given by the formula

$$\theta = \theta_0 + \omega t \tag{15.15}$$

2. *Uniformly Accelerated Rotation.* In this case, the angular acceleration is constant. The following formulas relating angular velocity, angular coordinate, and time can then be derived in a manner similar to that described in Sec. 11.5. The similarity between the formulas derived here and those obtained for the rectilinear uniformly accelerated motion of a particle is apparent.

$$\begin{aligned} \omega &= \omega_0 + \alpha t \\ \theta &= \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2 \\ \omega^2 &= \omega_0^2 + 2\alpha(\theta - \theta_0) \end{aligned} \tag{15.16}$$

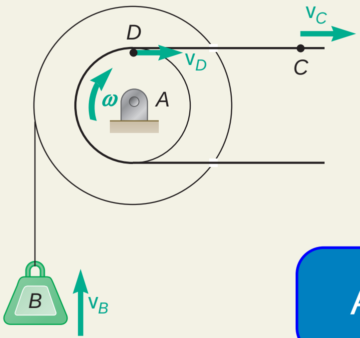
It should be emphasized that formula (15.15) can be used only when $\alpha = 0$, and formulas (15.16) can be used only when $\alpha =$ constant. In any other case, the general formulas (15.12) to (15.14) should be used.



SAMPLE PROBLEM 15.1

Load B is connected to a double pulley by one of the two inextensible cables shown. The motion of the pulley is controlled by cable C, which has a constant acceleration of 9 in./s^2 and an initial velocity of 12 in./s, both directed to the right. Determine (*a*) the number of revolutions executed by the pulley in 2 s, (*b*) the velocity and change in position of the load B after 2 s, and (*c*) the acceleration of point D on the rim of the inner pulley at $t = 0$.

SOLUTION



a. Motion of Pulley. Since the cable is inextensible, the velocity of point D is equal to the velocity of point C and the tangential component of the acceleration of D is equal to the acceleration of C.

$$(\mathbf{v}_D)_0 = (\mathbf{v}_C)_0 = 12 \text{ in./s y} \qquad (\mathbf{a}_D)_t = \mathbf{a}_C = 9 \text{ in./s}^2 \text{ y}$$

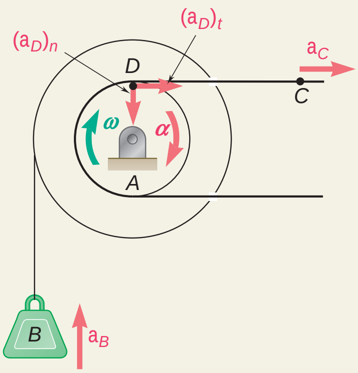
Noting that the distance from D to the center of the pulley is 3 in., we write

$$(v_D)_0 = r\text{V}_0 \qquad 12 \text{ in./s} = (3 \text{ in.})\text{V}_0 \qquad \text{V}_0 = 4 \text{ rad/s i}$$
$$(a_D)_t = r\text{a} \qquad 9 \text{ in./s}^2 = (3 \text{ in.})\text{a} \qquad \text{A} = 3 \text{ rad/s}^2 \text{ i}$$

...rated motion, we obtain, for $t = 2$ s,

$$\ldots(3 \text{ rad/s}^2)(2 \text{ s}) = 10 \text{ rad/s}$$
$$\text{V} = 10 \text{ rad/s i}$$
$$\text{u} = \text{V}_0 t + \tfrac{1}{2}\text{a}t^2 = (4 \text{ rad/s})(2 \text{ s}) + \tfrac{1}{2}(3 \text{ rad/s}^2)(2 \text{ s})^2 = 14 \text{ rad}$$
$$\text{u} = 14 \text{ rad i}$$

$$\text{Number of revolutions} = (14 \text{ rad})\left(\frac{1 \text{ rev}}{2\text{p rad}}\right) = 2.23 \text{ rev}$$ ◀



b. Motion of Load *B*. Using the following relations between linear and angular motion, with $r = 5$ in., we write

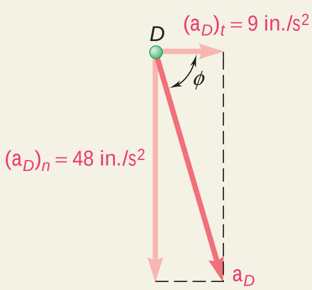
$$v_B = r\text{V} = (5 \text{ in.})(10 \text{ rad/s}) = 50 \text{ in./s} \qquad \mathbf{v}_B = 50 \text{ in./s x}$$ ◀
$$\Delta y_B = r\text{u} = (5 \text{ in.})(14 \text{ rad}) = 70 \text{ in.} \qquad \Delta y_B = 70 \text{ in. upward}$$ ◀

c. Acceleration of Point *D* at *t* = 0. The tangential component of the acceleration is

$$(\mathbf{a}_D)_t = \mathbf{a}_C = 9 \text{ in./s}^2 \text{ y}$$

Since, at $t = 0$, $\text{V}_0 = 4$ rad/s, the normal component of the acceleration is

$$(a_D)_n = r_D\text{V}_0^2 = (3 \text{ in.})(4 \text{ rad/s})^2 = 48 \text{ in./s}^2 \qquad (\mathbf{a}_D)_n = 48 \text{ in./s}^2 \text{ w}$$



The magnitude and direction of the total acceleration can be obtained by writing

$$\tan \text{f} = (48 \text{ in./s}^2)/(9 \text{ in./s}^2) \qquad \text{f} = 79.4°$$
$$a_D \sin 79.4° = 48 \text{ in./s}^2 \qquad a_D = 48.8 \text{ in./s}^2$$
$$\mathbf{a}_D = 48.8 \text{ in./s}^2 \text{ c } 79.4°$$ ◀

SOLVING PROBLEMS ON YOUR OWN

In this lesson we began the study of the motion of rigid bodies by considering two particular types of motion of rigid bodies: *translation* and *rotation* about a *fixed axis*.

1. Rigid body in translation. At any given instant, all the points of a rigid body in translation have the *same velocity* and the *same acceleration* (Fig. 15.7).

2. Rigid body rotating about a fixed axis. The position of a rigid body rotating about a fixed axis was defined at any given instant by the *angular coordinate* θ, which is usually measured in *radians*. Selecting the unit vector $\mathbf{k}$ along the fixed axis and in such a way that the rotation of the body appears counterclockwise as seen from the tip of $\mathbf{k}$, we defined the *angular velocity* $\boldsymbol{\omega}$ and the *angular acceleration* $\boldsymbol{\alpha}$ of the body:

$$\boldsymbol{\omega} = \dot{\theta}\mathbf{k} \qquad \boldsymbol{\alpha} = \ddot{\theta}\mathbf{k} \tag{15.6, 15.9}$$

In solving problems, keep in mind that the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are both directed along the fixed axis of rotation and that their sense can be obtained by the right-hand rule.

a. The velocity of a point *P* of a body rotating about a fixed axis was found to be

(15.5)

where $\boldsymbol{\omega}$ is the angular velocity of the body and $\mathbf{r}$ is the position vector drawn from any point on the axis of rotation to point P (Fig. 15.9).

b. The acceleration of point *P* was found to be

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{15.8}$$

Since vector products are not commutative, *be sure to write the vectors in the order shown* when using either of the above two equations.

3. Rotation of a representative slab. In many problems, you will be able to reduce the analysis of the rotation of a three-dimensional body about a fixed axis to the study of the rotation of a representative slab in a plane perpendicular to the fixed axis. The z axis should be directed along the axis of rotation and point out of the paper. Thus, the representative slab will be rotating in the xy plane about the origin O of the coordinate system (Fig. 15.10).

To solve problems of this type you should do the following:

a. Draw a diagram of the representative slab, showing its dimensions, its angular velocity and angular acceleration, as well as the vectors representing the velocities and accelerations of the points of the slab for which you have or seek information.

b. Relate the rotation of the slab and the motion of points of the slab by writing the equations

$$v = r\mathrm{V} \qquad (15.10')$$

$$a_t = r\mathrm{a} \qquad a_n = r\mathrm{V}^2 \qquad (15.11')$$

Remember that the velocity $\mathbf{v}$ and the component $\mathbf{a}_t$ of the acceleration of a point P of the slab are tangent to the circular path described by P. The directions of $\mathbf{v}$ and $\mathbf{a}_t$ are found by rotating the position vector $\mathbf{r}$ through 90° in the sense indicated by V and A, respectively. The normal component $\mathbf{a}_n$ of the acceleration of P is always directed toward the axis of rotation.

4. Equations defining the rotation of a rigid body. You must have been pleased to note the similarity existing between the equations defining the rotation of a rigid body about a fixed axis [Eqs. (15.12) through (15.16)] and those in Chap. 11 defining the rectilinear motion of a particle [Eqs. (11.1) through (11.8)]. All you have to do to obtain the new set of equations is to substitute u, V, and a for x, v, and a in the equations of Chap. 11.

PROBLEMS

CONCEPT QUESTIONS

15.CQ1 A rectangular plate swings from arms of equal length as shown. What is the magnitude of the angular velocity of the plate?

a. 0 rad/s
b. 1 rad/s
c. 2 rad/s
d. 3 rad/s
e. Need to know the location of the center of gravity.

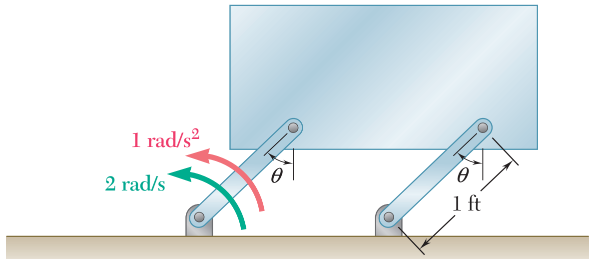


Fig. P15.CQ1

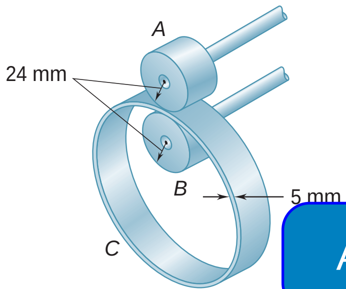


Fig. P15.CQ2

15.CQ2 Knowing that wheel A rotates with a constant angular velocity and [illegible] n ring C and wheel A and wheel B, [illegible] nts concerning the angular speeds [illegible]

a. [illegible]
b. $\omega_a > \omega_b$
c. $\omega_a < \omega_b$
d. $\omega_a = \omega_c$
e. The contact points between A and C have the same acceleration.

END-OF-SECTION PROBLEMS

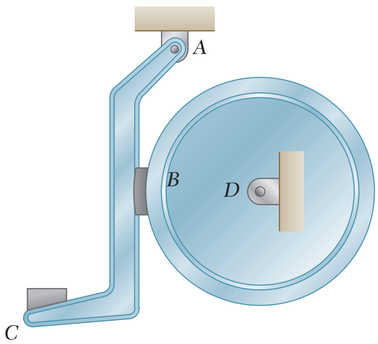


Fig. P15.1

15.1 The brake drum is attached to a larger flywheel that is not shown. The motion of the brake drum is defined by the relation $\theta = 36t - 1.6t^2$, where θ is expressed in radians and t in seconds. Determine (*a*) the angular velocity at $t = 2$ s, (*b*) the number of revolutions executed by the brake drum before coming to rest.

15.2 The motion of an oscillating crank is defined by the relation $\theta = \theta_0 \sin(\pi t/T) - (0.5\theta_0)\sin(2\pi t/T)$, where θ is expressed in radians and t in seconds. Knowing that $\theta_0 = 6$ rad and $T = 4$ s, determine the angular coordinate, the angular velocity, and the angular acceleration of the crank when (*a*) $t = 0$, (*b*) $t = 2$ s.

15.3 The motion of a disk rotating in an oil bath is defined by the relation $\theta = \theta_0(1 - e^{-t/4})$, where θ is expressed in radians and t in seconds. Knowing that $\theta_0 = 0.40$ rad, determine the angular coordinate, velocity, and acceleration of the disk when (*a*) $t = 0$, (*b*) $t = 3$ s, (*c*) $t = \infty$.

15.4 The rotor of a gas turbine is rotating at a speed of 6900 rpm when the turbine is shut down. It is observed that 4 min is required for the rotor to coast to rest. Assuming uniformly accelerated motion, determine (*a*) the angular acceleration, (*b*) the number of revolutions that the rotor executes before coming to rest.

15.5 A small grinding wheel is attached to the shaft of an electric motor which has a rated speed of 3600 rpm. When the power is turned on, the unit reaches its rated speed in 5 s, and when the power is turned off, the unit coasts to rest in 70 s. Assuming uniformly accelerated motion, determine the number of revolutions that the motor executes (*a*) in reaching its rated speed, (*b*) in coasting to rest.

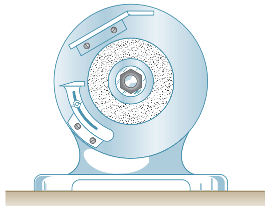


Fig. P15.5

15.6 A connecting rod is supported by a knife-edge at point *A*. For small oscillations the angular acceleration of the connecting rod is governed by the relation a = −6u where a is expressed in rad/s^2 and u in radians. Knowing that the connecting rod is released from rest when u = 20°, determine (*a*) the maximum angular velocity, (*b*) the angular position when t = 2 s.

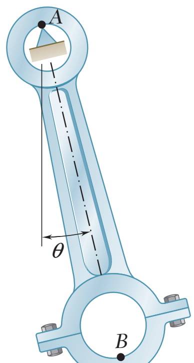


Fig.

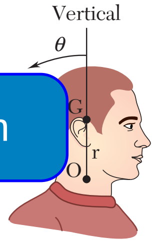


Fig. *P15.7*

15.7 When studying whiplash resulting from rear-end collisions, the rotation of the head is of primary interest. An impact test was performed, and it was found that the angular acceleration of the head is defined by the relation a = 700 cos u + 70 sin u, where a is expressed in rad/s^2 and u in radians. Knowing that the head is initially at rest, determine the angular velocity of the head when u = 30°.

15.8 The angular acceleration of an oscillating disk is defined by the relation a = −*k*u. Determine (*a*) the value of *k* for which v = 8 rad/s when u = 0 and u = 4 rad when v = 0, (*b*) the angular velocity of the disk when u = 3 rad.

15.9 The angular acceleration of a shaft is defined by the relation a = −0.25v, where a is expressed in rad/s^2 and v in rad/s. Knowing that at t = 0 the angular velocity of the shaft is 20 rad/s, determine (*a*) the number of revolutions the shaft will execute before coming to rest, (*b*) the time required for the shaft to come to rest, (*c*) the time required for the angular velocity of the shaft to be reduced to 1 percent of its initial value.

15.10 The bent rod *ABCDE* rotates about a line joining points *A* and *E* with a constant angular velocity of 9 rad/s. Knowing that the rotation is clockwise as viewed from *E*, determine the velocity and acceleration of corner *C*.

15.11 In Prob. 15.10, determine the velocity and acceleration of corner *B*, assuming that the angular velocity is 9 rad/s and increases at the rate of 45 rad/s^2.

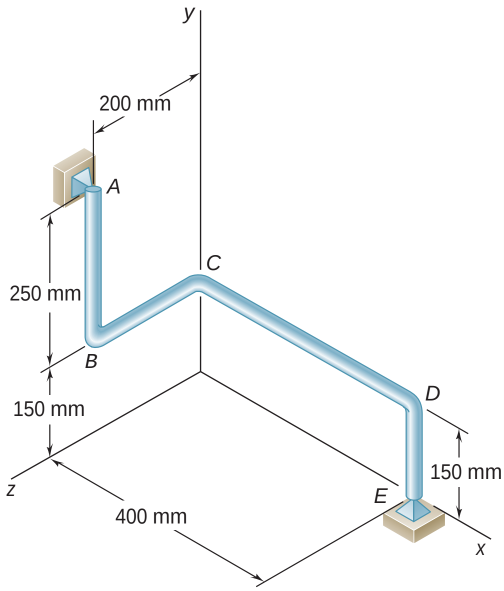


Fig. P15.10

15.12 The assembly shown consists of the straight rod *ABC* which passes through and is welded to the rectangular plate *DEFH*. The assembly rotates about the axis *AC* with a constant angular velocity of 9 rad/s. Knowing that the motion when viewed from *C* is counterclockwise, determine the velocity and acceleration of corner *F*.

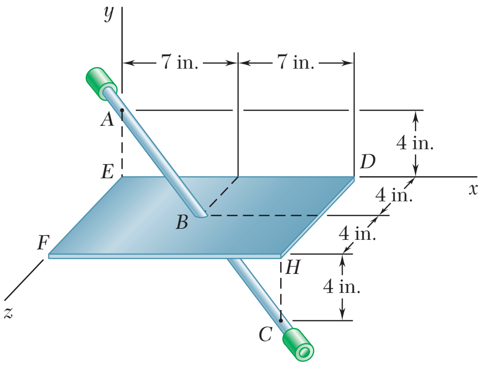


Fig. P15.12

15.13 In Prob. 15.12, determine the acceleration of corner *H*, assuming that the angular velocity is 9 rad/s and decreases at a rate of 18 rad/s^2.

15.14 A circular plate of 120-mm radius is supported by two bearings *A* and *B* as shown. The plate rotates about the rod joining *A* and *B* with a constant angular velocity of 26 rad/s. Knowing that, at the instant considered, the velocity of point *C* is directed to the right, determine the velocity and acceleration of point *E*.

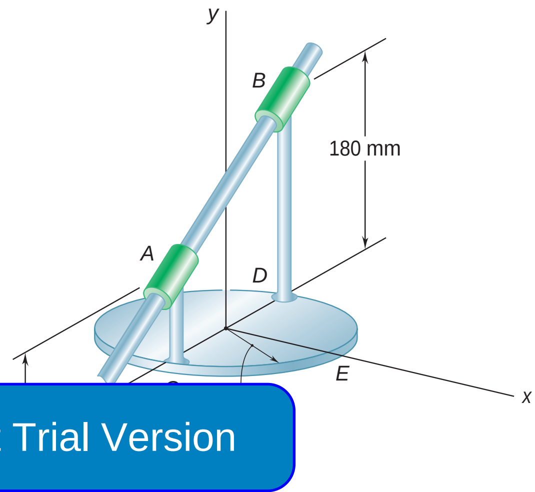


Fig. P15.14

15.15 In Prob. 15.14, determine the velocity and acceleration of point *E*, assuming that the angular velocity is 26 rad/s and increases at the rate of 65 rad/s^2.

15.16 The earth makes one complete revolution around the sun in 365.24 days. Assuming that the orbit of the earth is circular and has a radius of 93,000,000 mi, determine the velocity and acceleration of the earth.

15.17 The earth makes one complete revolution on its axis in 23 h 56 min. Knowing that the mean radius of the earth is 3960 mi, determine the linear velocity and acceleration of a point on the surface of the earth (*a*) at the equator, (*b*) at Philadelphia, latitude 40° north, (*c*) at the North Pole.

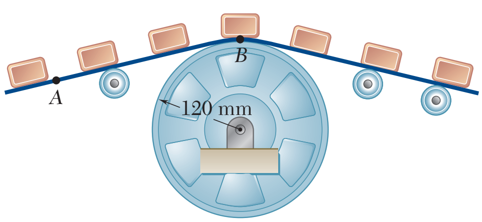


Fig. P15.18 and P15.19

15.18 A series of small machine components being moved by a conveyor belt pass over a 120-mm-radius idler pulley. At the instant shown, the velocity of point *A* is 300 mm/s to the left and its acceleration is 180 mm/s^2 to the right. Determine (*a*) the angular velocity and angular acceleration of the idler pulley, (*b*) the total acceleration of the machine component at *B*.

15.19 A series of small machine components being moved by a conveyor belt pass over a 120-mm-radius idler pulley. At the instant shown, the angular velocity of the idler pulley is 4 rad/s clockwise. Determine the angular acceleration of the pulley for which the magnitude of the total acceleration of the machine component at *B* is 2400 mm/s^2.

15.20 The belt sander shown is initially at rest. If the driving drum B has a constant angular acceleration of 120 rad/s^2 counterclockwise, determine the magnitude of the acceleration of the belt at point C when (*a*) $t = 0.5$ s, (*b*) $t = 2$ s.

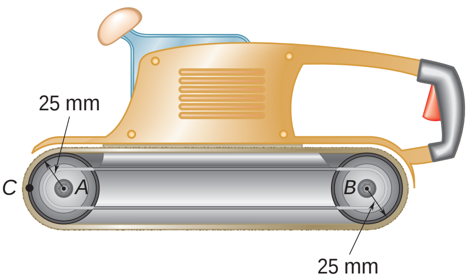


Fig. P15.20 and P15.21

15.21 The rated speed of drum B of the belt sander shown is 2400 rpm. When the power is turned off, it is observed that the sander coasts from its rated speed to rest in 10 s. Assuming uniformly decelerated motion, determine the velocity and acceleration of point C of the belt, (*a*) immediately before the power is turned off, (*b*) 9 s later.

15.22 The two pulleys shown may be operated with the V belt in any of three positions. If the angular acceleration of shaft A is 6 rad/s^2 and if the system is initially at rest, determine the time required for shaft B to reach a speed of 400 rpm with the belt in each of the three positions.

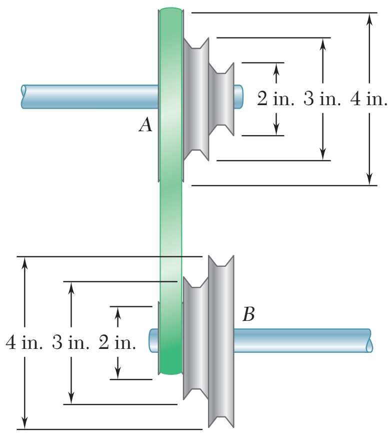


Fig. P15.22

15.23 Three belts move over t [illegible] reduction system shown. At the instant shown, the velocity of point A on the input belt is 2 ft/s to the right, decreasing at the rate of 6 ft/s^2. Determine, at this instant, (*a*) the velocity and acceleration of point C on the output belt, (*b*) the acceleration of point B on the output pulley.

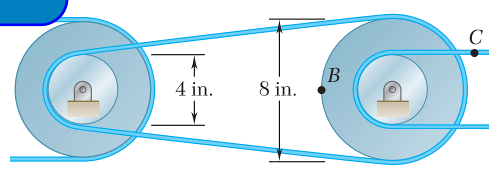


Fig. P15.23

15.24 A gear reduction system consists of three gears A, B, and C. Knowing that gear A rotates clockwise with a constant angular velocity $\omega_A = 600$ rpm, determine (*a*) the angular velocities of gears B and C, (*b*) the accelerations of the points on gears B and C which are in contact.

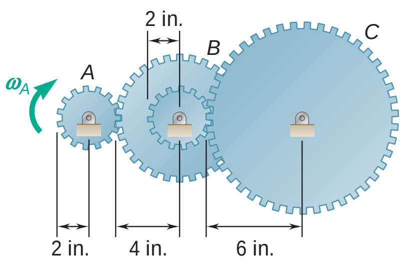


Fig. P15.24

15.25 A belt is pulled to the right between cylinders A and B. Knowing that the speed of the belt is a constant 5 ft/s and no slippage occurs, determine (*a*) the angular velocities of A and B, (*b*) the accelerations of the points which are in contact with the belt.

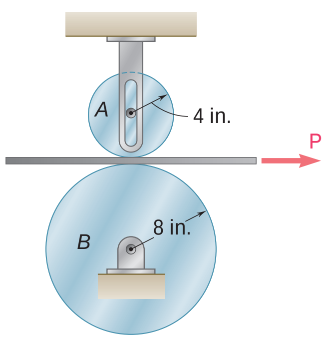


Fig. P15.25

15.26 Ring C has an inside radius of 55 mm and an outside radius of 60 mm and is positioned between two wheels A and B, each of 24-mm outside radius. Knowing that wheel A rotates with a constant angular velocity of 300 rpm and that no slipping occurs, determine (*a*) the angular velocity of ring C and of wheel B, (*b*) the acceleration of the points A and B which are in contact with C.

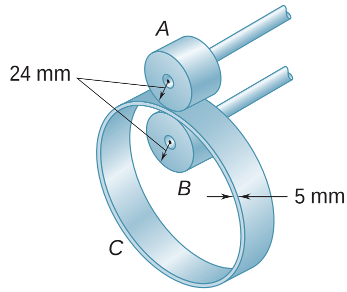


Fig. P15.26

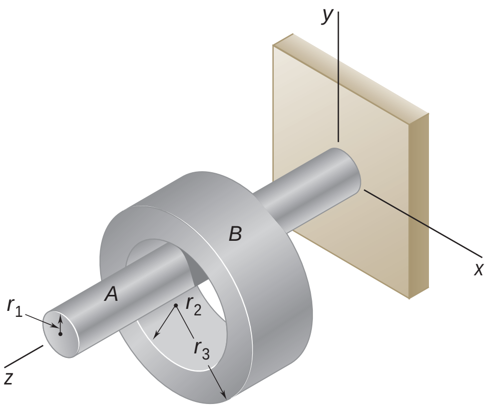


Fig. P15.27

15.27 Ring B has an inside radius r_2 and hangs from the horizontal shaft A as shown. Shaft A rotates with a constant angular velocity of 25 rad/s and no slipping occurs. Knowing that $r_1 = 12$ mm, $r_2 = 30$ mm, and $r_3 = 40$ mm, determine (*a*) the angular velocity of ring B, (*b*) the accelerations of the points of shaft A and ring B which are in contact, (*c*) the magnitude of the acceleration of a point on the outside surface of ring B.

[illegible] ms. During a 4-s interval the speed [illegible] y from $v_0 = 2$ ft/s to $v_1 = 4$ ft/s. [illegible] ip on the drums, determine (*a*) the angular acceleration of drum B, (*b*) the number of revolutions executed by drum B during the 4-s interval.

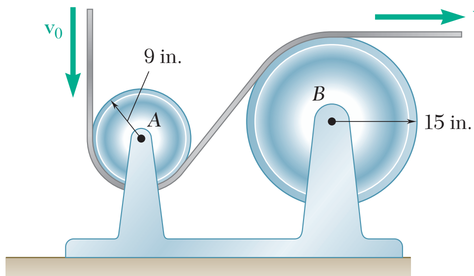


Fig. P15.28

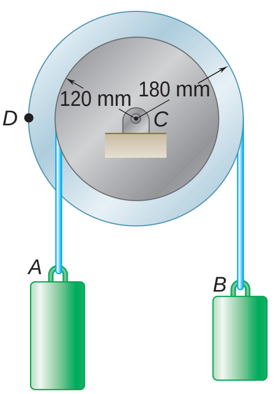


Fig. P15.29 and P15.30

15.29 A pulley and two loads are connected by inextensible cords as shown. Load A has a constant acceleration of 300 mm/s^2 and an initial velocity of 240 mm/s, both directed upward. Determine (*a*) the number of revolutions executed by the pulley in 3 s, (*b*) the velocity and position of load B after 3 s, (*c*) the acceleration of point D on the rim of the pulley at $t = 0$.

15.30 A pulley and two loads are connected by inextensible cords as shown. The pulley starts from rest at $t = 0$ and is accelerated at the uniform rate of 2.4 rad/s^2 clockwise. At $t = 4$ s, determine the velocity and position (*a*) of load A, (*b*) of load B.

15.31 A load is to be raised 20 ft by the hoisting system shown. Assuming gear A is initially at rest, accelerates uniformly to a speed of 120 rpm in 5 s, and then maintains a constant speed of 120 rpm, determine (*a*) the number of revolutions executed by gear A in raising the load, (*b*) the time required to raise the load.

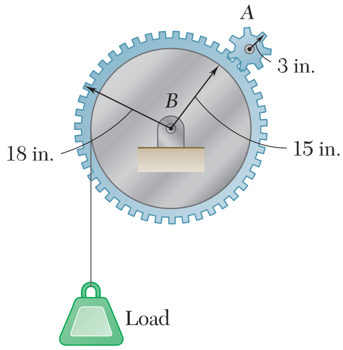


Fig. P15.31

15.32 Disk B is at rest when it is brought into contact with disk A which is rotating freely at 450 rpm clockwise. After 6 s of slippage, during which each disk has a constant angular acceleration, disk A reaches a final angular velocity of 140 rpm clockwise. Determine the angular acceleration of each disk during the period of slippage.

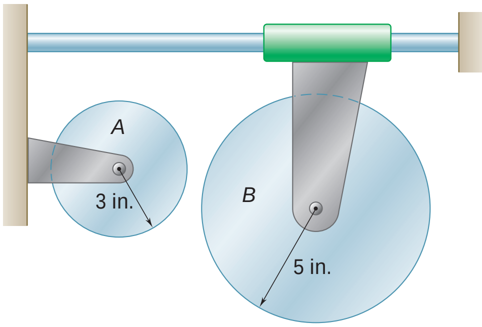


Fig. P15.32 and P15.33

15.33 and *15.34* A simple [illegible] B. Initially, disk A has a [illegible] disk B is at rest. It is known that disk A will coast to rest in 60 s. However, rather than waiting until both disks are at rest to bring them together, disk B is given a constant angular acceleration of 2.5 rad/s^2 counterclockwise. Determine (*a*) at what time the disks can be brought together if they are not to slip, (*b*) the angular velocity of each disk as contact is made.

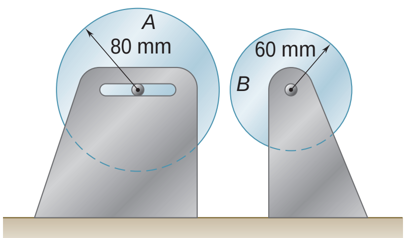


Fig. *P15.34* and *P15.35*

15.35 Two friction disks A and B are both rotating freely at 240 rpm counterclockwise when they are brought into contact. After 8 s of slippage, during which each disk has a constant angular acceleration, disk A reaches a final angular velocity of 60 rpm counterclockwise. Determine (*a*) the angular acceleration of each disk during the period of slippage, (*b*) the time at which the angular velocity of disk B is equal to zero.

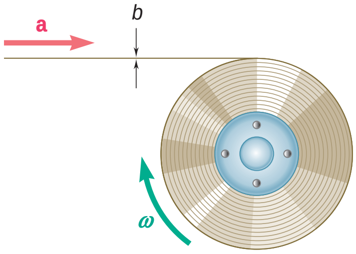


Fig. P15.36

***15.36** Steel tape is being wound onto a spool which rotates with a constant angular velocity V_0. Denoting by r the radius of the spool and tape at any given time and by b the thickness of the tape, derive an expression for the acceleration of the tape as it approaches the spool.

***15.37** In a continuous printing process, paper is drawn into the presses at a constant speed v. Denoting by r the radius of the paper roll at any given time and by b the thickness of the paper, derive an expression for the angular acceleration of the paper roll.

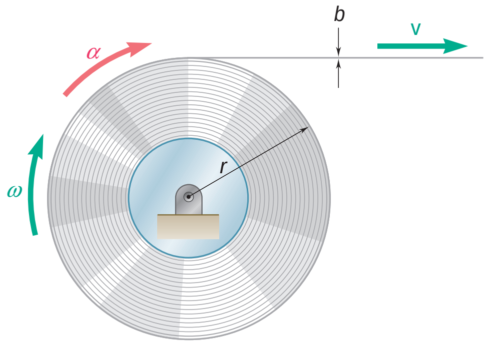
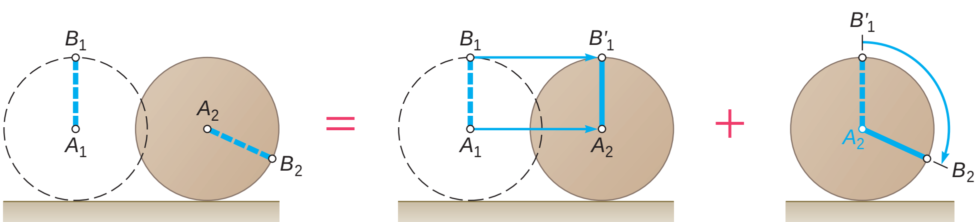


Fig. P15.37

N

As indicated in Sec. 15.1, we understand by general plane motion a plane motion which is neither a translation nor a rotation. As you will presently see, however, *a general plane motion can always be considered as the sum of a translation and a rotation*.

Consider, for example, a wheel rolling on a straight track (Fig. 15.12). Over a certain interval of time, two given points A and B will have moved, respectively, from A_1 to A_2 and from B_1 to B_2. The same result could be obtained through a translation which would bring A and B into A_2 and B'_1 (the line AB remaining vertical), followed by a rotation about A bringing B into B_2. Although the original rolling motion differs from the combination of translation and rotation when these motions are taken in succession, the original motion can be exactly duplicated by a combination of simultaneous translation and rotation.



Plane motion = Translation with A + Rotation about A

Fig. 15.12

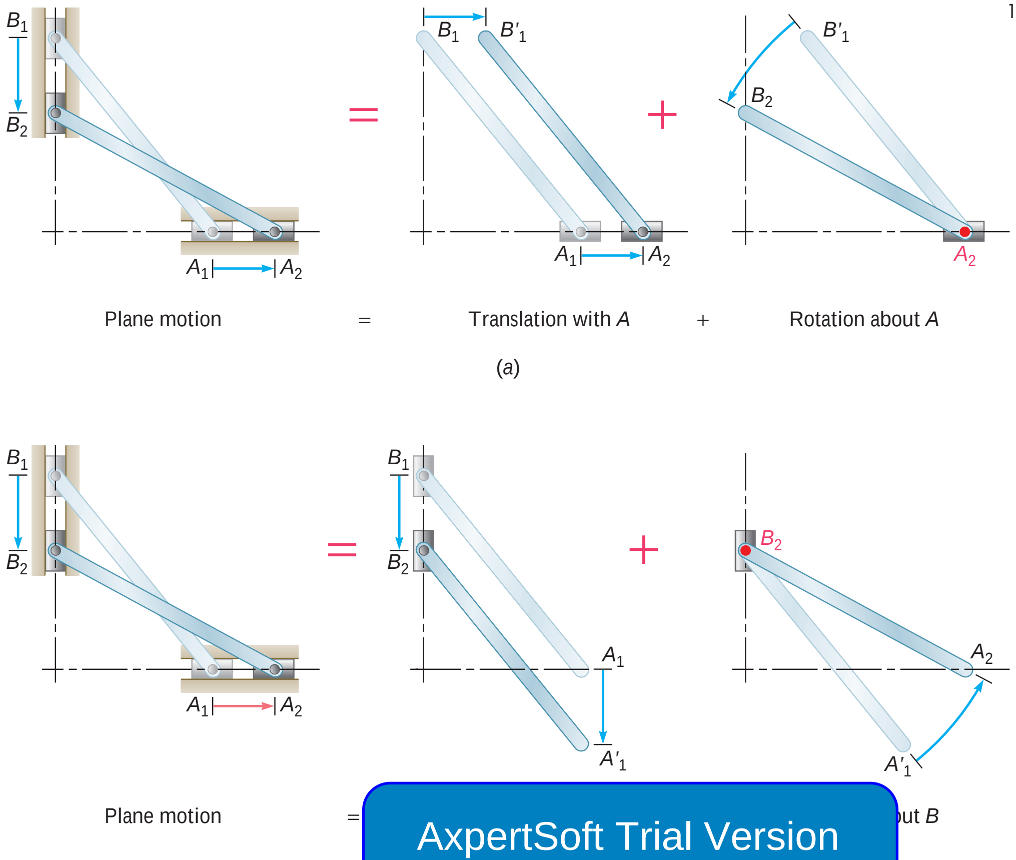


Fig. 15.13

Another example of plane motion is given in Fig. 15.13, which represents a rod whose extremities slide along a horizontal and a vertical track, respectively. This motion can be replaced by a translation in a horizontal direction and a rotation about A (Fig. 15.13a) or by a translation in a vertical direction and a rotation about B (Fig. 15.13b).

In the general case of plane motion, we will consider a small displacement which brings two particles A and B of a representative slab, respectively, from A_1 and B_1 into A_2 and B_2 (Fig. 15.14). This displacement can be divided into two parts: in one, the particles move into A_2 and B_1' while the line AB maintains the same direction; in the other, B moves into B_2 while A remains fixed. The first part of the motion is clearly a translation and the second part a rotation about A.

Recalling from Sec. 11.12 the definition of the relative motion of a particle with respect to a moving frame of reference—as opposed to its absolute motion with respect to a fixed frame of reference—we can restate as follows the result obtained above: Given two particles A and B of a rigid slab in plane motion, the relative motion of B with respect to a frame attached to A and of fixed orientation is a rotation. To an observer moving with A but not rotating, particle B will appear to describe an arc of circle centered at A.

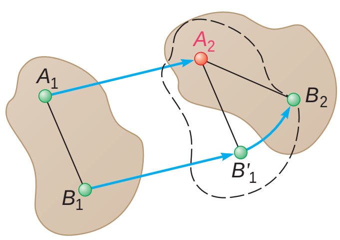


Fig. 15.14

Photo 15.4 Planetary gear systems are used to high reduction ratios with minimum space and weight. The small gears undergo general plane motion.

15.6 ABSOLUTE AND RELATIVE VELOCITY IN PLANE MOTION

We saw in the preceding section that any plane motion of a slab can be replaced by a translation defined by the motion of an arbitrary reference point A and a simultaneous rotation about A. The absolute velocity $\mathbf{v}_B$ of a particle B of the slab is obtained from the relative-velocity formula derived in Sec. 11.12,

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{15.17}$$

where the right-hand member represents a vector sum. The velocity $\mathbf{v}_A$ corresponds to the translation of the slab with A, while the relative velocity $\mathbf{v}_{B/A}$ is associated with the rotation of the slab about A and is measured with respect to axes centered at A and of fixed orientation (Fig. 15.15). Denoting by $\mathbf{r}_{B/A}$ the position vector of B relative to A, and by $\omega\mathbf{k}$ the angular velocity of the slab with respect to axes of fixed orientation, we have from (15.10) and (15.10′)

$$\mathbf{v}_{B/A} = \omega\mathbf{k} \times \mathbf{r}_{B/A} \qquad v_{B/A} = r\omega \tag{15.18}$$

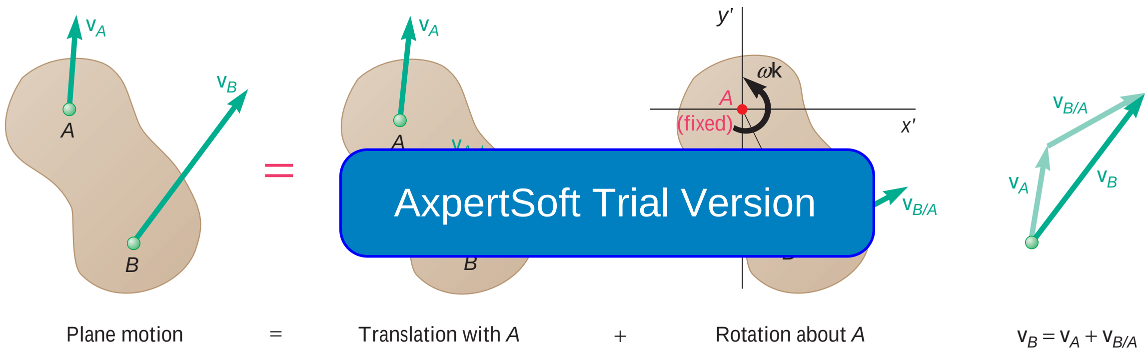


Fig. 15.15

where r is the distance from A to B. Substituting for $\mathbf{v}_{B/A}$ from (15.18) into (15.17), we can also write

$$\mathbf{v}_B = \mathbf{v}_A + \omega\mathbf{k} \times \mathbf{r}_{B/A} \tag{15.17′}$$

As an example, let us again consider the rod AB of Fig. 15.13. Assuming that the velocity $\mathbf{v}_A$ of end A is known, we propose to find the velocity $\mathbf{v}_B$ of end B and the angular velocity ω of the rod, in terms of the velocity $\mathbf{v}_A$, the length l, and the angle θ. Choosing A as a reference point, we express that the given motion is equivalent to a translation with A and a simultaneous rotation about A (Fig. 15.16). The absolute velocity of B must therefore be equal to the vector sum

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{15.17}$$

We note that while the direction of $\mathbf{v}_{B/A}$ is known, its magnitude $l\omega$ is unknown. However, this is compensated for by the fact that the direction of $\mathbf{v}_B$ is known. We can therefore complete the diagram of Fig. 15.16. Solving for the magnitudes v_B and ω, we write

$$v_B = v_A \tan\theta \qquad \omega = \frac{v_{B/A}}{l} = \frac{v_A}{l\cos\theta} \tag{15.19}$$

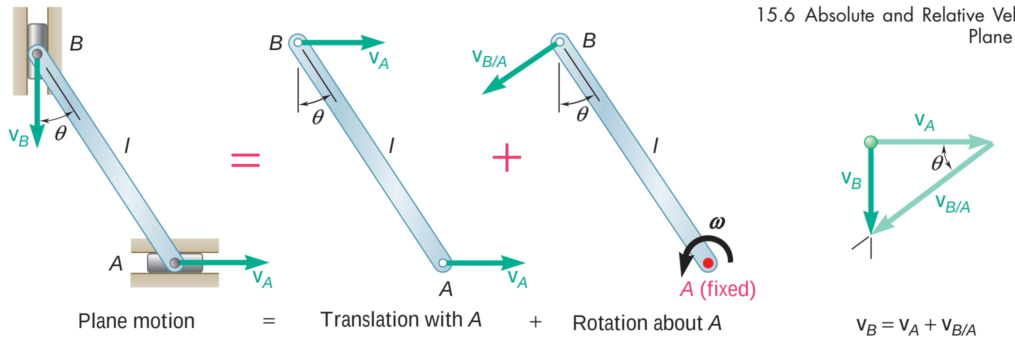


Fig. 15.16

The same result can be obtained by using B as a point of reference. Resolving the given motion into a translation with B and a simultaneous rotation about B (Fig. 15.17), we write the equation

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} \tag{15.20}$$

which is represented graphically in Fig. 15.17. We note that $\mathbf{v}_{A/B}$ and $\mathbf{v}_{B/A}$ have the same magnitude *l*V but opposite sense. The sense of the relative velocity depends, therefore, upon the point of reference which has been selected and should be carefully ascertained from the appropriate diagram (Fig. 15.16 or 15.17).

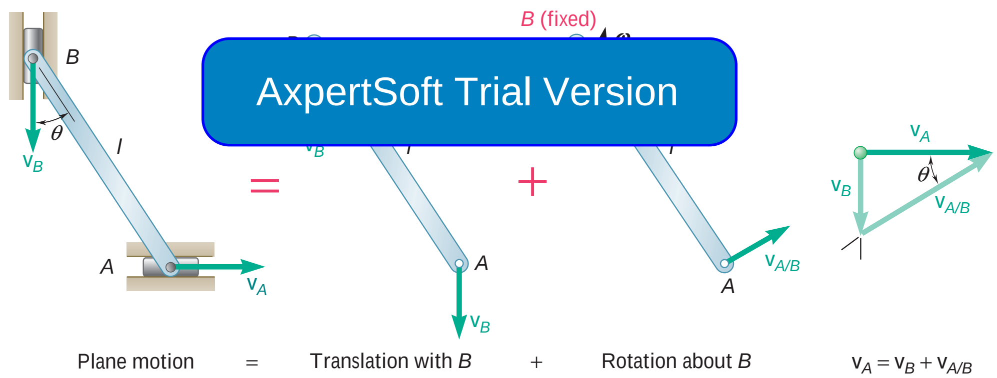
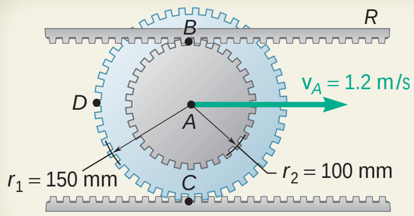


Fig. 15.17

Finally, we observe that the angular velocity V of the rod in its rotation about B is the same as in its rotation about A. It is measured in both cases by the rate of change of the angle u. This result is quite general; we should therefore bear in mind that *the angular velocity* V *of a rigid body in plane motion is independent of the reference point.*

Most mechanisms consist not of one but of *several* moving parts. When the various parts of a mechanism are pin-connected, the analysis of the mechanism can be carried out by considering each part as a rigid body, keeping in mind that the points where two parts are connected must have the same absolute velocity (see Sample Prob. 15.3). A similar analysis can be used when gears are involved, since the teeth in contact must also have the same absolute velocity. However, when a mechanism contains parts which slide on each other, the relative velocity of the parts in contact must be taken into account (see Secs. 15.10 and 15.11).



SAMPLE PROBLEM 15.2

The double gear shown rolls on the stationary lower rack; the velocity of its center A is 1.2 m/s directed to the right. Determine (*a*) the angular velocity of the gear, (*b*) the velocities of the upper rack R and of point D of the gear.

SOLUTION

a. Angular Velocity of the Gear. Since the gear rolls on the lower rack, its center A moves through a distance equal to the outer circumference $2\pi r_1$ for each full revolution of the gear. Noting that 1 rev $= 2\pi$ rad, and that when A moves to the right ($x_A > 0$) the gear rotates clockwise ($\theta < 0$), we write

$$\frac{x_A}{2\pi r_1} = -\frac{\theta}{2\pi} \qquad x_A = -r_1\theta$$

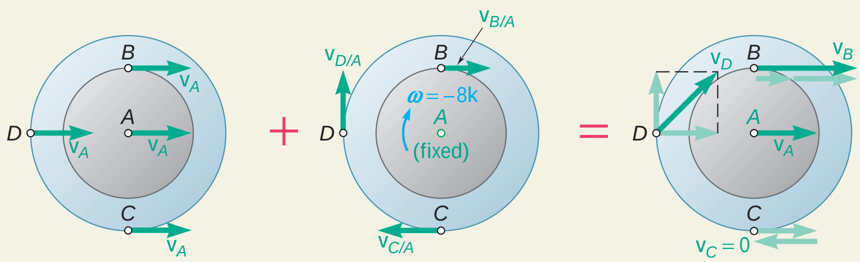
Differentiating with respect to the time t and substituting the known values $v_A = 1.2$ m/s and $r_1 = 150$ mm $= 0.150$ m, we obtain

$$v_A = -r_1\omega \qquad 1.2 \text{ m/s} = -(0.150 \text{ m})\omega \qquad \omega = -8 \text{ rad/s}$$

$$\boldsymbol{\omega} = \omega\mathbf{k} = -(8 \text{ rad/s})\mathbf{k} \quad ◀$$

where $\mathbf{k}$ is a unit vector pointing out of the paper.

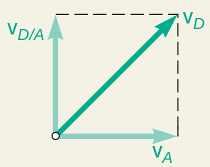
...d into two component motions: ...on about the center A. In the ... the same velocity $\mathbf{v}_A$. In the rotation, each point P of the gear moves about A with a relative velocity $\mathbf{v}_{P/A} = \omega\mathbf{k} \times \mathbf{r}_{P/A}$, where $\mathbf{r}_{P/A}$ is the position vector of P relative to A.



Velocity of Upper Rack. The velocity of the upper rack is equal to the velocity of point B; we write

$$\begin{aligned}\mathbf{v}_R = \mathbf{v}_B &= \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + \omega\mathbf{k} \times \mathbf{r}_{B/A}\\ &= (1.2 \text{ m/s})\mathbf{i} - (8 \text{ rad/s})\mathbf{k} \times (0.100 \text{ m})\mathbf{j}\\ &= (1.2 \text{ m/s})\mathbf{i} + (0.8 \text{ m/s})\mathbf{i} = (2 \text{ m/s})\mathbf{i}\end{aligned}$$

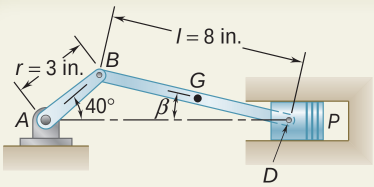
$$\mathbf{v}_R = 2 \text{ m/s} \rightarrow \quad ◀$$



Velocity of Point *D*

$$\begin{aligned}\mathbf{v}_D &= \mathbf{v}_A + \mathbf{v}_{D/A} = \mathbf{v}_A + \omega\mathbf{k} \times \mathbf{r}_{D/A}\\ &= (1.2 \text{ m/s})\mathbf{i} - (8 \text{ rad/s})\mathbf{k} \times (-0.150 \text{ m})\mathbf{i}\\ &= (1.2 \text{ m/s})\mathbf{i} + (1.2 \text{ m/s})\mathbf{j}\end{aligned}$$

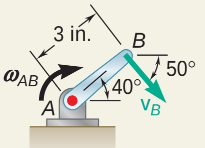
$$\mathbf{v}_D = 1.697 \text{ m/s} \nearrow 45° \quad ◀$$



SAMPLE PROBLEM 15.3

In the engine system shown, the crank AB has a constant clockwise angular velocity of 2000 rpm. For the crank position indicated, determine (*a*) the angular velocity of the connecting rod BD, (*b*) the velocity of the piston P.

SOLUTION



Motion of Crank *AB*. The crank AB rotates about point A. Expressing ω_{AB} in rad/s and writing $v_B = r\omega_{AB}$, we obtain

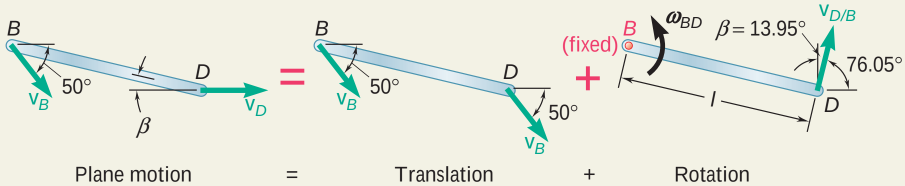
$$\omega_{AB} = \left(2000 \frac{\text{rev}}{\text{min}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right)\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 209.4 \text{ rad/s}$$

$$v_B = (AB)\omega_{AB} = (3 \text{ in.})(209.4 \text{ rad/s}) = 628.3 \text{ in./s}$$
$$\mathbf{v}_B = 628.3 \text{ in./s} \searrow 50°$$

Motion of Connecting Rod *BD*. We consider this motion as a general plane motion. Using the law of sines, we compute the angle β between the connecting rod and the horizontal:

$$\frac{\sin 40°}{\ } = \frac{\sin \beta}{\ } \qquad \beta = 13.95°$$

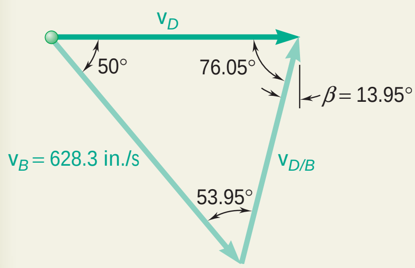
...he rod is attached to the piston must ... point B is equal to the velocity $\mathbf{v}_B$ obtained above. Resolving the motion of BD into a translation with B and a rotation about B, we obtain



Expressing the relation between the velocities $\mathbf{v}_D$, $\mathbf{v}_B$, and $\mathbf{v}_{D/B}$, we write

$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$$

We draw the vector diagram corresponding to this equation. Recalling that $\beta = 13.95°$, we determine the angles of the triangle and write



$$\frac{v_D}{\sin 53.95°} = \frac{v_{D/B}}{\sin 50°} = \frac{628.3 \text{ in./s}}{\sin 76.05°}$$

$$v_{D/B} = 495.9 \text{ in./s} \qquad \mathbf{v}_{D/B} = 495.9 \text{ in./s} \angle 76.05°$$
$$v_D = 523.4 \text{ in./s} = 43.6 \text{ ft/s} \qquad \mathbf{v}_D = 43.6 \text{ ft/s} \rightarrow$$
$$\mathbf{v}_P = \mathbf{v}_D = 43.6 \text{ ft/s} \rightarrow$$ ◀

Since $v_{D/B} = l\omega_{BD}$, we have

$$495.9 \text{ in./s} = (8 \text{ in.})\omega_{BD} \qquad \omega_{BD} = 62.0 \text{ rad/s} \circlearrowleft$$ ◀

SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to analyze the velocity of bodies in *general plane motion.* You found that a general plane motion can always be considered as the sum of the two motions you studied in the last lesson, namely, *a translation and a rotation.*

To solve a problem involving the velocity of a body in plane motion you should take the following steps.

1. Whenever possible determine the velocity of the points of the body where the body is connected to another body whose motion is known. That other body may be an arm or crank rotating with a given angular velocity [Sample Prob. 15.3].

2. Next start drawing a "diagram equation" to use in your solution (Figs. 15.15 and 15.16). This "equation" will consist of the following diagrams.

a. Plane motion diagram: Draw a diagram of the body including all dimensions and showing those points for which you know or seek the velocity.

b. Translation diagram: Select a reference point A for which you know the direction and/or the magnitude of the velocity $\mathbf{v}_A$, and draw a second diagram showing the body in [illegible] me velocity $\mathbf{v}_A$.

c. Rotation diag[illegible] draw a diagram showing the body in [illegible] $\mathbf{V} = v\mathbf{k}$ of the body and the relative velocities with respect to A of the other points, such as the velocity $\mathbf{v}_{B/A}$ of B relative to A.

3. Write the relative-velocity formula

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

While you can solve this vector equation analytically by writing the corresponding scalar equations, you will usually find it easier to solve it by using a vector triangle (Fig. 15.16).

4. A different reference point can be used to obtain an equivalent solution. For example, if point B is selected as the reference point, the velocity of point A is expressed as

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

Note that the relative velocities $\mathbf{v}_{B/A}$ and $\mathbf{v}_{A/B}$ have the same magnitude but opposite sense. Relative velocities, therefore, depend upon the reference point that has been selected. The angular velocity, however, is independent of the choice of reference point.

PROBLEMS

CONCEPT QUESTIONS

15.CQ3 The ball rolls without slipping on the fixed surface as shown. What is the direction of the velocity of point A?

a. y **b.** ↗ **c.** ↑ **d.** ↓ **e.** ↘

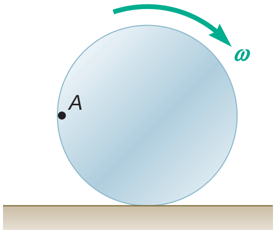


Fig. P15.CQ3

15.CQ4 Three uniform rods—ABC, DCE, and FGH—are connected as shown. Which of the following statements concerning the angular speed of the three objects is true?

a. $\mathrm{v}_{ABC} = \mathrm{v}_{DCE} = \mathrm{v}_{FGH}$

b. $\mathrm{v}_{DCE} > \mathrm{v}_{ABC} > \mathrm{v}_{FGH}$

c. $\mathrm{v}_{DCE} < \mathrm{v}_{ABC} < \mathrm{v}_{FGH}$

d. $\mathrm{v}_{ABC} > \mathrm{v}_{DCE} > \mathrm{v}_{FGH}$

e. $\mathrm{v}_{FGH} = \mathrm{v}_{DCE} < \mathrm{v}_{ABC}$

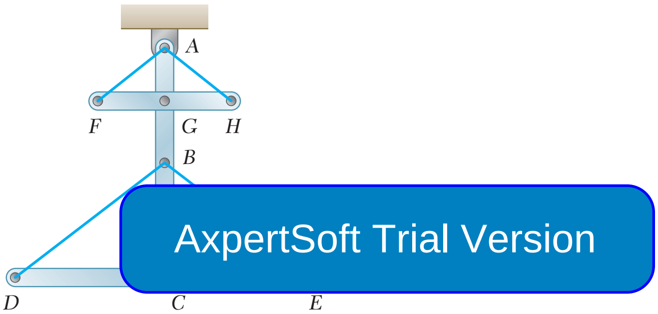


Fig. P15.CQ4

END-OF-SECTION PROBLEMS

15.38 An automobile travels to the right at a constant speed of 48 mi/h. If the diameter of a wheel is 22 in., determine the velocities of points B, C, D, and E on the rim of the wheel.

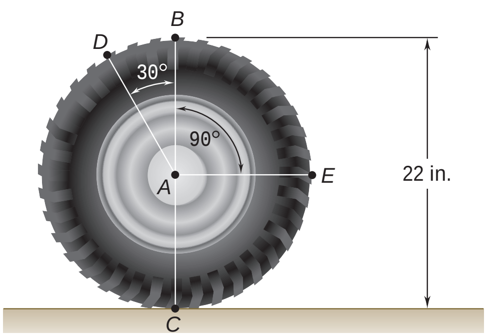


Fig. P15.38

15.39 The motion of rod AB is guided by pins attached at A and B which slide in the slots shown. At the instant shown, u = 40° and the pin at B moves upward to the left with a constant velocity of 6 in./s. Determine (*a*) the angular velocity of the rod, (*b*) the velocity of the pin at end A.

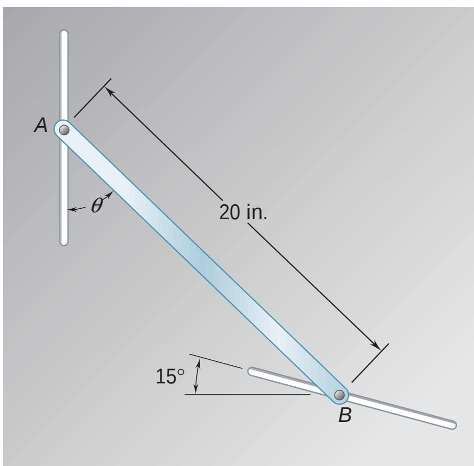


Fig. P15.39

15.40 Collar B moves upward with a constant velocity of 1.5 m/s. At the instant when u = 50°, determine (*a*) the angular velocity of rod AB, (*b*) the velocity of end A of the rod.

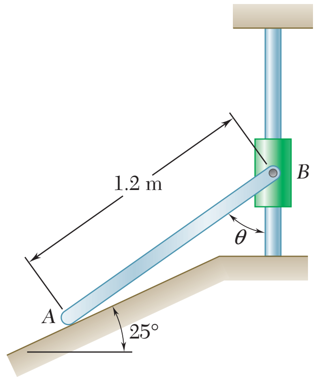


Fig. P15.40

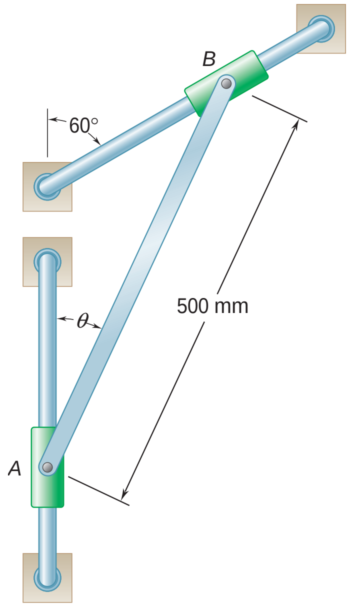


Fig. P15.41 and *P15.42*

15.41 Collar B moves downward to the left with a constant velocity of 1.6 m/s. At the instant shown when u = 40°, determine (*a*) the angular velocity of rod AB, (*b*) the velocity of collar A.

15.42 Collar A moves upward with a constant velocity of 1.2 m/s. At the instant shown when u = 25°, determine (*a*) the angular velocity of rod AB, (*b*) the velocity of collar B.

15.43 Rod AB moves over a small wheel at C while end A moves to the ... f 25 in./s. At the instant shown, ... ity of the rod, (*b*) the velocity of

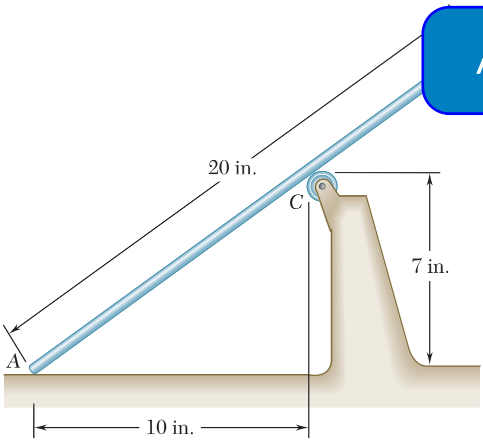


Fig. *P15.43*

15.44 The plate shown moves in the xy plane. Knowing that $(v_A)_x = 120$ mm/s, $(v_B)_y = 300$ mm/s, and $(v_C)_y = -60$ mm/s, determine (*a*) the angular velocity of the plate, (*b*) the velocity of point A.

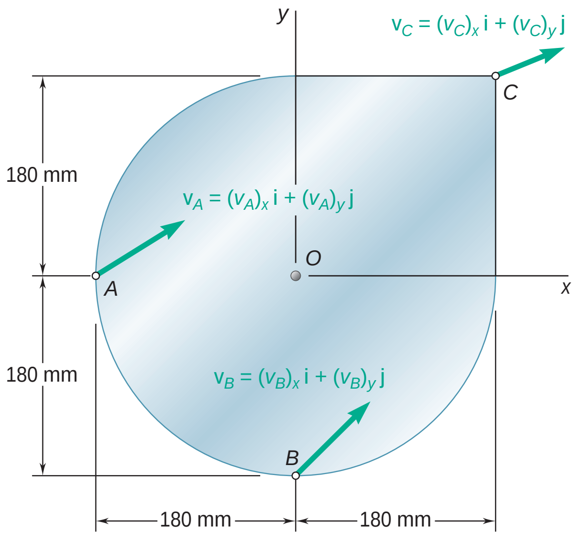


Fig. P15.44

15.45 In Prob. 15.44, determine (*a*) the velocity of point B, (*b*) the point of the plate with zero velocity.

15.46 The plate shown moves in the xy plane. Knowing that $(v_A)_x = 250$ mm/s, $(v_B)_y = -450$ mm/s, and $(v_C)_x = -500$ mm/s, determine (*a*) the angular velocity of the plate, (*b*) the velocity of point A.

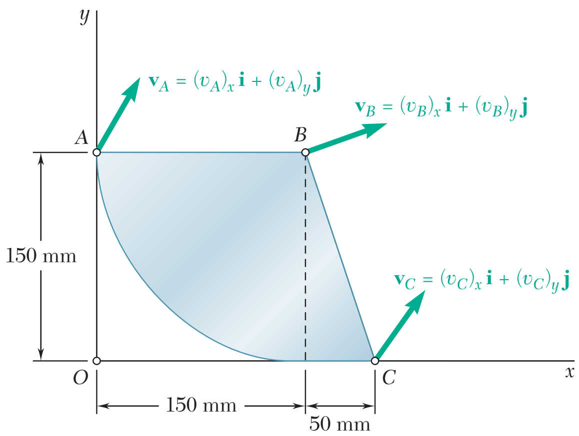


Fig. *P15.46*

15.47 The plate shown moves in the xy plane. Knowing that $(v_A)_x = 12$ in./s, $(v_B)_x = -4$ in./s, and $(v_C)_y = -24$ in./s, determine (*a*) the angular velocity of the plate, (*b*) the velocity of point B.

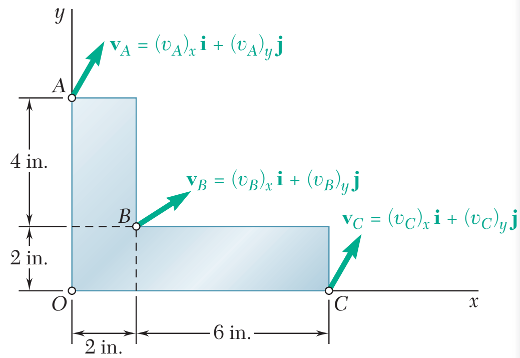


Fig. P15.47

15.48 In the planetary gear system shown, the radius of gears A, B, C, and D is a and the radius of th[illegible] angular velocity of gear A [illegible] is stationary, determine (*a* [illegible] (*b*) the angular velocity of [illegible]

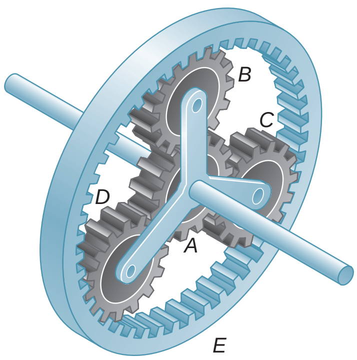


Fig. P15.48 and P15.49

15.49 In the planetary gear system shown, the radius of gears A, B, C, and D is 30 mm and the radius of the outer gear E is 90 mm. Knowing that gear E has an angular velocity of 180 rpm clockwise and that the central gear A has an angular velocity of 240 rpm clockwise, determine (*a*) the angular velocity of each planetary gear, (*b*) the angular velocity of the spider connecting the planetary gears.

15.50 Arm AB rotates with an angular velocity of 20 rad/s counterclockwise. Knowing that the outer gear C is stationary, determine (*a*) the angular velocity of gear B, (*b*) the velocity of the gear tooth located at point D.

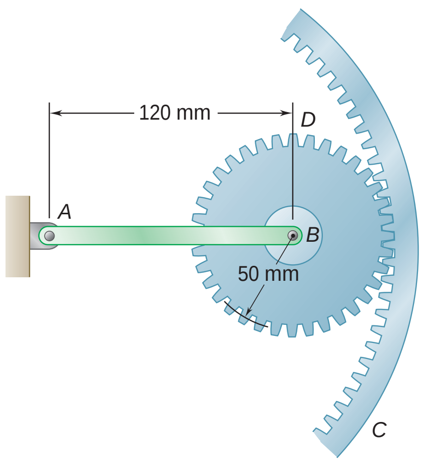


Fig. P15.50

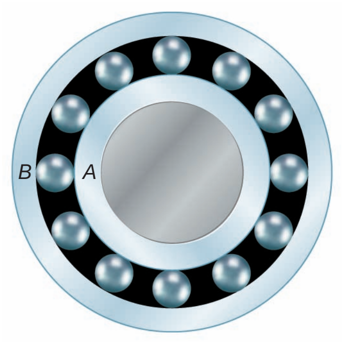
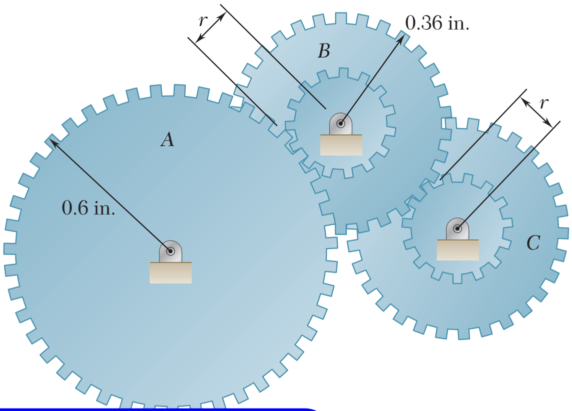


Fig. P15.51

15.51 In the simplified sketch of a ball bearing shown, the diameter of the inner race *A* is 60 mm and the diameter of each ball is 12 mm. The outer race *B* is stationary while the inner race has an angular velocity of 3600 rpm. Determine (*a*) the speed of the center of each ball, (*b*) the angular velocity of each ball, (*c*) the number of times per minute each ball describes a complete circle.

15.52 A simplified gear system for a mechanical watch is shown. Knowing that gear *A* has a constant angular velocity of 1 rev/h and gear *C* has a constant angular velocity of 1 rpm, determine (*a*) the radius *r*, (*b*) the magnitudes of the accelerations of the points on gear *B* that are in contact with gears *A* and *C*.



[illegible]s about point *C* with an angular velocity of 40 rad/s counterclockwise. Two friction disks *A* and *B* are pinned at their centers to arm *ACB* as shown. Knowing that the disks roll without slipping at surfaces of contact, determine the angular velocity of (*a*) disk *A*, (*b*) disk *B*.

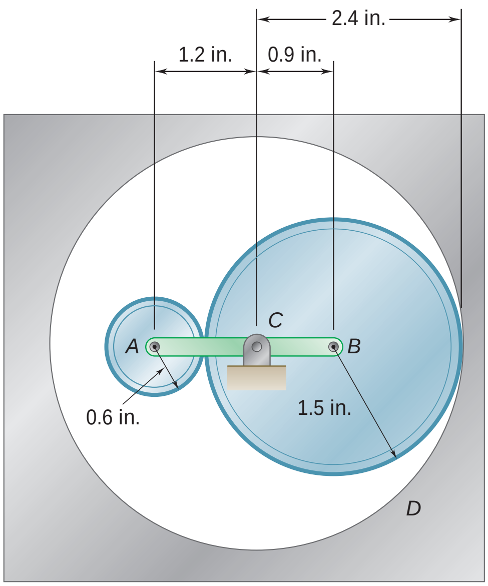


Fig. P15.53

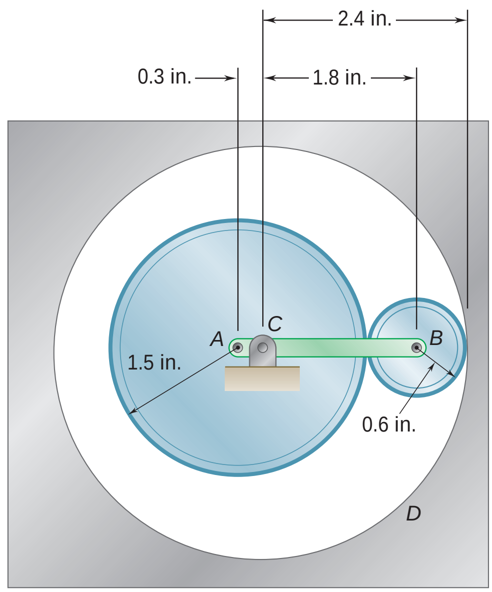


Fig. P15.54

15.55 Knowing that at the instant shown the velocity of collar *A* is 900 mm/s to the left, determine (*a*) the angular velocity of rod *ADB*, (*b*) the velocity of point *B*.

15.56 Knowing that at the instant shown the angular velocity of rod *DE* is 2.4 rad/s clockwise, determine (*a*) the velocity of collar *A*, (*b*) the velocity of point *B*.

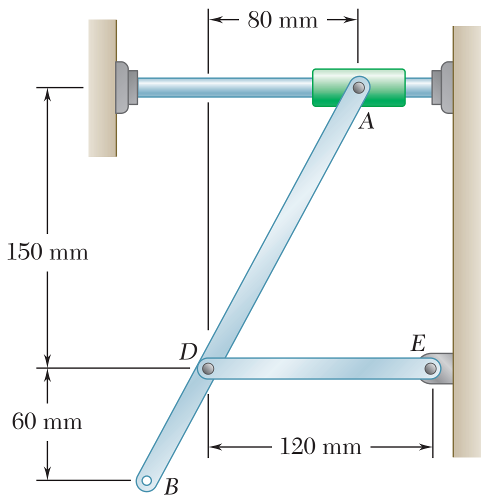


Fig. P15.55 and P15.56

15.57 A straight rack rests on a gear of radius *r* and is attached to a block *B* as shown. Denoting by v_D the clockwise angular velocity of gear *D* and by u the angle formed by the rack and the horizontal, derive expressions for the velocity of block *B* and the angular velocity of the rack in terms of *r*, u, and v_D.

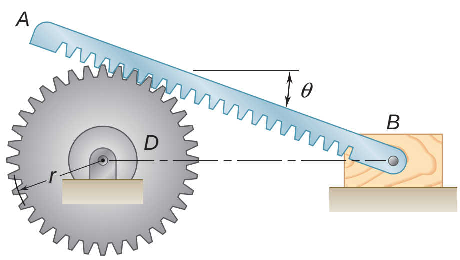


Fig. P15.57 and P15.58

15.58 A straight rack rests o[illegible] attached to a block *B* as shown. Knowing that at the instant shown the velocity of block *B* is 8 in./s to the right and u = 25°, determine (*a*) the angular velocity of gear *D*, (*b*) the angular velocity of the rack.

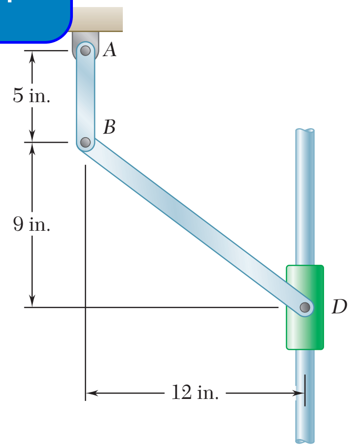


Fig. P15.59

15.59 Knowing that at the instant shown the angular velocity of crank *AB* is 2.7 rad/s clockwise, determine (*a*) the angular velocity of link *BD*, (*b*) the velocity of collar *D*, (*c*) the velocity of the midpoint of link *BD*.

15.60 In the eccentric shown, a disk of 2-in. radius revolves about shaft *O* that is located 0.5 in. from the center *A* of the disk. The distance between the center *A* of the disk and the pin at *B* is 8 in. Knowing that the angular velocity of the disk is 900 rpm clockwise, determine the velocity of the block when u = 30°.

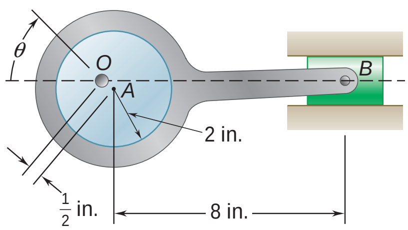


Fig. *P15.60*

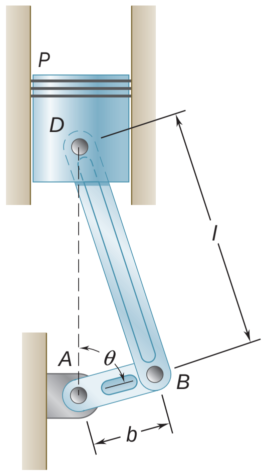


Fig. P15.61 and P15.62

15.61 In the engine system shown, l = 160 mm and b = 60 mm. Knowing that the crank AB rotates with a constant angular velocity of 1000 rpm clockwise, determine the velocity of the piston P and the angular velocity of the connecting rod when (*a*) u = 0, (*b*) u = 90°.

15.62 In the engine system shown, l = 160 mm and b = 60 mm. Knowing that crank AB rotates with a constant angular velocity of 1000 rpm clockwise, determine the velocity of the piston P and the angular velocity of the connecting rod when u = 60°.

15.63 Knowing that at the instant shown the angular velocity of rod AB is 15 rad/s clockwise, determine (*a*) the angular velocity of rod BD, (*b*) the velocity of the midpoint of rod BD.

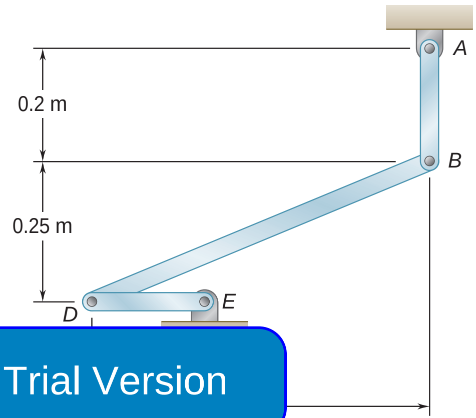


Fig. P15.63

15.64 and 15.65 In the position shown, bar AB has an angular velocity of 4 rad/s clockwise. Determine the angular velocity of bars BD and DE.

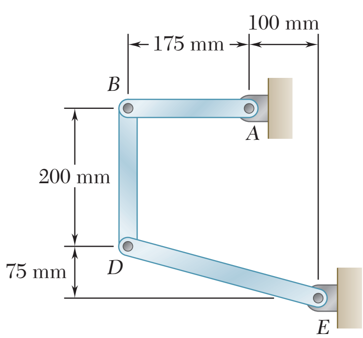


Fig. P15.64

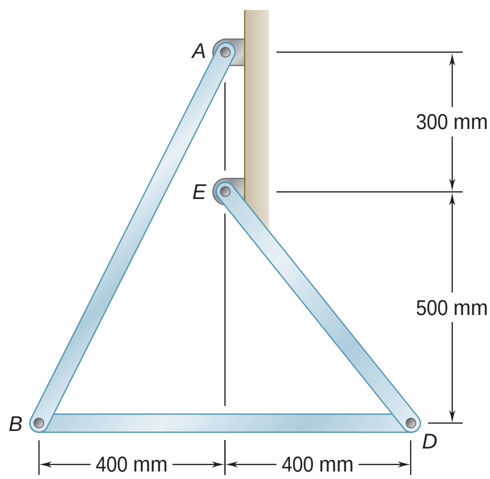


Fig. P15.65

15.66 Roberts linkage is named after Richard Roberts (1789–1864) and can be used to draw a close approximation to a straight line by locating a pen at point *F*. The distance *AB* is the same as *BF*, *DF*, and *DE*. Knowing that the angular velocity of bar *AB* is 5 rad/s clockwise in the position shown, determine (*a*) the angular velocity of bar *DE*, (*b*) the velocity of point *F*.

15.67 Roberts linkage is named after Richard Roberts (1789–1864) and can be used to draw a close approximation to a straight line by locating a pen at point *F*. The distance *AB* is the same as *BF*, *DF*, and *DE*. Knowing that the angular velocity of plate *BDF* is 2 rad/s counterclockwise when u = 90°, determine (*a*) the angular velocities of bars *AB* and *DE*, (*b*) the velocity of point *F*. When u = 90°, point *F* may be assumed to coincide with point *E*, with negligible error in the velocity analysis.

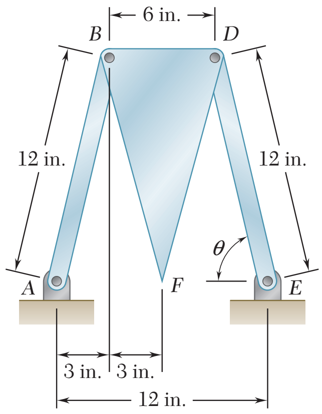


Fig. *P15.66* and *P15.67*

15.68 In the position shown, bar *DE* has a constant angular velocity of 10 rad/s clockwise. Knowing that $h = 500$ mm, determine (*a*) the angular velocity of bar *FBD*, (*b*) the velocity of point *F*.

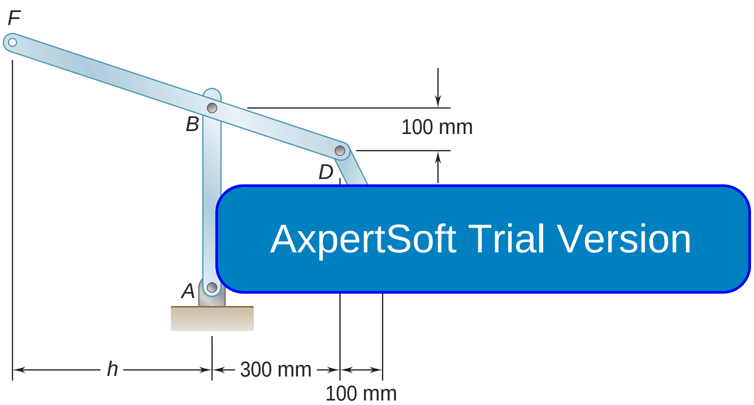


Fig. P15.68 and P15.69

15.69 In the position shown, bar *DE* has a constant angular velocity of 10 rad/s clockwise. Determine (*a*) the distance *h* for which the velocity of point *F* is vertical, (*b*) the corresponding velocity of point *F*.

15.70 Both 6-in.-radius wheels roll without slipping on the horizontal surface. Knowing that the distance *AD* is 5 in., the distance *BE* is 4 in., and *D* has a velocity of 6 in./s to the right, determine the velocity of point *E*.

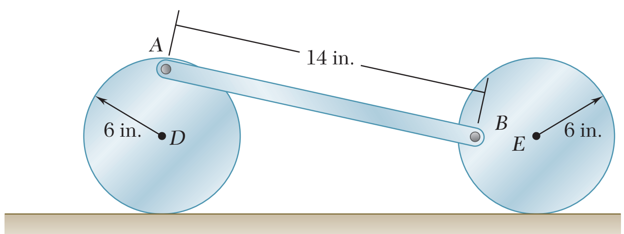


Fig. P15.70

15.71 The 80-mm-radius wheel shown rolls to the left with a velocity of 900 mm/s. Knowing that the distance *AD* is 50 mm, determine the velocity of the collar and the angular velocity of rod *AB* when (*a*) $\beta = 0$, (*b*) $\beta = 90°$.

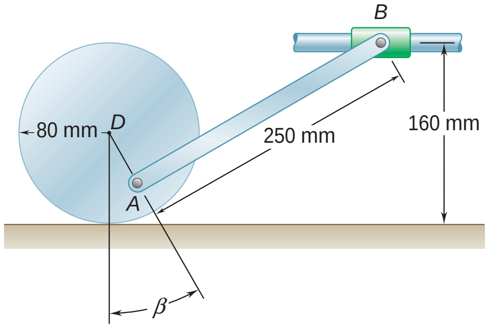


Fig. P15.71

***15.72** For the gearing shown, derive an expression for the angular velocity ω_C of gear *C* and show that ω_C is independent of the radius of gear *B*. Assume that point *A* is fixed and denote the angular velocities of rod *ABC* and gear *A* by ω_{ABC} and ω_A, respectively.

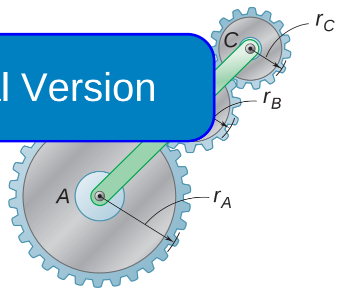


Fig. P15.72

15.7 INSTANTANEOUS CENTER OF ROTATION IN PLANE MOTION

Photo 15.5 If the tires of this car are rolling without sliding, the instantaneous center of rotation of a tire is the point of contact between the road and the tire.

Consider the general plane motion of a slab. We propose to show that at any given instant the velocities of the various particles of the slab are the same as if the slab were rotating about a certain axis perpendicular to the plane of the slab, called the *instantaneous axis of rotation.* This axis intersects the plane of the slab at a point *C*, called the *instantaneous center of rotation* of the slab.

We first recall that the plane motion of a slab can always be replaced by a translation defined by the motion of an arbitrary reference point *A* and by a rotation about *A*. As far as the velocities are concerned, the translation is characterized by the velocity $\mathbf{v}_A$ of the reference point *A* and the rotation is characterized by the angular velocity ω of the slab (which is independent of the choice of *A*). Thus, the velocity $\mathbf{v}_A$ of point *A* and the angular velocity ω of the slab define

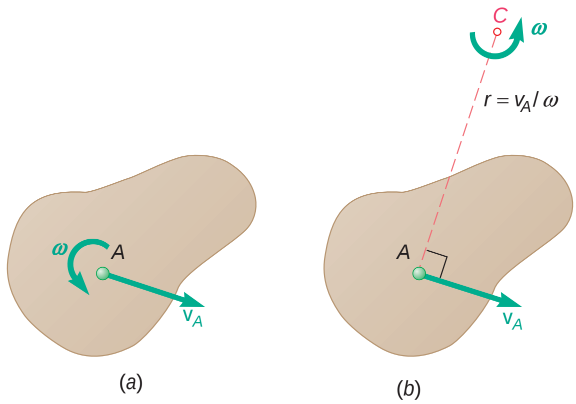


Fig. 15.18

completely the velocities of all the other particles of the slab (Fig. 15.18*a*). Now let us assume that $\mathbf{v}_A$ and ω are known and that they are both different from zero. (If $\mathbf{v}_A = 0$, point A is itself the instantaneous center of rotation, and if $\omega = 0$, all the particles have the same velocity $\mathbf{v}_A$.) These velocities could be obtained by letting the slab rotate with the angular velocity ω about a point C located on the perpendicular to $\mathbf{v}_A$ at a distance $r = v_A/\omega$ from A as shown in Fig. 15.18*b*. We check that the velocity of A would be perpendicular to AC and that its magnitude would be $r\omega = (v_A/\omega)\omega = v_A$. Thus the velocities of all the other particles of the slab would be the same as originally defined. Therefore, *as far as the velocitie* [illegible] *about the instantaneous center* [illegible]

The position of the instan[illegible] other ways. If the directions of the velocities of two particles A and B of the slab are known and if they are different, the instantaneous center C is obtained by drawing the perpendicular to $\mathbf{v}_A$ through A and the perpendicular to $\mathbf{v}_B$ through B and determining the point in which these two lines intersect (Fig. 15.19*a*). If the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of two particles A and B are perpendicular to the line AB and if their magnitudes are known, the instantaneous center can be found by intersecting the line AB with the line joining the extremities of the vectors $\mathbf{v}_A$ and $\mathbf{v}_B$ (Fig. 15.19*b*). Note that if $\mathbf{v}_A$ and $\mathbf{v}_B$ were parallel

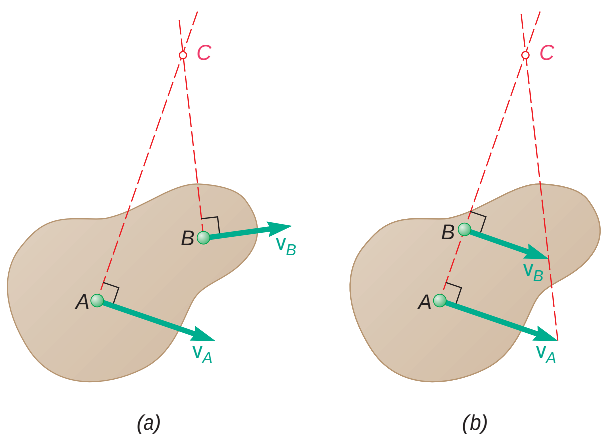


Fig. 15.19

in Fig. 15.19*a* or if $\mathbf{v}_A$ and $\mathbf{v}_B$ had the same magnitude in Fig. 15.19*b*, the instantaneous center C would be at an infinite distance and ω would be zero; all points of the slab would have the same velocity.

To see how the concept of instantaneous center of rotation can be put to use, let us consider again the rod of Sec. 15.6. Drawing the perpendicular to $\mathbf{v}_A$ through A and the perpendicular to $\mathbf{v}_B$ through B (Fig. 15.20), we obtain the instantaneous center C. At the

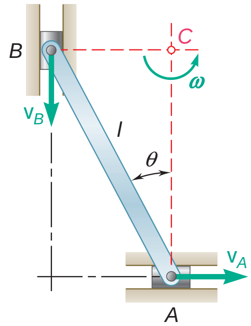


Fig. 15.20

instant considered, the velocities of all the particles of the rod are thus the same as if the rod rotated about C. Now, if the magnitude v_A of the velocity of A is known, the magnitude ω of the angular velocity of the rod can be obtained by writing

$$[illegible] \frac{v_A}{[illegible] \cos\theta}$$

[illegible] an then be obtained by writing

$$v_B = (BC)\omega = l \sin\theta \frac{v_A}{l\cos\theta} = v_A \tan\theta$$

Note that only *absolute* velocities are involved in the computation.

The instantaneous center of a slab in plane motion can be located either on the slab or outside the slab. If it is located on the slab, the particle C coinciding with the instantaneous center at a given instant t must have zero velocity at that instant. However, it should be noted that the instantaneous center of rotation is valid only at a given instant. Thus, the particle C of the slab which coincides with the instantaneous center at time t will generally not coincide with the instantaneous center at time $t + \Delta t$; while its velocity is zero at time t, it will probably be different from zero at time $t + \Delta t$. This means that, in general, the particle C *does not have zero acceleration* and, therefore, that the *accelerations* of the various particles of the slab *cannot* be determined as if the slab were rotating about C.

As the motion of the slab proceeds, the instantaneous center moves in space. But it was just pointed out that the position of the instantaneous center on the slab keeps changing. Thus, the instantaneous center describes one curve in space, called the *space centrode*, and another curve on the slab, called the *body centrode* (Fig. 15.21). It can be shown that at any instant, these two curves are tangent at C and that as the slab moves, the body centrode appears to *roll* on the space centrode.

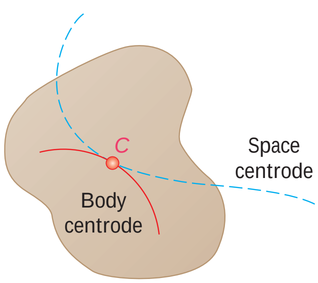
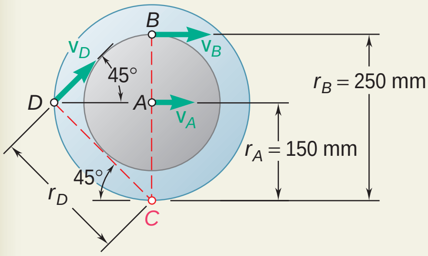


Fig. 15.21

SAMPLE PROBLEM 15.4

Solve Sample Prob. 15.2, using the method of the instantaneous center of rotation.



SOLUTION

a. Angular Velocity of the Gear. Since the gear rolls on the stationary lower rack, the point of contact C of the gear with the rack has no velocity; point C is therefore the instantaneous center of rotation. We write

$$v_A = r_A\omega \qquad 1.2 \text{ m/s} = (0.150 \text{ m})\omega$$

$$\omega = 8 \text{ rad/s i} \quad ◀$$

b. Velocities. As far as velocities are concerned, all points of the gear seem to rotate about the instantaneous center.

Velocity of Upper Rack. Recalling that $v_R = v_B$, we write

$$v_R = v_B = r_B\omega \qquad v_R = (0.250 \text{ m})(8 \text{ rad/s}) = 2 \text{ m/s}$$

$$\mathbf{v}_R = 2 \text{ m/s y} \quad ◀$$

Velocity of Point *D*. Since $r_D = (0.150 \text{ m})\sqrt{2} = 0.2121$ m, we write

$$v_D = r_D\omega \qquad v_D = (0.2121 \text{ m})(8 \text{ rad/s}) = 1.697 \text{ m/s}$$

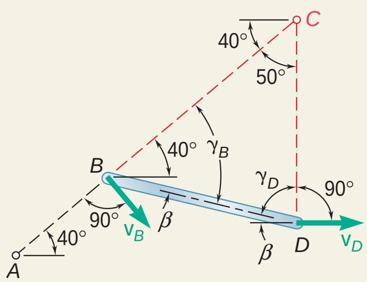
$$\mathbf{v}_D = 1.697 \text{ m/s a } 45° \quad ◀$$

Solve Sample Prob. 15.3, using the method of the instantaneous center of rotation.

SOLUTION

Motion of Crank *AB*. Referring to Sample Prob. 15.3, we obtain the velocity of point B; $\mathbf{v}_B = 628.3$ in./s c 50°.

Motion of the Connecting Rod *BD*. We first locate the instantaneous center C by drawing lines perpendicular to the absolute velocities $\mathbf{v}_B$ and $\mathbf{v}_D$. Recalling from Sample Prob. 15.3 that $\beta = 13.95°$ and that $BD = 8$ in., we solve the triangle BCD.



$$\gamma_B = 40° + \beta = 53.95° \qquad \gamma_D = 90° - \beta = 76.05°$$

$$\frac{BC}{\sin 76.05°} = \frac{CD}{\sin 53.95°} = \frac{8 \text{ in.}}{\sin 50°}$$

$$BC = 10.14 \text{ in.} \qquad CD = 8.44 \text{ in.}$$

Since the connecting rod BD seems to rotate about point C, we write

$$v_B = (BC)\omega_{BD}$$
$$628.3 \text{ in./s} = (10.14 \text{ in.})\omega_{BD}$$

$$\omega_{BD} = 62.0 \text{ rad/s l} \quad ◀$$

$$v_D = (CD)\omega_{BD} = (8.44 \text{ in.})(62.0 \text{ rad/s})$$
$$= 523 \text{ in./s} = 43.6 \text{ ft/s}$$

$$\mathbf{v}_P = \mathbf{v}_D = 43.6 \text{ ft/s y} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we introduced the *instantaneous center of rotation* in plane motion. This provides us with an alternative way for solving problems involving the *velocities* of the various points of a body in plane motion.

As its name suggests, the *instantaneous center of rotation* is the point about which you can assume a body is rotating at a given instant, as you determine the velocities of the points of the body at that instant.

A. To determine the instantaneous center of rotation of a body in plane motion, you should use one of the following procedures.

1. If the velocity $\mathbf{v}_A$ of a point *A* and the angular velocity ω of the body are both known (Fig. 15.18):

a. Draw a sketch of the body, showing point A, its velocity $\mathbf{v}_A$, and the angular velocity ω of the body.

b. From *A* draw a line perpendicular to $\mathbf{v}_A$ on the side of $\mathbf{v}_A$ from which this velocity is viewed as having *the same sense as* ω.

c. Locate the instantaneous center *C* on this line, at a distance $r = v_A/\omega$ from point A.

2. If the directions [illegible] re known and are different (Fig. 1[illegible]

a. Draw a sketch of the body, showing points A and B and their velocities $\mathbf{v}_A$ and $\mathbf{v}_B$.

b. From *A* and *B* draw lines perpendicular to $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively. The instantaneous center C is located at the point where the two lines intersect.

c. If the velocity of one of the two points is known, you can determine the angular velocity of the body. For example, if you know v_A, you can write $\omega = v_A/AC$, where AC is the distance from point A to the instantaneous center C.

3. If the velocities of two points *A* and *B* are known and are both perpendicular to the line *AB* (Fig. 15.19*b*):

a. Draw a sketch of the body, showing points A and B with their velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ *drawn to scale.*

b. Draw a line through points *A* and *B*, and another line through the tips of the vectors $\mathbf{v}_A$ and $\mathbf{v}_B$. The instantaneous center C is located at the point where the two lines intersect.

c. The angular velocity of the body is obtained by either dividing $\mathbf{v}_A$ by AC or $\mathbf{v}_B$ by BC.

d. If the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ have the same magnitude, the two lines drawn in part b do not intersect; the instantaneous center C is at an infinite distance. The angular velocity V is zero and *the body is in translation.*

B. Once you have determined the instantaneous center and the angular velocity of a body, you can determine the velocity $\mathbf{v}_P$ of any point P of the body in the following way.

1. Draw a sketch of the body, showing point P, the instantaneous center of rotation C, and the angular velocity V.

2. Draw a line from P to the instantaneous center C and measure or calculate the distance from P to C.

3. The velocity $\mathbf{v}_P$ is a vector perpendicular to the line PC, of the same sense as V, and of magnitude $v_P = (PC)$v.

Finally, keep i[illegible]ion can be used *only* to determine ve[illegible]*erations.*

PROBLEMS

CONCEPT QUESTIONS

15.CQ5 The disk rolls without sliding on the fixed horizontal surface. At the instant shown, the instantaneous center of zero velocity for rod *AB* would be located in which region?

a. Region 1
b. Region 2
c. Region 3
d. Region 4
e. Region 5
f. Region 6

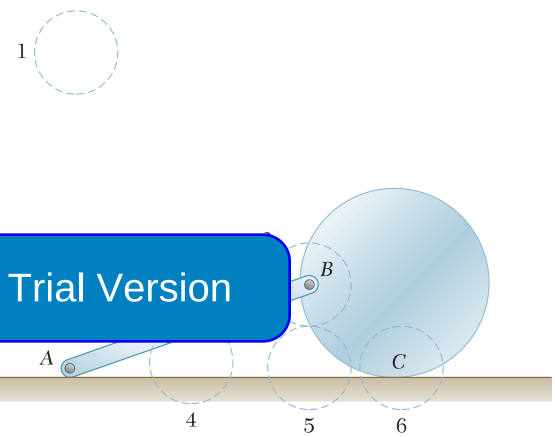


Fig. P15.CQ5

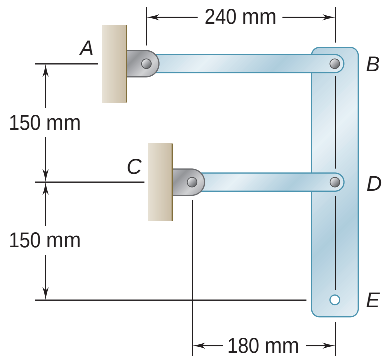


Fig. P15.CQ6

15.CQ6 Bar *BDE* is pinned to two links, *AB* and *CD*. At the instant shown, the angular velocities of link *AB*, link *CD*, and bar *BDE* are ω_{AB}, ω_{CD}, and ω_{BDE}, respectively. Which of the following statements concerning the angular speeds of the three objects is true at this instant?

a. $\omega_{AB} = \omega_{CD} = \omega_{BDE}$
b. $\omega_{BDE} > \omega_{AB} > \omega_{CD}$
c. $\omega_{AB} = \omega_{CD} > \omega_{BDE}$
d. $\omega_{AB} > \omega_{CD} > \omega_{BDE}$
e. $\omega_{CD} > \omega_{AB} > \omega_{BDE}$

END-OF-SECTION PROBLEMS

15.73 A juggling club is thrown vertically into the air. The center of gravity *G* of the 20-in. club is located 12 in. from the knob. Knowing that at the instant shown, *G* has a velocity of 4 ft/s upwards and the club has an angular velocity of 30 rad/s counterclockwise, determine (*a*) the speeds of points *A* and *B*, (*b*) the location of the instantaneous center of rotation.

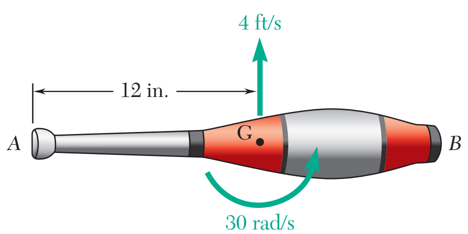


Fig. *P15.73*

15.74 A 10-ft beam *AE* is being lowered by means of two overhead cranes. At the instant shown, it is known that the velocity of point *D* is 24 in./s downward and the velocity of point *E* is 36 in./s downward. Determine (*a*) the instantaneous center of rotation of the beam, (*b*) the velocity of point *A*.

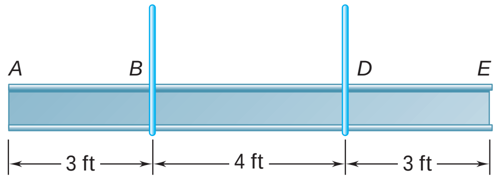


Fig. P15.74

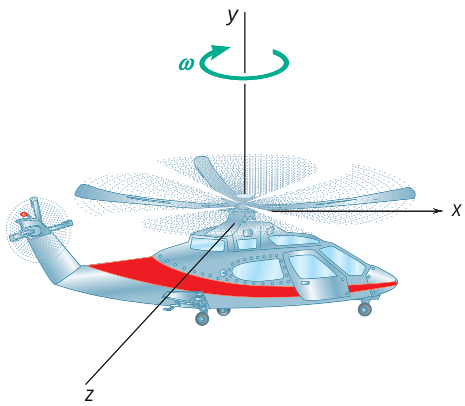


Fig. P15.75

15.75 A helicopter moves horizontally in the *x* direction at a speed of 120 mi/h. Knowing that the main blades rotate clockwise with an angular velocity of 180 rpm, determine the instantaneous axis of rotation of the main blades.

15.76 and 15.77 A 60-mm-radius drum is rigidly attached to a 100-mm-radius drum as shown. One of the drums rolls without sliding on the surface shown, and a cord is wound around the other drum. Knowing that end *E* of the cord is pulled to the left with a velocity of 120 mm/s, determine (*a*) the angular velocity of the drums, (*b*) the velocity of the center of the drums, (*c*) the length of cord wound or unwou

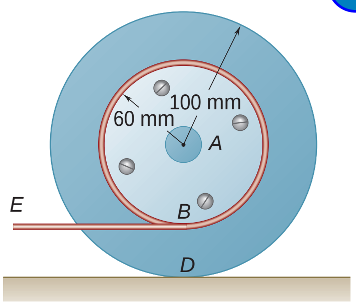


Fig. P15.76

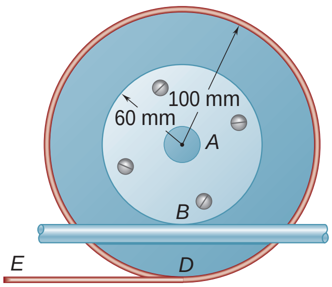


Fig. P15.77

15.78 The spool of tape shown and its frame assembly are pulled upward at a speed $v_A = 750$ mm/s. Knowing that the 80-mm-radius spool has an angular velocity of 15 rad/s clockwise and that at the instant shown the total thickness of the tape on the spool is 20 mm, determine (*a*) the instantaneous center of rotation of the spool, (*b*) the velocities of points *B* and *D*.

15.79 The spool of tape shown and its frame assembly are pulled upward at a speed $v_A = 100$ mm/s. Knowing that end *B* of the tape is pulled downward with a velocity of 300 mm/s and that at the instant shown the total thickness of the tape on the spool is 20 mm, determine (*a*) the instantaneous center of rotation of the spool, (*b*) the velocity of point *D* of the spool.

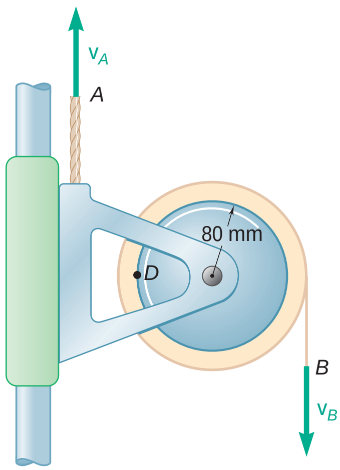


Fig. P15.78 and P15.79

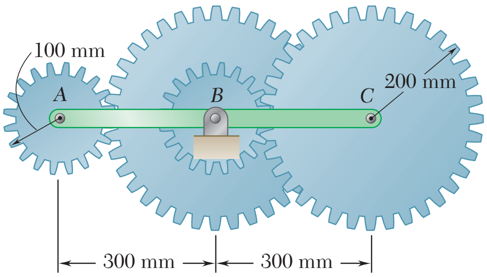


Fig. P15.80

15.80 The arm ABC rotates with an angular velocity of 4 rad/s counterclockwise. Knowing that the angular velocity of the intermediate gear B is 8 rad/s counterclockwise, determine (*a*) the instantaneous centers of rotation of gears A and C, (*b*) the angular velocities of gears A and C.

15.81 The double gear rolls on the stationary left rack R. Knowing that the rack on the right has a constant velocity of 2 ft/s, determine (*a*) the angular velocity of the gear, (*b*) the velocities of points A and D.

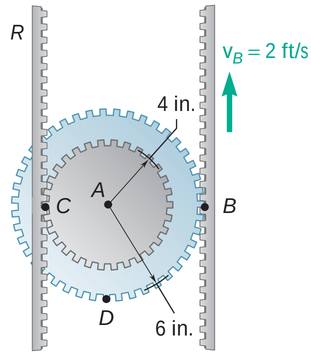


Fig. *P15.81*

…heels at A and B that roll in hori-
…ng that when u = 40° the velocity
… determine (*a*) the angular velocity of the door, (*b*) the velocity of end D of the door.

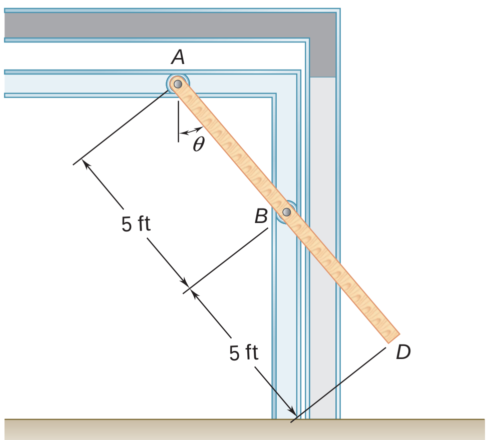


Fig. P15.82

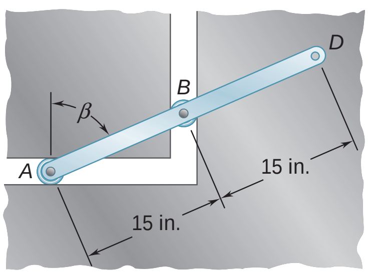


Fig. P15.83

15.83 Rod ABD is guided by wheels at A and B that roll in horizontal and vertical tracks. Knowing that at the instant b = 60° and the velocity of wheel B is 40 in./s downward, determine (*a*) the angular velocity of the rod, (*b*) the velocity of point D.

15.84 Rod BDE is partially guided by a roller at D which moves in a vertical track. Knowing that at the instant shown the angular velocity of crank AB is 5 rad/s clockwise and that b = 25°, determine (*a*) the angular velocity of the rod, (*b*) the velocity of point E.

15.85 Rod BDE is partially guided by a roller at D which moves in a vertical track. Knowing that at the instant shown b = 30°, point E has a velocity of 2 m/s down and to the right, determine the angular velocities of rod BDE and crank AB.

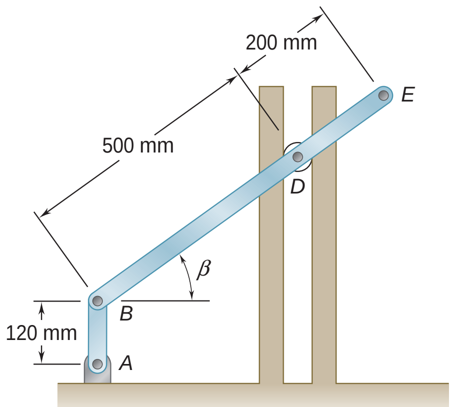


Fig. *P15.84* and *P15.85*

15.86 Knowing that at the instant shown, the velocity of collar D is 1.6 m/s upward, determine (*a*) the angular velocity of rod AD, (*b*) the velocity of point B, (*c*) the velocity of point A.

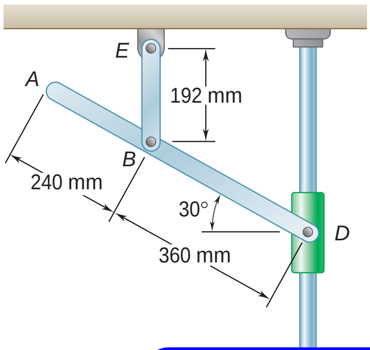


Fig. P15.86

15.87 Knowing that at the instant shown, the angular velocity of rod BE is 4 rad/s counterclockwise, determine (*a*) the angular velocity of rod AD, (*b*) the velocity of collar D, (*c*) the velocity of point A.

15.88 Rod AB can slide freely along the floor and the inclined plane. Denoting by $\mathbf{v}_A$ the velocity of point A, derive an expression for (*a*) the angular velocity of the rod, (*b*) the velocity of end B.

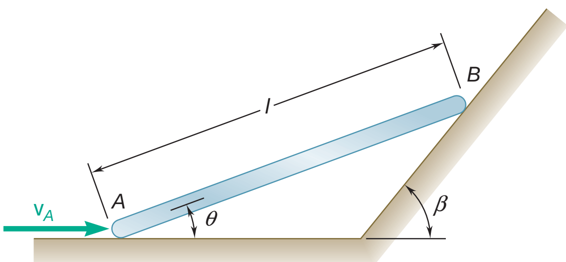


Fig. P15.88 and P15.89

15.89 Rod AB can slide freely along the floor and the inclined plane. Knowing that u = 20°, b = 50°, l = 2 ft, and v_A = 8 ft/s, determine (*a*) the angular velocity of the rod, (*b*) the velocity of end B.

15.90 Two slots have been cut in plate FG and the plate has been placed so that the slots fit two fixed pins A and B. Knowing that at the instant shown the angular velocity of crank DE is 6 rad/s clockwise, determine (*a*) the velocity of point F, (*b*) the velocity of point G.

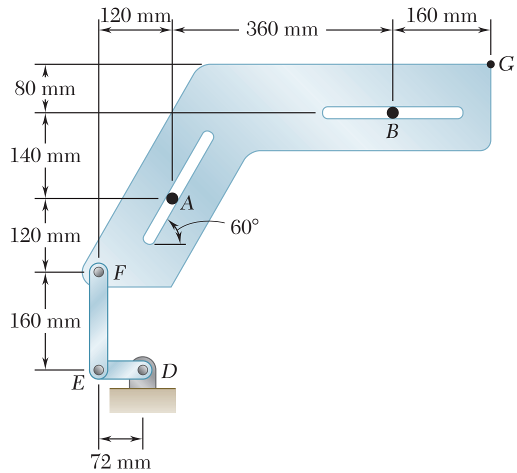


Fig. P15.90

15.91 The disk is released from rest and rolls down the incline. Knowing that the speed of A is 1.2 m/s when $u = 0°$, determine at that instant (*a*) the angular velocity of the rod, (*b*) the velocity of B. (Only portions of the two tracks are shown.)

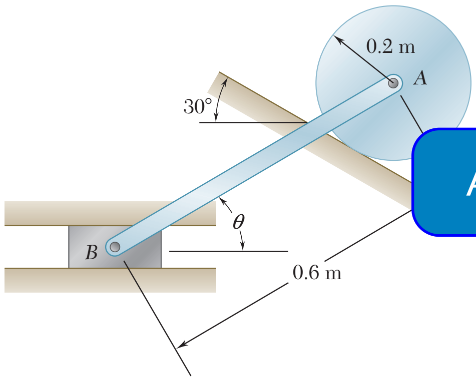


Fig. *P15.91*

s to a collar at B and to crank of collar B is 400 mm/s upward, ity of arm ABD, (*b*) the velocity of point A.

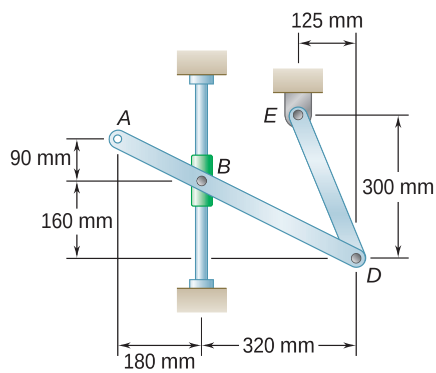


Fig. *P15.92* and *P15.93*

15.93 Arm ABD is connected by pins to a collar at B and to crank DE. Knowing that the angular velocity of crank DE is 1.2 rad/s counterclockwise, determine (*a*) the angular velocity of arm ABD, (*b*) the velocity of point A.

15.94 Two links AB and BD, each 25 in. long, are connected at B and guided by hydraulic cylinders attached at A and D. Knowing that D is stationary and that the velocity of A is 30 in./s to the right, determine at the instant shown (*a*) the angular velocity of each link, (*b*) the velocity of B.

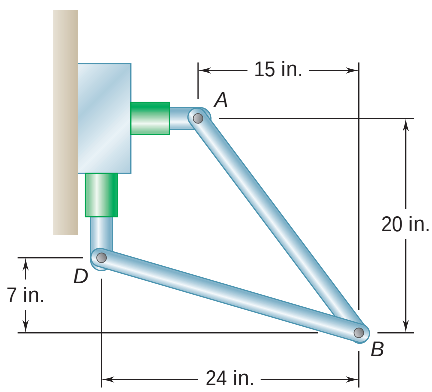


Fig. P15.94

15.95 Two 25-in. rods are pin-connected at D as shown. Knowing that B moves to the left with a constant velocity of 24 in./s, determine at the instant shown (*a*) the angular velocity of each rod, (*b*) the velocity of E.

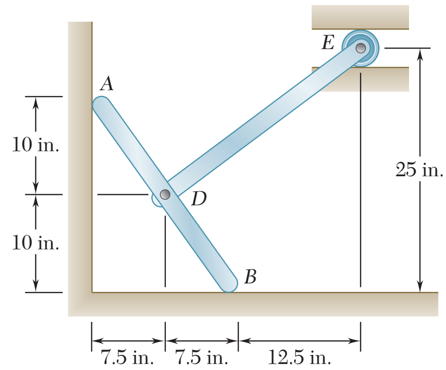


Fig. P15.95

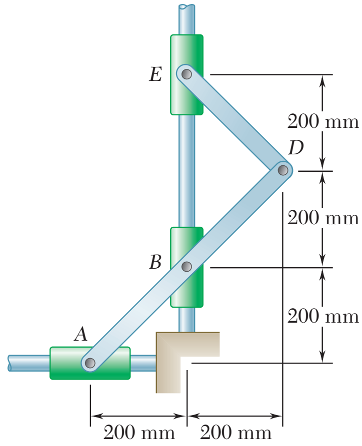


Fig. P15.96

15.96 Two rods ABD and DE are connected to three collars as shown. Knowing that the angular velocity of ABD is 5 rad/s clockwise, determine at the instant shown (*a*) the angular velocity of DE, (*b*) the velocity of collar E.

15.97 Two collars C and D move along the vertical rod shown. Knowing that the velocity of collar C is 660 mm/s downward, determine (*a*) the velocity of collar D, (*b*) the angular velocity of member AB.

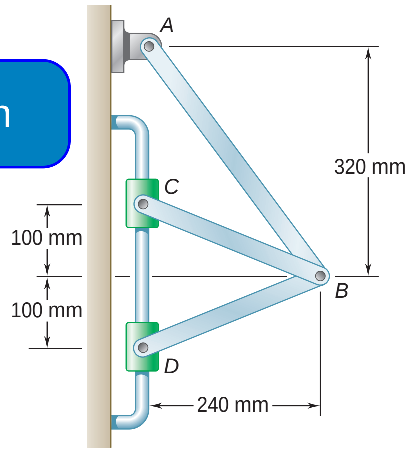


Fig. P15.97

15.98 Two rods AB and DE are [illegible] D moves to the left wit[illegible] angular velocity of each r[illegible]

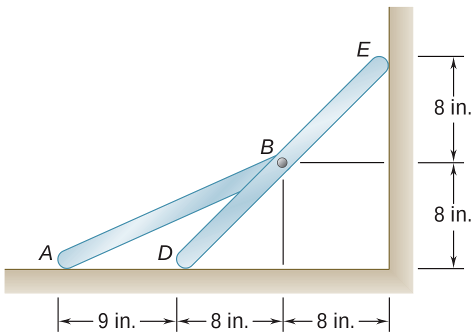


Fig. P15.98

15.99 Describe the space centrode and the body centrode of rod ABD of Prob. 15.83. (*Hint:* The body centrode need not lie on a physical portion of the rod.)

15.100 Describe the space centrode and the body centrode of the gear of Sample Prob. 15.2 as the gear rolls on the stationary horizontal rack.

15.101 Using the method of Sec. 15.7, solve Prob. 15.60.

15.102 Using the method of Sec. 15.7, solve Prob. 15.64.

15.103 Using the method of Sec. 15.7, solve Prob. 15.65.

15.104 Using the method of Sec. 15.7, solve Prob. 15.38.

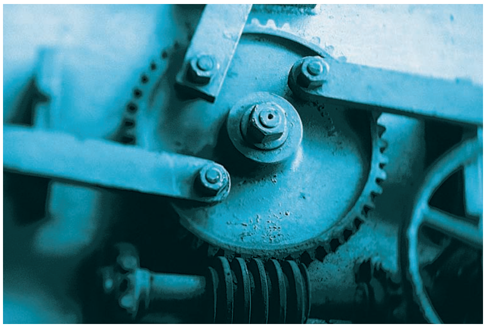


Photo 15.6 The central gear rotates about a fixed axis and is pin-connected to three bars which are in general plane motion.

15.8 ABSOLUTE AND RELATIVE ACCELERATION IN PLANE MOTION

We saw in Sec. 15.5 that any plane motion can be replaced by a translation defined by the motion of an arbitrary reference point A and a simultaneous rotation about A. This property was used in Sec. 15.6 to determine the velocity of the various points of a moving slab. The same property will now be used to determine the acceleration of the points of the slab.

We first recall that the absolute acceleration $\mathbf{a}_B$ of a particle of the slab can be obtained from the relative-acceleration formula derived in Sec. 11.12,

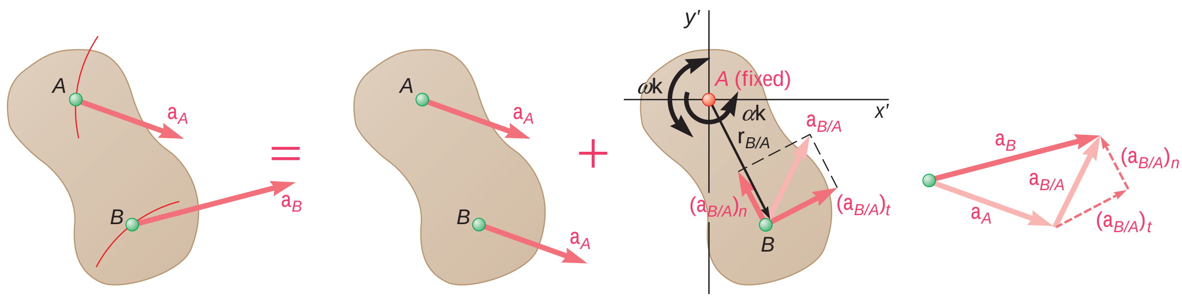
$$\boxed{\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}} \qquad (15.21)$$

where the right-hand member represents a vector sum. The acceleration $\mathbf{a}_A$ corresponds to the translation of the slab with A, while the relative acceleration $\mathbf{a}_{B/A}$ is associated with the rotation of the slab about A and is measured with respect to axes centered at A and of fixed orientation. We recall from Sec. 15.3 that the relative acceleration $\mathbf{a}_{B/A}$ can be resolved into two components, a *tangential component* $(\mathbf{a}_{B/A})_t$ perpendicular to the line AB, and a *normal component* $(\mathbf{a}_{B/A})_n$ directed toward A (Fig. 15.22). Denoting by $\mathbf{r}_{B/A}$ the position ... ively, by $v\mathbf{k}$ and $a\mathbf{k}$ the angular ... he slab with respect to axes of

$$\begin{aligned} (\mathbf{a}_{B/A})_t &= a\mathbf{k} \times \mathbf{r}_{B/A} \qquad & (a_{B/A})_t &= ra \\ (\mathbf{a}_{B/A})_n &= -v^2\mathbf{r}_{B/A} \qquad & (a_{B/A})_n &= rv^2 \end{aligned} \qquad (15.22)$$

where r is the distance from A to B. Substituting into (15.21) the expressions obtained for the tangential and normal components of $\mathbf{a}_{B/A}$, we can also write

$$\mathbf{a}_B = \mathbf{a}_A + a\mathbf{k} \times \mathbf{r}_{B/A} - v^2\mathbf{r}_{B/A} \qquad (15.21')$$



Plane motion = Translation with *A* + Rotation about *A*

Fig. 15.22

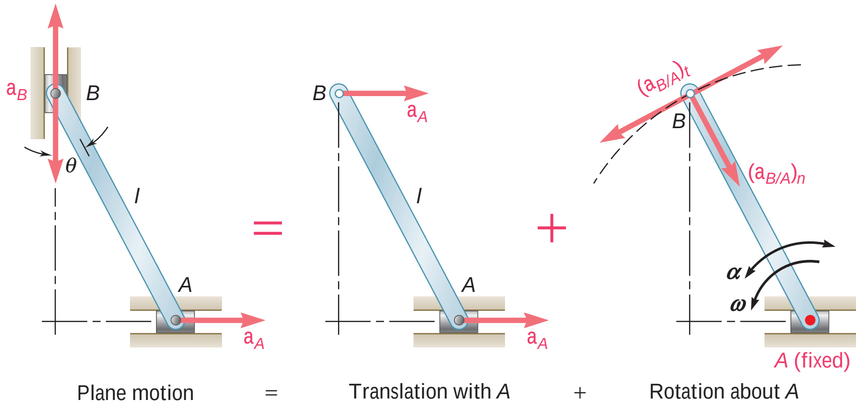


Fig. 15.23

As an example, let us again consider the rod AB whose extremities slide, respectively, along a horizontal and a vertical track (Fig. 15.23). Assuming that the velocity $\mathbf{v}_A$ and the acceleration $\mathbf{a}_A$ of A are known, we propose to determine the acceleration $\mathbf{a}_B$ of B and the angular acceleration A of the rod. Choosing A as a reference point, we express that the given motion is equivalent to a translation with A and a rotation about A. The absolute acceleration of B must be equal to the sum

$$\mathbf{a}_B = \mathbf{a}_A +$$
$$= \mathbf{a}_A +$$

where $(\mathbf{a}_{B/A})_n$ has the magnitude lv^2 and is *directed toward A*, while $(\mathbf{a}_{B/A})_t$ has the magnitude la and is perpendicular to AB. Students should note that there is no way to tell whether the tangential component $(\mathbf{a}_{B/A})_t$ is directed to the left or to the right, and therefore both possible directions for this component are indicated in Fig. 15.23. Similarly, both possible senses for $\mathbf{a}_B$ are indicated, since it is not known whether point B is accelerated upward or downward.

Equation (15.23) has been expressed geometrically in Fig. 15.24. Four different vector polygons can be obtained, depending upon the sense of $\mathbf{a}_A$ and the relative magnitude of a_A and $(a_{B/A})_n$. If we are to determine a_B and a from one of these diagrams, we must know not only a_A and u but also v. The angular velocity of the rod should therefore be separately determined by one of the methods indicated in Secs. 15.6 and 15.7. The values of a_B and a can then be obtained by considering successively the x and y components of the vectors shown in Fig. 15.24. In the case of polygon a, for example, we write

y x components: $\quad 0 = a_A + l$v^2 sin u $- l$a cos u
+x y components: $\quad -a_B = -l$v^2 cos u $- l$a sin u

and solve for a_B and a. The two unknowns can also be obtained by direct measurement on the vector polygon. In that case, care should be taken to draw first the known vectors $\mathbf{a}_A$ and $(\mathbf{a}_{B/A})_n$.

It is quite evident that the determination of accelerations is considerably more involved than the determination of velocities. Yet

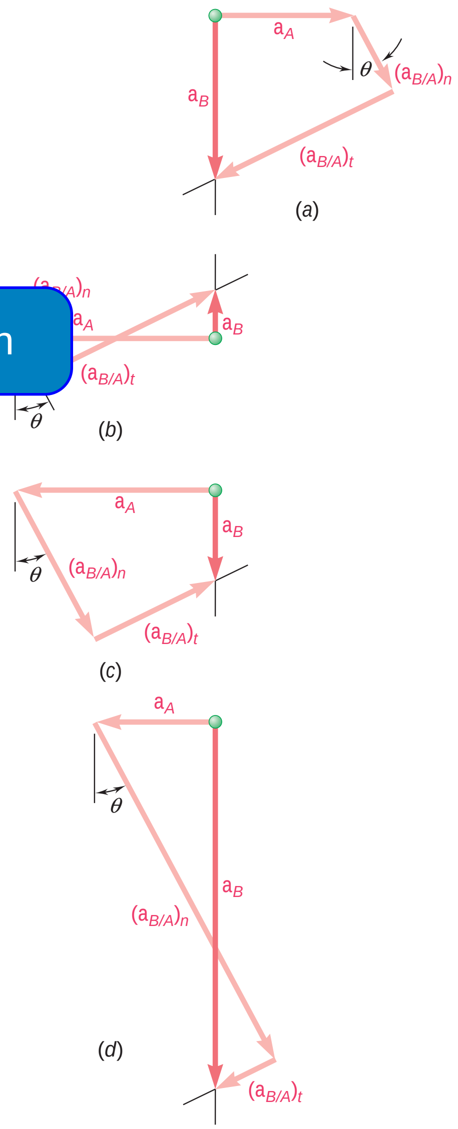


Fig. 15.24

in the example considered here, the extremities A and B of the rod were moving along straight tracks, and the diagrams drawn were relatively simple. If A and B had moved along curved tracks, it would have been necessary to resolve the accelerations $\mathbf{a}_A$ and $\mathbf{a}_B$ into normal and tangential components and the solution of the problem would have involved six different vectors.

When a mechanism consists of several moving parts which are pin-connected, the analysis of the mechanism can be carried out by considering each part as a rigid body, keeping in mind that the points at which two parts are connected must have the same absolute acceleration (see Sample Prob. 15.7). In the case of meshed gears, the tangential components of the accelerations of the teeth in contact are equal, but their normal components are different.

*15.9 ANALYSIS OF PLANE MOTION IN TERMS OF A PARAMETER

In the case of certain mechanisms, it is possible to express the coordinates x and y of all the significant points of the mechanism by means of simple analytic expressions containing a single parameter. It is sometimes advantageous in such a case to determine the absolute velocity and the absolute acceleration of the various points of the mechanism directly, since the components of the velocity and [illegible] be obtained by differentiating [illegible] AB whose extremities slide, respectively, in a horizontal and a vertical track (Fig. 15.25). The coordinates x_A and y_B of the extremities of the rod can be expressed in terms of the angle θ the rod forms with the vertical:

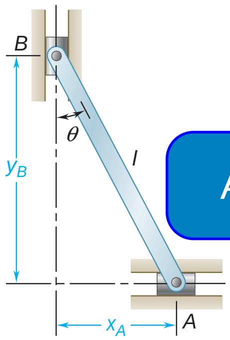


Fig. 15.25

$$x_A = l \sin \theta \qquad y_B = l \cos \theta \tag{15.24}$$

Differentiating Eqs. (15.24) twice with respect to t, we write

$$\begin{aligned} v_A &= \dot{x}_A = l\dot{\theta} \cos \theta \\ a_A &= \ddot{x}_A = -l\dot{\theta}^2 \sin \theta + l\ddot{\theta} \cos \theta \end{aligned}$$

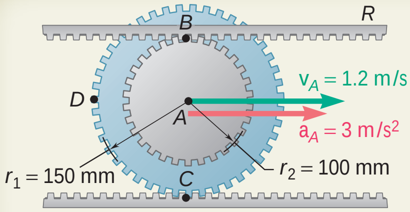
$$\begin{aligned} v_B &= \dot{y}_B = -l\dot{\theta} \sin \theta \\ a_B &= \ddot{y}_B = -l\dot{\theta}^2 \cos \theta - l\ddot{\theta} \sin \theta \end{aligned}$$

Recalling that $\dot{\theta} = \omega$ and $\ddot{\theta} = \alpha$, we obtain

$$v_A = l\omega \cos \theta \qquad v_B = -l\omega \sin \theta \tag{15.25}$$

$$a_A = -l\omega^2 \sin \theta + l\alpha \cos \theta \qquad a_B = -l\omega^2 \cos \theta - l\alpha \sin \theta \tag{15.26}$$

We note that a positive sign for v_A or a_A indicates that the velocity $\mathbf{v}_A$ or the acceleration $\mathbf{a}_A$ is directed to the right; a positive sign for v_B or a_B indicates that $\mathbf{v}_B$ or $\mathbf{a}_B$ is directed upward. Equations (15.25) can be used, for example, to determine v_B and ω when v_A and θ are known. Substituting for ω in (15.26), we can then determine a_B and α if a_A is known.



SAMPLE PROBLEM 15.6

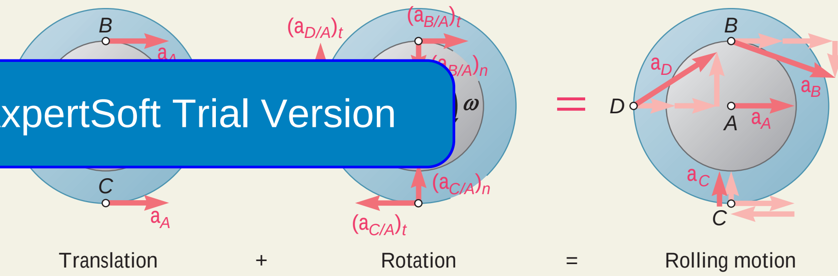
The center of the double gear of Sample Prob. 15.2 has a velocity of 1.2 m/s to the right and an acceleration of 3 m/s^2 to the right. Recalling that the lower rack is stationary, determine (*a*) the angular acceleration of the gear, (*b*) the acceleration of points *B*, *C*, and *D* of the gear.

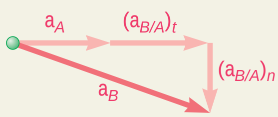
SOLUTION

a. Angular Acceleration of the Gear. In Sample Prob. 15.2, we found that $x_A = -r_1 u$ and $v_A = -r_1 v$. Differentiating the latter with respect to time, we obtain $a_A = -r_1 a$.

$$v_A = -r_1 v \qquad 1.2 \text{ m/s} = -(0.150 \text{ m})v \qquad v = -8 \text{ rad/s}$$
$$a_A = -r_1 a \qquad 3 \text{ m/s}^2 = -(0.150 \text{ m})a \qquad a = -20 \text{ rad/s}^2$$
$$A = a\mathbf{k} = -(20 \text{ rad/s}^2)\mathbf{k} \quad ◀$$

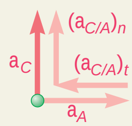
b. Accelerations. The rolling motion of the gear is resolved into a translation with *A* and a rotation about *A*.





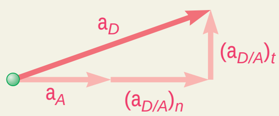
Acceleration of Point *B*. Adding vectorially the accelerations corresponding to the translation and to the rotation, we obtain

$$\begin{aligned} \mathbf{a}_B &= \mathbf{a}_A + \mathbf{a}_{B/A} = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n \\ &= \mathbf{a}_A + a\mathbf{k} \times \mathbf{r}_{B/A} - v^2\mathbf{r}_{B/A} \\ &= (3 \text{ m/s}^2)\mathbf{i} - (20 \text{ rad/s}^2)\mathbf{k} \times (0.100 \text{ m})\mathbf{j} - (8 \text{ rad/s})^2(0.100 \text{ m})\mathbf{j} \\ &= (3 \text{ m/s}^2)\mathbf{i} + (2 \text{ m/s}^2)\mathbf{i} - (6.40 \text{ m/s}^2)\mathbf{j} \end{aligned}$$
$$\mathbf{a}_B = 8.12 \text{ m/s}^2 \text{ c } 52.0° \quad ◀$$



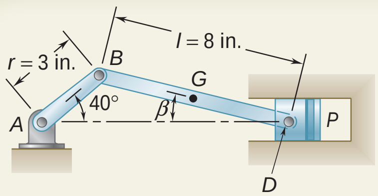
Acceleration of Point *C*

$$\begin{aligned} \mathbf{a}_C &= \mathbf{a}_A + \mathbf{a}_{C/A} = \mathbf{a}_A + a\mathbf{k} \times \mathbf{r}_{C/A} - v^2\mathbf{r}_{C/A} \\ &= (3 \text{ m/s}^2)\mathbf{i} - (20 \text{ rad/s}^2)\mathbf{k} \times (-0.150 \text{ m})\mathbf{j} - (8 \text{ rad/s})^2(-0.150 \text{ m})\mathbf{j} \\ &= (3 \text{ m/s}^2)\mathbf{i} - (3 \text{ m/s}^2)\mathbf{i} + (9.60 \text{ m/s}^2)\mathbf{j} \end{aligned}$$
$$\mathbf{a}_C = 9.60 \text{ m/s}^2 \text{ x} \quad ◀$$



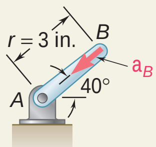
Acceleration of Point *D*

$$\begin{aligned} \mathbf{a}_D &= \mathbf{a}_A + \mathbf{a}_{D/A} = \mathbf{a}_A + a\mathbf{k} \times \mathbf{r}_{D/A} - v^2\mathbf{r}_{D/A} \\ &= (3 \text{ m/s}^2)\mathbf{i} - (20 \text{ rad/s}^2)\mathbf{k} \times (-0.150 \text{ m})\mathbf{i} - (8 \text{ rad/s})^2(-0.150 \text{ m})\mathbf{i} \\ &= (3 \text{ m/s}^2)\mathbf{i} + (3 \text{ m/s}^2)\mathbf{j} + (9.60 \text{ m/s}^2)\mathbf{i} \end{aligned}$$
$$\mathbf{a}_D = 12.95 \text{ m/s}^2 \text{ a } 13.4° \quad ◀$$



SAMPLE PROBLEM 15.7

Crank *AB* of the engine system of Sample Prob. 15.3 has a constant clockwise angular velocity of 2000 rpm. For the crank position shown, determine the angular acceleration of the connecting rod *BD* and the acceleration of point *D*.



SOLUTION

Motion of Crank *AB*. Since the crank rotates about *A* with constant V$_{AB}$ = 2000 rpm = 209.4 rad/s, we have a$_{AB}$ = 0. The acceleration of *B* is therefore directed toward *A* and has a magnitude

$$a_B = r\text{V}_{AB}^2 = (\tfrac{3}{12}\text{ ft})(209.4\text{ rad/s})^2 = 10{,}962\text{ ft/s}^2$$
$$\mathbf{a}_B = 10{,}962\text{ ft/s}^2\ \text{d}\ 40°$$

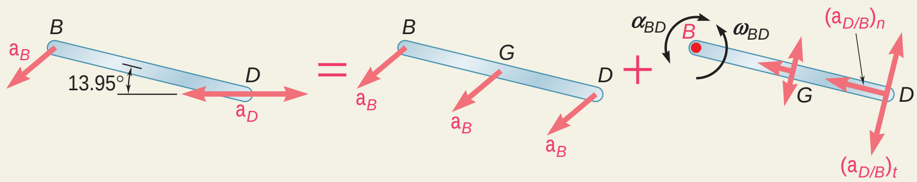
Motion of the Connecting Rod *BD*. The angular velocity V$_{BD}$ and the value of b were obtained in Sample Prob. 15.3:

$$\text{V}_{BD} = 62.0\text{ rad/s l} \qquad \text{b} = 13.95°$$

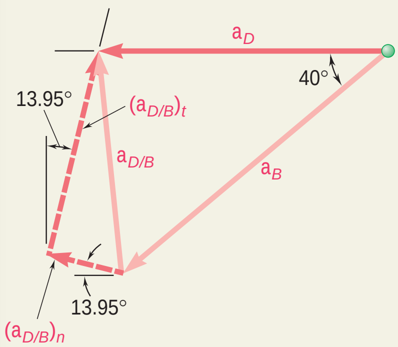
The motion of *BD* is resolved into a translation with *B* and a rotation about *B*. The relative acceleration $\mathbf{a}_{D/B}$ is resolved into normal and tangential components:

$$\ldots)^2 = 2563\text{ ft/s}^2$$
$$\ldots = 2563\text{ ft/s}^2\ \text{b}\ 13.95°$$
$$\ldots 6667\text{a}_{BD}$$
$$\ldots .6667\text{a}_{BD}\ \text{za}\ 76.05°$$

While $(\mathbf{a}_{D/B})_t$ must be perpendicular to *BD*, its sense is not known.



Plane motion = Translation + Rotation



Noting that the acceleration $\mathbf{a}_D$ must be horizontal, we write

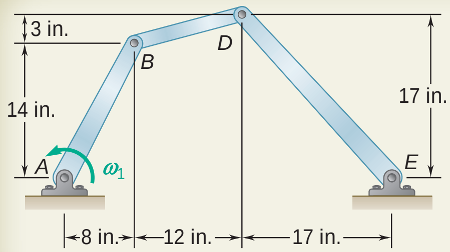
$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B} = \mathbf{a}_B + (\mathbf{a}_{D/B})_n + (\mathbf{a}_{D/B})_t$$
$$[a_D\ \text{G}\,] = [10{,}962\ \text{d}\ 40°] + [2563\ \text{b}\ 13.95°] + [0.6667\text{a}_{BD}\ \text{za}\ 76.05°]$$

Equating *x* and *y* components, we obtain the following scalar equations:

 y$^+$ *x* components:
$$-a_D = -10{,}962\cos 40° - 2563\cos 13.95° + 0.6667\text{a}_{BD}\sin 13.95°$$
+x*y* components:
$$0 = -10{,}962\sin 40° + 2563\sin 13.95° + 0.6667\text{a}_{BD}\cos 13.95°$$

Solving the equations simultaneously, we obtain a$_{BD}$ = +9940 rad/s^2 and a_D = +9290 ft/s^2. The positive signs indicate that the senses shown on the vector polygon are correct; we write

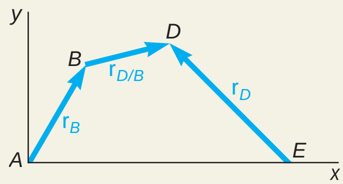
$$\text{a}_{BD} = 9940\text{ rad/s}^2\text{ l} \quad ◀$$
$$\mathbf{a}_D = 9290\text{ ft/s}^2\text{ z} \quad ◀$$



SAMPLE PROBLEM 15.8

The linkage *ABDE* moves in the vertical plane. Knowing that in the position shown crank *AB* has a constant angular velocity $\boldsymbol{\omega}_1$ of 20 rad/s counterclockwise, determine the angular velocities and angular accelerations of the connecting rod *BD* and of the crank *DE*.

SOLUTION



$\mathbf{r}_B = 8\mathbf{i} + 14\mathbf{j}$
$\mathbf{r}_D = -17\mathbf{i} + 17\mathbf{j}$
$\mathbf{r}_{D/B} = 12\mathbf{i} + 3\mathbf{j}$

This problem could be solved by the method used in Sample Prob. 15.7. In this case, however, the vector approach will be used. The position vectors $\mathbf{r}_B$, $\mathbf{r}_D$, and $\mathbf{r}_{D/B}$ are chosen as shown in the sketch.

Velocities. Since the motion of each element of the linkage is contained in the plane of the figure, we have

$$\boldsymbol{\omega}_{AB} = \omega_{AB}\mathbf{k} = (20 \text{ rad/s})\mathbf{k} \qquad \boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k} \qquad \boldsymbol{\omega}_{DE} = \omega_{DE}\mathbf{k}$$

where $\mathbf{k}$ is a unit vector pointing out of the paper. We now write

$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$$
$$\omega_{DE}\mathbf{k} \times \mathbf{r}_D = \omega_{AB}\mathbf{k} \times \mathbf{r}_B + \omega_{BD}\mathbf{k} \times \mathbf{r}_{D/B}$$
$$\ldots(8\mathbf{i} + 14\mathbf{j}) + \omega_{BD}\mathbf{k} \times (12\mathbf{i} + 3\mathbf{j})$$
$$\ldots 280\mathbf{i} + 12\omega_{BD}\mathbf{j} - 3\omega_{BD}\mathbf{i}$$

…ctors $\mathbf{i}$ and $\mathbf{j}$, we obtain the following two scalar equations:

$$-17\omega_{DE} = -280 - 3\omega_{BD}$$
$$-17\omega_{DE} = +160 + 12\omega_{BD}$$
$$\boldsymbol{\omega}_{BD} = -(29.33 \text{ rad/s})\mathbf{k} \qquad \boldsymbol{\omega}_{DE} = (11.29 \text{ rad/s})\mathbf{k} \quad ◀$$

Accelerations. Noting that at the instant considered crank *AB* has a constant angular velocity, we write

$$\boldsymbol{\alpha}_{AB} = 0 \qquad \boldsymbol{\alpha}_{BD} = \alpha_{BD}\mathbf{k} \qquad \boldsymbol{\alpha}_{DE} = \alpha_{DE}\mathbf{k}$$
$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B} \tag{1}$$

Each term of Eq. (1) is evaluated separately:

$$\begin{aligned}
\mathbf{a}_D &= \alpha_{DE}\mathbf{k} \times \mathbf{r}_D - \omega_{DE}^2\mathbf{r}_D \\
&= \alpha_{DE}\mathbf{k} \times (-17\mathbf{i} + 17\mathbf{j}) - (11.29)^2(-17\mathbf{i} + 17\mathbf{j}) \\
&= -17\alpha_{DE}\mathbf{j} - 17\alpha_{DE}\mathbf{i} + 2170\mathbf{i} - 2170\mathbf{j} \\
\mathbf{a}_B &= \alpha_{AB}\mathbf{k} \times \mathbf{r}_B - \omega_{AB}^2\mathbf{r}_B = 0 - (20)^2(8\mathbf{i} + 14\mathbf{j}) \\
&= -3200\mathbf{i} - 5600\mathbf{j} \\
\mathbf{a}_{D/B} &= \alpha_{BD}\mathbf{k} \times \mathbf{r}_{D/B} - \omega_{BD}^2\mathbf{r}_{D/B} \\
&= \alpha_{BD}\mathbf{k} \times (12\mathbf{i} + 3\mathbf{j}) - (29.33)^2(12\mathbf{i} + 3\mathbf{j}) \\
&= 12\alpha_{BD}\mathbf{j} - 3\alpha_{BD}\mathbf{i} - 10{,}320\mathbf{i} - 2580\mathbf{j}
\end{aligned}$$

Substituting into Eq. (1) and equating the coefficients of $\mathbf{i}$ and $\mathbf{j}$, we obtain

$$-17\alpha_{DE} + 3\alpha_{BD} = -15{,}690$$
$$-17\alpha_{DE} - 12\alpha_{BD} = -6010$$
$$\boldsymbol{\alpha}_{BD} = -(645 \text{ rad/s}^2)\mathbf{k} \qquad \boldsymbol{\alpha}_{DE} = (809 \text{ rad/s}^2)\mathbf{k} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

This lesson was devoted to the determination of the *accelerations* of the points of a *rigid body in plane motion.* As you did previously for velocities, you will again consider the plane motion of a rigid body as the sum of two motions, namely, *a translation and a rotation.*

To solve a problem involving accelerations in plane motion you should use the following steps:

1. Determine the angular velocity of the body. To find V you can either

a. Consider the motion of the body as the sum of a translation and a rotation as you did in Sec. 15.6, or

b. Use the instantaneous center of rotation of the body as you did in Sec. 15.7. However, *keep in mind that you cannot use the instantaneous center to determine accelerations.*

2. Start drawing a "diagram equation" to use in your solution. This "equation" will involve the following diagrams (Fig. 15.22).

a. Plane motion diagram. Draw a sketch of the body, including all dimensions, as well as the a[illegible]ration A with its magnitude and sens[illegible] for which you know or seek the [illegible]w about these accelerations.

b. Translation diagram. Select a reference point A for which you know the direction, the magnitude, or a component of the acceleration $\mathbf{a}_A$. Draw a second diagram showing the body in translation with each point having the same acceleration as point A.

c. Rotation diagram. Considering point A as a fixed reference point, draw a third diagram showing the body in rotation about A. Indicate the normal and tangential components of the relative accelerations of other points, such as the components $(\mathbf{a}_{B/A})_n$ and $(\mathbf{a}_{B/A})_t$ of the acceleration of point B with respect to point A.

3. Write the relative-acceleration formula

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \qquad \text{or} \qquad \mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_n + (\mathbf{a}_{B/A})_t$$

The sample problems illustrate three different ways to use this vector equation:

a. If A is given or can easily be determined, you can use this equation to determine the accelerations of various points of the body [Sample Prob. 15.6].

b. If A cannot easily be determined, select for point B a point for which you know the direction, the magnitude, or a component of the acceleration $\mathbf{a}_B$ and draw a vector diagram of the equation. Starting at the same point, draw all known acceleration components in tip-to-tail fashion for each member of the equation. Complete the diagram by drawing the two remaining vectors in appropriate directions and in such a way that the two sums of vectors end at a common point.

The magnitudes of the two remaining vectors can be found either graphically or analytically. Usually an analytic solution will require the solution of two simultaneous equations [Sample Prob. 15.7]. However, by first considering the components of the various vectors in a direction perpendicular to one of the unknown vectors, you may be able to obtain an equation in a single unknown.

One of the two vectors obtained by the method just described will be $(\mathbf{a}_{B/A})_t$, from which you can compute α. Once α has been found, the vector equation can be used to determine the acceleration of any other point of the body.

c. A full vector approach can also be used to solve the vector equation. This is illustrated in Sample Prob. 15.8.

4. The analysi[illegible]ompleted this lesson. This method sh[illegible]he coordinates x and y of all significa[illegible]arameter (Sec. 15.9). By differentiating twice with respect to t the coordinates x and y of a given point, you can determine the rectangular components of the absolute velocity and absolute acceleration of that point.

PROBLEMS

CONCEPT QUESTION

15.CQ7 A rear-wheel-drive car starts from rest and accelerates to the left so that the tires do not slip on the road. What is the direction of the acceleration of the point on the tire in contact with the road, that is, point A?

a. ← **b.** ↖ **c.** ↑ **d.** ↓ **e.** ↙

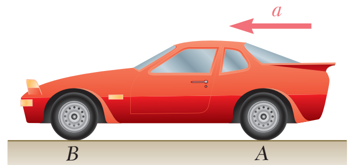


Fig. P15.CQ7

END-OF-SECTION PROBLEMS

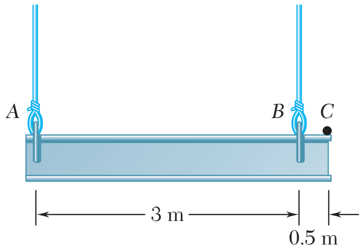


Fig. P15.105 and P15.106

15.105 A 3.5-m steel beam is lowered by means of two cables unwinding at the same speed from overhead cranes. As the beam approaches the ground, the crane operators apply brakes to slow down the unwinding motion. At the instant considered, the deceleration of the cable attached at A is 4 m/s^2, while that of the cable at B is 1.5 m/s^2. Determine [illegible] beam, (b) the acceleration of point C.

[illegible] 3 m/s^2 downward and the angular [illegible] rad/s^2 clockwise. Knowing that the angular velocity of the beam is zero at the instant considered, determine the acceleration of each cable.

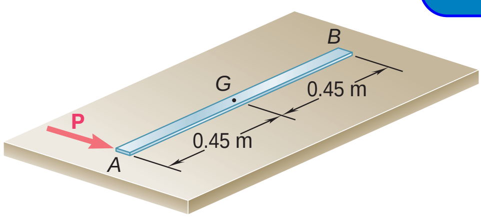


Fig. P15.107 and P15.108

15.107 A 900-mm rod rests on a horizontal table. A force **P** applied as shown produces the following accelerations: $\mathbf{a}_A = 3.6$ m/s^2 to the right, $\alpha = 6$ rad/s^2 counterclockwise as viewed from above. Determine the acceleration (a) of point G, (b) of point B.

15.108 In Prob. 15.107, determine the point of the rod that (a) has no acceleration, (b) has an acceleration of 2.4 m/s^2 to the right.

15.109 Knowing that at the instant shown crank BC has a constant angular velocity of 45 rpm clockwise, determine the acceleration (a) of point A, (b) of point D.

15.110 End A of rod AB moves to the right with a constant velocity of 6 ft/s. For the position shown, determine (a) the angular acceleration of rod AB, (b) the acceleration of the midpoint G of rod AB.

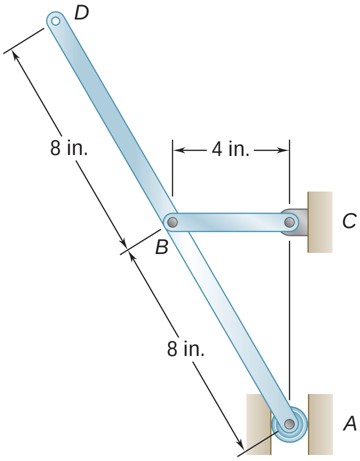


Fig. P15.109

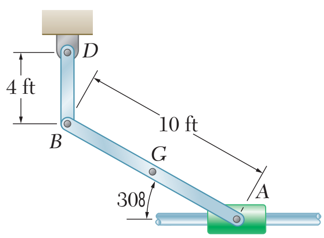


Fig. P15.110

15.111 An automobile travels to the left at a constant speed of 72 km/h. Knowing that the diameter of the wheel is 560 mm, determine the acceleration (*a*) of point *B*, (*b*) of point *C*, (*c*) of point *D*.

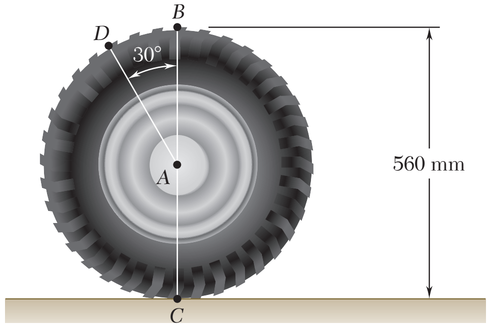


Fig. P15.111

15.112 The 18-in.-radius flywheel is rigidly attached to a 1.5-in.-radius shaft that can roll along parallel rails. Knowing that at the instant shown the center of the shaft has a velocity of 1.2 in./s and an acceleration of 0.5 in./s^2, both directed down to the left, determine the acceleration (*a*) of point *A*, (*b*) of point *B*.

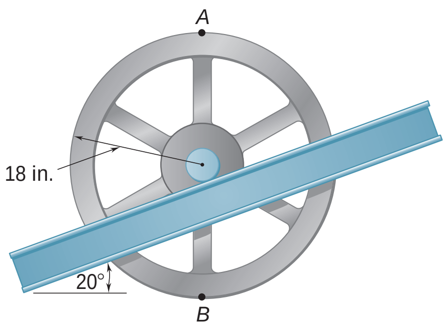


Fig. P15.112

15.113 and 15.114 A 3-in.-radius drum is rigidly attached to a 5-in.-radius drum as shown. One of the drums rolls without sliding on the surface shown, and a cord is wound around the other drum. Knowing that at th[illegible] a velocity of 8 in./s and [illegible] to the left, determine th[illegible] the drums.

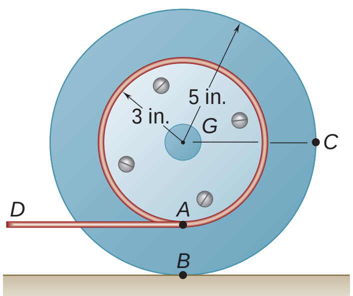


Fig. P15.113

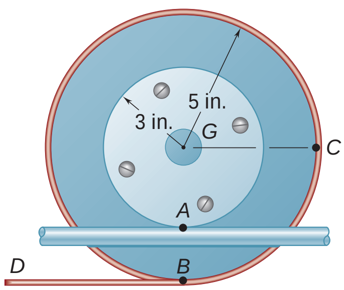


Fig. P15.114

15.115 A carriage *C* is supported by a caster *A* and a cylinder *B*, each of 50-mm diameter. Knowing that at the instant shown the carriage has an acceleration of 2.4 m/s^2 and a velocity of 1.5 m/s, both directed to the left, determine (*a*) the angular accelerations of the caster and of the cylinder, (*b*) the accelerations of the centers of the caster and of the cylinder.

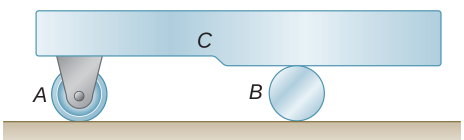


Fig. P15.115

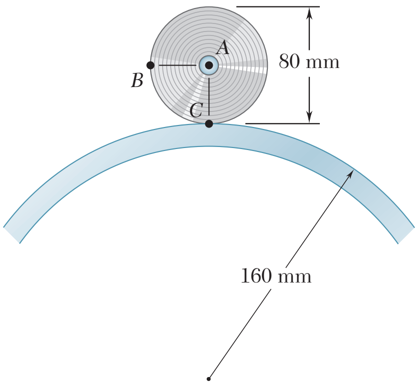


Fig. *P15.116*

15.116 A wheel rolls without slipping on a fixed cylinder. Knowing that at the instant shown the angular velocity of the wheel is 10 rad/s clockwise and its angular acceleration is 30 rad/s^2 counterclockwise, determine the acceleration of (*a*) point *A*, (*b*) point *B*, (*c*) point *C*.

15.117 The 100-mm-radius drum rolls without slipping on a portion of a belt which moves downward to the left with a constant velocity of 120 mm/s. Knowing that at a given instant the velocity and acceleration of the center *A* of the drum are as shown, determine the acceleration of point *D*.

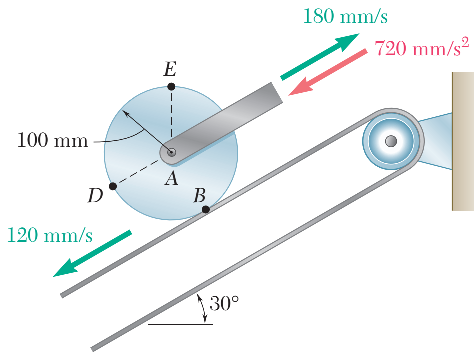


Fig. *P15.117*

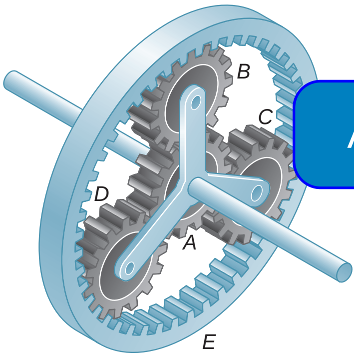


Fig. P15.118

…own, the radius of gears *A*, *B*, *C*, … the outer gear *E* is 9 in. Knowing … lar velocity of 150 rpm clockwise and that the outer gear *E* is stationary, determine the magnitude of the acceleration of the tooth of gear *D* that is in contact with (*a*) gear *A*, (*b*) gear *E*.

15.119 The 200-mm-radius disk rolls without sliding on the surface shown. Knowing that the distance *BG* is 160 mm and that at the instant shown the disk has an angular velocity of 8 rad/s counterclockwise and an angular acceleration of 2 rad/s^2 clockwise, determine the acceleration of *A*.

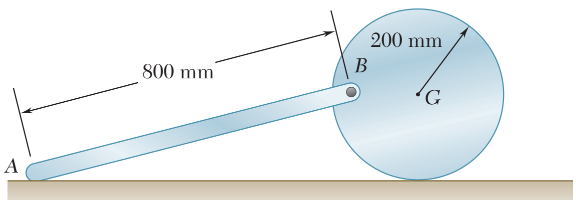


Fig. *P15.119*

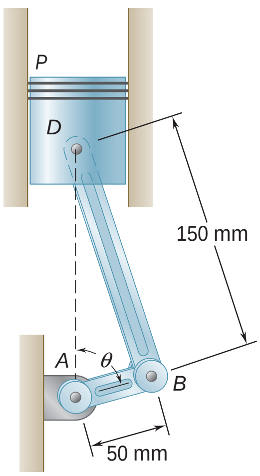


Fig. P15.120 and P15.121

15.120 Knowing that crank *AB* rotates about point *A* with a constant angular velocity of 900 rpm clockwise, determine the acceleration of the piston *P* when u = 60°.

15.121 Knowing that crank *AB* rotates about point *A* with a constant angular velocity of 900 rpm clockwise, determine the acceleration of the piston *P* when u = 120°.

15.122 In the two-cylinder air compressor shown the connecting rods *BD* and *BE* are each 190 mm long and crank *AB* rotates about the fixed point *A* with a constant angular velocity of 1500 rpm clockwise. Determine the acceleration of each piston when u = 0.

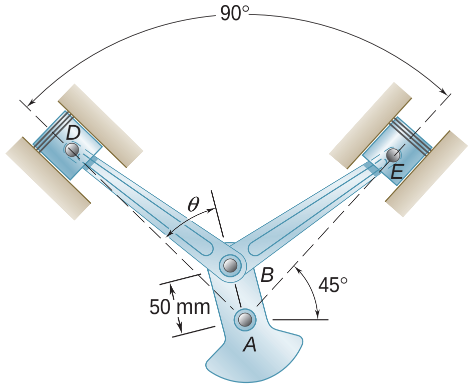


Fig. P15.122

15.123 The disk shown has a constant angular velocity of 500 rpm counterclockwise. Knowing that rod *BD* is 10 in. long, determine the acceleration of collar *D* when (*a*) u = 90°, (*b*) u = 180°.

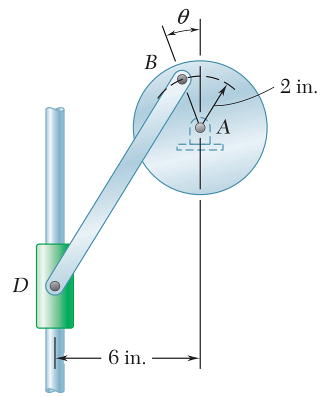


Fig. *P15.123*

15.124 Arm *AB* has a constan

clockwise. At the instant

(*a*) of collar *D*, (*b*) of the

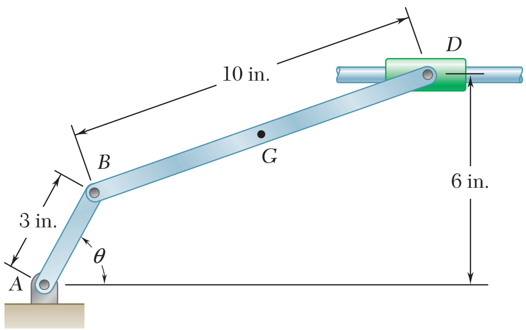


Fig. P15.124 and P15.125

15.125 Arm *AB* has a constant angular velocity of 16 rad/s counterclockwise. At the instant when u = 60°, determine the acceleration of collar *D*.

15.126 A straight rack rests on a gear of radius r = 3 in. and is attached to a block *B* as shown. Knowing that at the instant shown u = 20°, the angular velocity of gear *D* is 3 rad/s clockwise, and it is speeding up at a rate of 2 rad/s^2, determine (*a*) the angular acceleration of *AB*, (*b*) the acceleration of block *B*.

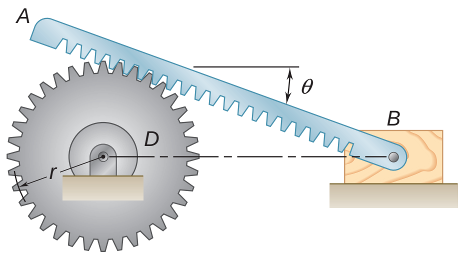


Fig. *P15.126*

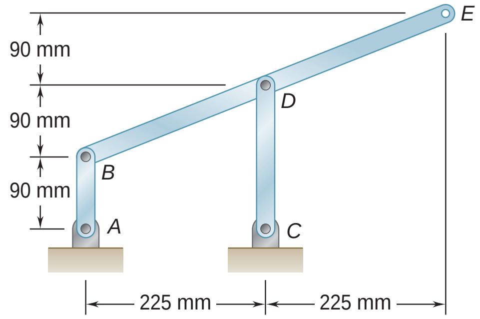


Fig. P15.127 and P15.128

15.127 Knowing that at the instant shown rod AB has a constant angular velocity of 6 rad/s clockwise, determine the acceleration of point D.

15.128 Knowing that at the instant shown rod AB has a constant angular velocity of 6 rad/s clockwise, determine (*a*) the angular acceleration of member BDE, (*b*) the acceleration of point E.

15.129 Knowing that at the instant shown bar AB has a constant angular velocity of 19 rad/s clockwise, determine (*a*) the angular acceleration of bar BGD, (*b*) the angular acceleration of bar DE.

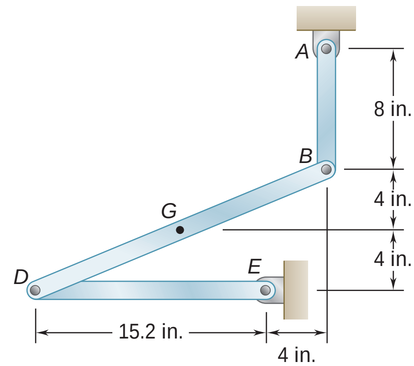


Fig. P15.129 and P15.130

...n bar DE has a constant angular ... determine (*a*) the acceleration of point B, (*b*) the acceleration of point G.

15.131 and 15.132 Knowing that at the instant shown bar AB has a constant angular velocity of 4 rad/s clockwise, determine the angular acceleration (*a*) of bar BD, (*b*) of bar DE.

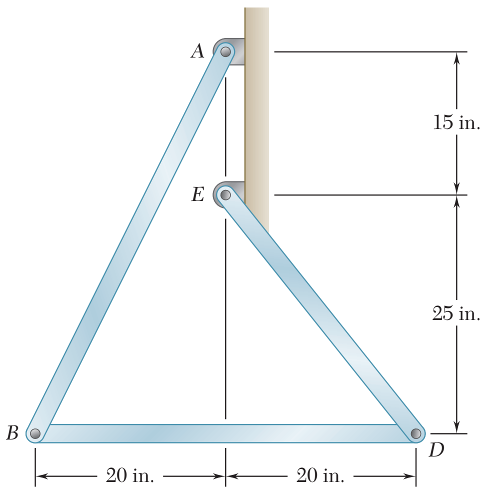


Fig. P15.131 and P15.133

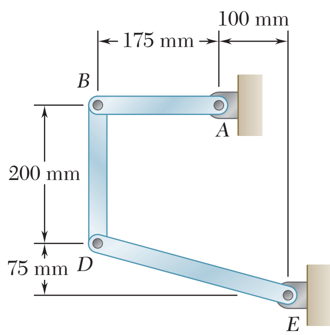


Fig. P15.132 and P15.134

15.133 and 15.134 Knowing that at the instant shown bar AB has an angular velocity of 4 rad/s and an angular acceleration of 2 rad/s^2, both clockwise, determine the angular acceleration (*a*) of bar BD, (*b*) of bar DE by using the vector approach as is done in Sample Prob. 15.8.

15.135 Roberts linkage is named after Richard Roberts (1789–1864) and can be used to draw a close approximation to a straight line by locating a pen at point *F*. The distance *AB* is the same as *BF*, *DF*, and *DE*. Knowing that at the instant shown, bar *AB* has a constant angular velocity of 4 rad/s clockwise, determine (*a*) the angular acceleration of bar *DE*, (*b*) the acceleration of point *F*.

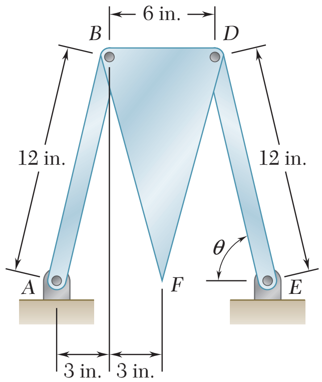


Fig. P15.135

15.136 For the oil pump rig shown, link *AB* causes the beam *BCE* to oscillate as the crank *OA* revolves. Knowing that *OA* has a radius of 0.6 m and a constant clockwise angular velocity of 20 rpm, determine the velocity and acceleration of point *D* at the instant shown.

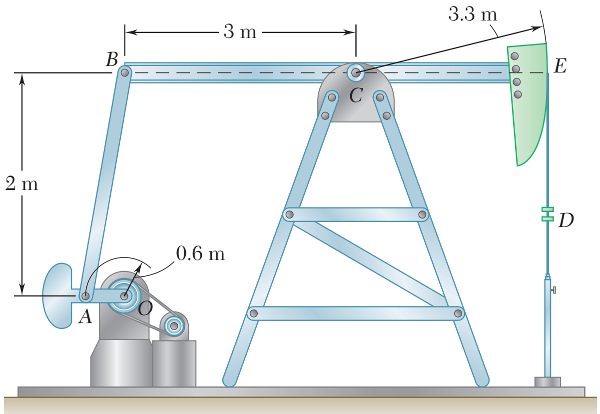


Fig. P15.136

15.137 Denoting by $\mathbf{r}_A$ the posit is in plane motion, show instantaneous center of rotation is

$$\mathbf{r}_C = \mathbf{r}_A + \frac{\mathsf{V} \times \mathbf{v}_A}{\mathsf{V}^2}$$

where V is the angular velocity of the slab and $\mathbf{v}_A$ is the velocity of point *A*, (*b*) the acceleration of the instantaneous center of rotation is zero if, and only if,

$$\mathbf{a}_A = \frac{\mathsf{a}}{\mathsf{V}}\mathbf{v}_A + \mathsf{V} \times \mathbf{v}_A$$

where A = a**k** is the angular acceleration of the slab.

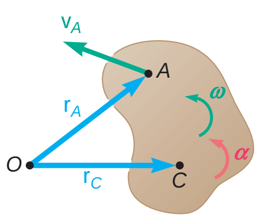


Fig. P15.137

***15.138** The drive disk of the Scotch crosshead mechanism shown has an angular velocity V and an angular acceleration A, both directed counterclockwise. Using the method of Sec. 15.9, derive expressions for the velocity and acceleration of point *B*.

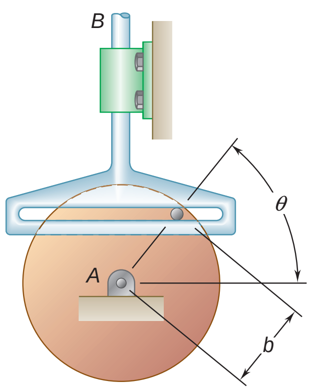


Fig. P15.138

***15.139** The wheels attached to the ends of rod AB roll along the surfaces shown. Using the method of Sec. 15.9, derive an expression for the angular velocity of the rod in terms of v_B, u, l, and b.

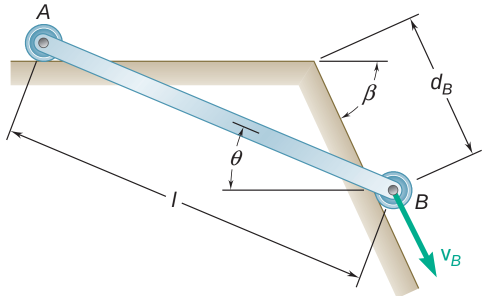


Fig. P15.139 and P15.140

***15.140** The wheels attached to the ends of rod AB roll along the surfaces shown. Using the method of Sec. 15.9 and knowing that the acceleration of wheel B is zero, derive an expression for the angular acceleration of the rod in terms of v_B, u, l, and b.

***15.141** A disk of radius r rolls to the right with a constant velocity **v**. Denoting by P the point of the rim in contact with the ground at $t = 0$, derive expressions for the horizontal and vertical components of the velocity of P at any time t.

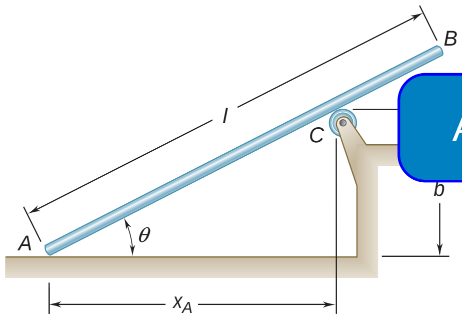


Fig. P15.142 and P15.143

...el at C while end A moves to the ... . Using the method of Sec. 15.9, ...ar velocity and angular acceleration of the rod.

***15.143** Rod AB moves over a small wheel at C while end A moves to the right with a constant velocity $\mathbf{v}_A$. Using the method of Sec. 15.9, derive expressions for the horizontal and vertical components of the velocity of point B.

15.144 Crank AB rotates with a constant clockwise angular velocity $\boldsymbol{\omega}$. Using the method of Sec. 15.9, derive expressions for the angular velocity of rod BD and the velocity of the point on the rod coinciding with point E in terms of u, v, b, and l.

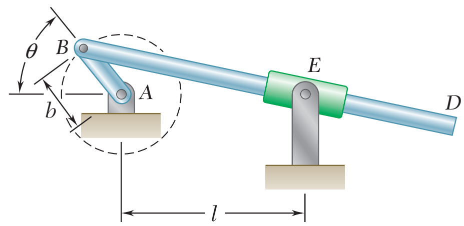


Fig. P15.144 and P15.145

15.145 Crank AB rotates with a constant clockwise angular velocity $\boldsymbol{\omega}$. Using the method of Sec. 15.9, derive an expression for the angular acceleration of rod BD in terms of u, v, b, and l.

15.146 Pin C is attached to rod CD and slides in a slot cut in arm AB. Knowing that rod CD moves vertically upward with a constant velocity $\mathbf{v}_0$, derive an expression for (*a*) the angular velocity of arm AB, (*b*) the components of the velocity of point A, (*c*) an expression for the angular acceleration of arm AB.

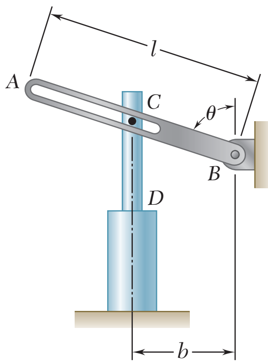


Fig. P15.146

***15.147** The position of rod AB is controlled by a disk of radius r which is attached to yoke CD. Knowing that the yoke moves vertically upward with a constant velocity $\mathbf{v}_0$, derive expressions for the angular velocity and angular acceleration of rod AB.

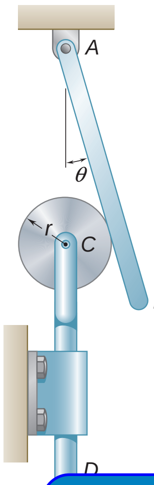


Fig.

***15.148** A wheel of radius r rolls without slipping along the inside of a fixed cylinder of radius R with a constant angular velocity V. Denoting by P the point of the wheel in contact with the cylinder at $t = 0$, derive expressions for the horizontal and vertical components of the velocity of P at any time t. (The curve described by point P is a *hypocycloid*.)

***15.149** In Prob. 15.148, show that the path of P is a vertical straight line when $r = R/2$. Derive expressions for the corresponding velocity and acceleration of P at any time t.

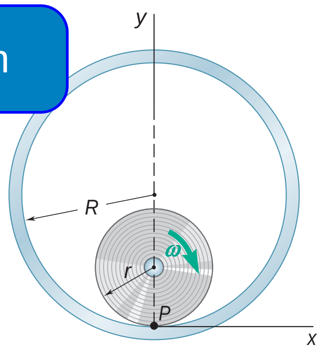


Fig. P15.148

15.10 RATE OF CHANGE OF A VECTOR WITH RESPECT TO A ROTATING FRAME

We saw in Sec. 11.10 that the rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation. In this section, the rates of change of a vector $\mathbf{Q}$ with respect to a fixed frame and with respect to a rotating frame of reference will be considered.† You will learn to determine the rate of change of $\mathbf{Q}$ with respect to one frame of reference when $\mathbf{Q}$ is defined by its components in another frame.

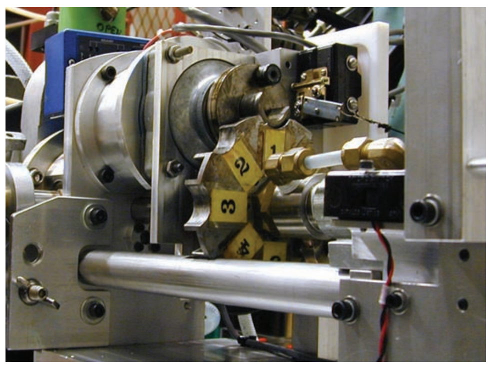


Photo 15.7 A Geneva mechanism is used to convert rotary motion into intermittent motion.

†It is recalled that the selection of a fixed frame of reference is arbitrary. Any frame may be designated as "fixed"; all others will then be considered as moving.

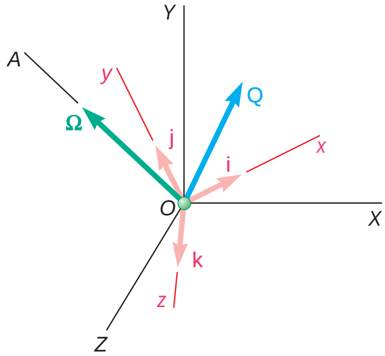


Fig. 15.26

Consider two frames of reference centered at O, a fixed frame $OXYZ$ and a frame $Oxyz$ which rotates about the fixed axis OA; let $\mathbf{\Omega}$ denote the angular velocity of the frame $Oxyz$ at a given instant (Fig. 15.26). Consider now a vector function $\mathbf{Q}(t)$ represented by the vector $\mathbf{Q}$ attached at O; as the time t varies, both the direction and the magnitude of $\mathbf{Q}$ change. Since the variation of $\mathbf{Q}$ is viewed differently by an observer using $OXYZ$ as a frame of reference and by an observer using $Oxyz$, we should expect the rate of change of $\mathbf{Q}$ to depend upon the frame of reference which has been selected. Therefore, the rate of change of $\mathbf{Q}$ with respect to the fixed frame $OXYZ$ will be denoted by $(\dot{\mathbf{Q}})_{OXYZ}$, and the rate of change of $\mathbf{Q}$ with respect to the rotating frame $Oxyz$ will be denoted by $(\dot{\mathbf{Q}})_{Oxyz}$. We propose to determine the relation existing between these two rates of change.

Let us first resolve the vector $\mathbf{Q}$ into components along the x, y, and z axes of the rotating frame. Denoting by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ the corresponding unit vectors, we write

$$\mathbf{Q} = Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k} \tag{15.27}$$

Differentiating (15.27) with respect to t and considering the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as fixed, we obtain the rate of change of $\mathbf{Q}$ *with respect to the rotating frame Oxyz:*

$$(\dot{\mathbf{Q}})_{Oxyz} = \dot{Q}_x\mathbf{i} + \dot{Q}_y\mathbf{j} + \dot{Q}_z\mathbf{k} \tag{15.28}$$

To obtain the rate of change of $\mathbf{Q}$ *with respect to the fixed* [illegible] unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as variable [illegible]efore write

$$(\dot{\mathbf{Q}})_{OXYZ} = \dot{Q}_x\mathbf{i} + \dot{Q}_y\mathbf{j} + \dot{Q}_z\mathbf{k} + Q_x\frac{d\mathbf{i}}{dt} + Q_y\frac{d\mathbf{j}}{dt} + Q_z\frac{d\mathbf{k}}{dt} \tag{15.29}$$

Recalling (15.28), we observe that the sum of the first three terms in the right-hand member of (15.29) represents the rate of change $(\dot{\mathbf{Q}})_{Oxyz}$. We note, on the other hand, that the rate of change $(\dot{\mathbf{Q}})_{OXYZ}$ would reduce to the last three terms in (15.29) if the vector $\mathbf{Q}$ were fixed within the frame $Oxyz$, since $(\dot{\mathbf{Q}})_{Oxyz}$ would then be zero. But in that case, $(\dot{\mathbf{Q}})_{OXYZ}$ would represent the velocity of a particle located at the tip of $\mathbf{Q}$ and belonging to a body rigidly attached to the frame $Oxyz$. Thus, the last three terms in (15.29) represent the velocity of that particle; since the frame $Oxyz$ has an angular velocity $\mathbf{\Omega}$ with respect to $OXYZ$ at the instant considered, we write, by (15.5),

$$Q_x\frac{d\mathbf{i}}{dt} + Q_y\frac{d\mathbf{j}}{dt} + Q_z\frac{d\mathbf{k}}{dt} = \mathbf{\Omega} \times \mathbf{Q} \tag{15.30}$$

Substituting from (15.28) and (15.30) into (15.29), we obtain the fundamental relation

$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \mathbf{\Omega} \times \mathbf{Q} \tag{15.31}$$

We conclude that the rate of change of the vector $\mathbf{Q}$ with respect to the fixed frame $OXYZ$ is made of two parts: The first part represents the rate of change of $\mathbf{Q}$ with respect to the rotating frame $Oxyz$; the second part, $\mathbf{\Omega} \times \mathbf{Q}$, is induced by the rotation of the frame $Oxyz$.

The use of relation (15.31) simplifies the determination of the rate of change of a vector $\mathbf{Q}$ with respect to a fixed frame of reference $OXYZ$ when the vector $\mathbf{Q}$ is defined by its components along the axes of a rotating frame $Oxyz$, since this relation does not require the separate computation of the derivatives of the unit vectors defining the orientation of the rotating frame.

15.11 PLANE MOTION OF A PARTICLE RELATIVE TO A ROTATING FRAME. CORIOLIS ACCELERATION

Consider two frames of reference, both centered at O and both in the plane of the figure, a fixed frame OXY and a rotating frame Oxy (Fig. 15.27). Let P be a particle moving in the plane of the figure. The position vector $\mathbf{r}$ of P is the same in both frames, but its rate of change depends upon the frame of reference which has been selected.

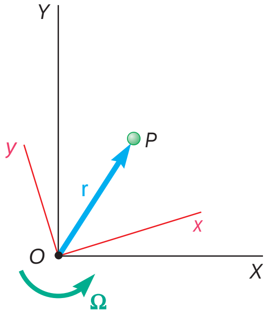


Fig. 15.27

The absolute velocity $\mathbf{v}_P$ of the particle is defined as the velocity observed from the fixed frame OXY and is equal to the rate of change $(\dot{\mathbf{r}})_{OXY}$ of $\mathbf{r}$ with respect to that frame. We can, however, express $\mathbf{v}_P$ in terms of the rate of change $(\dot{\mathbf{r}})_{Oxy}$ observed from the rotating frame if we make use of Eq. (15.31). Denoting by $\boldsymbol{\Omega}$ the angular velocity of the frame Oxy with respect to OXY at the instant considered, we write

$$\mathbf{v}_P = (\dot{\mathbf{r}})_{OXY} = \boldsymbol{\Omega} \times \mathbf{r} + (\dot{\mathbf{r}})_{Oxy} \tag{15.32}$$

But $(\dot{\mathbf{r}})_{Oxy}$ defines the velocity o frame Oxy. Denoting the rotat the velocity $(\dot{\mathbf{r}})_{Oxy}$ of P relative imagine that a rigid slab has been attached to the rotating frame. Then $v_{P/\mathcal{F}}$ represents the velocity of P along the path that it describes on that slab (Fig. 15.28), and the term $\boldsymbol{\Omega} \times \mathbf{r}$ in (15.32) represents the velocity $\mathbf{v}_{P'}$ of the point P' of the slab—or rotating frame—which coincides with P at the instant considered. Thus, we have

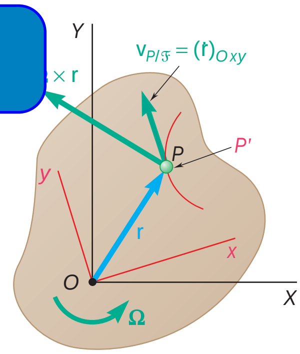


Fig. 15.28

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \tag{15.33}$$

where $\mathbf{v}_P$ = absolute velocity of particle P
$\mathbf{v}_{P'}$ = velocity of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{v}_{P/\mathcal{F}}$ = velocity of P relative to moving frame $\mathcal{F}$

The absolute acceleration $\mathbf{a}_P$ of the particle is defined as the rate of change of $\mathbf{v}_P$ with respect to the fixed frame OXY. Computing the rates of change with respect to OXY of the terms in (15.32), we write

$$\mathbf{a}_P = \dot{\mathbf{v}}_P = \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times \dot{\mathbf{r}} + \frac{d}{dt}[(\dot{\mathbf{r}})_{Oxy}] \tag{15.34}$$

where all derivatives are defined with respect to OXY, except where indicated otherwise. Referring to Eq. (15.31), we note that the last term in (15.34) can be expressed as

$$\frac{d}{dt}[(\dot{\mathbf{r}})_{Oxy}] = (\ddot{\mathbf{r}})_{Oxy} + \boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxy}$$

On the other hand, $\dot{\mathbf{r}}$ represents the velocity $\mathbf{v}_P$ and can be replaced by the right-hand member of Eq. (15.32). After completing these two substitutions into (15.34), we write

$$\mathbf{a}_P = \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxy} + (\ddot{\mathbf{r}})_{Oxy} \tag{15.35}$$

Referring to the expression (15.8) obtained in Sec. 15.3 for the acceleration of a particle in a rigid body rotating about a fixed axis, we note that the sum of the first two terms represents the acceleration $\mathbf{a}_{P'}$ of the point P' of the rotating frame which coincides with P at the instant considered. On the other hand, the last term defines the acceleration $\mathbf{a}_{P/f}$ of P relative to the rotating frame. If it were not for the third term, which has not been accounted for, a relation similar to (15.33) could be written for the accelerations, and $\mathbf{a}_P$ could be expressed as the sum of $\mathbf{a}_{P'}$ and $\mathbf{a}_{P/f}$. However, it is clear that *such a relation would be incorrect* and that we must include the additional term. This term, which will be denoted by $\mathbf{a}_c$, is called the *complementary acceleration*, or *Coriolis acceleration*, after the French mathematician de Coriolis (1792–1843). We write

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \tag{15.36}$$

where $\mathbf{a}_P$ = absolute acceleration of particle P
[illegible] of moving frame $\mathcal{F}$ coinciding
[illegible]
[illegible] to moving frame $\mathcal{F}$
$\mathbf{a}_c$ [illegible] $(\dot{\mathbf{r}})_{Oxy}$ [illegible] $\mathbf{v}_{P/\mathcal{F}}$
= complementary, or Coriolis, acceleration†

We note that since point P' moves in a circle about the origin O, its acceleration $\mathbf{a}_{P'}$ has, in general, two components: a component $(\mathbf{a}_{P'})_t$ tangent to the circle, and a component $(\mathbf{a}_{P'})_n$ directed toward O. Similarly, the acceleration $\mathbf{a}_{P/\mathcal{F}}$ generally has two components: a component $(\mathbf{a}_{P/\mathcal{F}})_t$ tangent to the path that P describes on the rotating slab, and a component $(\mathbf{a}_{P/\mathcal{F}})_n$ directed toward the center of curvature of that path. We further note that since the vector $\boldsymbol{\Omega}$ is perpendicular to the plane of motion, and thus to $\mathbf{v}_{P/\mathcal{F}}$, the magnitude of the Coriolis acceleration $\mathbf{a}_c = 2\boldsymbol{\Omega} \times \mathbf{v}_{P/\mathcal{F}}$ is equal to $2\Omega v_{P/\mathcal{F}}$, and its direction can be obtained by rotating the vector $\mathbf{v}_{P/\mathcal{F}}$ through 90° in the sense of rotation of the moving frame (Fig. 15.29). The Coriolis acceleration reduces to zero when either $\boldsymbol{\Omega}$ or $\mathbf{v}_{P/\mathcal{F}}$ is zero.

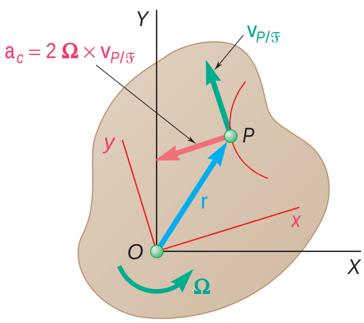


Fig. 15.29

The following example will help in understanding the physical meaning of the Coriolis acceleration. Consider a collar P which is

†It is important to note the difference between Eq. (15.36) and Eq. (15.21) of Sec. 15.8. When we wrote

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \tag{15.21}$$

in Sec. 15.8, we were expressing the absolute acceleration of point B as the sum of its acceleration $\mathbf{a}_{B/A}$ relative to a *frame in translation* and of the acceleration $\mathbf{a}_A$ of a point of that frame. We are now trying to relate the absolute acceleration of point P to its acceleration $\mathbf{a}_{P/f}$ relative to a *rotating frame* f and to the acceleration $\mathbf{a}_{P'}$ of the point P' of that frame which coincides with P; Eq. (15.36) shows that because the frame is rotating, it is necessary to include an additional term representing the Coriolis acceleration $\mathbf{a}_c$.

made to slide at a constant relative speed u along a rod OB rotating at a constant angular velocity V about O (Fig. 15.30*a*). According to formula (15.36), the absolute acceleration of P can be obtained by adding vectorially the acceleration $\mathbf{a}_A$ of the point A of the rod coinciding with P, the relative acceleration $\mathbf{a}_{P/OB}$ of P with respect to the rod, and the Coriolis acceleration $\mathbf{a}_c$. Since the angular velocity V of the rod is constant, $\mathbf{a}_A$ reduces to its normal component $(\mathbf{a}_A)_n$ of magnitude $r\mathrm{V}^2$; and since u is constant, the relative acceleration $\mathbf{a}_{P/OB}$ is zero. According to the definition given above, the Coriolis acceleration is a vector perpendicular to OB, of magnitude $2\mathrm{V}\,u$, and directed as shown in the figure. The acceleration of the collar P consists, therefore, of the two vectors shown in Fig. 15.30*a*. Note that the result obtained can be checked by applying the relation (11.44).

To understand better the significance of the Coriolis acceleration, let us consider the absolute velocity of P at time t and at time $t + \Delta t$ (Fig. 15.30*b*). The velocity at time t can be resolved into its components $\mathbf{u}$ and $\mathbf{v}_A$; the velocity at time $t + \Delta t$ can be resolved into its components $\mathbf{u}'$ and $\mathbf{v}_{A'}$. Drawing these components from the same origin (Fig. 15.30*c*), we note that the change in velocity during the time Δt can be represented by the sum of three vectors, $\overrightarrow{RR'}$, $\overrightarrow{TT''}$, and $\overrightarrow{T''T'}$. The vector $\overrightarrow{TT''}$ measures the change in direction of the velocity $\mathbf{v}_A$, and the quotient $\overrightarrow{TT''}/\Delta t$ represents the acceleration $\mathbf{a}_A$ when Δt approaches zero. We check that the direction of $\overrightarrow{TT''}$ is that of $\mathbf{a}_A$ when Δt approaches zero and that

$$\lim_{\Delta t\ \mathrm{y}\ 0} \frac{TT''}{\Delta t} = \lim_{\Delta t\ \mathrm{y}\ 0} \ldots$$

The vector $\overrightarrow{RR'}$ measures the change in direction of $\mathbf{u}$ due to the rotation of the rod; the vector $\overrightarrow{T''T'}$ measures the change in magnitude of $\mathbf{v}_A$ due to the motion of P on the rod. The vectors $\overrightarrow{RR'}$ and $\overrightarrow{T''T'}$ result from the *combined effect* of the relative motion of P and of the rotation of the rod; they would vanish if *either* of these two motions stopped. It is easily verified that the sum of these two vectors defines the Coriolis acceleration. Their direction is that of $\mathbf{a}_c$ when Δt approaches zero, and since $RR' = u\ \Delta\mathrm{u}$ and $T''T' = v_{A'} - v_A = (r + \Delta r)\mathrm{V} - r\mathrm{V} = \mathrm{V}\ \Delta r$, we check that a_c is equal to

$$\lim_{\Delta t\ \mathrm{y}\ 0}\left(\frac{RR'}{\Delta t} + \frac{T''T'}{\Delta t}\right) = \lim_{\Delta t\ \mathrm{y}\ 0}\left(u\frac{\Delta\mathrm{u}}{\Delta t} + \mathrm{V}\frac{\Delta r}{\Delta t}\right) = u\mathrm{V} + \mathrm{V}u = 2\mathrm{V}u$$

Formulas (15.33) and (15.36) can be used to analyze the motion of mechanisms which contain parts sliding on each other. They make it possible, for example, to relate the absolute and relative motions of sliding pins and collars (see Sample Probs. 15.9 and 15.10). The concept of Coriolis acceleration is also very useful in the study of long-range projectiles and of other bodies whose motions are appreciably affected by the rotation of the earth. As was pointed out in Sec. 12.2, a system of axes attached to the earth does not truly constitute a newtonian frame of reference; such a system of axes should actually be considered as rotating. The formulas derived in this section will therefore facilitate the study of the motion of bodies with respect to axes attached to the earth.

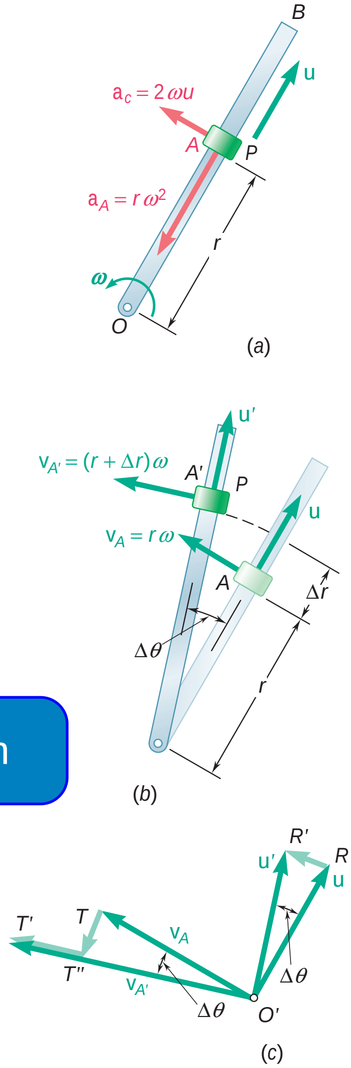
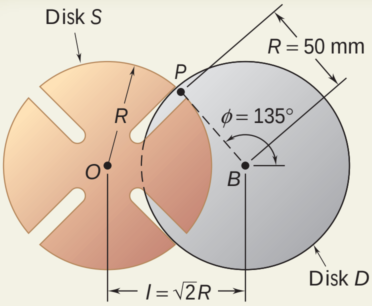


Fig. 15.30

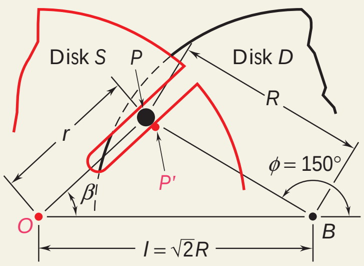


SAMPLE PROBLEM 15.9

The Geneva mechanism shown is used in many counting instruments and in other applications where an intermittent rotary motion is required. Disk D rotates with a constant counterclockwise angular velocity ω_D of 10 rad/s. A pin P is attached to disk D and slides along one of several slots cut in disk S. It is desirable that the angular velocity of disk S be zero as the pin enters and leaves each slot; in the case of four slots, this will occur if the distance between the centers of the disks is $l = \sqrt{2}R$.

At the instant when $\phi = 150°$, determine (*a*) the angular velocity of disk S, (*b*) the velocity of pin P relative to disk S.

SOLUTION



We solve triangle OPB, which corresponds to the position $\phi = 150°$. Using the law of cosines, we write

$$r^2 = R^2 + l^2 - 2Rl \cos 30° = 0.551R^2 \qquad r = 0.742R = 37.1 \text{ mm}$$

From the law of sines,

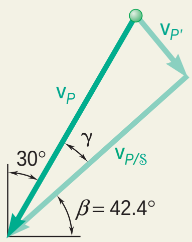
$$\frac{\sin \beta}{R} = \frac{\sin 30°}{r} \qquad \sin \beta = \frac{\sin 30°}{0.742} \qquad \beta = 42.4°$$

disk D rotates about point B, ... s

$$= 500 \text{ mm/s}$$
$$\mathbf{v}_P = 500 \text{ mm/s} \measuredangle 60°$$

We consider now the motion of pin P along the slot in disk S. Denoting by P' the point of disk S which coincides with P at the instant considered and selecting a rotating frame $\mathcal{S}$ attached to disk S, we write

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{S}}$$



Noting that $\mathbf{v}_{P'}$ is perpendicular to the radius OP and that $\mathbf{v}_{P/\mathcal{S}}$ is directed along the slot, we draw the velocity triangle corresponding to the equation above. From the triangle, we compute

$$\gamma = 90° - 42.4° - 30° = 17.6°$$
$$v_{P'} = v_P \sin \gamma = (500 \text{ mm/s}) \sin 17.6°$$
$$\mathbf{v}_{P'} = 151.2 \text{ mm/s} \ \angle\ 42.4°$$
$$v_{P/\mathcal{S}} = v_P \cos \gamma = (500 \text{ mm/s}) \cos 17.6°$$
$$\mathbf{v}_{P/S} = \mathbf{v}_{P/\mathcal{S}} = 477 \text{ mm/s} \measuredangle 42.4° \quad ◀$$

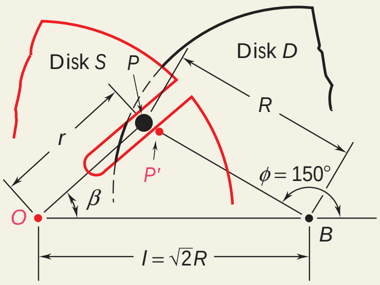
Since $\mathbf{v}_{P'}$ is perpendicular to the radius OP, we write

$$v_{P'} = r\omega_{\mathcal{S}} \qquad 151.2 \text{ mm/s} = (37.1 \text{ mm})\omega_{\mathcal{S}}$$
$$\omega_S = \omega_{\mathcal{S}} = 4.08 \text{ rad/s} \circlearrowleft \quad ◀$$

SAMPLE PROBLEM 15.10

In the Geneva mechanism of Sample Prob. 15.9, disk D rotates with a constant counterclockwise angular velocity V_D of 10 rad/s. At the instant when $\mathsf{f} = 150°$, determine the angular acceleration of disk S.

SOLUTION



Referring to Sample Prob. 15.9, we obtain the angular velocity of the frame $\mathcal{S}$ attached to disk S and the velocity of the pin relative to $\mathcal{S}$:

$$\mathsf{V}_{\mathcal{S}} = 4.08 \text{ rad/s i}$$

$$\mathsf{b} = 42.4° \qquad \mathbf{v}_{P/\mathcal{S}} = 477 \text{ mm/s d } 42.4°$$

Since pin P moves with respect to the rotating frame $\mathcal{S}$, we write

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{S}} + \mathbf{a}_c \qquad (1)$$

Each term of this vector equation is investigated separately.

Absolute Acceleration $\mathbf{a}_P$. Since disk D rotates with a constant angular velocity, the absolute acceleration $\mathbf{a}_P$ is directed toward B. We have

$$a_P = R\mathsf{V}_D^2 = (500 \text{ mm})(10 \text{ rad/s})^2 = 5000 \text{ mm/s}^2$$

$$\mathbf{a}_P = 5000 \text{ mm/s}^2 \text{ c } 30°$$

… **oint P'.** The acceleration $\mathbf{a}_{P'}$ of the … es with P at the instant considered is … omponents. (We recall from Sample Prob. 15.9 that $r = 37.1$ mm.)

$$(a_{P'})_n = r\mathsf{V}_{\mathcal{S}}^2 = (37.1 \text{ mm})(4.08 \text{ rad/s})^2 = 618 \text{ mm/s}^2$$

$$(\mathbf{a}_{P'})_n = 618 \text{ mm/s}^2 \text{ d } 42.4°$$

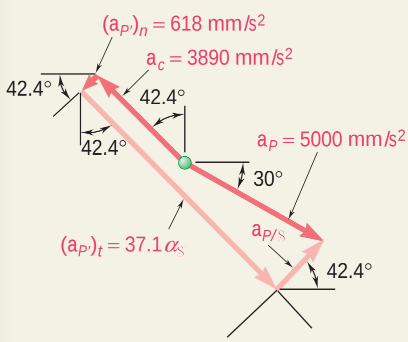
$$(a_{P'})_t = r\mathsf{a}_{\mathcal{S}} = 37.1\mathsf{a}_{\mathcal{S}} \qquad (\mathbf{a}_{P'})_t = 37.1\mathsf{a}_{\mathcal{S}} \text{ ⤒ } 42.4°$$

Relative Acceleration $\mathbf{a}_{P/\mathcal{S}}$. Since the pin P moves in a straight slot cut in disk S, the relative acceleration $\mathbf{a}_{P/\mathcal{S}}$ must be parallel to the slot; i.e., its direction must be ⤡ 42.4°.

Coriolis Acceleration $\mathbf{a}_c$. Rotating the relative velocity $\mathbf{v}_{P/\mathcal{S}}$ through 90° in the sense of $\mathsf{V}_{\mathcal{S}}$, we obtain the direction of the Coriolis component of the acceleration: h 42.4°. We write

$$a_c = 2\mathsf{V}_{\mathcal{S}}v_{P/\mathcal{S}} = 2(4.08 \text{ rad/s})(477 \text{ mm/s}) = 3890 \text{ mm/s}^2$$

$$\mathbf{a}_c = 3890 \text{ mm/s}^2 \text{ h } 42.4°$$



We rewrite Eq. (1) and substitute the accelerations found above:

$$\mathbf{a}_P = (\mathbf{a}_{P'})_n + (\mathbf{a}_{P'})_t + \mathbf{a}_{P/\mathcal{S}} + \mathbf{a}_c$$

$$[5000 \text{ c } 30°] = [618 \text{ d } 42.4°] + [37.1\mathsf{a}_{\mathcal{S}} \text{ ⤒ } 42.4°] + [a_{P/\mathcal{S}} \text{ ⤡ } 42.4°] + [3890 \text{ h } 42.4°]$$

Equating components in a direction perpendicular to the slot,

$$5000 \cos 17.6° = 37.1\mathsf{a}_{\mathcal{S}} - 3890$$

$$\mathsf{A}_S = \mathsf{A}_{\mathcal{S}} = 233 \text{ rad/s}^2\text{i} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you studied the rate of change of a vector with respect to a rotating frame and then applied your knowledge to the analysis of the plane motion of a particle relative to a rotating frame.

1. Rate of change of a vector with respect to a fixed frame and with respect to a rotating frame. Denoting by $(\dot{\mathbf{Q}})_{OXYZ}$ the rate of change of a vector $\mathbf{Q}$ with respect to a fixed frame $OXYZ$ and by $(\dot{\mathbf{Q}})_{Oxyz}$ its rate of change with respect to a rotating frame $Oxyz$, we obtained the fundamental relation

$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{Q} \qquad (15.31)$$

where $\boldsymbol{\Omega}$ is the angular velocity of the rotating frame.

This fundamental relation will now be applied to the solution of two-dimensional problems.

2. Plane motion of a particle relative to a rotating frame. Using the above fundamental relation and designating by $\mathcal{F}$ the rotating frame, we obtained the following expressions for the velocity and the acceleration of a particle P:

(15.33)

(15.36)

In these equations:

a. The subscript *P* refers to the absolute motion of the particle P, that is, to its motion with respect to a fixed frame of reference OXY.

b. The subscript *P′* refers to the motion of the point P' of the rotating frame $\mathcal{F}$ which coincides with P at the instant considered.

c. The subscript *P/𝓕* refers to the motion of the particle P relative to the rotating frame $\mathcal{F}$.

d. The term $\mathbf{a}_c$ represents the Coriolis acceleration of point *P*. Its magnitude is $2\Omega v_{P/\mathcal{F}}$, and its direction is found by rotating $\mathbf{v}_{P/\mathcal{F}}$ through 90° in the sense of rotation of the frame $\mathcal{F}$.

You should keep in mind that the Coriolis acceleration should be taken into account whenever a part of the mechanism you are analyzing is moving with respect to another part that is rotating. The problems you will encounter in this lesson involve collars that slide on rotating rods, booms that extend from cranes rotating in a vertical plane, etc.

When solving a problem involving a rotating frame, you will find it convenient to draw vector diagrams representing Eqs. (15.33) and (15.36), respectively, and use these diagrams to obtain either an analytical or a graphical solution.

PROBLEMS

CONCEPT QUESTION

15.CQ8 A person walks radially inward on a platform that is rotating counterclockwise about its center. Knowing that the platform has a constant angular velocity V and the person walks with a constant speed **u** relative to the platform, what is the direction of the acceleration of the person at the instant shown?

a. Negative x
b. Negative y
c. Negative x and positive y
d. Positive x and positive y
e. Negative x and negative y

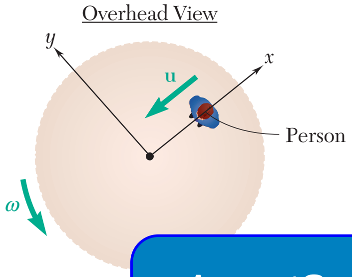


Fig. P15.CQ

END-OF-SECTION PROBLEMS

15.150 and 15.151 Pin P is attached to the collar shown; the motion of the pin is guided by a slot cut in rod BD and by the collar that slides on rod AE. Knowing that at the instant considered the rods rotate clockwise with constant angular velocities, determine for the given data the velocity of pin P.

15.150 $v_{AE} = 8$ rad/s, $v_{BD} = 3$ rad/s
15.151 $v_{AE} = 7$ rad/s, $v_{BD} = 4.8$ rad/s

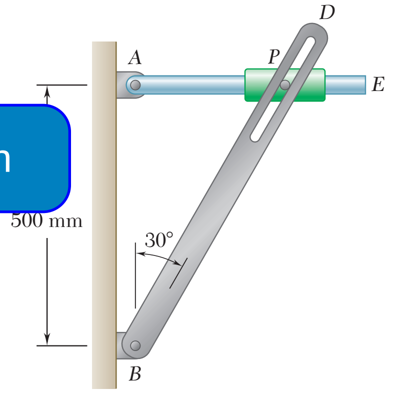


Fig. P15.150 and P15.151

15.152 and 15.153 Two rotating rods are connected by slider block P. The rod attached at A rotates with a constant clockwise angular velocity v_A. For the given data, determine for the position shown (*a*) the angular velocity of the rod attached at B, (*b*) the relative velocity of slider block P with respect to the rod on which it slides.

15.152 $b = 8$ in., $v_A = 6$ rad/s
15.153 $b = 300$ mm, $v_A = 10$ rad/s

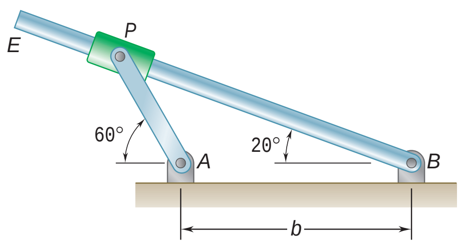


Fig. P15.152

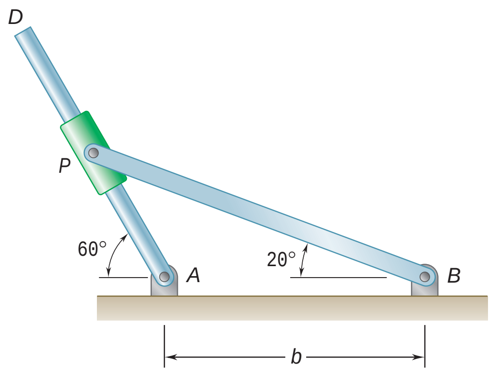


Fig. P15.153

15.154 Pin P is attached to the wheel shown and slides in a slot cut in bar BD. The wheel rolls to the right without slipping with a constant angular velocity of 20 rad/s. Knowing that $x = 480$ mm when u $= 0$, determine the angular velocity of the bar and the relative velocity of pin P with respect to the rod when *(a)* u $= 0$, *(b)* u $= 90°$.

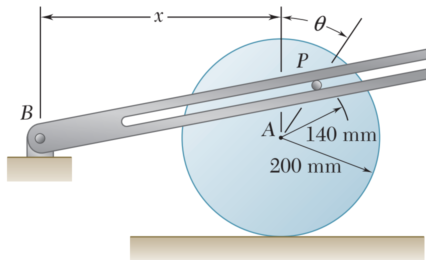


Fig. P15.154

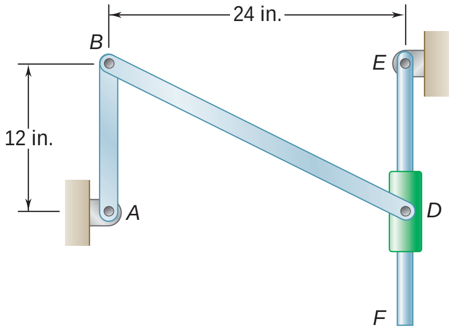


Fig. P15.155 and *P15.156*

15.155 Bar AB rotates clockwise with a constant angular velocity of 8 rad/s and rod EF rotates clockwise with a constant angular velocity of 6 rad/s. Determine at the instant shown (*a*) the angular velocity of bar BD, (*b*) the relative velocity of collar D with respect to rod EF.

15.156 Bar AB rotates clockwise with a constant angular velocity of 4 rad/s. Knowing that the magnitude of the velocity of collar D is 20 ft/s and that the angular velocity of bar BD is counterclockwise at the instant shown, determine (*a*) the angular velocity of bar EF, (*b*) the relative velocity of collar D with respect to rod EF.

... by slots cut in rods AD and BE. ... nstant angular velocity of 4 rad/s ... angular velocity of 5 rad/s counterclockwise and is slowing down at a rate of 2 rad/s², determine the velocity of P for the position shown.

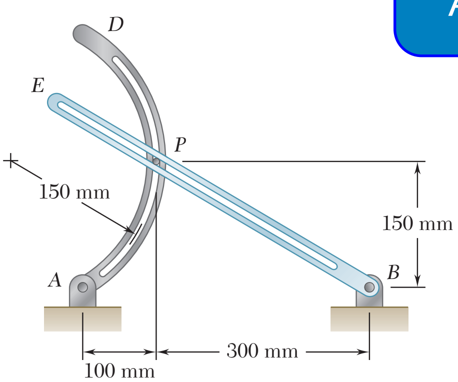


Fig. *P15.157*

15.158 Four pins slide in four separate slots cut in a circular plate as shown. When the plate is at rest, each pin has a velocity directed as shown and of the same constant magnitude u. If each pin maintains the same velocity relative to the plate when the plate rotates about O with a constant counterclockwise angular velocity V, determine the acceleration of each pin.

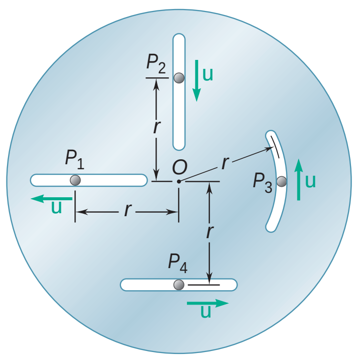


Fig. *P15.158*

15.159 Solve Prob. 15.158, assuming that the plate rotates about O with a constant clockwise angular velocity V.

15.160 Pin P slides in the circular slot cut in the plate shown at a constant relative speed u = 500 mm/s. Assuming that at the instant shown the angular velocity of the plate is 6 rad/s and is increasing at the rate of 20 rad/s^2, determine the acceleration of pin P when u = 90°.

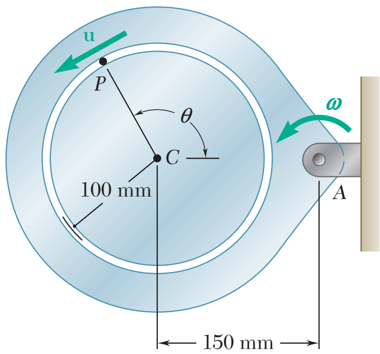
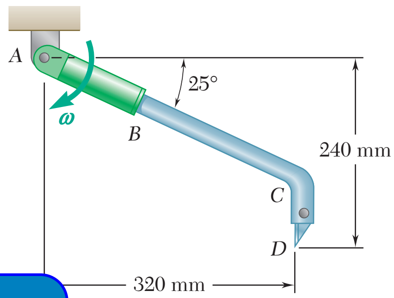


Fig. P15.160

15.161 The cage of a mine elevator moves downward at a constant speed of 40 ft/s. Determine the magnitude and direction of the Coriolis acceleration of the cage if the elevator is located (*a*) at the equator, (*b*) at latitude 40° north, (*c*) at latitude 40° south.

15.162 A rocket sled is tested on a straight track that is built along a meridian. Knowing that the track is located at latitude 40° north, determine the Coriolis acceleration of the sled when it is moving north at a speed of 900 km/h.

15.163 The motion of blade D is controlled by the robot arm ABC. At the instant shown the arm is rotating clockwise at the constant rate V = 1.8 rad/s and the length of portion BC of the arm is being decreased at the constant rate of 250 mm/s. Determine (*a*) the velocity of D, (*b*) the acceleration of D.



15.163

15.164 At the instant shown the length of the boom AB is being *decreased* at the constant rate of 0.2 m/s and the boom is being lowered at the constant rate of 0.08 rad/s. Determine (*a*) the velocity of point B, (*b*) the acceleration of point B.

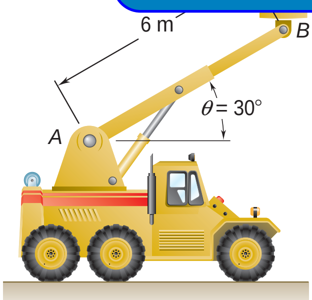


Fig. P15.164 and P15.165

15.165 At the instant shown the length of the boom AB is being *increased* at the constant rate of 0.2 m/s and the boom is being lowered at the constant rate of 0.08 rad/s. Determine (*a*) the velocity of point B, (*b*) the acceleration of point B.

15.166 and 15.167 The sleeve BC is welded to an arm that rotates about A with a constant angular velocity V. In the position shown rod DF is being moved to the left at a constant speed u = 400 mm/s relative to the sleeve. For the given angular velocity V, determine the acceleration (*a*) of point D, (*b*) of the point of rod DF that coincides with point E.

15.166 V = (3 rad/s) **i**
15.167 V = (3 rad/s) **j**

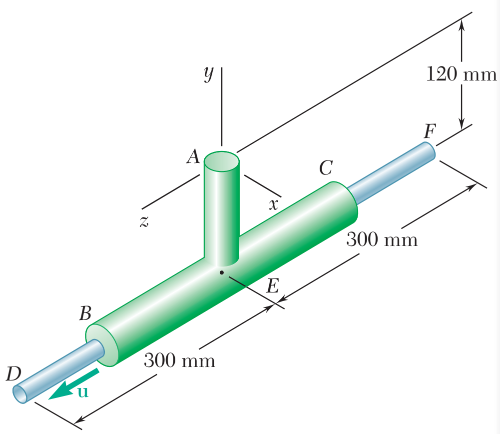


Fig. P15.166 and P15.167

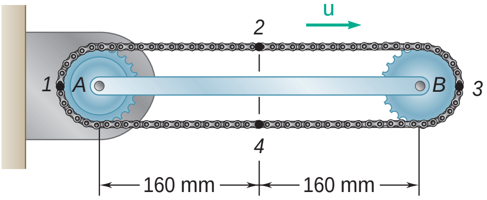


Fig. P15.168 and P15.169

15.168 and 15.169 A chain is looped around two gears of radius 40 mm that can rotate freely with respect to the 320-mm arm *AB*. The chain moves about arm *AB* in a clockwise direction at the constant rate of 80 mm/s relative to the arm. Knowing that in the position shown arm *AB* rotates clockwise about *A* at the constant rate v = 0.75 rad/s, determine the acceleration of each of the chain links indicated.

15.168 Links 1 and 2
15.169 Links 3 and 4

15.170 A basketball player shoots a free throw in such a way that his shoulder can be considered a pin joint at the moment of release as shown. Knowing that at the instant shown the upper arm *SE* has a constant angular velocity of 2 rad/s counterclockwise and the forearm *EW* has a constant clockwise angular velocity of 4 rad/s with respect to *SE*, determine the velocity and acceleration of the wrist *W*.

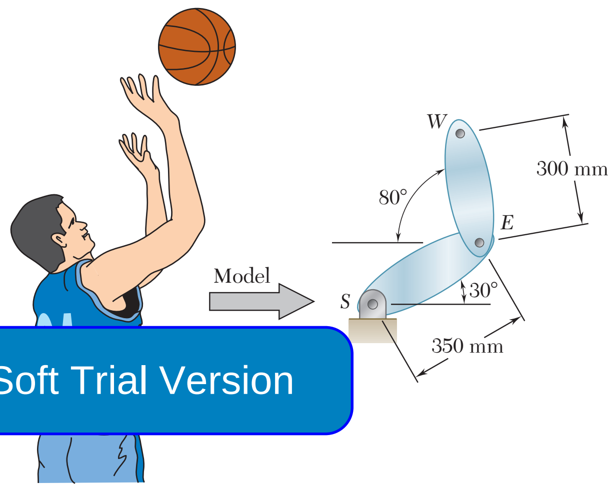


Fig. P15.170

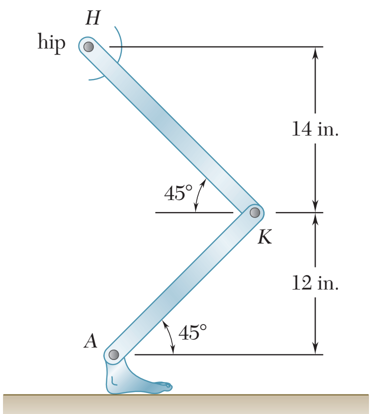


Fig. P15.171

15.171 The human leg can be crudely approximated as two rigid bars (the femur and the tibia) connected with a pin joint. At the instant shown, the velocity of the ankle *A* is zero, the tibia *AK* has an angular velocity of 1.5 rad/s counterclockwise and an angular acceleration of 1 rad/s^2 counterclockwise. Determine the relative angular velocity and relative angular acceleration of the femur *KH* with respect to *AK* so that the velocity and acceleration of *H* are both straight up at this instant.

15.172 The collar *P* slides outward at a constant relative speed *u* along rod *AB*, which rotates counterclockwise with a constant angular velocity of 20 rpm. Knowing that r = 250 mm when u = 0 and that the collar reaches *B* when u = 90°, determine the magnitude of the acceleration of the collar *P* just as it reaches *B*.

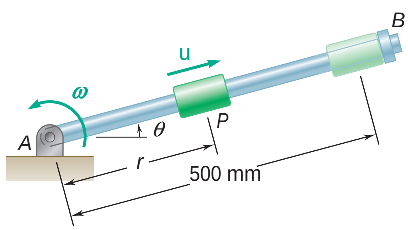


Fig. *P15.172*

15.173 Pin P slides in a circular slot cut in the plate shown at a constant relative speed $u = 90$ mm/s. Knowing that at the instant shown the plate rotates clockwise about A at the constant rate $V = 3$ rad/s, determine the acceleration of the pin if it is located at (*a*) point A, (*b*) point B, (*c*) point C.

15.174 Pin P slides in a circular slot cut in the plate shown at a constant relative speed $u = 90$ mm/s. Knowing that at the instant shown the angular velocity V of the plate is 3 rad/s clockwise and is decreasing at the rate of 5 rad/s^2, determine the acceleration of the pin if it is located at (*a*) point A, (*b*) point B, (*c*) point C.

15.175 Pin P is attached to the wheel shown and slides in a slot cut in bar BD. The wheel rolls to the right without slipping with a constant angular velocity of 20 rad/s. Knowing that $x = 480$ mm when $u = 0$, determine (*a*) the angular acceleration of the bar, (*b*) the relative acceleration of pin P with respect to the bar when $u = 0$.

15.176 Knowing that at the instant shown the rod attached at A has an angular velocity of 5 rad/s counterclockwise and an angular acceleration of 2 rad/s^2 clockwise, determine the angular velocity and the angular acceleration of the rod attached at B.

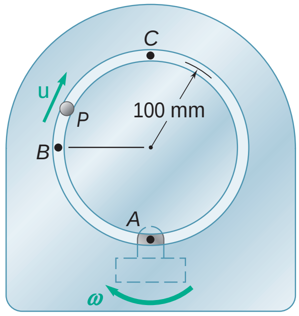


Fig. P15.173 and P15.174

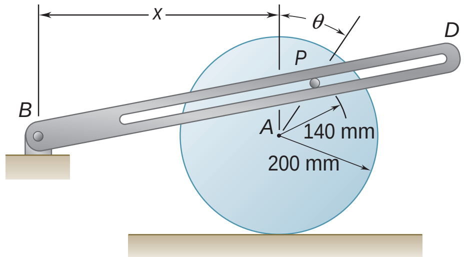


Fig. P15.175

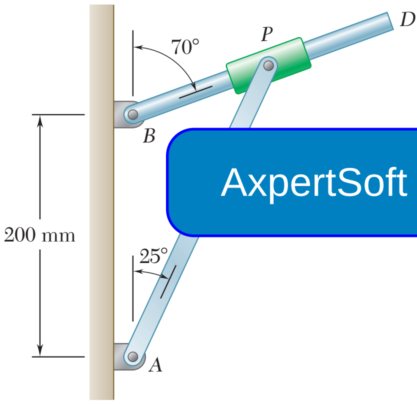


Fig. P15.176

15.177 The Geneva mechanism shown is used to provide an intermittent rotary motion of disk S. Disk D rotates with a constant counterclockwise angular velocity V_D of 8 rad/s. A pin P is attached to disk D and can slide in one of the six equally spaced slots cut in disk S. It is desirable that the angular velocity of disk S be zero as the pin enters and leaves each of the six slots; this will occur if the distance between the centers of the disks and the radii of the disks are related as shown. Determine the angular velocity and angular acceleration of disk S at the instant when $f = 150°$.

15.178 In Prob. 15.177, determine the angular velocity and angular acceleration of disk S at the instant when $f = 135°$.

15.179 At the instant shown bar BC has an angular velocity of 3 rad/s and an angular acceleration of 2 rad/s^2, both counterclockwise; determine the angular acceleration of the plate.

15.180 At the instant shown bar BC has an angular velocity of 3 rad/s and an angular acceleration of 2 rad/s^2, both clockwise; determine the angular acceleration of the plate.

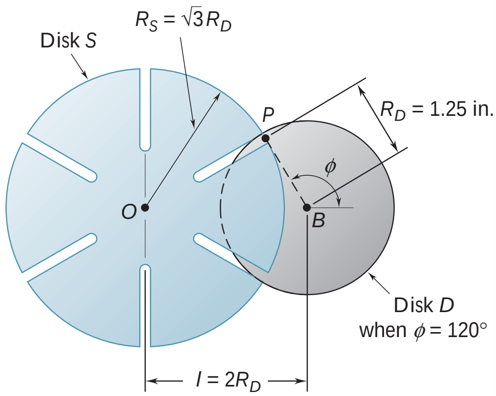


Fig. P15.177

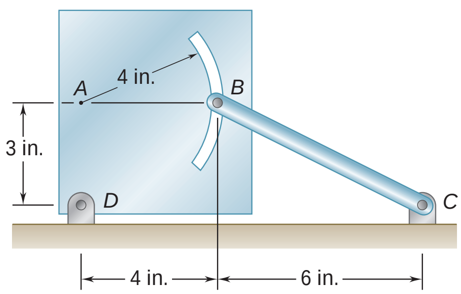


Fig. P15.179 and P15.180

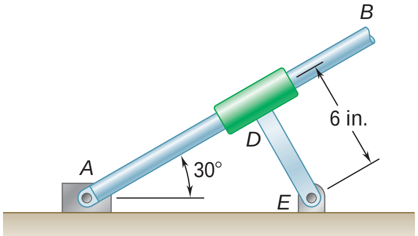


Fig. P15.181

***15.181** Rod AB passes through a collar which is welded to link DE. Knowing that at the instant shown block A moves to the right at a constant speed of 75 in./s, determine (*a*) the angular velocity of rod AB, (*b*) the velocity relative to the collar of the point of the rod in contact with the collar, (*c*) the acceleration of the point of the rod in contact with the collar. (*Hint:* Rod AB and link DE have the same $\boldsymbol{\omega}$ and the same $\boldsymbol{\alpha}$.)

***15.182** Solve Prob. 15.181 assuming block A moves to the left at a constant speed of 75 in./s.

***15.183** In Prob. 15.157, determine the acceleration of pin P.

*15.12 MOTION ABOUT A FIXED POINT

In Sec. 15.3 the motion of a rigid body constrained to rotate about a fixed axis was considered. The more general case of the motion of a rigid body which has a fixed point O will now be examined.

First, it will be proved that *the most general displacement of a rigid body with a fixed point O is equivalent to a rotation of the body about an axis through O.*† Instead of considering the rigid body itself, we can detach a sphere of center O from the body and analyze

he motion of the sphere com- given body. Since three points , the center O and two points A and B on the surface of the sphere will define the position of the sphere and thus the position of the body. Let A_1 and B_1 characterize the position of the sphere at one instant, and let A_2 and B_2 characterize its position at a later instant (Fig. 15.31*a*). Since the sphere is rigid, the lengths of the arcs of great circle A_1B_1 and A_2B_2 must be equal, but except for this requirement, the positions of A_1, A_2, B_1, and B_2 are arbitrary. We propose to prove that the points A and B can be brought, respectively, from A_1 and B_1 into A_2 and B_2 by a single rotation of the sphere about an axis.

For convenience, and without loss of generality, we select point B so that its initial position coincides with the final position of A; thus, $B_1 = A_2$ (Fig. 15.31*b*). We draw the arcs of great circle A_1A_2, A_2B_2 and the arcs bisecting, respectively, A_1A_2 and A_2B_2. Let C be the point of intersection of these last two arcs; we complete the construction by drawing A_1C, A_2C, and B_2C. As pointed out above, because of the rigidity of the sphere, $A_1B_1 = A_2B_2$. Since C is by construction equidistant from A_1, A_2, and B_2, we also have $A_1C = A_2C = B_2C$. As a result, the spherical triangles A_1CA_2 and B_1CB_2 are congruent and the angles A_1CA_2 and B_1CB_2 are equal. Denoting by θ the common value of these angles, we conclude that the sphere can be brought from its initial position into its final position by a single rotation through θ about the axis OC.

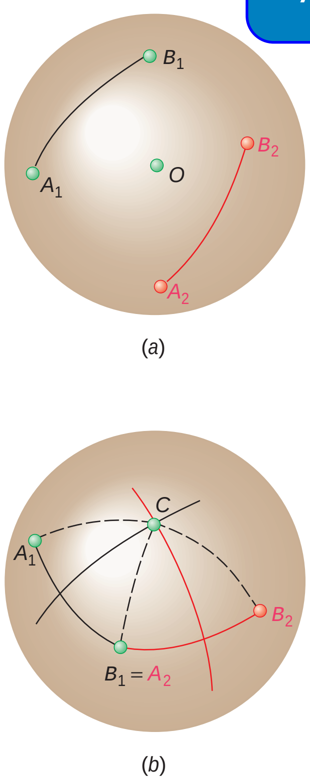


Fig. 15.31

†This is known as *Euler's theorem.*

It follows that the motion during a time interval Δt of a rigid body with a fixed point O can be considered as a rotation through Δu about a certain axis. Drawing along that axis a vector of magnitude $\Delta u/\Delta t$ and letting Δt approach zero, we obtain at the limit the *instantaneous axis of rotation* and the angular velocity V of the body at the instant considered (Fig. 15.32). The velocity of a particle P of the body can then be obtained, as in Sec. 15.3, by forming the vector product of V and of the position vector **r** of the particle:

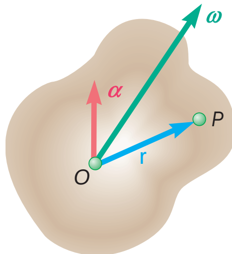


Fig. 15.32

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathrm{V} \times \mathbf{r} \tag{15.37}$$

The acceleration of the particle is obtained by differentiating (15.37) with respect to t. As in Sec. 15.3 we have

$$\mathbf{a} = \mathrm{A} \times \mathbf{r} + \mathrm{V} \times (\mathrm{V} \times \mathbf{r}) \tag{15.38}$$

where the angular acceleration A is defined as the derivative

$$\mathrm{A} = \frac{d\mathrm{V}}{dt} \tag{15.39}$$

of the angular velocity V.

In the case of the motion of a rigid body with a fixed point, the direction of V and of the instantaneous axis of rotation changes from one instant to the next. The angular acceleration A therefore reflects the change in direction of V as well as its change in magnitude and, in general, *is not directed along the instantaneous axis of rotation.* While the particles of the body located on the instantaneous axis of rotation have zero velocity at the instant considered, they do not have zero acceleration. Also, the accelerations of the various particles of the body *cannot* be determined as if the body were rotating permanently about the instantaneous axis.

Recalling the definition of the velocity of a particle with position vector **r**, we note that the angular acceleration A, as expressed in (15.39), represents the velocity of the tip of the vector V. This property may be useful in the determination of the angular acceleration of a rigid body. For example, it follows that the vector A is tangent to the curve described in space by the tip of the vector V.

We should note that the vector V moves within the body, as well as in space. It thus generates two cones called, respectively, the *body cone* and the *space cone* (Fig. 15.33).† It can be shown that at any given instant, the two cones are tangent along the instantaneous axis of rotation and that as the body moves, the body cone appears to *roll* on the space cone.

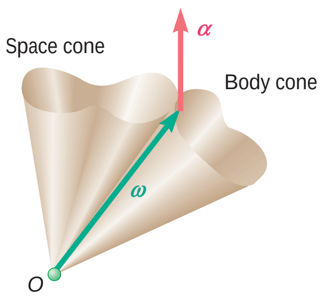


Fig. 15.33

†It is recalled that a *cone* is, by definition, a surface generated by a straight line passing through a fixed point. In general, the cones considered here *will not be circular cones.*

Photo 15.8 When the ladder rotates about its fixed base, its angular velocity can be obtained by adding the angular velocities which correspond to simultaneous rotations about two different axes.

Before concluding our analysis of the motion of a rigid body with a fixed point, we should prove that angular velocities are actually vectors. As indicated in Sec. 2.3, some quantities, such as the *finite rotations* of a rigid body, have magnitude and direction but do not obey the parallelogram law of addition; these quantities cannot be considered as vectors. In contrast, angular velocities (and also *infinitesimal rotations*), as will be demonstrated presently, *do obey* the parallelogram law and thus are truly vector quantities.

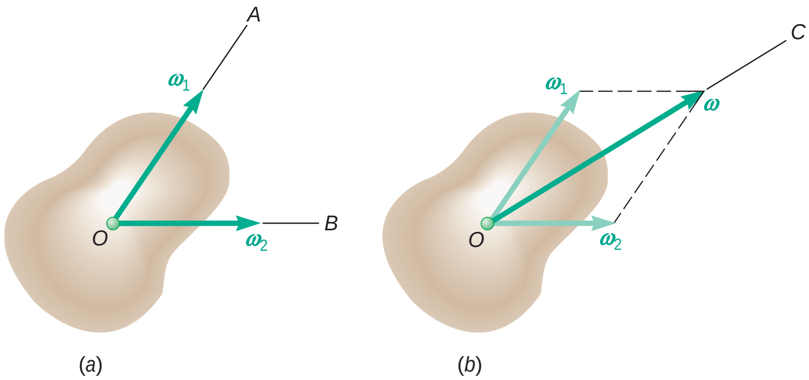


Fig. 15.34

Consider a rigid body with a fixed point O which at a given

…e axes OA and OB with angu-
…We know that this motion must
…d to a single rotation of angular
velocity V. We propose to show that

$$\mathrm{V} = \mathrm{V}_1 + \mathrm{V}_2 \tag{15.40}$$

i.e., that the resulting angular velocity can be obtained by adding V_1 and V_2 by the parallelogram law (Fig. 15.34*b*).

Consider a particle P of the body, defined by the position vector $\mathbf{r}$. Denoting, respectively, by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}$ the velocity of P when the body rotates about OA only, about OB only, and about both axes simultaneously, we write

$$\mathbf{v} = \mathrm{V} \times \mathbf{r} \qquad \mathbf{v}_1 = \mathrm{V}_1 \times \mathbf{r} \qquad \mathbf{v}_2 = \mathrm{V}_2 \times \mathbf{r} \tag{15.41}$$

But the vectorial character of *linear* velocities is well established (since they represent the derivatives of position vectors). We therefore have

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$

where the plus sign indicates vector addition. Substituting from (15.41), we write

$$\mathrm{V} \times \mathbf{r} = \mathrm{V}_1 \times \mathbf{r} + \mathrm{V}_2 \times \mathbf{r}$$
$$\mathrm{V} \times \mathbf{r} = (\mathrm{V}_1 + \mathrm{V}_2) \times \mathbf{r}$$

where the plus sign still indicates vector addition. Since the relation obtained holds for an arbitrary $\mathbf{r}$, we conclude that (15.40) must be true.

*15.13 GENERAL MOTION

The most general motion of a rigid body in space will now be considered. Let A and B be two particles of the body. We recall from Sec. 11.12 that the velocity of B with respect to the fixed frame of reference $OXYZ$ can be expressed as

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{15.42}$$

where $\mathbf{v}_{B/A}$ is the velocity of B relative to a frame $AX'Y'Z'$ attached to A and of fixed orientation (Fig. 15.35). Since A is fixed in this frame, the motion of the body relative to $AX'Y'Z'$ is the motion of a body with a fixed point. The relative velocity $\mathbf{v}_{B/A}$ can therefore be obtained from (15.37) after $\mathbf{r}$ has been replaced by the position vector $\mathbf{r}_{B/A}$ of B relative to A. Substituting for $\mathbf{v}_{B/A}$ into (15.42), we write

$$\mathbf{v}_B = \mathbf{v}_A + \mathsf{V} \times \mathbf{r}_{B/A} \tag{15.43}$$

where V is the angular velocity of the body at the instant considered.

The acceleration of B is obtained by a similar reasoning. We first write

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

and, recalling Eq. (15.38),

$$\mathbf{a}_B = \mathbf{a}_A + \mathsf{A} \times$$

where A is the angular acceleration of the body at the instant considered.

Equations (15.43) and (15.44) show that *the most general motion of a rigid body is equivalent, at any given instant, to the sum of a translation,* in which all the particles of the body have the same velocity and acceleration as a reference particle A, *and of a motion in which particle A is assumed to be fixed.*†

It is easily shown, by solving (15.43) and (15.44) for $\mathbf{v}_A$ and $\mathbf{a}_A$, that the motion of the body with respect to a frame attached to B would be characterized by the same vectors V and A as its motion relative to $AX'Y'Z'$. The angular velocity and angular acceleration of a rigid body at a given instant are thus independent of the choice of reference point. On the other hand, one should keep in mind that whether the moving frame is attached to A or to B, it should maintain a fixed orientation; that is, it should remain parallel to the fixed reference frame $OXYZ$ throughout the motion of the rigid body. In many problems it will be more convenient to use a moving frame which is allowed to rotate as well as to translate. The use of such moving frames will be discussed in Secs. 15.14 and 15.15.

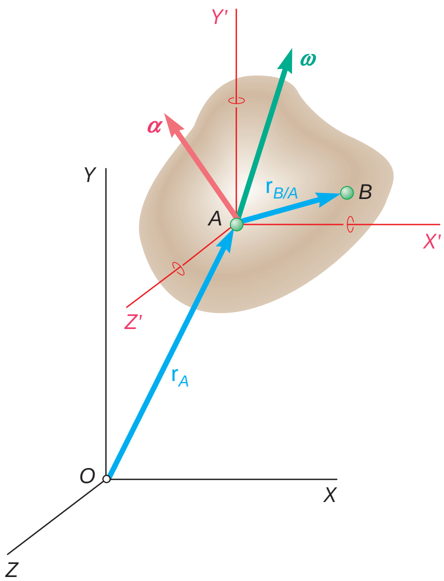
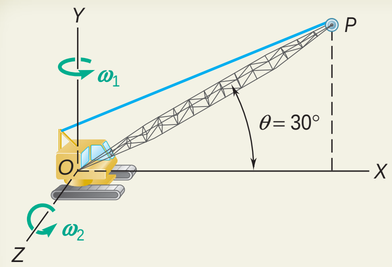


Fig. 15.35

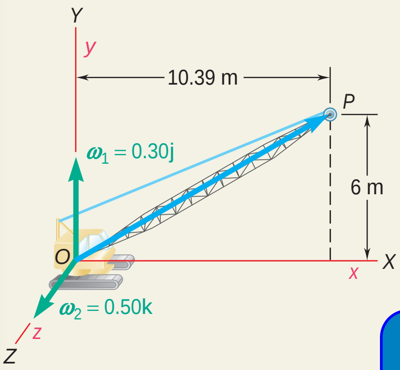
†It is recalled from Sec. 15.12 that, in general, the vectors V and A are not collinear, and that the accelerations of the particles of the body in their motion relative to the frame $AX'Y'Z'$ cannot be determined as if the body were rotating permanently about the instantaneous axis through A.



SAMPLE PROBLEM 15.11

The crane shown rotates with a constant angular velocity V_1 of 0.30 rad/s. Simultaneously, the boom is being raised with a constant angular velocity V_2 of 0.50 rad/s relative to the cab. Knowing that the length of the boom *OP* is $l = 12$ m, determine (*a*) the angular velocity V of the boom, (*b*) the angular acceleration A of the boom, (*c*) the velocity **v** of the tip of the boom, (*d*) the acceleration **a** of the tip of the boom.

SOLUTION



a. Angular Velocity of Boom. Adding the angular velocity V_1 of the cab and the angular velocity V_2 of the boom relative to the cab, we obtain the angular velocity V of the boom at the instant considered:

$$V = V_1 + V_2 \qquad V = (0.30 \text{ rad/s})\mathbf{j} + (0.50 \text{ rad/s})\mathbf{k} \quad ◀$$

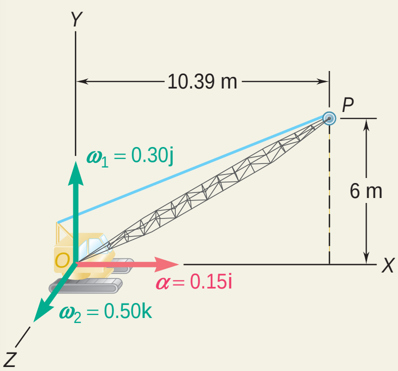
b. Angular Acceleration of Boom. The angular acceleration A of the boom is obtained by differentiating V. Since the vector V_1 is constant in magnitude and direction, we have

$$A = \dot{V} = \dot{V}_1 + \dot{V}_2 = 0 + \dot{V}_2$$

...uted with respect to the fixed ...t to use a frame *Oxyz* attached ...or V_2 also rotates with the cab ...espect to that frame. Using Eq. (15.31) with $\mathbf{Q} = V_2$ and $\mathbf{\Omega} = V_1$, we write

$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \mathbf{\Omega} \times \mathbf{Q}$$
$$(\dot{V}_2)_{OXYZ} = (\dot{V}_2)_{Oxyz} + V_1 \times V_2$$
$$A = (\dot{V}_2)_{OXYZ} = 0 + (0.30 \text{ rad/s})\mathbf{j} \times (0.50 \text{ rad/s})\mathbf{k}$$

$$A = (0.15 \text{ rad/s}^2)\mathbf{i} \quad ◀$$



c. Velocity of Tip of Boom. Noting that the position vector of point *P* is $\mathbf{r} = (10.39 \text{ m})\mathbf{i} + (6 \text{ m})\mathbf{j}$ and using the expression found for V in part *a*, we write

$$\mathbf{v} = V \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.30 \text{ rad/s} & 0.50 \text{ rad/s} \\ 10.39 \text{ m} & 6 \text{ m} & 0 \end{vmatrix}$$

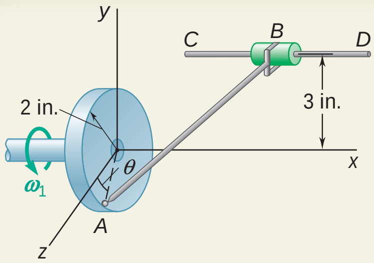
$$\mathbf{v} = -(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j} - (3.12 \text{ m/s})\mathbf{k} \quad ◀$$

d. Acceleration of Tip of Boom. Recalling that $\mathbf{v} = V \times \mathbf{r}$, we write

$$\mathbf{a} = A \times \mathbf{r} + V \times (V \times \mathbf{r}) = A \times \mathbf{r} + V \times \mathbf{v}$$

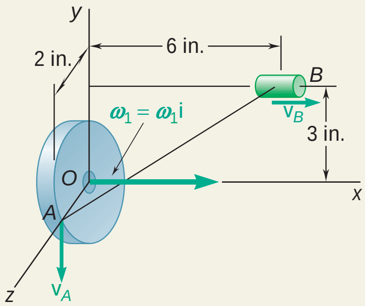
$$\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.15 & 0 & 0 \\ 10.39 & 6 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.30 & 0.50 \\ -3 & 5.20 & -3.12 \end{vmatrix}$$
$$= 0.90\mathbf{k} - 0.94\mathbf{i} - 2.60\mathbf{i} - 1.50\mathbf{j} + 0.90\mathbf{k}$$

$$\mathbf{a} = -(3.54 \text{ m/s}^2)\mathbf{i} - (1.50 \text{ m/s}^2)\mathbf{j} + (1.80 \text{ m/s}^2)\mathbf{k} \quad ◀$$



SAMPLE PROBLEM 15.12

The rod AB, of length 7 in., is attached to the disk by a ball-and-socket connection and to the collar B by a clevis. The disk rotates in the yz plane at a constant rate $\omega_1 = 12$ rad/s, while the collar is free to slide along the horizontal rod CD. For the position $\theta = 0$, determine (a) the velocity of the collar, (b) the angular velocity of the rod.



$\boldsymbol{\omega}_1 = 12\,\mathbf{i}$
$\mathbf{r}_A = 2\mathbf{k}$
$\mathbf{r}_B = 6\mathbf{i} + 3\mathbf{j}$
$\mathbf{r}_{B/A} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

SOLUTION

a. Velocity of Collar. Since point A is attached to the disk and since collar B moves in a direction parallel to the x axis, we have

$$\mathbf{v}_A = \boldsymbol{\omega}_1 \times \mathbf{r}_A = 12\mathbf{i} \times 2\mathbf{k} = -24\mathbf{j} \qquad \mathbf{v}_B = v_B\mathbf{i}$$

Denoting by $\boldsymbol{\omega}$ the angular velocity of the rod, we write

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

$$v_B\mathbf{i} = -24\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 6 & 3 & -2 \end{vmatrix}$$

$$\ldots + (6\omega_z + 2\omega_x)\mathbf{j} + (3\omega_x - 6\omega_y)\mathbf{k}$$

...ectors, we obtain

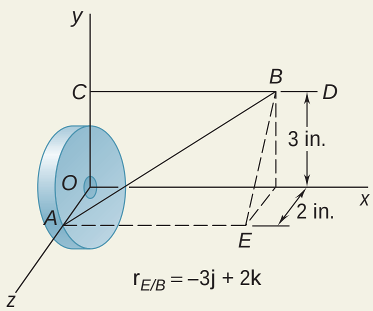
$$\ldots 2\omega_y \quad -3\omega_z \qquad (1)$$
$$24 = 2\omega_x \qquad +6\omega_z \qquad (2)$$
$$0 = 3\omega_x \quad -6\omega_y \qquad (3)$$

Multiplying Eqs. (1), (2), (3), respectively, by 6, 3, −2 and adding, we write

$$6v_B + 72 = 0 \qquad v_B = -12 \qquad \mathbf{v}_B = -(12 \text{ in./s})\mathbf{i} \quad ◀$$

b. Angular Velocity of Rod *AB*. We note that the angular velocity cannot be determined from Eqs. (1), (2), and (3), since the determinant formed by the coefficients of ω_x, ω_y, and ω_z is zero. We must therefore obtain an additional equation by considering the constraint imposed by the clevis at B.

The collar-clevis connection at B permits rotation of AB about the rod CD and also about an axis perpendicular to the plane containing AB and CD. It prevents rotation of AB about the axis EB, which is perpendicular to CD and lies in the plane containing AB and CD. Thus the projection of $\boldsymbol{\omega}$ on $\mathbf{r}_{E/B}$ must be zero and we write†



$$\boldsymbol{\omega} \cdot \mathbf{r}_{E/B} = 0 \qquad (\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \cdot (-3\mathbf{j} + 2\mathbf{k}) = 0$$
$$-3\omega_y + 2\omega_z = 0 \qquad (4)$$

Solving Eqs. (1) through (4) simultaneously, we obtain

$$v_B = -12 \qquad \omega_x = 3.69 \qquad \omega_y = 1.846 \qquad \omega_z = 2.77$$
$$\boldsymbol{\omega} = (3.69 \text{ rad/s})\mathbf{i} + (1.846 \text{ rad/s})\mathbf{j} + (2.77 \text{ rad/s})\mathbf{k} \quad ◀$$

†We could also note that the direction of EB is that of the vector triple product $\mathbf{r}_{B/C} \times (\mathbf{r}_{B/C} \times \mathbf{r}_{B/A})$ and write $\boldsymbol{\omega} \cdot [\mathbf{r}_{B/C} \times (\mathbf{r}_{B/C} \times \mathbf{r}_{B/A})] = 0$. This formulation would be particularly useful if the rod CD were skew.

SOLVING PROBLEMS ON YOUR OWN

In this lesson you started the study of the *kinematics of rigid bodies in three dimensions.* You first studied the *motion of a rigid body about a fixed point* and then the *general motion of a rigid body.*

A. Motion of a rigid body about a fixed point. To analyze the motion of a point B of a body rotating about a fixed point O you may have to take some or all of the following steps.

1. Determine the position vector r connecting the fixed point O to point B.

2. Determine the angular velocity V of the body with respect to a fixed frame of reference. The angular velocity V will often be obtained by adding two component angular velocities V_1 and V_2 [Sample Prob. 15.11].

3. Compute the velocity of *B* by using the equation

$$\mathbf{v} = \mathsf{V} \times \mathbf{r} \tag{15.37}$$

Your computation will usually be facilitated if you express the vector product as a determinant.

4. Determine the an[illegible]lar acceleration A represents the rat[illegible]spect to a fixed *frame of reference OXYZ* and reflects both *a change in magnitude* and *a change in direction* of the angular velocity. However, when computing A you may find it convenient to first compute the rate of change $(\dot{\mathsf{V}})_{Oxyz}$ of V with respect to a rotating frame of reference $Oxyz$ of your choice and use Eq. (15.31) of the preceding lesson to obtain A. You will write

$$\mathsf{A} = (\dot{\mathsf{V}})_{OXYZ} = (\dot{\mathsf{V}})_{Oxyz} + \mathbf{\Omega} \times \mathsf{V}$$

where $\mathbf{\Omega}$ is the angular velocity of the rotating frame $Oxyz$ [Sample Prob. 15.11].

5. Compute the acceleration of *B* by using the equation

$$\mathbf{a} = \mathsf{A} \times \mathbf{r} + \mathsf{V} \times (\mathsf{V} \times \mathbf{r}) \tag{15.38}$$

Note that the vector product $(\mathsf{V} \times \mathbf{r})$ represents the velocity of point B and was computed in step 3. Also, the computation of the first vector product in (15.38) will be facilitated if you express this product in determinant form. Remember that, as was the case with the plane motion of a rigid body, the instantaneous axis of rotation *cannot* be used to determine accelerations.

B. General motion of a rigid body. The general motion of a rigid body may be considered as *the sum of a translation and a rotation.* Keep the following in mind:

a. In the translation part of the motion, all the points of the body have the *same velocity* $\mathbf{v}_A$ *and the same acceleration* $\mathbf{a}_A$ as the point A of the body that has been selected as the reference point.

b. In the rotation part of the motion, the same reference point A is assumed to be a *fixed point.*

1. To determine the velocity of a point *B* of the rigid body when you know the velocity $\mathbf{v}_A$ of the reference point A and the angular velocity V of the body, you simply add $\mathbf{v}_A$ to the velocity $\mathbf{v}_{B/A} = \mathsf{V} \times \mathbf{r}_{B/A}$ of B in its rotation about A:

$$\mathbf{v}_B = \mathbf{v}_A + \mathsf{V} \times \mathbf{r}_{B/A} \tag{15.43}$$

As indicated earlier, the computation of the vector product will usually be facilitated if you express this product in determinant form.

Equation (15.43) can also be used to determine the magnitude of $\mathbf{v}_B$ when its direction is known, even if V is not known. While the corresponding three scalar equations are li[illegible] V are indeterminate, these compone[illegible] using an appropriate linear combinat[illegible].12, part *a*]. Alternatively, you can a[illegible]nents of V and solve the equations for $\mathbf{v}_A$. However, an additional equation must be sought in order to determine the true values of the components of V [Sample Prob. 15.12, part *b*].

2. To determine the acceleration of a point *B* of the rigid body when you know the acceleration $\mathbf{a}_A$ of the reference point A and the angular acceleration A of the body, you simply add $\mathbf{a}_A$ to the acceleration of B in its rotation about A, as expressed by Eq. (15.38):

$$\mathbf{a}_B = \mathbf{a}_A + \mathsf{A} \times \mathbf{r}_{B/A} + \mathsf{V} \times (\mathsf{V} \times \mathbf{r}_{B/A}) \tag{15.44}$$

Note that the vector product $(\mathsf{V} \times \mathbf{r}_{B/A})$ represents the velocity $\mathbf{v}_{B/A}$ of B relative to A and may already have been computed as part of your calculation of $\mathbf{v}_B$. We also remind you that the computation of the other two vector products will be facilitated if you express these products in determinant form.

The three scalar equations associated with Eq. (15.44) can also be used to determine the magnitude of $\mathbf{a}_B$ when its direction is known, even if V and A are not known. While the components of V and A are indeterminate, you can assign arbitrary values to one of the components of V and to one of the components of A and solve the equations for $\mathbf{a}_B$.

PROBLEMS

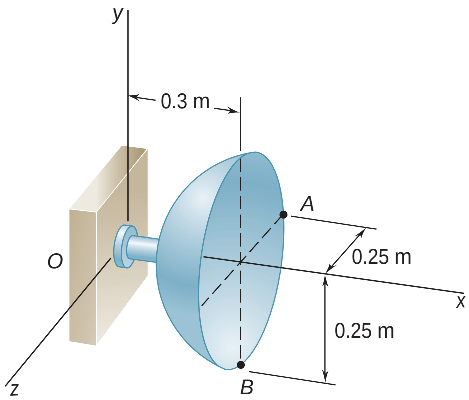


Fig. P15.184 and P15.185

END-OF-SECTION PROBLEMS

15.184 At the instant considered the radar antenna shown rotates about the origin of coordinates with an angular velocity $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$. Knowing that $(v_A)_y = 300$ mm/s, $(v_B)_y = 180$ mm/s, and $(v_B)_z = 360$ mm/s, determine (*a*) the angular velocity of the antenna, (*b*) the velocity of point *A*.

15.185 At the instant considered the radar antenna shown rotates about the origin of coordinates with an angular velocity $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$. Knowing that $(v_A)_x = 100$ mm/s, $(v_A)_y = -90$ mm/s, and $(v_B)_z = 120$ mm/s, determine (*a*) the angular velocity of the antenna, (*b*) the velocity of point *A*.

15.186 Plate *ABD* and rod *OB* are rigidly connected and rotate about the ball-and-socket joint *O* with an angular velocity $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$. Knowing that $\mathbf{v}_A = (80\text{ mm/s})\mathbf{i} + (360\text{ mm/s})\mathbf{j} + (v_A)_z\mathbf{k}$ and $\omega_x = 1.5$ rad/s, determine (*a*) the angular velocity of the assembly, (*b*) the velocity of point *D*.

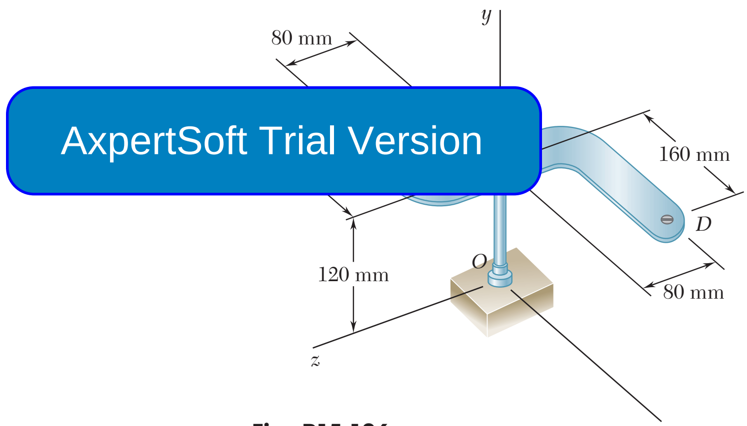


Fig. P15.186

15.187 The bowling ball shown rolls without slipping on the horizontal *xz* plane with an angular velocity $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$. Knowing that $\mathbf{v}_A = (14.4\text{ ft/s})\mathbf{i} - (14.4\text{ ft/s})\mathbf{j} + (10.8\text{ ft/s})\mathbf{k}$ and $\mathbf{v}_D = (28.8\text{ ft/s})\mathbf{i} + (21.6\text{ ft/s})\mathbf{k}$, determine (*a*) the angular velocity of the bowling ball, (*b*) the velocity of its center *C*.

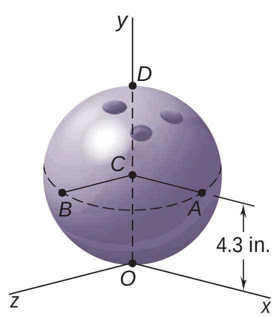


Fig. P15.187

15.188 The rotor of an electric motor rotates at the constant rate $\omega_1 = 1800$ rpm. Determine the angular acceleration of the rotor as the motor is rotated about the y axis with a constant angular velocity ω_2 of 6 rpm counterclockwise when viewed from the positive y axis.

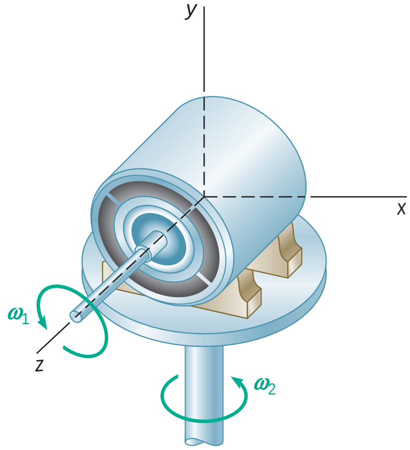


Fig. P15.188

15.189 The disk of a portable sander rotates at the constant rate $\omega_1 = 4400$ rpm as shown. Determine the angular acceleration of the disk as a worker rotates the sander about the z axis with an angular velocity of 0.5 rad/s and an angular acceleration of 2.5 rad/s^2, both clockwise when viewed from the positive z axis.

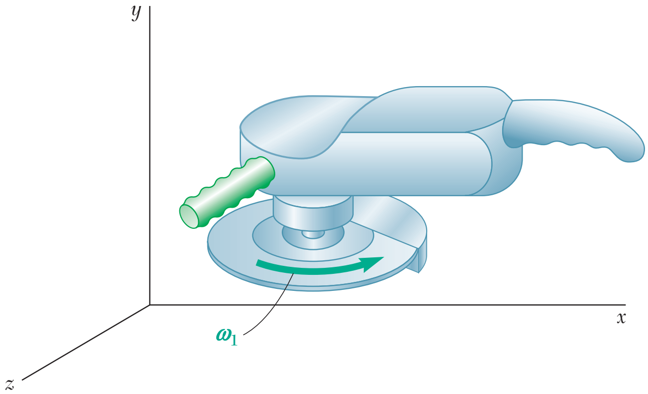


Fig. P15.189

15.190 Knowing that the turbi[illegible] $\omega_1 = 9000$ rpm, determi[illegible] the turbine housing has a constant angular velocity of 2.4 rad/s clockwise as viewed from (*a*) the positive y axis, (*b*) the positive z axis.

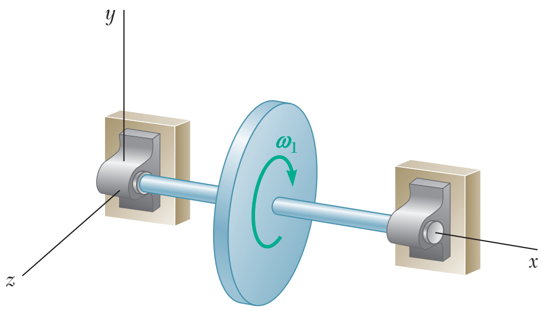


Fig. P15.190

15.191 In the system shown, disk A is free to rotate about the horizontal rod OA. Assuming that disk B is stationary ($\omega_2 = 0$), and that shaft OC rotates with a constant angular velocity ω_1, determine (*a*) the angular velocity of disk A, (*b*) the angular acceleration of disk A.

15.192 In the system shown, disk A is free to rotate about the horizontal rod OA. Assuming that shaft OC and disk B rotate with constant angular velocities ω_1 and ω_2, respectively, both counterclockwise, determine (*a*) the angular velocity of disk A, (*b*) the angular acceleration of disk A.

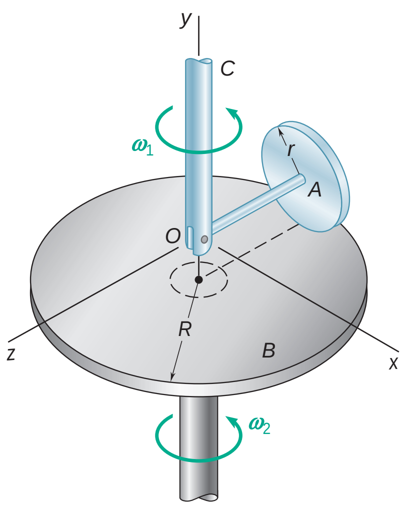


Fig. *P15.191* and *P15.192*

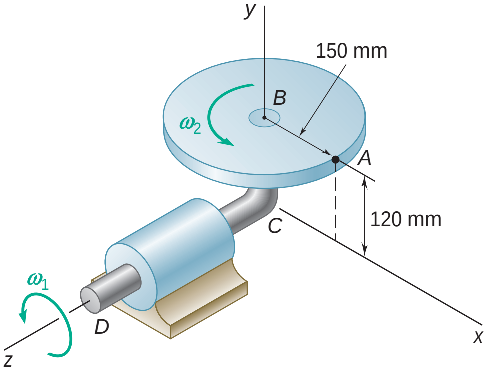


Fig. P15.193

15.193 The L-shaped arm *BCD* rotates about the *z* axis with a constant angular velocity $\omega_1 = 5$ rad/s. Knowing that the 150-mm-radius disk rotates about *BC* with a constant angular velocity $\omega_2 = 4$ rad/s, determine (*a*) the velocity of point *A*, (*b*) the acceleration of point *A*.

15.194 A gun barrel of length $OP = 4$ m is mounted on a turret as shown. To keep the gun aimed at a moving target the azimuth angle β is being increased at the rate $d\beta/dt = 30°/\text{s}$ and the elevation angle γ is being increased at the rate $d\gamma/dt = 10°/\text{s}$. For the position $\beta = 90°$ and $\gamma = 30°$, determine (*a*) the angular velocity of the barrel, (*b*) the angular acceleration of the barrel, (*c*) the velocity and acceleration of point *P*.

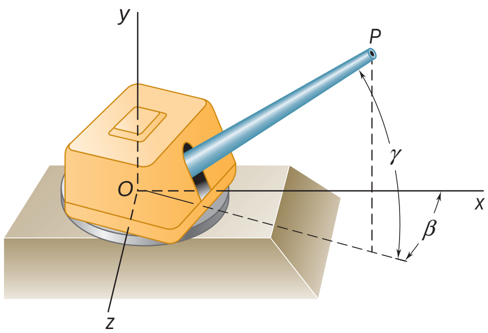


Fig. *P15.194*

constant rate $\omega_2 = 4$ rad/s about ed to a horizontal rod that rotates at the constant rate $\omega_1 = 5$ rad/s. For the position shown, determine (*a*) the angular acceleration of the disk, (*b*) the acceleration of point *P* on the rim of the disk if $\theta = 0$, (*c*) the acceleration of point *P* on the rim of the disk if $\theta = 90°$.

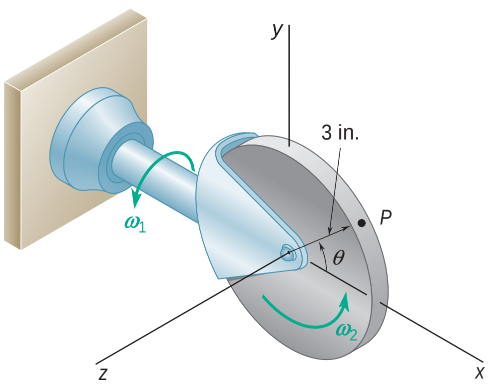


Fig. P15.195 and P15.196

15.196 A 3-in.-radius disk spins at the constant rate $\omega_2 = 4$ rad/s about an axis held by a housing attached to a horizontal rod that rotates at the constant rate $\omega_1 = 5$ rad/s. Knowing that $\theta = 30°$, determine the acceleration of point *P* on the rim of the disk.

15.197 A 30-mm-radius wheel is mounted on an axle *OB* of length 100 mm. The wheel rolls without sliding on the horizontal floor, and the axle is perpendicular to the plane of the wheel. Knowing that the system rotates about the *y* axis at a constant rate $\omega_1 = 2.4$ rad/s, determine (*a*) the angular velocity of the wheel, (*b*) the angular acceleration of the wheel, (*c*) the acceleration of point *C* located at the highest point on the rim of the wheel.

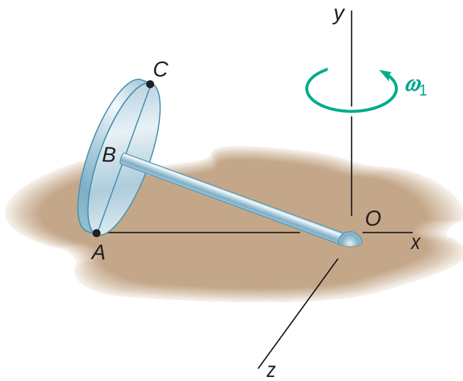


Fig. P15.197

15.198 At the instant shown, the robotic arm *ABC* is being rotated simultaneously at the constant rate $\omega_1 = 0.15$ rad/s about the *y* axis, and at the constant rate $\omega_2 = 0.25$ rad/s about the *z* axis. Knowing that the length of arm *ABC* is 1 m, determine (*a*) the angular acceleration of the arm, (*b*) the velocity of point *C*, (*c*) the acceleration of point *C*.

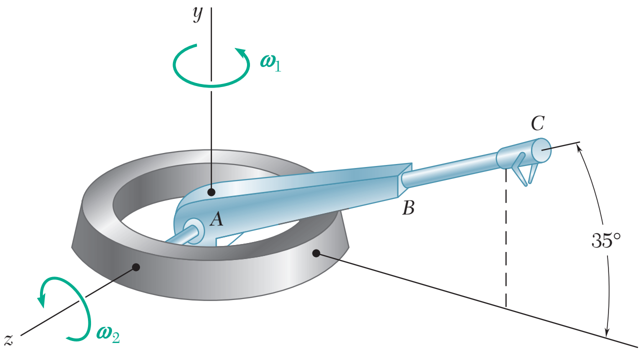


Fig. P15.198

15.199 In the planetary gear sys nected to each other and rotate as a unit about the inclined shaft. Gears *C* and *D* rotate with constant angular velocities of 30 rad/s and 20 rad/s, respectively (both counterclockwise when viewed from the right). Choosing the *x* axis to the right, the *y* axis upward, and the *z* axis pointing out of the plane of the figure, determine (*a*) the common angular velocity of gears *A* and *B*, (*b*) the angular velocity of shaft *FH*, which is rigidly attached to the inclined shaft.

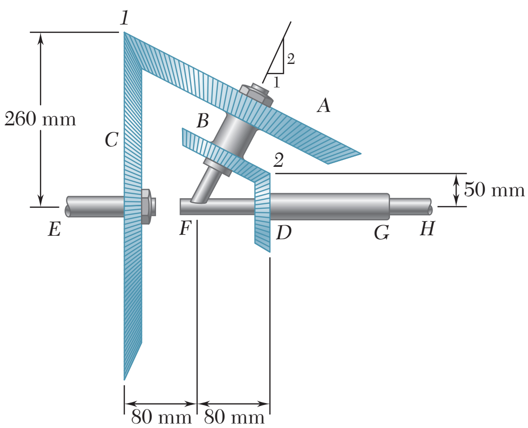


Fig. P15.199

15.200 In Prob. 15.199, determine (*a*) the common angular acceleration of gears *A* and *B*, (*b*) the acceleration of the tooth of gear *A* which is in contact with gear *C* at point *l*.

15.201 Several rods are brazed together to form the robotic guide arm shown which is attached to a ball-and-socket joint at O. Rod OA slides in a straight inclined slot while rod OB slides in a slot parallel to the z axis. Knowing that at the instant shown $\mathbf{v}_B = (9 \text{ in./s})\mathbf{k}$, determine (*a*) the angular velocity of the guide arm, (*b*) the velocity of point A, (*c*) the velocity of point C.

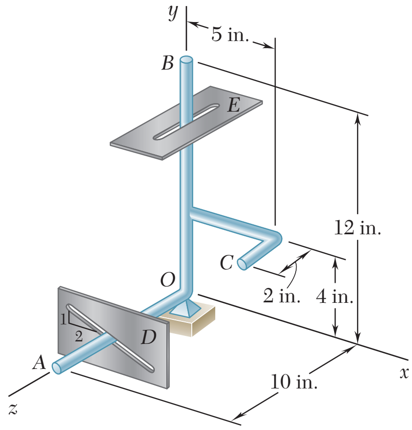


Fig. P15.201

15.202 In Prob. 15.201, the speed of point B is known to be constant. For [illegible] (*a*) the angular acceleration of the [illegible] of point C.

[illegible]ected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Knowing that collar B moves toward point E at a constant speed of 20 in./s, determine the velocity of collar A as collar B passes through point D.

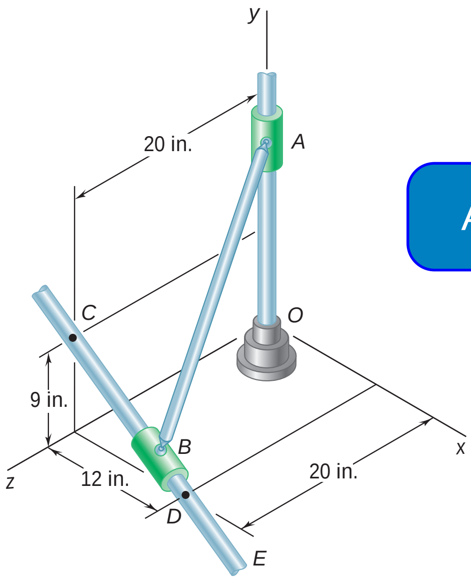


Fig. P15.203

15.204 Rod AB, of length 11 in., is connected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Knowing that collar B moves downward at a constant speed of 54 in./s, determine the velocity of collar A when $c = 2$ in.

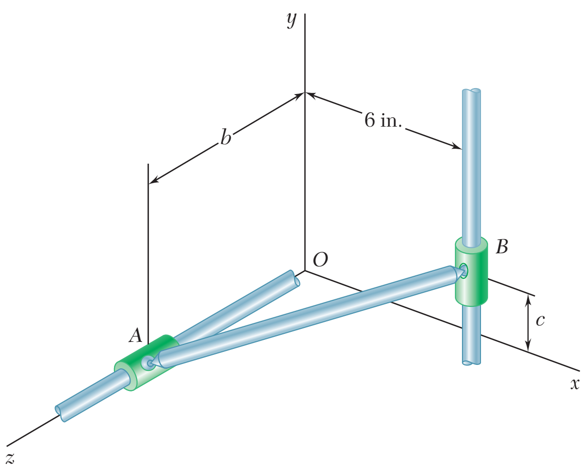


Fig. P15.204

15.205 Rod BC and BD are each 840 mm long and are connected by ball-and-socket joints to collars which may slide on the fixed rods shown. Knowing that collar B moves toward A at a constant speed of 390 mm/s, determine the velocity of collar C for the position shown.

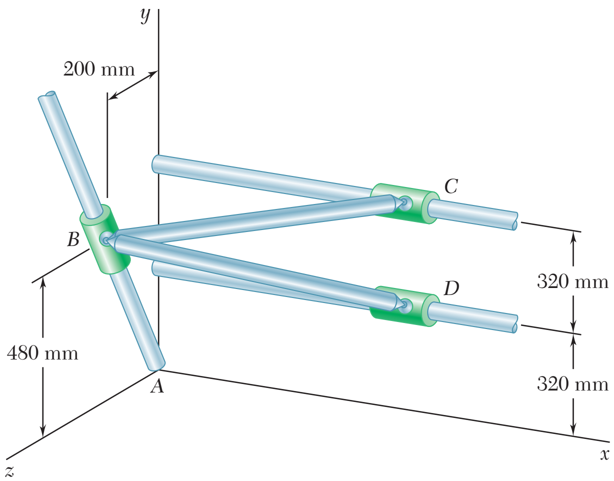


Fig. P15.205

15.206 Rod AB is connected by ball-and-socket joints to collar A and to the 16-in.-diameter disk C. Knowing that disk C rotates counterclockwise at the constant rate $\omega_0 = 3$ rad/s in the zx plane, determine the velocity of collar A for the position shown.

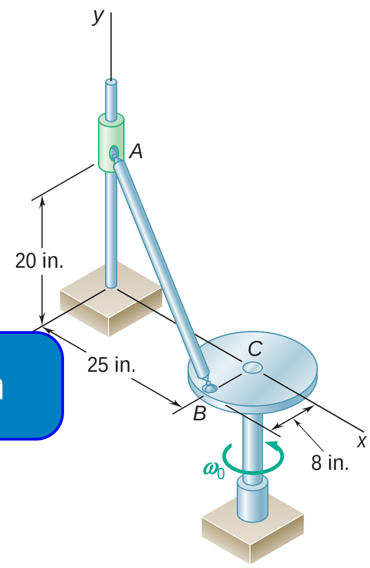


Fig. P15.206

15.207 Rod AB of length 29 in. [...] the rotating crank BC a[...] 8 in. and rotates in the [...] $\omega_0 = 10$ rad/s. At the instant shown, when crank BC is parallel to the z axis, determine the velocity of collar A.

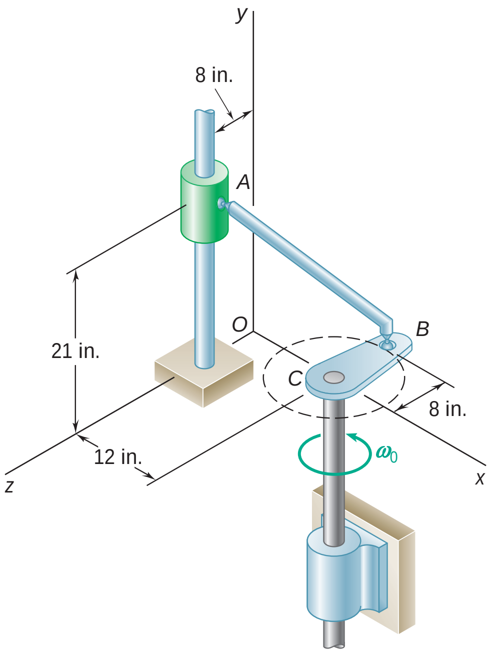


Fig. P15.207

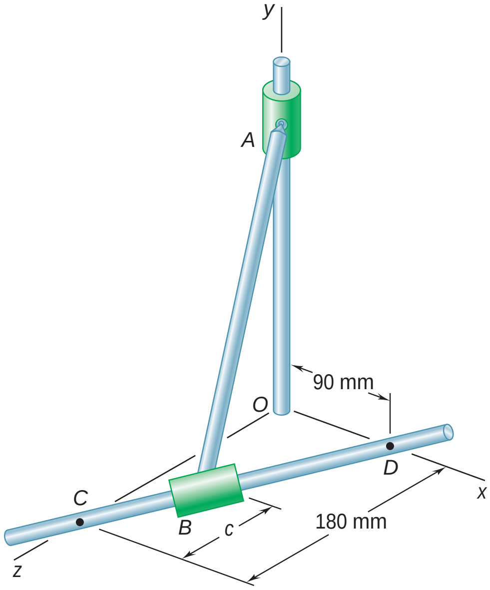


Fig. *P15.208* and *P15.209*

15.208 Rod AB of length 300 mm is connected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Knowing that collar B moves toward point D at a constant speed of 50 mm/s, determine the velocity of collar A when $c = 80$ mm.

15.209 Rod AB of length 300 mm is connected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Knowing that collar B moves toward point D at a constant speed of 50 mm/s, determine the velocity of collar A when $c = 120$ mm.

15.210 Two shafts AC and EG, which lie in the vertical yz plane, are connected by a universal joint at D. Shaft AC rotates with a constant angular velocity ω_1 as shown. At a time when the arm of the crosspiece attached to shaft AC is vertical, determine the angular velocity of shaft EG.

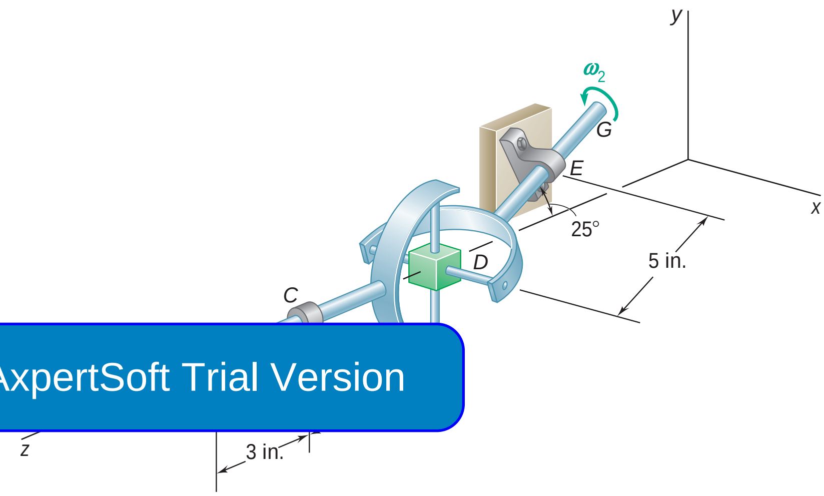


Fig. P15.210

15.211 Solve Prob. 15.210, assuming that the arm of the crosspiece attached to shaft AC is horizontal.

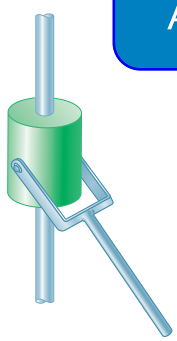


Fig. P15.212

15.212 In Prob. 15.206, the ball-and-socket joint between the rod and collar A is replaced by the clevis shown. Determine (a) the angular velocity of the rod, (b) the velocity of collar A.

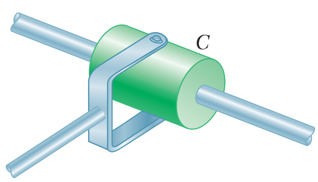


Fig. P15.213

15.213 In Prob. 15.205, the ball-and-socket joint between the rod and collar C is replaced by the clevis connection shown. Determine (a) the angular velocity of the rod, (b) the velocity of collar C.

15.214 In Prob. 15.204, determine the acceleration of collar A when $c = 2$ in.

***15.215** In Prob. 15.205, determine the acceleration of collar C.

15.216 In Prob. 15.206, determine the acceleration of collar A.

15.217 In Prob. 15.207, determine the acceleration of collar A.

15.218 In Prob. 15.208, determine the acceleration of collar A.

15.219 In Prob. 15.209, determine the acceleration of collar A.

*15.14 THREE-DIMENSIONAL MOTION OF A PARTICLE RELATIVE TO A ROTATING FRAME. CORIOLIS ACCELERATION

We saw in Sec. 15.10 that given a vector function $\mathbf{Q}(t)$ and two frames of reference centered at O—a fixed frame $OXYZ$ and a rotating frame $Oxyz$—the rates of change of $\mathbf{Q}$ with respect to the two frames satisfy the relation

$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{Q} \qquad (15.31)$$

We had assumed at the time that the frame $Oxyz$ was constrained to rotate about a fixed axis OA. However, the derivation given in Sec. 15.10 remains valid when the frame $Oxyz$ is constrained only to have a fixed point O. Under this more general assumption, the axis OA represents the *instantaneous* axis of rotation of the frame $Oxyz$ (Sec. 15.12) and the vector $\boldsymbol{\Omega}$, its angular velocity at the instant considered (Fig. 15.36).

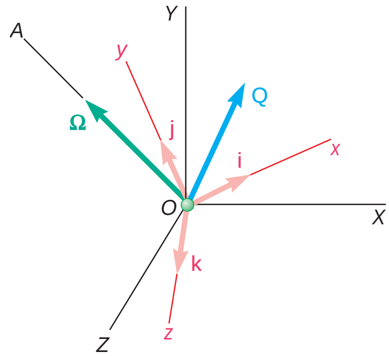


Fig. 15.36

Let us now consider the three-dimensional motion of a particle P relative to a rotating frame $Oxyz$ constrained to have a fixed origin O. Let $\mathbf{r}$ be the position vector of P at a given instant and $\boldsymbol{\Omega}$ be the angular velocity of the frame $Oxyz$ with respect to the fixed frame $OXYZ$ at the same instant (Fig. 15.37). The derivations given in Sec. 15.11 for the two-dimensional motion of a particle can be readily extended to the three-dimensional case, and the absolute velocity $\mathbf{v}_P$ of P (i.e., its velocity with resp

expressed as

$$\mathbf{v}_P = \boldsymbol{\Omega} \times \mathbf{r} + (\dot{\mathbf{r}})_{Oxyz} \qquad (15.45)$$

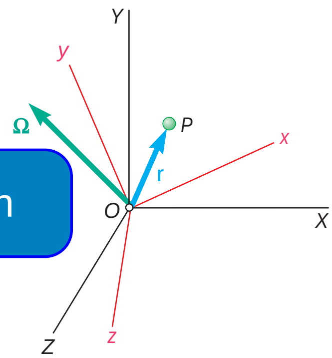


Fig. 15.37

Denoting by $\mathcal{F}$ the rotating frame $Oxyz$, we write this relation in the alternative form

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \qquad (15.46)$$

where $\mathbf{v}_P$ = absolute velocity of particle P
$\mathbf{v}_{P'}$ = velocity of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{v}_{P/\mathcal{F}}$ = velocity of P relative to moving frame $\mathcal{F}$

The absolute acceleration $\mathbf{a}_P$ of P can be expressed as

$$\mathbf{a}_P = \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) + 2\boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxyz} + (\ddot{\mathbf{r}})_{Oxyz} \qquad (15.47)$$

An alternative form is

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \qquad (15.48)$$

where $\mathbf{a}_P$ = absolute acceleration of particle P

$\mathbf{a}_{P'}$ = acceleration of point P' of moving frame $\mathcal{F}$ coinciding with P

$\mathbf{a}_{P/\mathcal{F}}$ = acceleration of P relative to moving frame $\mathcal{F}$

$\mathbf{a}_c = 2\boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxyz} = 2\boldsymbol{\Omega} \times \mathbf{v}_{P/\mathcal{F}}$

= complementary, or Coriolis, acceleration†

We note that the Coriolis acceleration is perpendicular to the vectors $\boldsymbol{\Omega}$ and $\mathbf{v}_{P/\mathcal{F}}$. However, since these vectors are usually not perpendicular to each other, the magnitude of $\mathbf{a}_c$ is in general *not* equal to $2\Omega v_{P/\mathcal{F}}$, as was the case for the plane motion of a particle. We further note that the Coriolis acceleration reduces to zero when the vectors $\boldsymbol{\Omega}$ and $\mathbf{v}_{P/\mathcal{F}}$ are parallel, or when either of them is zero.

Rotating frames of reference are particularly useful in the study of the three-dimensional motion of rigid bodies. If a rigid body has a fixed point O, as was the case for the crane of Sample Prob. 15.11, we can use a frame $Oxyz$ which is neither fixed nor rigidly attached to the rigid body. Denoting by $\boldsymbol{\Omega}$ the angular velocity of the frame $Oxyz$, we then resolve the angular velocity V of the body into the components $\boldsymbol{\Omega}$ and $\mathsf{V}_{B/\mathcal{F}}$, where the second component represents the angular velocity of the body relative to the frame $Oxyz$ (see Sample Prob. 15.14). An appropriate choice of the rotating frame often leads to a simpler analysis of the motion of the rigid body than would be possible with axes of fixed orientation. This is especially true in the case of the general three-dimensional motion of a rigid onsideration has no fixed point

*15.15 FRAME OF REFERENCE IN GENERAL MOTION

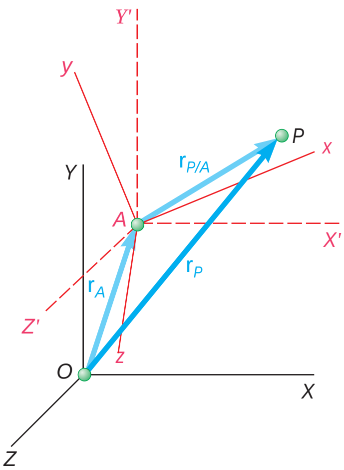


Fig. 15.38

Consider a fixed frame of reference $OXYZ$ and a frame $Axyz$ which moves in a known, but arbitrary, fashion with respect to $OXYZ$ (Fig. 15.38). Let P be a particle moving in space. The position of P is defined at any instant by the vector $\mathbf{r}_P$ in the fixed frame, and by the vector $\mathbf{r}_{P/A}$ in the moving frame. Denoting by $\mathbf{r}_A$ the position vector of A in the fixed frame, we have

$$\mathbf{r}_P = \mathbf{r}_A + \mathbf{r}_{P/A} \tag{15.49}$$

The absolute velocity $\mathbf{v}_P$ of the particle is obtained by writing

$$\mathbf{v}_P = \dot{\mathbf{r}}_P = \dot{\mathbf{r}}_A + \dot{\mathbf{r}}_{P/A} \tag{15.50}$$

where the derivatives are defined with respect to the fixed frame $OXYZ$. The first term in the right-hand member of (15.50) thus represents the velocity $\mathbf{v}_A$ of the origin A of the moving axes. On the other hand, since the rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation (Sec. 11.10), the second term can be regarded as the velocity $\mathbf{v}_{P/A}$ of

†It is important to note the difference between Eq. (15.48) and Eq. (15.21) of Sec. 15.8. See the footnote on page 988.

P relative to the frame $AX'Y'Z'$ of the same orientation as $OXYZ$ and the same origin as $Axyz$. We therefore have

$$\mathbf{v}_P = \mathbf{v}_A + \mathbf{v}_{P/A} \tag{15.51}$$

But the velocity $\mathbf{v}_{P/A}$ of P relative to $AX'Y'Z'$ can be obtained from (15.45) by substituting $\mathbf{r}_{P/A}$ for $\mathbf{r}$ in that equation. We write

$$\mathbf{v}_P = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{P/A} + (\dot{\mathbf{r}}_{P/A})_{Axyz} \tag{15.52}$$

where $\boldsymbol{\Omega}$ is the angular velocity of the frame $Axyz$ at the instant considered.

The absolute acceleration $\mathbf{a}_P$ of the particle is obtained by differentiating (15.51) and writing

$$\mathbf{a}_P = \dot{\mathbf{v}}_P = \dot{\mathbf{v}}_A + \dot{\mathbf{v}}_{P/A} \tag{15.53}$$

where the derivatives are defined with respect to either of the frames $OXYZ$ or $AX'Y'Z'$. Thus, the first term in the right-hand member of (15.53) represents the acceleration $\mathbf{a}_A$ of the origin A of the moving axes and the second term represents the acceleration $\mathbf{a}_{P/A}$ of P relative to the frame $AX'Y'Z'$. This acceleration can be obtained from (15.47) by substituting $\mathbf{r}_{P/A}$ for $\mathbf{r}$. We therefore write

$$\mathbf{a}_P = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{P/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{P/A})$$
$$+ 2\boldsymbol{\Omega} \text{ [illegible]}$$

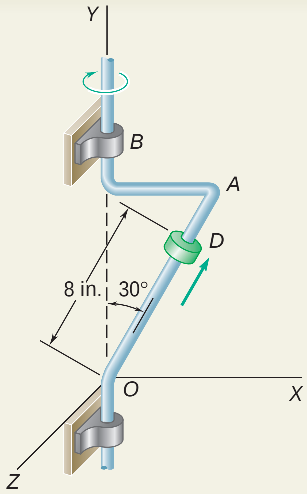
Formulas (15.52) and (15.54) make it possible to determine the velocity and acceleration of a given particle with respect to a fixed frame of reference, when the motion of the particle is known with respect to a moving frame. These formulas become more significant, and considerably easier to remember, if we note that the sum of the first two terms in (15.52) represents the velocity of the point P' of the moving frame which coincides with P at the instant considered, and that the sum of the first three terms in (15.54) represents the acceleration of the same point. Thus, the relations (15.46) and (15.48) of the preceding section are still valid in the case of a reference frame in general motion, and we can write

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \tag{15.46}$$
$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \tag{15.48}$$

where the various vectors involved have been defined in Sec. 15.14.

It should be noted that if the moving reference frame $\mathcal{F}$ (or $Axyz$) is in translation, the velocity and acceleration of the point P' of the frame which coincides with P become, respectively, equal to the velocity and acceleration of the origin A of the frame. On the other hand, since the frame maintains a fixed orientation, $\mathbf{a}_c$ is zero, and the relations (15.46) and (15.48) reduce, respectively, to the relations (11.33) and (11.34) derived in Sec. 11.12.

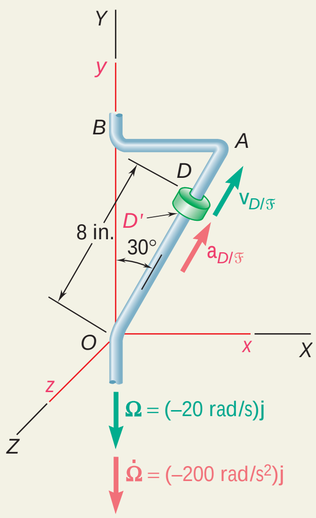
Photo 15.9 The motion of air particles in a hurricane can be considered as motion relative to a frame of reference attached to the Earth and rotating with it.



SAMPLE PROBLEM 15.13

The bent rod *OAB* rotates about the vertical *OB*. At the instant considered, its angular velocity and angular acceleration are, respectively, 20 rad/s and 200 rad/s^2, both clockwise when viewed from the positive *Y* axis. The collar *D* moves along the rod, and at the instant considered, $OD = 8$ in. The velocity and acceleration of the collar relative to the rod are, respectively, 50 in./s and 600 in./s^2, both upward. Determine (*a*) the velocity of the collar, (*b*) the acceleration of the collar.

SOLUTION



Frames of Reference. The frame *OXYZ* is fixed. We attach the rotating frame *Oxyz* to the bent rod. Its angular velocity and angular acceleration relative to *OXYZ* are therefore $\boldsymbol{\Omega} = (-20 \text{ rad/s})\mathbf{j}$ and $\dot{\boldsymbol{\Omega}} = (-200 \text{ rad/s}^2)\mathbf{j}$,

$(4 \text{ in.})\mathbf{i} + (6.93 \text{ in.})\mathbf{j}$

a. Velocity $\mathbf{v}_D$. Denoting by D' the point of the rod which coincides with *D* and by $\mathcal{F}$ the rotating frame *Oxyz*, we write from Eq. (15.46)

$$\mathbf{v}_D = \mathbf{v}_{D'} + \mathbf{v}_{D/\mathcal{F}} \tag{1}$$

where

$$\mathbf{v}_{D'} = \boldsymbol{\Omega} \times \mathbf{r} = (-20 \text{ rad/s})\mathbf{j} \times [(4 \text{ in.})\mathbf{i} + (6.93 \text{ in.})\mathbf{j}] = (80 \text{ in./s})\mathbf{k}$$
$$\mathbf{v}_{D/\mathcal{F}} = (50 \text{ in./s})(\sin 30°\mathbf{i} + \cos 30°\mathbf{j}) = (25 \text{ in./s})\mathbf{i} + (43.3 \text{ in./s})\mathbf{j}$$

Substituting the values obtained for $\mathbf{v}_{D'}$ and $\mathbf{v}_{D/\mathcal{F}}$ into (1), we find

$$\mathbf{v}_D = (25 \text{ in./s})\mathbf{i} + (43.3 \text{ in./s})\mathbf{j} + (80 \text{ in./s})\mathbf{k} \quad ◀$$

b. Acceleration $\mathbf{a}_D$. From Eq. (15.48) we write

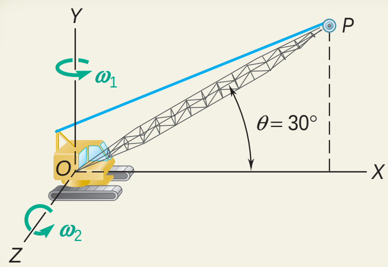
$$\mathbf{a}_D = \mathbf{a}_{D'} + \mathbf{a}_{D/\mathcal{F}} + \mathbf{a}_c \tag{2}$$

where

$$\begin{aligned}
\mathbf{a}_{D'} &= \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \\
&= (-200 \text{ rad/s}^2)\mathbf{j} \times [(4 \text{ in.})\mathbf{i} + (6.93 \text{ in.})\mathbf{j}] - (20 \text{ rad/s})\mathbf{j} \times (80 \text{ in./s})\mathbf{k} \\
&= +(800 \text{ in./s}^2)\mathbf{k} - (1600 \text{ in./s}^2)\mathbf{i} \\
\mathbf{a}_{D/\mathcal{F}} &= (600 \text{ in./s}^2)(\sin 30°\mathbf{i} + \cos 30°\mathbf{j}) = (300 \text{ in./s}^2)\mathbf{i} + (520 \text{ in./s}^2)\mathbf{j} \\
\mathbf{a}_c &= 2\boldsymbol{\Omega} \times \mathbf{v}_{D/\mathcal{F}} \\
&= 2(-20 \text{ rad/s})\mathbf{j} \times [(25 \text{ in./s})\mathbf{i} + (43.3 \text{ in./s})\mathbf{j}] = (1000 \text{ in./s}^2)\mathbf{k}
\end{aligned}$$

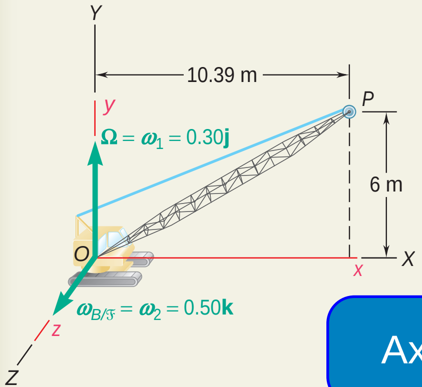
Substituting the values obtained for $\mathbf{a}_{D'}$, $\mathbf{a}_{D/\mathcal{F}}$, and $\mathbf{a}_c$ into (2),

$$\mathbf{a}_D = -(1300 \text{ in./s}^2)\mathbf{i} + (520 \text{ in./s}^2)\mathbf{j} + (1800 \text{ in./s}^2)\mathbf{k} \quad ◀$$



SAMPLE PROBLEM 15.14

The crane shown rotates with a constant angular velocity V_1 of 0.30 rad/s. Simultaneously, the boom is being raised with a constant angular velocity V_2 of 0.50 rad/s relative to the cab. Knowing that the length of the boom *OP* is $l = 12$ m, determine (*a*) the velocity of the tip of the boom, (*b*) the acceleration of the tip of the boom.



SOLUTION

Frames of Reference. The frame *OXYZ* is fixed. We attach the rotating frame *Oxyz* to the cab. Its angular velocity with respect to the frame *OXYZ* is therefore $\boldsymbol{\Omega} = V_1 = (0.30 \text{ rad/s})\mathbf{j}$. The angular velocity of the boom relative to the cab and the rotating frame *Oxyz* (or $\mathcal{F}$, for short) is $V_{B/\mathcal{F}} = V_2 = (0.50 \text{ rad/s})\mathbf{k}$.

a. Velocity $\mathbf{v}_P$. From Eq. (15.46) we write

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \qquad (1)$$

where $\mathbf{v}_{P'}$ is the velocity of the point P' of the rotating frame which coincides with *P*:

$$39 \text{ m})\mathbf{i} + (6 \text{ m})\mathbf{j}] = -(3.12 \text{ m/s})\mathbf{k}$$

tive to the rotating frame *Oxyz*. But the angular velocity of the boom relative to *Oxyz* was found to be $V_{B/\mathcal{F}} = (0.50 \text{ rad/s})\mathbf{k}$. The velocity of its tip *P* relative to *Oxyz* is therefore

$$\begin{aligned}\mathbf{v}_{P/\mathcal{F}} = V_{B/\mathcal{F}} \times \mathbf{r} &= (0.50 \text{ rad/s})\mathbf{k} \times [(10.39 \text{ m})\mathbf{i} + (6 \text{ m})\mathbf{j}] \\ &= -(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j}\end{aligned}$$

Substituting the values obtained for $\mathbf{v}_{P'}$ and $\mathbf{v}_{P/\mathcal{F}}$ into (1), we find

$$\mathbf{v}_P = -(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j} - (3.12 \text{ m/s})\mathbf{k} \quad ◀$$

b. Acceleration $\mathbf{a}_P$. From Eq. (15.48) we write

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \qquad (2)$$

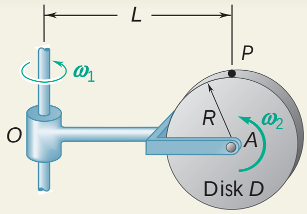
Since $\boldsymbol{\Omega}$ and $V_{B/\mathcal{F}}$ are both constant, we have

$$\begin{aligned}\mathbf{a}_{P'} &= \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = (0.30 \text{ rad/s})\mathbf{j} \times (-3.12 \text{ m/s})\mathbf{k} = -(0.94 \text{ m/s}^2)\mathbf{i} \\ \mathbf{a}_{P/\mathcal{F}} &= V_{B/\mathcal{F}} \times (V_{B/\mathcal{F}} \times \mathbf{r}) \\ &= (0.50 \text{ rad/s})\mathbf{k} \times [-(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j}] \\ &= -(1.50 \text{ m/s}^2)\mathbf{j} - (2.60 \text{ m/s}^2)\mathbf{i} \\ \mathbf{a}_c &= 2\boldsymbol{\Omega} \times \mathbf{v}_{P/\mathcal{F}} \\ &= 2(0.30 \text{ rad/s})\mathbf{j} \times [-(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j}] = (1.80 \text{ m/s}^2)\mathbf{k}\end{aligned}$$

Substituting for $\mathbf{a}_{P'}$, $\mathbf{a}_{P/\mathcal{F}}$, and $\mathbf{a}_c$ into (2), we find

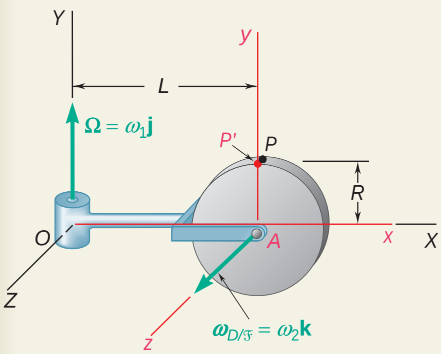
$$\mathbf{a}_P = -(3.54 \text{ m/s}^2)\mathbf{i} - (1.50 \text{ m/s}^2)\mathbf{j} + (1.80 \text{ m/s}^2)\mathbf{k} \quad ◀$$

SAMPLE PROBLEM 15.15



Disk D, of radius R, is pinned to end A of the arm OA of length L located in the plane of the disk. The arm rotates about a vertical axis through O at the constant rate ω_1, and the disk rotates about A at the constant rate ω_2. Determine (*a*) the velocity of point P located directly above A, (*b*) the acceleration of P, (*c*) the angular velocity and angular acceleration of the disk.

SOLUTION



Frames of Reference. The frame $OXYZ$ is fixed. We attach the moving frame $Axyz$ to the arm OA. Its angular velocity with respect to the frame $OXYZ$ is therefore $\mathbf{\Omega} = \omega_1\mathbf{j}$. The angular velocity of disk D relative to the moving frame $Axyz$ (or $\mathcal{F}$, for short) is $\boldsymbol{\omega}_{D/\mathcal{F}} = \omega_2\mathbf{k}$. The position vector of P relative to O is $\mathbf{r} = L\mathbf{i} + R\mathbf{j}$, and its position vector relative to A is $\mathbf{r}_{P/A} = R\mathbf{j}$.

a. Velocity $\mathbf{v}_P$. Denoting by P' the point of the moving frame which coincides with P, we write from Eq. (15.46)

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \qquad (1)$$

where $\mathbf{v}_{P'} = \mathbf{\Omega} \times \mathbf{r} = \omega_1\mathbf{j} \times (L\mathbf{i} + R\mathbf{j}) = -\omega_1 L\mathbf{k}$

$$\mathbf{v}_{P/\mathcal{F}} = \boldsymbol{\omega}_{D/\mathcal{F}} \times \mathbf{r}_{P/A} = \omega_2\mathbf{k} \times R\mathbf{j} = -\omega_2 R\mathbf{i}$$

$\mathbf{v}_{P/\mathcal{F}}$ into (1), we find

$$\mathbf{v}_P = -\omega_2 R\mathbf{i} - \omega_1 L\mathbf{k}$$

b. Acceleration $\mathbf{a}_P$. From Eq. (15.48) we write

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \qquad (2)$$

Since $\mathbf{\Omega}$ and $\boldsymbol{\omega}_{D/\mathcal{F}}$ are both constant, we have

$$\begin{aligned}
\mathbf{a}_{P'} &= \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \omega_1\mathbf{j} \times (-\omega_1 L\mathbf{k}) = -\omega_1^2 L\mathbf{i} \\
\mathbf{a}_{P/\mathcal{F}} &= \boldsymbol{\omega}_{D/\mathcal{F}} \times (\boldsymbol{\omega}_{D/\mathcal{F}} \times \mathbf{r}_{P/A}) = \omega_2\mathbf{k} \times (-\omega_2 R\mathbf{i}) = -\omega_2^2 R\mathbf{j} \\
\mathbf{a}_c &= 2\mathbf{\Omega} \times \mathbf{v}_{P/\mathcal{F}} = 2\omega_1\mathbf{j} \times (-\omega_2 R\mathbf{i}) = 2\omega_1\omega_2 R\mathbf{k}
\end{aligned}$$

Substituting the values obtained into (2), we find

$$\mathbf{a}_P = -\omega_1^2 L\mathbf{i} - \omega_2^2 R\mathbf{j} + 2\omega_1\omega_2 R\mathbf{k}$$

c. Angular Velocity and Angular Acceleration of Disk.

$$\boldsymbol{\omega} = \mathbf{\Omega} + \boldsymbol{\omega}_{D/\mathcal{F}} \qquad \boldsymbol{\omega} = \omega_1\mathbf{j} + \omega_2\mathbf{k}$$

Using Eq. (15.31) with $\mathbf{Q} = \boldsymbol{\omega}$, we write

$$\begin{aligned}
\boldsymbol{\alpha} = (\dot{\boldsymbol{\omega}})_{OXYZ} &= (\dot{\boldsymbol{\omega}})_{Axyz} + \mathbf{\Omega} \times \boldsymbol{\omega} \\
&= 0 + \omega_1\mathbf{j} \times (\omega_1\mathbf{j} + \omega_2\mathbf{k})
\end{aligned}$$

$$\boldsymbol{\alpha} = \omega_1\omega_2\mathbf{i}$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you concluded your study of the kinematics of rigid bodies by learning how to use an auxiliary frame of reference $\mathcal{F}$ to analyze the three-dimensional motion of a rigid body. This auxiliary frame may be a *rotating frame* with a fixed origin O, or it may be a *frame in general motion.*

A. Using a rotating frame of reference. As you approach a problem involving the use of a rotating frame $\mathcal{F}$ you should take the following steps.

1. Select the rotating frame $\mathcal{F}$ that you wish to use and draw the corresponding coordinate axes x, y, and z from the fixed point O.

2. Determine the angular velocity Ω of the frame $\mathcal{F}$ with respect to a fixed frame $OXYZ$. In most cases, you will have selected a frame which is attached to some rotating element of the system; $\boldsymbol{\Omega}$ will then be the angular velocity of that element.

3. Designate as P' the point of the rotating frame $\mathcal{F}$ that coincides with the point P of interest at the instant you are considering. Determine the velocity $\mathbf{v}_{P'}$ and the acceleration $\mathbf{a}_{P'}$ of point P'. Since P' is part of $\mathcal{F}$ and has the same position vector $\mathbf{r}$ as

$$\mathbf{v}_{P'} \quad \ldots \quad (\boldsymbol{\Omega} \times \mathbf{r})$$

where A is the angular acceleration of $\mathcal{F}$. However, in many of the problems that you will encounter, the angular velocity of $\mathcal{F}$ is constant in both magnitude and direction, and $\mathsf{A} = 0$.

4. Determine the velocity and acceleration of point P with respect to the frame $\mathcal{F}$. As you are trying to determine $\mathbf{v}_{P/\mathcal{F}}$ and $\mathbf{a}_{P/\mathcal{F}}$ you will find it useful to visualize the motion of P on frame $\mathcal{F}$ when the frame is not rotating. If P is a point of a rigid body $\mathcal{B}$ which has an angular velocity $\mathsf{V}_{\mathcal{B}}$ and an angular acceleration $\mathsf{A}_{\mathcal{B}}$ relative to $\mathcal{F}$ [Sample Prob. 15.14], you will find that

$$\mathbf{v}_{P/\mathcal{F}} = \mathsf{V}_{\mathcal{B}} \times \mathbf{r} \qquad \text{and} \qquad \mathbf{a}_{P/\mathcal{F}} = \mathsf{A}_{\mathcal{B}} \times \mathbf{r} + \mathsf{V}_{\mathcal{B}} \times (\mathsf{V}_{\mathcal{B}} \times \mathbf{r})$$

In many of the problems that you will encounter, the angular velocity of body $\mathcal{B}$ relative to frame $\mathcal{F}$ is constant in both magnitude and direction, and $\mathsf{A}_{\mathcal{B}} = 0$.

5. Determine the Coriolis acceleration. Considering the angular velocity $\boldsymbol{\Omega}$ of frame $\mathcal{F}$ and the velocity $\mathbf{v}_{P/\mathcal{F}}$ of point P relative to that frame, which was computed in the previous step, you write

$$\mathbf{a}_c = 2\boldsymbol{\Omega} \times \mathbf{v}_{P/\mathcal{F}}$$

(continued)

6. The velocity and the acceleration of *P* with respect to the fixed frame *OXYZ* can now be obtained by adding the expressions you have determined:

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \qquad (15.46)$$
$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \qquad (15.48)$$

B. Using a frame of reference in general motion. The steps that you will take differ only slightly from those listed under *A*. They consist of the following:

1. Select the frame $\mathcal{F}$ that you wish to use and a reference point *A* in that frame, from which you will draw the coordinate axes, x, y, and z defining that frame. You will consider the motion of the frame as the sum of a *translation with A and a rotation about A.*

2. Determine the velocity $\mathbf{v}_A$ of point *A* and the angular velocity $\boldsymbol{\Omega}$ of the frame. In most cases, you will have selected a frame which is attached to some element of the system; $\boldsymbol{\Omega}$ will then be the angular velocity of that element.

3. Designate as *P′* the point of frame $\mathcal{F}$ that coincides with the point *P* of interest at the instan [illegible] velocity $\mathbf{v}_{P'}$ and the acceleration $\mathbf{a}_{P'}$ [illegible]e by visualizing the motion of *P* if [illegible]h respect to $\mathcal{F}$ [Sample Prob. 15.15]. [illegible]he motion of *P′* is the sum of a translation with the reference point *A* and a rotation about *A*. The velocity $\mathbf{v}_{P'}$ and the acceleration $\mathbf{a}_{P'}$ of *P′*, therefore, can be obtained by adding $\mathbf{v}_A$ and $\mathbf{a}_A$, respectively, to the expressions found in paragraph A3 and replacing the position vector $\mathbf{r}$ by the vector $\mathbf{r}_{P/A}$ drawn from *A* to *P*:

$$\mathbf{v}_{P'} = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{P/A} \qquad \mathbf{a}_{P'} = \mathbf{a}_A + \mathrm{A} \times \mathbf{r}_{P/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{P/A})$$

Steps 4, 5, and 6 are the same as in Part *A*, except that the vector $\mathbf{r}$ should again be replaced by $\mathbf{r}_{P/A}$. Thus, Eqs. (15.46) and (15.48) can still be used to obtain the velocity and the acceleration of *P* with respect to the fixed frame of reference *OXYZ*.

PROBLEMS

END-OF-SECTION PROBLEMS

15.220 A square plate of side 18 in. is hinged at A and B to a clevis. The plate rotates at the constant rate v_2 = 4 rad/s with respect to the clevis, which itself rotates at the constant rate v_1 = 3 rad/s about the Y axis. For the position shown, determine (*a*) the velocity of point C, (*b*) the acceleration of point C.

15.221 A square plate of side 18 in. is hinged at A and B to a clevis. The plate rotates at the constant rate v_2 = 4 rad/s with respect to the clevis, which itself rotates at the constant rate v_1 = 3 rad/s about the Y axis. For the position shown, determine (*a*) the velocity of corner D, (*b*) the acceleration of corner D.

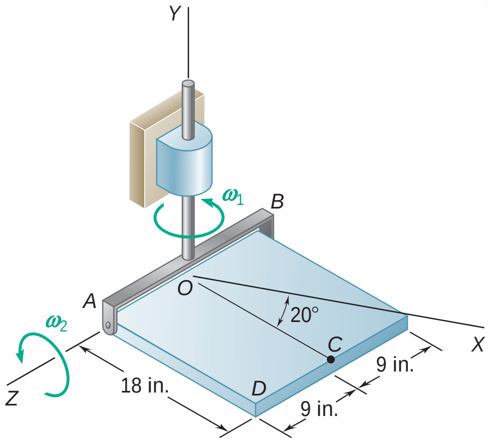


Fig. P15.220 and P15.221

15.222 and 15.223 The rectangular plate shown rotates at the constant rate v_2 = 12 rad/s with respect to arm AE, which itself rotates at the constant rate v_1 = 9 rad/s about the Z axis. For the position shown, determine the velocity and acceleration of the point of the plate indicated.

15.222 Corner B
15.223 Corner C

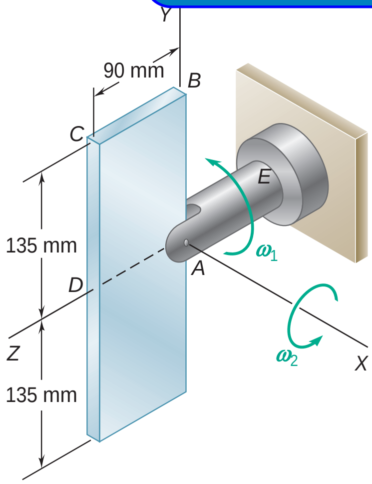


Fig. P15.222 and P15.223

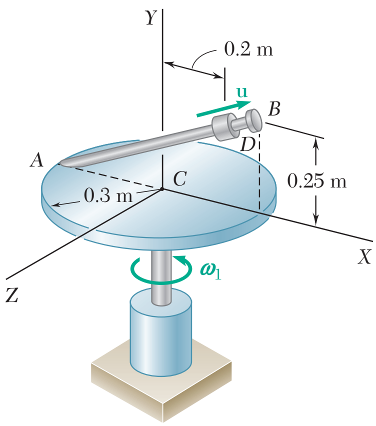


Fig. P15.224

15.224 Rod AB is welded to the 0.3-m-radius plate which rotates at the constant rate v_1 = 6 rad/s. Knowing that collar D moves toward end B of the rod at a constant speed u = 1.3 m/s, determine, for the position shown, (*a*) the velocity of D, (*b*) the acceleration of D.

15.225 The bent rod ABC rotates at the constant rate $v_1 = 4$ rad/s. Knowing that collar D moves downward along the rod at a constant relative speed $u = 65$ in./s, determine, for the position shown, (*a*) the velocity of D, (*b*) the acceleration of D.

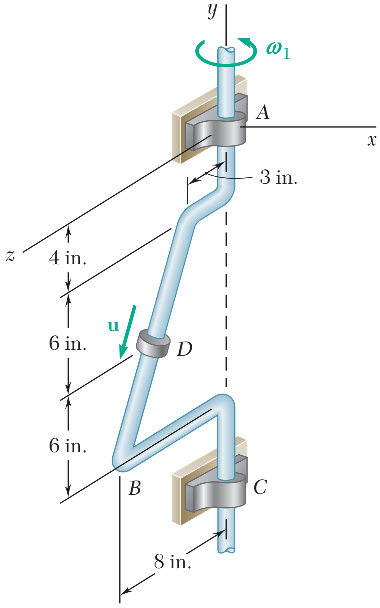
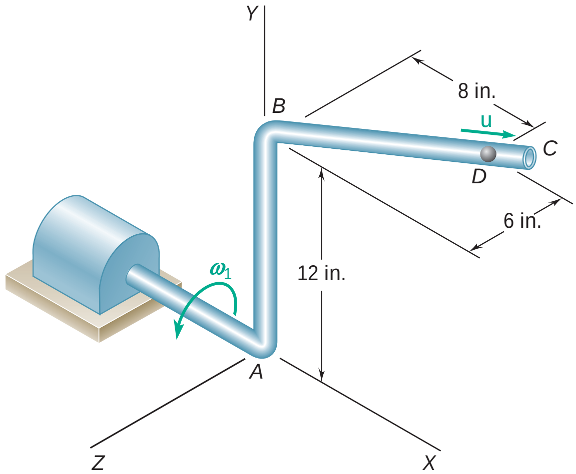


Fig. P15.225

15.226 The bent pipe shown rotates at the constant rate $v_1 = 10$ rad/s. Knowing that a ball bearing D moves in portion BC of the pipe toward end C at a constant relative speed $u = 2$ ft/s, determine at the instant shown (*a*) the velocity of D, (*b*) the acceleration of D.



about its vertical diameter at the constant rate $v_1 = 10$ rad/s. Knowing that in the position shown the disk lies in the XY plane and point D of strap CD moves upward at a constant relative speed $u = 1.5$ m/s, determine (*a*) the velocity of D, (b) the acceleration of D.

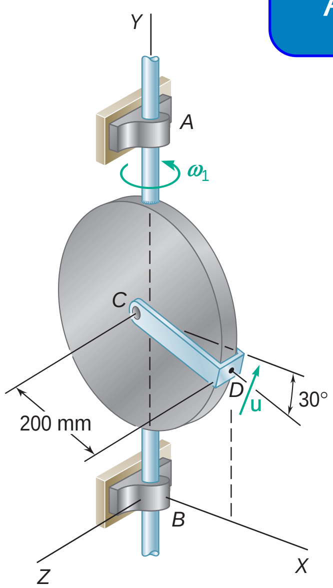


Fig. P15.227

15.228 Manufactured items are spray-painted as they pass through the automated work station shown. Knowing that the bent pipe ACE rotates at the constant rate $v_1 = 0.4$ rad/s and that at point D the paint moves through the pipe at a constant relative speed $u = 150$ mm/s, determine, for the position shown, (*a*) the velocity of the paint at D, (*b*) the acceleration of the paint at D.

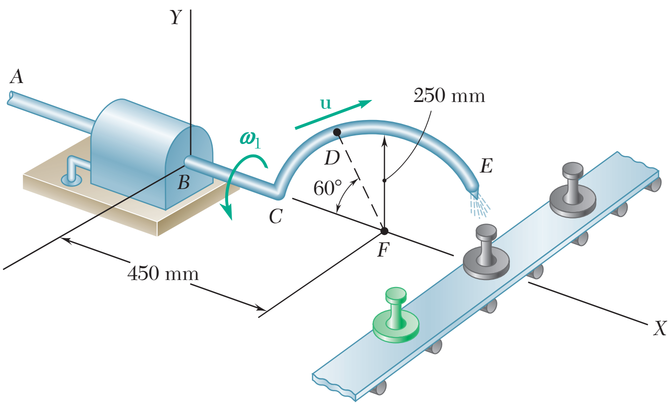


Fig. P15.228

15.229 Solve Prob. 15.227, assuming that at the instant shown the angular velocity v_1 of the plate is 10 rad/s and is decreasing at the rate of 25 rad/s^2, while the relative speed u of point D of strap CD is 1.5 m/s and is decreasing at the rate of 3 m/s^2.

15.230 Solve Prob. 15.226 assuming that at the instant shown the angular velocity v_1 of the pipe is 10 rad/s and is decreasing at the rate of 15 rad/s^2, while the relative speed u of the ball bearing is 2 ft/s and is increasing at the rate of 10 ft/s^2.

15.231 Using the method of Sec. 15.14, solve Prob. 15.192.

15.232 Using the method of Sec. 15.14, solve Prob. 15.196.

15.233 Using the method of Sec. 15.14, solve Prob. 15.198.

15.234 A disk of radius 120 mm rotates at the constant rate $\mathrm{v}_2 = 5$ rad/s with respect to the arm AB, which itself rotates at the constant rate $\mathrm{v}_1 = 3$ rad/s. For the position shown, determine the velocity and acceleration of point C.

15.235 A disk of radius 120 mm rotates at the constant rate $\mathrm{v}_2 = 5$ rad/s with respect to the arm AB, which itself rotates at the constant rate $\mathrm{v}_1 = 3$ rad/s. For the position shown, determine the velocity and acceleration of point D.

15.236 The arm AB of length 16 ft is used to provide an elevated platform for construction workers. In the position shown, arm AB is being raised at the constant rate [illegible] unit is being rotated ab[illegible] 0.15 rad/s. Knowing that [illegible] eration of point B.

15.237 The remote manipulator system (RMS) shown is used to deploy payloads from the cargo bay of space shuttles. At the instant shown, the whole RMS is rotating at the constant rate $\mathrm{v}_1 = 0.03$ rad/s about the axis AB. At the same time, portion BCD rotates as a rigid body at the constant rate $\mathrm{v}_2 = d\mathrm{b}/dt = 0.04$ rad/s about an axis through B parallel to the X axis. Knowing that $\mathrm{b} = 30°$, determine (*a*) the angular acceleration of BCD, (*b*) the velocity of D, (*c*) the acceleration of D.

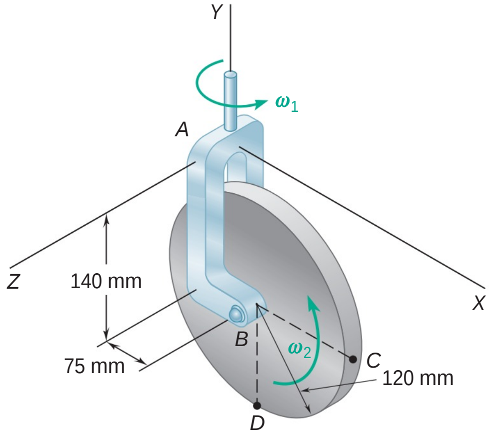


Fig. P15.234 and P15.235

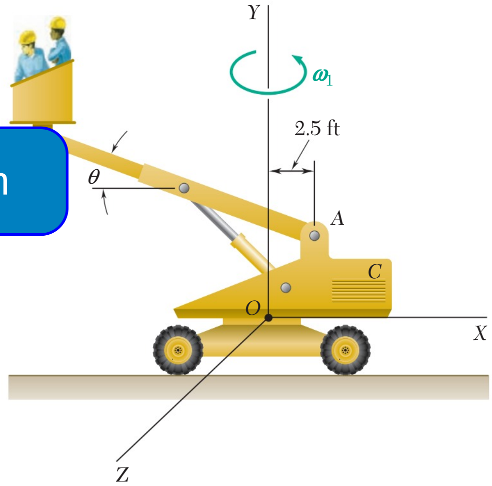


Fig. P15.236

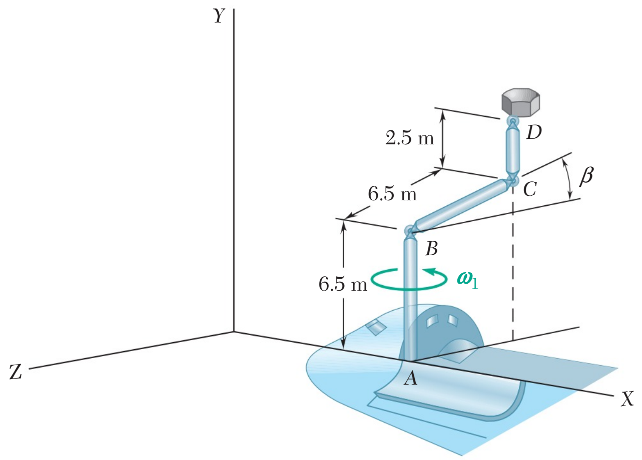


Fig. *P15.237*

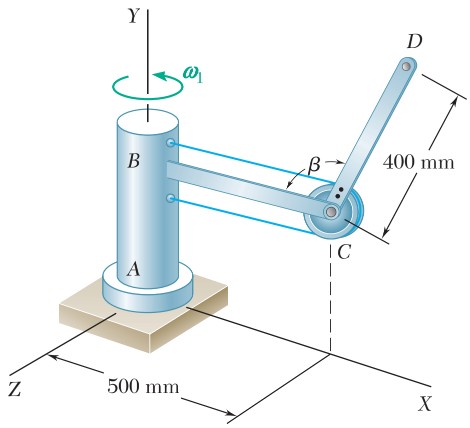


Fig. P15.238

15.238 The body AB and rod BC of the robotic component shown rotate at the constant rate $\omega_1 = 0.60$ rad/s about the Y axis. Simultaneously a wire-and-pulley control causes arm CD to rotate about C at the constant rate $\omega = d\beta/dt = 0.45$ rad/s. Knowing $\beta = 120°$, determine (*a*) the angular acceleration of arm CD, (*b*) the velocity of D, (*c*) the acceleration of D.

15.239 The crane shown rotates at the constant rate $\omega_1 = 0.25$ rad/s; simultaneously, the telescoping boom is being lowered at the constant rate $\omega_2 = 0.40$ rad/s. Knowing that at the instant shown the length of the boom is 20 ft and is increasing at the constant rate $u = 1.5$ ft/s, determine the velocity and acceleration of point B.

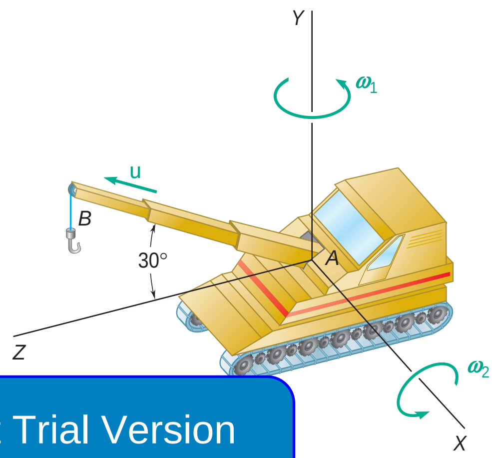


Fig. P15.239

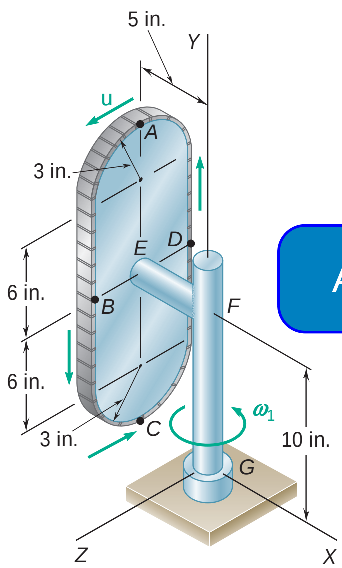


Fig. P15.240 and P15.241

15.240 The vertical plate shown is welded to arm EFG, and the entire unit rotates at the constant rate $\omega_1 = 1.6$ rad/s about the Y axis. At the same time, a continuous link belt moves around the perimeter of the plate at a constant speed $u = 4.5$ in./s. For the position shown, determine the acceleration of the link of the belt located (*a*) at point A, (*b*) at point B.

15.241 The vertical plate shown is welded to arm EFG, and the entire unit rotates at the constant rate $\omega_1 = 1.6$ rad/s about the Y axis. At the same time, a continuous link belt moves around the perimeter of the plate at a constant speed $u = 4.5$ in./s. For the position shown, determine the acceleration of the link of the belt located (*a*) at point C, (*b*) at point D.

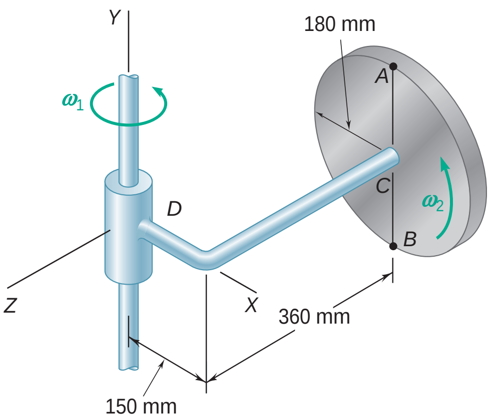


Fig. P15.242 and P15.243

15.242 A disk of 180-mm radius rotates at the constant rate $\omega_2 = 12$ rad/s with respect to arm CD, which itself rotates at the constant rate $\omega_1 = 8$ rad/s about the Y axis. Determine at the instant shown the velocity and acceleration of point A on the rim of the disk.

15.243 A disk of 180-mm radius rotates at the constant rate $\omega_2 = 12$ rad/s with respect to arm CD, which itself rotates at the constant rate $\omega_1 = 8$ rad/s about the Y axis. Determine at the instant shown the velocity and acceleration of point B on the rim of the disk.

15.244 A square plate of side $2r$ is welded to a vertical shaft which rotates with a constant angular velocity V_1. At the same time, rod AB of length r rotates about the center of the plate with a constant angular velocity V_2 with respect to the plate. For the position of the plate shown, determine the acceleration of end B of the rod if (*a*) u = 0, (*b*) u = 90°, (*c*) u = 180°.

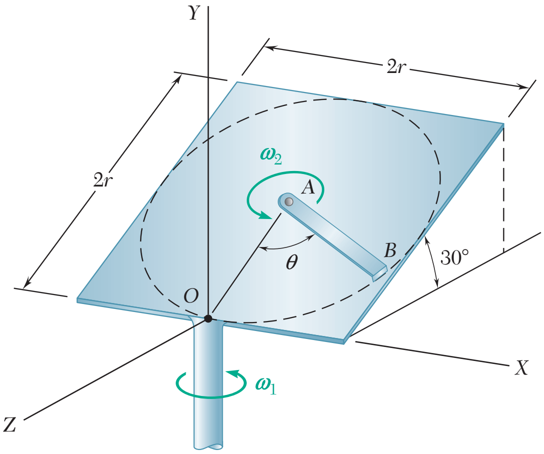


Fig. P15.244

15.245 Two disks, each of 130-mm radius, are welded to the 500-mm rod CD. The rod-and-disks unit rotates at the constant rate $V_2 = 3$ rad/s with respect to arm AB. 4 rad/s, determine the (*b*) point F.

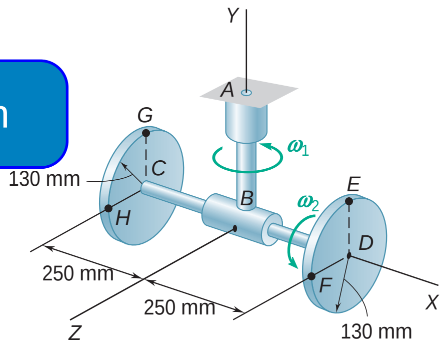


Fig. P15.245

15.246 In Prob. 15.245, determine the velocity and acceleration of (*a*) point G, (*b*) point H.

15.247 The position of the stylus tip A is controlled by the robot shown. In the position shown, the stylus moves at a constant speed u = 180 mm/s relative to the solenoid BC. At the same time, arm CD rotates at the constant rate $V_2 = 1.6$ rad/s with respect to component DEG. Knowing that the entire robot rotates about the X axis at the constant rate $V_1 = 1.2$ rad/s, determine (*a*) the velocity of A, (*b*) the acceleration of A.

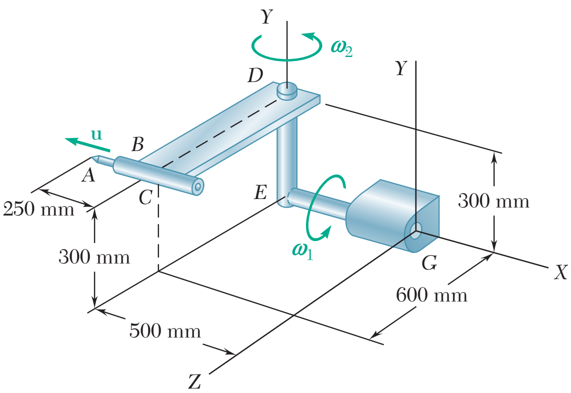


Fig. P15.247

REVIEW AND SUMMARY

This chapter was devoted to the study of the kinematics of rigid bodies.

Rigid body in translation

We first considered the *translation* of a rigid body [Sec. 15.2] and observed that in such a motion, *all points of the body have the same velocity and the same acceleration at any given instant.*

Rigid body in rotation about a fixed axis

We next considered the *rotation* of a rigid body about a fixed axis [Sec. 15.3]. The position of the body is defined by the angle θ that the line BP, drawn from the axis of rotation to a point P of the body, forms with a fixed plane (Fig. 15.39). We found that the magnitude of the velocity of P is

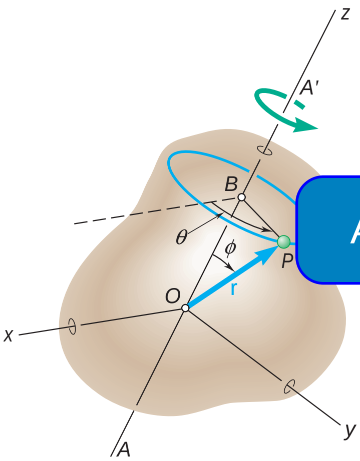


Fig. 15.39

$$v = \frac{ds}{dt} = r\dot{\theta}\sin\phi \qquad (15.4)$$

We then expressed the velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r} \qquad (15.5)$$

where the vector

$$\boldsymbol{\omega} = \omega\mathbf{k} = \dot{\theta}\mathbf{k} \qquad (15.6)$$

is directed along the fixed axis of rotation and represents the *angular velocity* of the body.

Denoting by $\boldsymbol{\alpha}$ the derivative $d\boldsymbol{\omega}/dt$ of the angular velocity, we expressed the acceleration of P as

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (15.8)$$

Differentiating (15.6), and recalling that $\mathbf{k}$ is constant in magnitude and direction, we found that

$$\boldsymbol{\alpha} = \alpha\mathbf{k} = \dot{\omega}\mathbf{k} = \ddot{\theta}\mathbf{k} \qquad (15.9)$$

The vector $\boldsymbol{\alpha}$ represents the *angular acceleration* of the body and is directed along the fixed axis of rotation.

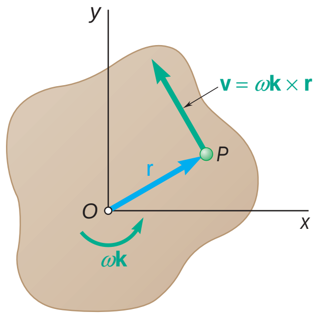


Fig. 15.40

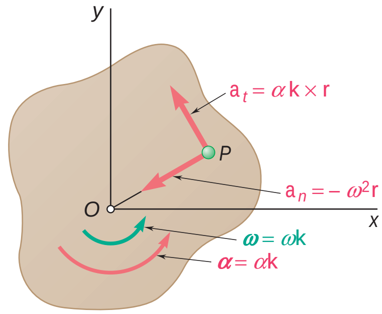


Fig. 15.41

Rotation of a representative slab

Next we considered the motion of a representative slab located in a plane perpendicular to the axis of rotation of the body (Fig. 15.40). Since the angular velocity is perpendicular to the slab, the velocity of a point P of the slab was expressed as

$$\mathbf{v} = \mathsf{v}\mathbf{k} \times \mathbf{r} \tag{15.10}$$

Tangential and normal components

where $\mathbf{v}$ is contained in the plane of the slab. Substituting $\mathsf{V} = \mathsf{v}\mathbf{k}$ and $\mathsf{A} = \mathsf{a}\mathbf{k}$ into (15.8), we found that the acceleration of P could be resolved into tangential and normal components (Fig. 15.41) respectively equal to

$$\mathbf{a}_t = \mathsf{a}\mathbf{k}$$
$$\mathbf{a}_n = -\mathsf{v}^2$$

Angular velocity and angular acceleration of rotating slab

Recalling Eqs. (15.6) and (15.9), we obtained the following expressions for the *angular velocity* and the *angular acceleration* of the slab [Sec. 15.4]:

$$\mathsf{v} = \frac{d\mathsf{u}}{dt} \tag{15.12}$$

$$\mathsf{a} = \frac{d\mathsf{v}}{dt} = \frac{d^2\mathsf{u}}{dt^2} \tag{15.13}$$

or

$$\mathsf{a} = \mathsf{v}\frac{d\mathsf{v}}{d\mathsf{u}} \tag{15.14}$$

We noted that these expressions are similar to those obtained in Chap. 11 for the rectilinear motion of a particle.

Two particular cases of rotation are frequently encountered: *uniform rotation* and *uniformly accelerated rotation.* Problems involving either of these motions can be solved by using equations similar to those used in Secs. 11.4 and 11.5 for the uniform rectilinear motion and the uniformly accelerated rectilinear motion of a particle, but where x, v, and a are replaced by u, v, and a, respectively [Sample Prob. 15.1].

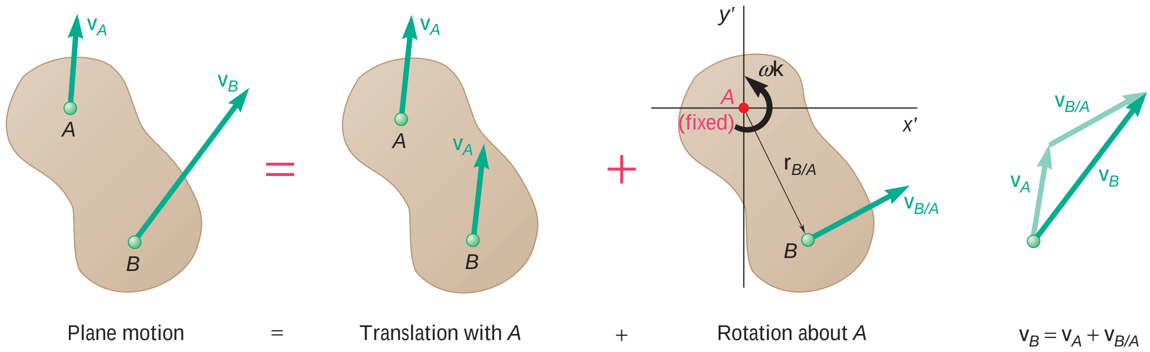


Fig. 15.42

Velocities in plane motion

The *most general plane motion* of a rigid slab can be considered as the *sum of a translation and a rotation* [Sec. 15.5]. For example, the slab shown in Fig. 15.42 can be assumed to translate with point A, while simultaneously rotating about A. It follows [Sec. 15.6] that the velocity of any point B of the slab can be expressed as

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{15.17}$$

where $\mathbf{v}_A$ is the velocity of A and $\mathbf{v}_{B/A}$ the relative velocity of B with respect to A or, more precisely, with respect to axes $x'y'$ translating with A. Denoting by $\mathbf{r}_{B/A}$ the position vector of B relative to A, we found that

$$v_{B/A} = r\omega \tag{15.18}$$

...lating the absolute velocities of points A and B and the relative velocity of B with respect to A was expressed in the form of a vector diagram and used to solve problems involving the motion of various types of mechanisms [Sample Probs. 15.2 and 15.3].

Instantaneous center of rotation

Another approach to the solution of problems involving the velocities of the points of a rigid slab in plane motion was presented in Sec. 15.7 and used in Sample Probs. 15.4 and 15.5. It is based on the determination of the *instantaneous center of rotation* C of the slab (Fig. 15.43).

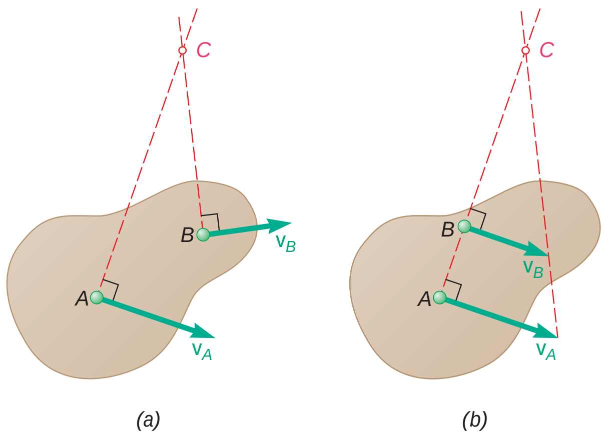


Fig. 15.43

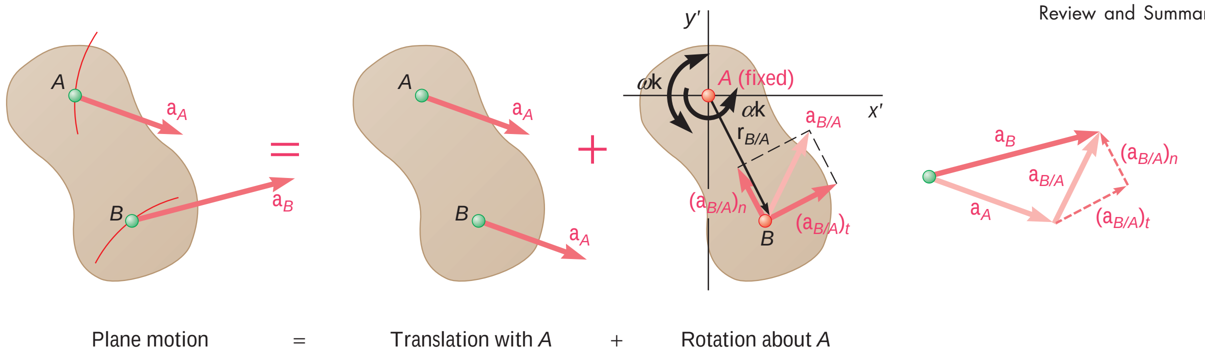


Fig. 15.44

Accelerations in plane motion

The fact that any plane motion of a rigid slab can be considered as the sum of a translation of the slab with a reference point A and a rotation about A was used in Sec. 15.8 to relate the absolute accelerations of any two points A and B of the slab and the relative acceleration of B with respect to A. We had

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \qquad (15.21)$$

where $\mathbf{a}_{B/A}$ consisted of a *normal component* $(\mathbf{a}_{B/A})_n$ of magnitude rv^2 directed toward A, and a *tangential component* $(\mathbf{a}_{B/A})_t$ of magnitude ra perpendicular to the line AB (Fig. 15.44). The fundamental relation (15.21) was expressed in terms of vector diagrams or vector equations and used to determi [...] various mechanisms [Sample P [...] noted that the instantaneous [...] Sec. 15.7 cannot be used for the determination of accelerations, since point C, in general, does *not* have zero acceleration.

Coordinates expressed in terms of a parameter

In the case of certain mechanisms, it is possible to express the coordinates x and y of all significant points of the mechanism by means of simple analytic expressions containing a *single parameter.* The components of the absolute velocity and acceleration of a given point are then obtained by differentiating twice with respect to the time t the coordinates x and y of that point [Sec. 15.9].

Rate of change of a vector with respect to a rotating frame

While the rate of change of a vector is the same with respect to a fixed frame of reference and with respect to a frame in translation, the rate of change of a vector with respect to a rotating frame is different. Therefore, in order to study the motion of a particle relative to a rotating frame we first had to compare the rates of change of a general vector $\mathbf{Q}$ with respect to a fixed frame $OXYZ$ and with respect to a frame $Oxyz$ rotating with an angular velocity $\boldsymbol{\Omega}$ [Sec. 15.10] (Fig. 15.45). We obtained the fundamental relation

$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{Q} \qquad (15.31)$$

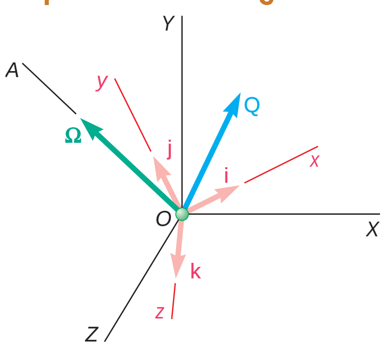


Fig. 15.45

and we concluded that the rate of change of the vector $\mathbf{Q}$ with respect to the fixed frame $OXYZ$ is made of two parts: The first part represents the rate of change of $\mathbf{Q}$ with respect to the rotating frame $Oxyz$; the second part, $\boldsymbol{\Omega} \times \mathbf{Q}$, is induced by the rotation of the frame $Oxyz$.

Plane motion of a particle relative to a rotating frame

The next part of the chapter [Sec. 15.11] was devoted to the two-dimensional kinematic analysis of a particle P moving with respect to a frame $\mathcal{F}$ rotating with an angular velocity $\boldsymbol{\Omega}$ about a fixed axis (Fig. 15.46). We found that the absolute velocity of P could be expressed as

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \tag{15.33}$$

where $\mathbf{v}_P$ = absolute velocity of particle P
$\mathbf{v}_{P'}$ = velocity of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{v}_{P/\mathcal{F}}$ = velocity of P relative to moving frame $\mathcal{F}$

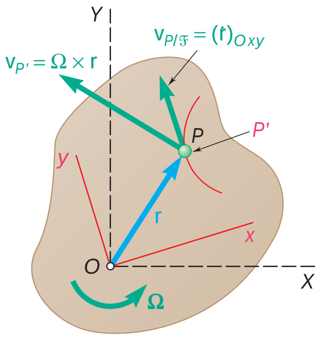


Fig. 15.46

We noted that the same expression for $\mathbf{v}_P$ is obtained if the frame is in translation rather than in rotation. However, when the frame is in rotation, the expression for the acceleration of P is found to contain an additional term $\mathbf{a}_c$ called the *complementary acceleration* or *Coriolis acceleration*. We wrote

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \tag{15.36}$$

where $\mathbf{a}_P$ = absolute acceleration of particle P
$\mathbf{a}_{P'}$ = acceleration of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{a}_{P/\mathcal{F}}$ = acceleration of P relative to moving frame $\mathcal{F}$
$\mathbf{a}_c = 2\boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxy} = 2\boldsymbol{\Omega} \times \mathbf{v}_{P/\mathcal{F}}$
= complementary, or Coriolis, acceleration

... each other in the case of plane ... s found to have a magnitude ... ection obtained by rotating the vector $\mathbf{v}_{P/\mathcal{F}}$ through 90° in the sense of rotation of the moving frame. Formulas (15.33) and (15.36) can be used to analyze the motion of mechanisms which contain parts sliding on each other [Sample Probs. 15.9 and 15.10].

Motion of a rigid body with a fixed point

The last part of the chapter was devoted to the study of the kinematics of rigid bodies in three dimensions. We first considered the motion of a rigid body with a fixed point [Sec. 15.12]. After proving that the most general displacement of a rigid body with a fixed point O is equivalent to a rotation of the body about an axis through O, we were able to define the angular velocity $\boldsymbol{\omega}$ and the *instantaneous axis of rotation* of the body at a given instant. The velocity of a point P of the body (Fig. 15.47) could again be expressed as

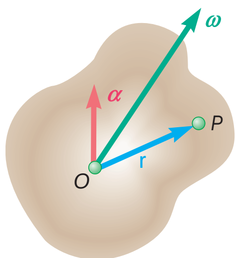


Fig. 15.47

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r} \tag{15.37}$$

Differentiating this expression, we also wrote

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{15.38}$$

However, since the direction of $\boldsymbol{\omega}$ changes from one instant to the next, the angular acceleration $\boldsymbol{\alpha}$ is, in general, not directed along the instantaneous axis of rotation [Sample Prob. 15.11].

General motion in space

It was shown in Sec. 15.13 that *the most general motion of a rigid body in space is equivalent, at any given instant, to the sum of a translation and a rotation.* Considering two particles A and B of the body, we found that

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \qquad (15.42)$$

where $\mathbf{v}_{B/A}$ is the velocity of B relative to a frame $AX'Y'Z'$ attached to A and of fixed orientation (Fig. 15.48). Denoting by $\mathbf{r}_{B/A}$ the position vector of B relative to A, we wrote

$$\mathbf{v}_B = \mathbf{v}_A + \mathsf{V} \times \mathbf{r}_{B/A} \qquad (15.43)$$

where V is the angular velocity of the body at the instant considered [Sample Prob. 15.12]. The acceleration of B was obtained by a similar reasoning. We first wrote

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

and, recalling Eq. (15.38),

$$\mathbf{a}_B = \mathbf{a}_A + \mathsf{A} \times \mathbf{r}_{B/A} + \mathsf{V} \times (\mathsf{V} \times \mathbf{r}_{B/A}) \qquad (15.44)$$

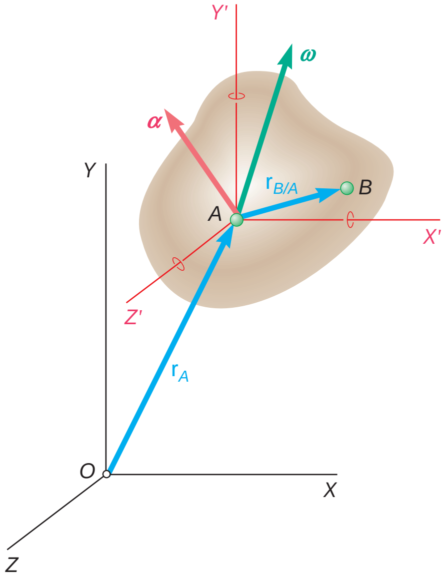


Fig. 15.48

[illegible]e-dimensional motion [illegible] particle relative to a rotating frame

In the final two sections of t[illegible] dimensional motion of a partic[illegible] with an angular velocity $\mathbf{\Omega}$ with respect to a fixed frame $OXYZ$ (Fig. 15.49). In Sec. 15.14 we expressed the absolute velocity $\mathbf{v}_P$ of P as

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \qquad (15.46)$$

where $\mathbf{v}_P$ = absolute velocity of particle P
$\mathbf{v}_{P'}$ = velocity of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{v}_{P/\mathcal{F}}$ = velocity of P relative to moving frame $\mathcal{F}$

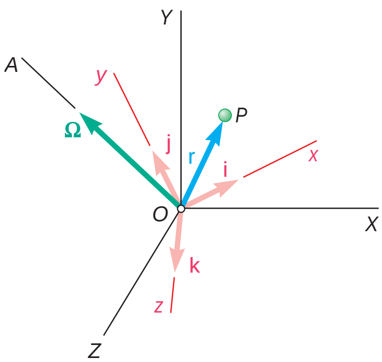


Fig. 15.49

The absolute acceleration $\mathbf{a}_P$ of P was then expressed as

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \tag{15.48}$$

where $\mathbf{a}_P$ = absolute acceleration of particle P
$\mathbf{a}_{P'}$ = acceleration of point P' of moving frame $\mathcal{F}$ coinciding with P
$\mathbf{a}_{P/\mathcal{F}}$ = acceleration of P relative to moving frame $\mathcal{F}$
$\mathbf{a}_c = 2\boldsymbol{\Omega} \times (\dot{\mathbf{r}})_{Oxyz} = 2\boldsymbol{\Omega} \times \mathbf{v}_{P/\mathcal{F}}$
= complementary, or Coriolis, acceleration

It was noted that the magnitude a_c of the Coriolis acceleration is not equal to $2\Omega v_{P/\mathcal{F}}$ [Sample Prob. 15.13] except in the special case when $\boldsymbol{\Omega}$ and $\mathbf{v}_{P/\mathcal{F}}$ are perpendicular to each other.

Frame of reference in general motion

We also observed [Sec. 15.15] that Eqs. (15.46) and (15.48) remain valid when the frame $Axyz$ moves in a known, but arbitrary, fashion with respect to the fixed frame $OXYZ$ (Fig. 15.50), provided that the motion of A is included in the terms $\mathbf{v}_{P'}$ and $\mathbf{a}_{P'}$ representing the absolute velocity and acceleration of the coinciding point P'.

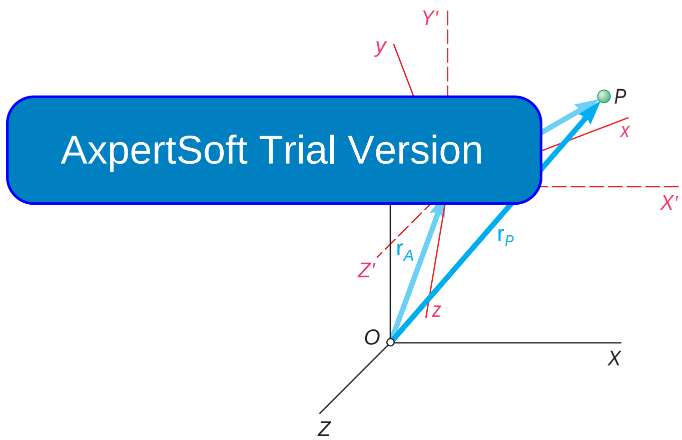


Fig. 15.50

Rotating frames of reference are particularly useful in the study of the three-dimensional motion of rigid bodies. Indeed, there are many cases where an appropriate choice of the rotating frame will lead to a simpler analysis of the motion of the rigid body than would be possible with axes of fixed orientation [Sample Probs. 15.14 and 15.15].

15.248 The angular acceleration of the 600-mm-radius circular plate shown is defined by the relation $\alpha = \alpha_0 e^{-t}$. Knowing that the plate is at rest when $t = 0$ and that $\alpha_0 = 10$ rad/s^2, determine the magnitude of the total acceleration of point B when (*a*) $t = 0$, (*b*) $t = 0.5$ s, (*c*) $t = \infty$.

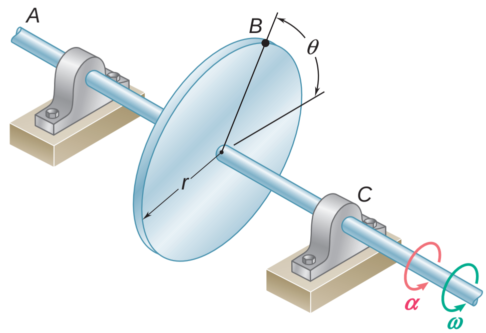


Fig. P15.248

15.249 Cylinder A is moving do[illegible] brake is suddenly applie[illegible] moves 18 ft downward be[illegible] accelerated motion, determine (*a*) the angular acceleration of the drum, (*b*) the time required for the cylinder to come to rest.

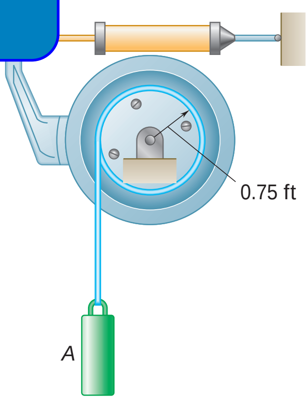


Fig. P15.249

15.250 A baseball pitching machine is designed to deliver a baseball with a ball speed of 70 mph and a ball rotation of 300 rpm clockwise. Knowing that there is no slipping between the wheels and the baseball during the ball launch, determine the angular velocities of wheels A and B.

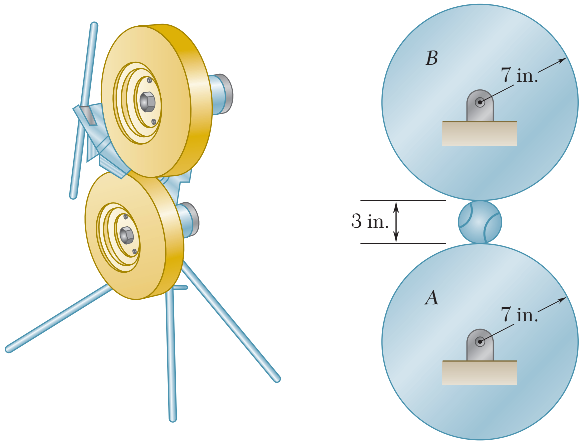


Fig. *P15.250*

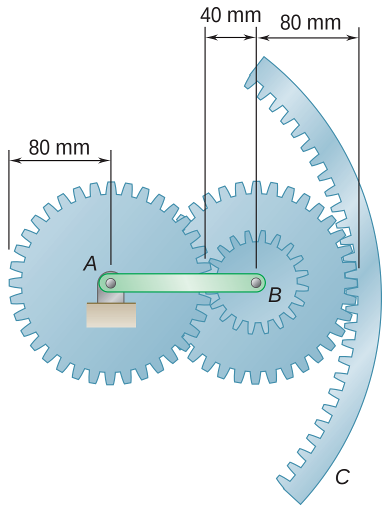


Fig. *P15.251*

15.251 Knowing that inner gear *A* is stationary and outer gear *C* starts from rest and has a constant angular acceleration of 4 rad/s^2 clockwise, determine at $t = 5$ s (*a*) the angular velocity of arm *AB*, (*b*) the angular velocity of gear *B*, (*c*) the acceleration of the point on gear *B* that is in contact with gear *A*.

15.252 Knowing that at the instant shown bar *AB* has an angular velocity of 10 rad/s clockwise and it is slowing down at a rate of 2 rad/s^2, determine the angular accelerations of bar *BD* and bar *DE*.

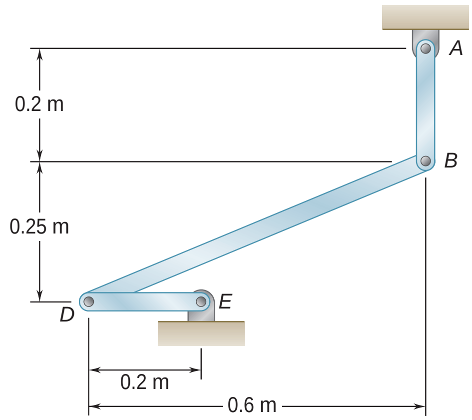


Fig. P15.252

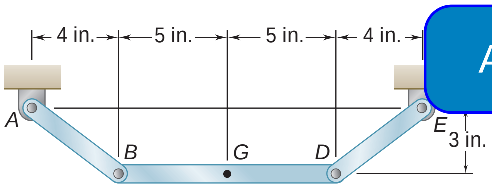


Fig. P15.253

...n rod *AB* has zero angular accel-...f 15 rad/s counterclockwise, deter-...n of arm *DE*, (*b*) the acceleration of point *D*.

15.254 Rod *AB* is attached to a collar at *A* and is fitted with a wheel at *B* that has a radius $r = 15$ mm. Knowing that when $\theta = 60°$ the collar has a velocity of 250 mm/s upward and it is slowing down at a rate of 150 mm/s^2, determine (*a*) the angular acceleration of rod *AB*, (*b*) the angular acceleration of the wheel.

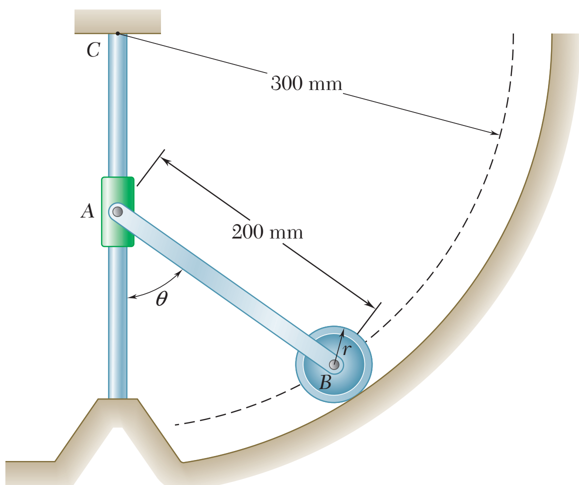


Fig. *P15.254*

15.255 Water flows through a curved pipe AB that rotates with a constant clockwise angular velocity of 90 rpm. If the velocity of the water relative to the pipe is 8 m/s, determine the total acceleration of a particle of water at point P.

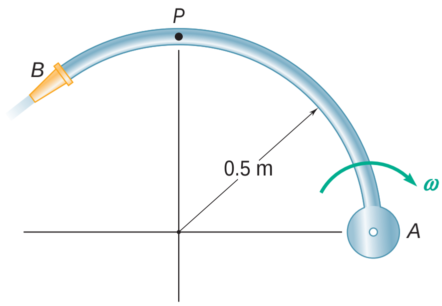


Fig. P15.255

15.256 A disk of 0.15-m radius rotates at the constant rate v_2 with respect to plate BC, which itself rotates at the constant rate v_1 about the y axis. Knowing that $v_1 = v_2 = 3$ rad/s, determine, for the position shown, the velocity and acceleration (*a*) of point D, (*b*) of point F.

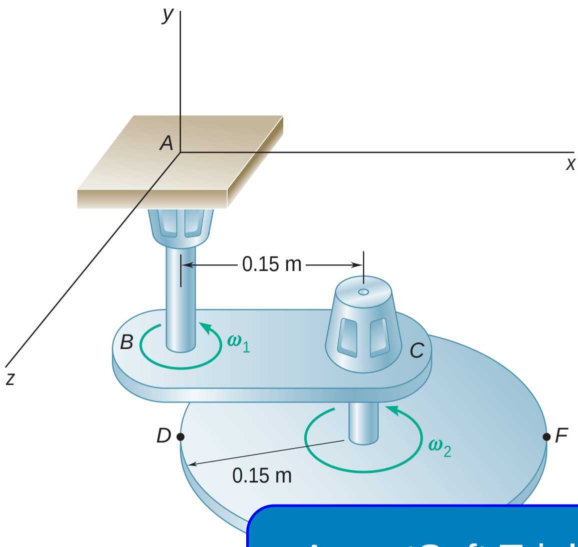


Fig. P15.256

15.257 Two rods AE and BD pass through holes drilled into a hexagonal block. (The holes are drilled in different planes so that the rods will not touch each other.) Knowing that rod AE has an angular velocity of 20 rad/s clockwise and an angular acceleration of 4 rad/s^2 counterclockwise when u = 90°, determine (*a*) the relative velocity of the block with respect to each rod, (*b*) the relative acceleration of the block with respect to each rod.

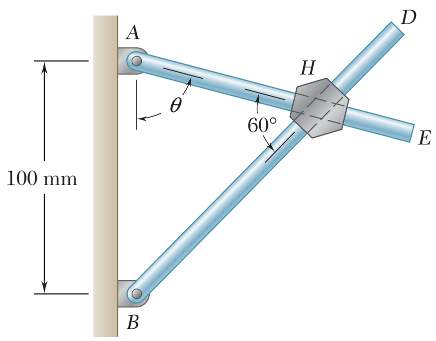


Fig. *P15.257*

15.258 Rod BC of length 24 in. is connected by ball-and-socket joints to a rotating arm AB and to a collar C that slides on the fixed rod DE. Knowing that the length of arm AB is 4 in. and that it rotates at the constant rate $v_1 = 10$ rad/s, determine the velocity of collar C when u = 0.

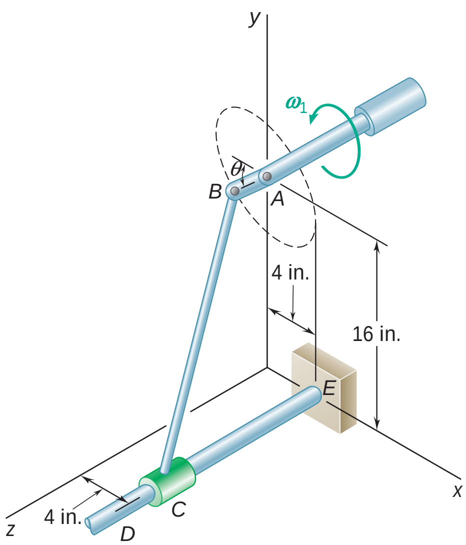


Fig. P15.258

15.259 In the position shown the thin rod moves at a constant speed $u = 3$ in./s out of the tube BC. At the same time tube BC rotates at the constant rate $v_2 = 1.5$ rad/s with respect to arm CD. Knowing that the entire assembly rotates about the X axis at the constant rate $v_1 = 1.2$ rad/s, determine the velocity and acceleration of end A of the rod.

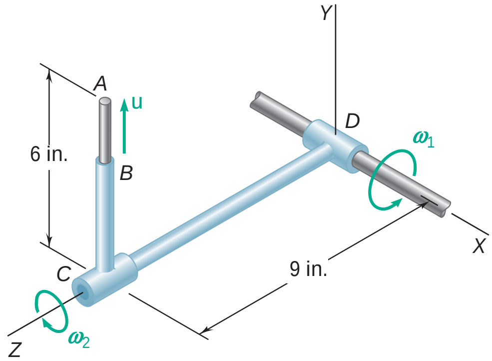


Fig. P15.259

COMPUTER PROBLEMS

15.C1 The disk shown has a constant angular velocity of 500 rpm counterclockwise. Knowing that rod *BD* is 250 mm long, use computational software to determine and plot for values of θ from 0 to 360° and using 30° increments, the velocity of collar *D* and the angular velocity of rod *BD*. Determine the two values of θ for which the speed of collar *D* is zero.

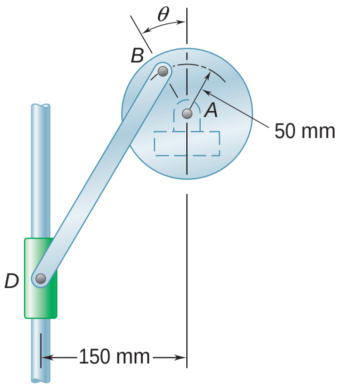


Fig. P15.C1

15.C2 Two rotating rods are connected by a slider block *P* as shown. Knowing that rod *BP* rotates with a constant angular velocity of 6 rad/s counterclockwise, use computational software to determine and plot for values of θ from 0 to 180° the angular velocity and angular acceleration of rod *AE*. Determine the value of θ for which the angular acceleration α_{AE} of rod *AE* is maximum and the corresponding value of α_{AE}.

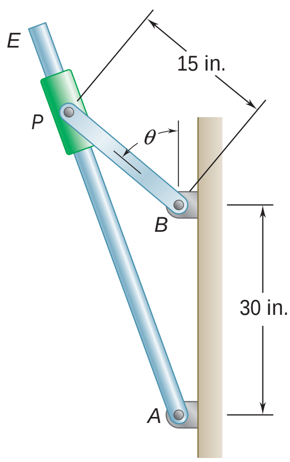


Fig. P15.C2

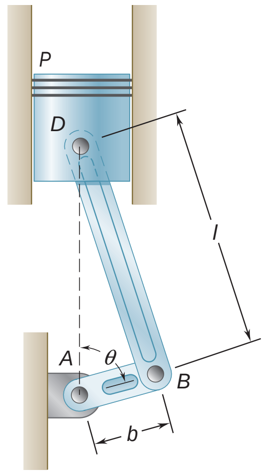


Fig. P15.C3

15.C3 In the engine system shown, l = 160 mm and b = 60 mm. Knowing that crank AB rotates with a constant angular velocity of 1000 rpm clockwise, use computational software to determine and plot for values of u from 0 to 180° and using 10° increments, (*a*) the angular velocity and angular acceleration of rod BD, (*b*) the velocity and acceleration of the piston P.

15.C4 Rod AB moves over a small wheel at C while end A moves to the right with a constant velocity of 180 mm/s. Use computational software to determine and plot for values of u from 20° to 90° and using 5° increments, the velocity of point B and the angular acceleration of the rod. Determine the value of u for which the angular acceleration a of the rod is maximum and the corresponding value of a.

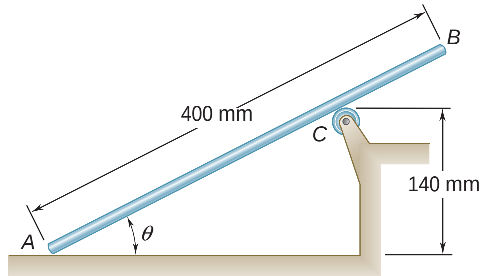


Fig. P15.C4

[illegible] ected by ball-and-socket joints to [illegible] at slides on the fixed rod DE. Arm AB of length 4 in. rotates in the XY plane with a constant angular velocity of 10 rad/s. Use computational software to determine and plot for values of u from 0 to 360° the velocity of collar C. Determine the two values of u for which the velocity of collar C is zero.

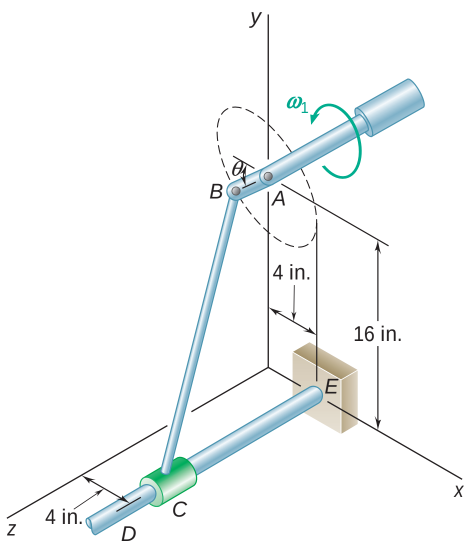


Fig. P15.C5

15.C6 Rod AB of length 25 in. is connected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Collar B moves toward support E at a constant speed of 20 in./s. Denoting by d the distance from point C to collar B, use computational software to determine and plot the velocity of collar A for values of d from 0 to 15 in.

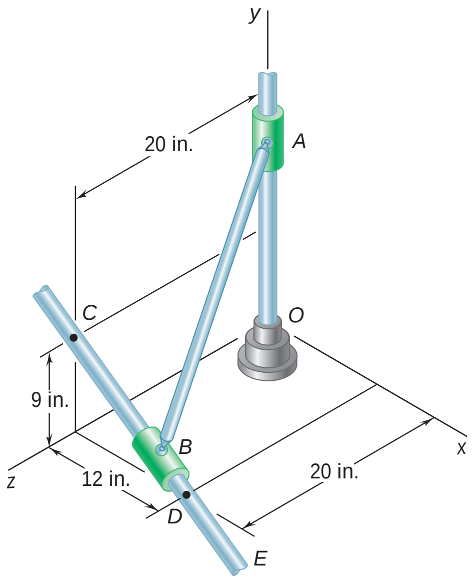
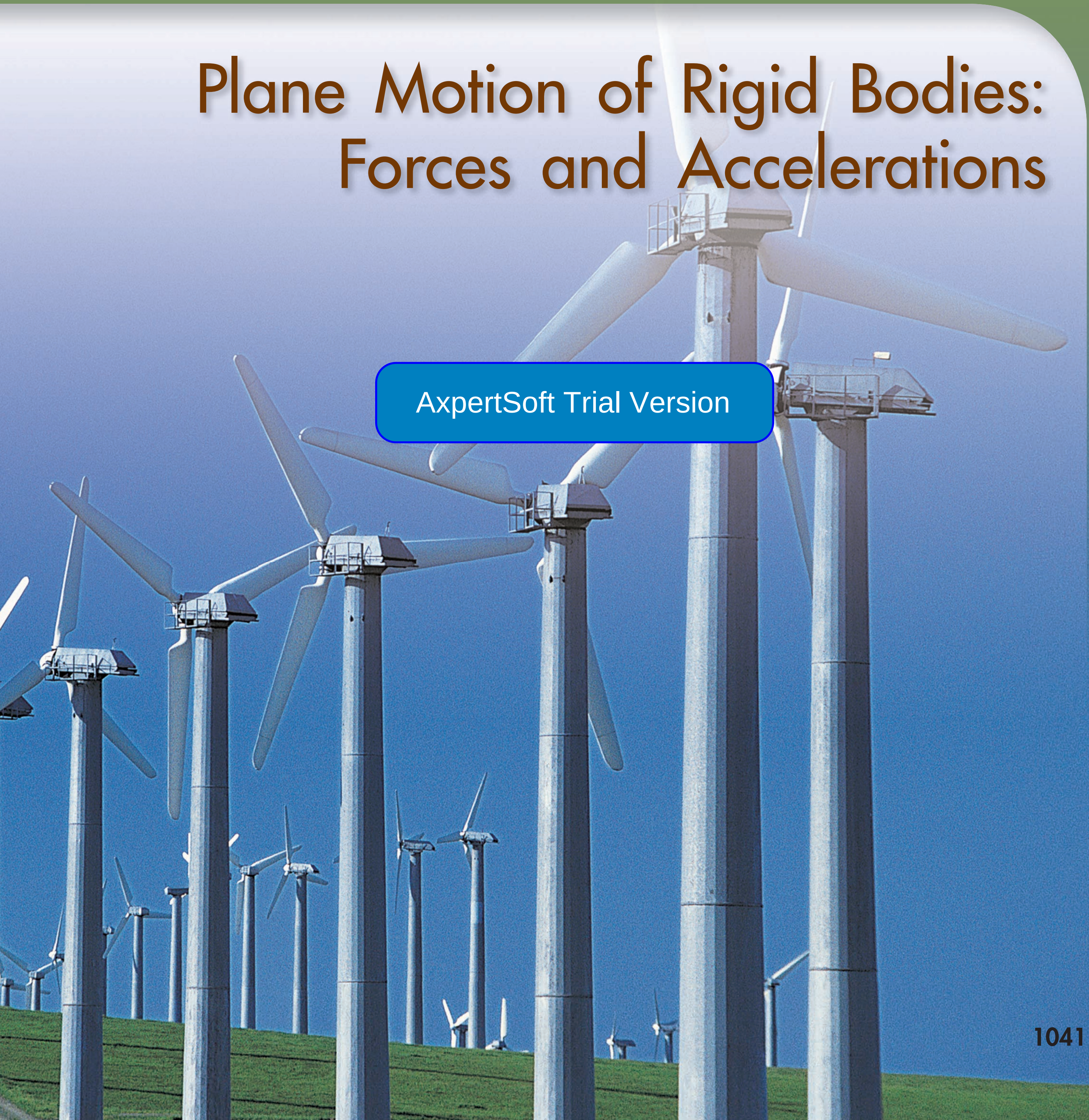


Fig. P15.C6

Three-bladed wind turbines, similar to the ones shown in this picture of a wind farm, are currently the most common design. In this chapter you will learn to analyze the motion of a rigid body by considering the motion of its mass center, the motion relative to its mass [illegible] and the external forces acting on it.

CHAPTER 16

Plane Motion of Rigid Bodies: Forces and Accelerations



Chapter 16 Plane Motion of Rigid Bodies: Forces and Accelerations

16.1 INTRODUCTION

In this chapter and in Chaps. 17 and 18, you will study the *kinetics of rigid bodies,* i.e., the relations existing between the forces acting on a rigid body, the shape and mass of the body, and the motion produced. In Chaps. 12 and 13, you studied similar relations, assuming then that the body could be considered as a particle, i.e., that its mass could be concentrated in one point and that all forces acted at that point. The shape of the body, as well as the exact location of the points of application of the forces, will now be taken into account. You will also be concerned not only with the motion of the body as a whole but also with the motion of the body about its mass center.

Our approach will be to consider rigid bodies as made of large numbers of particles and to use the results obtained in Chap. 14 for the motion of systems of particles. Specifically, two equations from Chap. 14 will be used: Eq. (14.16), $\Sigma\mathbf{F} = m\bar{\mathbf{a}}$, which relates the resultant of the external forces and the acceleration of the mass center G of the system of particles, and Eq. (14.23), $\Sigma\mathbf{M}_G = \dot{\mathbf{H}}_G$, which relates the moment resultant of the external forces and the angular momentum of the system of particles about G.

Except for Sec. 16.2, which applies to the most general case of the motion of a rigid body, the results derived in this chapter will be limited in two ways: (1) They will be restricted to the *plane motion* of rigid bodies, i.e., to a motion in which each particle of the body remains at a constant distance from a fixed reference plane. (2) The [illegible]ly of plane slabs and of bodies [illegible] to the reference plane.† The [illegible]etrical three-dimensional bodies and, more generally, the motion of rigid bodies in three-dimensional space will be postponed until Chap. 18.

In Sec. 16.3, we define the angular momentum of a rigid body in plane motion and show that the rate of change of the angular momentum $\dot{\mathbf{H}}_G$ about the mass center is equal to the product $\bar{I}\boldsymbol{\alpha}$ of the centroidal mass moment of inertia $\bar{I}$ and the angular acceleration $\boldsymbol{\alpha}$ of the body. D'Alembert's principle, introduced in Sec. 16.4, is used to prove that the external forces acting on a rigid body are equivalent to a vector $m\bar{\mathbf{a}}$ attached at the mass center and a couple of moment $\bar{I}\boldsymbol{\alpha}$.

In Sec. 16.5, we derive the principle of transmissibility using only the parallelogram law and Newton's laws of motion, allowing us to remove this principle from the list of axioms (Sec. 1.2) required for the study of the statics and dynamics of rigid bodies.

Free-body-diagram equations are introduced in Sec. 16.6 and will be used in the solution of all problems involving the plane motion of rigid bodies.

After considering the plane motion of connected rigid bodies in Sec. 16.7, you will be prepared to solve a variety of problems involving the translation, centroidal rotation, and unconstrained motion of rigid bodies. In Sec. 16.8 and in the remaining part of the chapter, the solution of problems involving noncentroidal rotation, rolling motion, and other partially constrained plane motions of rigid bodies will be considered.

†Or, more generally, bodies which have a principal centroidal axis of inertia perpendicular to the reference plane.

16.2 EQUATIONS OF MOTION FOR A RIGID BODY

Consider a rigid body acted upon by several external forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, . . . (Fig. 16.1). We can assume that the body is made of a large number n of particles of mass Δm_i ($i = 1, 2, \ldots, n$) and apply the results obtained in Chap. 14 for a system of particles (Fig. 16.2). Considering first the motion of the mass center G of the body with respect to the newtonian frame of reference $Oxyz$, we recall Eq. (14.16) and write

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \tag{16.1}$$

where m is the mass of the body and $\bar{\mathbf{a}}$ is the acceleration of the mass center G. Turning now to the motion of the body relative to the centroidal frame of reference $Gx'y'z'$, we recall Eq. (14.23) and write

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{16.2}$$

where $\dot{\mathbf{H}}_G$ represents the rate of change of $\mathbf{H}_G$, the angular momentum about G of the system of particles forming the rigid body. In the following, $\mathbf{H}_G$ will simply be referred to as the *angular momentum of the rigid body about its mass center G*. Together Eqs. (16.1) and (16.2) express that *the system of the external forces is equipollent to the system consisting of the vector $m\bar{\mathbf{a}}$ attached at G and the couple of moment $\dot{\mathbf{H}}_G$* (Fig. 16.3).†

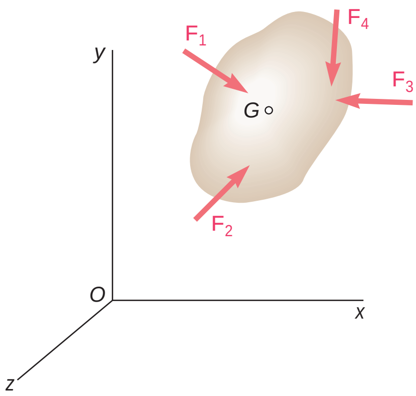


Fig. 16.1

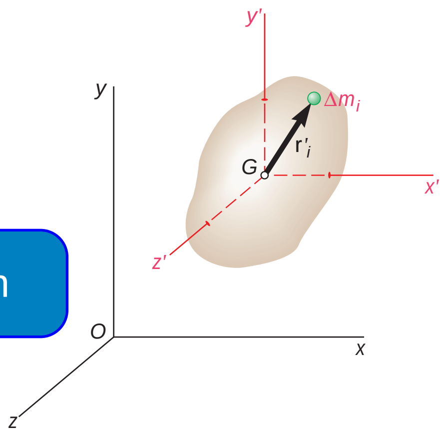


Fig. 16.2

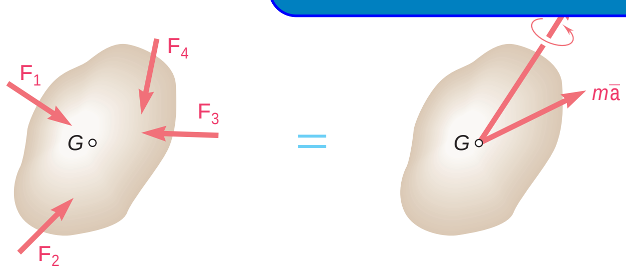


Fig. 16.3

Equations (16.1) and (16.2) apply in the most general case of the motion of a rigid body. In the rest of this chapter, however, our analysis will be limited to the *plane motion* of rigid bodies, i.e., to a motion in which each particle remains at a constant distance from a fixed reference plane, and it will be assumed that the rigid bodies considered consist only of plane slabs and of bodies which are symmetrical with respect to the reference plane. Further study of the plane motion of nonsymmetrical three-dimensional bodies and of the motion of rigid bodies in three-dimensional space will be postponed until Chap. 18.


Photo 16.1 The system of external forces acting on the man and wakeboard includes the weights, the tension in the tow rope, and the forces exerted by the water and the air.

†Since the systems involved act on a rigid body, we could conclude at this point, by referring to Sec. 3.19, that the two systems are *equivalent* as well as equipollent and use red rather than blue equals signs in Fig. 16.3. However, by postponing this conclusion, we will be able to arrive at it independently (Secs. 16.4 and 18.5), thereby eliminating the necessity of including the principle of transmissibility among the axioms of mechanics (Sec. 16.5).

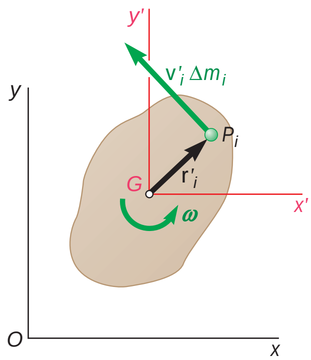


Fig. 16.4

16.3 ANGULAR MOMENTUM OF A RIGID BODY IN PLANE MOTION

Consider a rigid slab in plane motion. Assuming that the slab is made of a large number n of particles P_i of mass Δm_i and recalling Eq. (14.24) of Sec. 14.5, we note that the angular momentum $\mathbf{H}_G$ of the slab about its mass center G can be computed by taking the moments about G of the momenta of the particles of the slab in their motion with respect to either of the frames Oxy or $Gx'y'$ (Fig. 16.4). Choosing the latter course, we write

$$\mathbf{H}_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times \mathbf{v}'_i \, \Delta m_i) \tag{16.3}$$

where $\mathbf{r}'_i$ and $\mathbf{v}'_i \, \Delta m_i$ denote, respectively, the position vector and the linear momentum of the particle P_i relative to the centroidal frame of reference $Gx'y'$. But since the particle belongs to the slab, we have $\mathbf{v}'_i = \mathrm{V} \times \mathbf{r}'_i$, where V is the angular velocity of the slab at the instant considered. We write

$$\mathbf{H}_G = \sum_{i=1}^{n} [\mathbf{r}'_i \times (\mathrm{V} \times \mathbf{r}'_i) \, \Delta m_i]$$

Referring to Fig. 16.4, we easily verify that the expression obtained [illegible] tion as V (i.e., perpendicular to [illegible] $\mathrm{v}\Sigma r_i'^2 \, \Delta m_i$. Recalling that the [illegible] of inertia $\bar{I}$ of the slab about a centroidal axis perpendicular to the slab, we conclude that the angular momentum $\mathbf{H}_G$ of the slab about its mass center is

$$\boxed{\mathbf{H}_G = \bar{I}\mathrm{V}} \tag{16.4}$$

Differentiating both members of Eq. (16.4) we obtain

$$\boxed{\dot{\mathbf{H}}_G = \bar{I}\dot{\mathrm{V}} = \bar{I}\mathrm{A}} \tag{16.5}$$

Thus the rate of change of the angular momentum of the slab is represented by a vector of the same direction as A (i.e., perpendicular to the slab) and of magnitude $\bar{I}\mathrm{a}$.

It should be kept in mind that the results obtained in this section have been derived for a rigid slab in plane motion. As you will see in Chap. 18, they remain valid in the case of the plane motion of rigid bodies which are symmetrical with respect to the reference plane.† However, they do not apply in the case of nonsymmetrical bodies or in the case of three-dimensional motion.

Photo 16.2 The hard disk and pick-up arms of a hard disk computer undergo centroidal rotation.

†Or, more generally, bodies which have a principal centroidal axis of inertia perpendicular to the reference plane.

16.4 PLANE MOTION OF A RIGID BODY. D'ALEMBERT'S PRINCIPLE

Consider a rigid slab of mass m moving under the action of several external forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, . . . , contained in the plane of the slab (Fig. 16.5). Substituting for $\dot{\mathbf{H}}_G$ from Eq. (16.5) into Eq. (16.2) and writing the fundamental equations of motion (16.1) and (16.2) in scalar form, we have

$$\Sigma F_x = m\bar{a}_x \qquad \Sigma F_y = m\bar{a}_y \qquad \Sigma M_G = \bar{I}\alpha \tag{16.6}$$

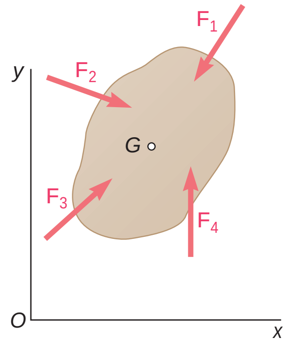


Fig. 16.5

Equations (16.6) show that the acceleration of the mass center G of the slab and its angular acceleration α are easily obtained once the resultant of the external forces acting on the slab and their moment resultant about G have been determined. Given appropriate initial conditions, the coordinates $\bar{x}$ and $\bar{y}$ of the mass center and the angular coordinate θ of the slab can then be obtained by integration at any instant t. Thus *the motion of the slab is completely defined by the resultant and moment resultant about G of the external forces acting on it*.

This property, which will be extended in Chap. 18 to the case of the three-dimensional motion of a rigid body, is characteristic of the motion of a rigid body. Indeed, as we saw in Chap. 14, the motion of a system of particles which are not rigidly connected will in general depend upon the specific external forces acting on the various particles, as well as upon the i[illegible]

Since the motion of a rigi[illegible] and moment resultant of the ex[illegible] *two systems of forces which are equipollent,* i.e., which have the same resultant and the same moment resultant, *are also equivalent;* that is, they have exactly the same effect on a given rigid body.†

Consider in particular the system of the external forces acting on a rigid body (Fig. 16.6*a*) and the system of the effective forces associated with the particles forming the rigid body (Fig. 16.6*b*). It was shown in Sec. 14.2 that the two systems thus defined are equipollent. But since the particles considered now form a rigid body, it follows from the discussion above that the two systems are also equivalent. We can thus state that *the external forces acting on a rigid body are equivalent to the effective forces of the various particles forming the body*. This statement is referred to as *d'Alembert's principle* after the French mathematician Jean le Rond d'Alembert (1717–1783), even though d'Alembert's original statement was written in a somewhat different form.

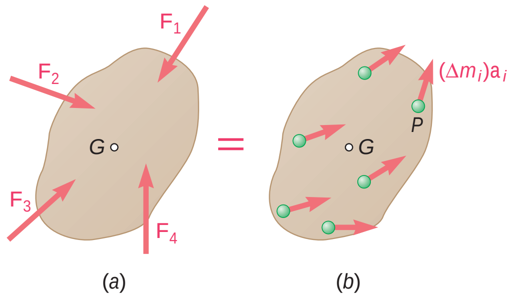


Fig. 16.6

The fact that the system of external forces is *equivalent* to the system of the effective forces has been emphasized by the use of a red equals sign in Fig. 16.6 and also in Fig. 16.7, where using results obtained earlier in this section, we have replaced the effective forces by a vector $m\bar{\mathbf{a}}$ attached at the mass center G of the slab and a couple of moment $\bar{I}\alpha$.

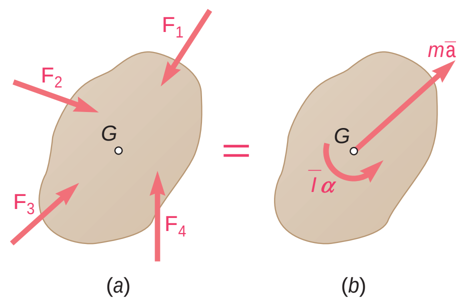


Fig. 16.7

†This result has already been derived in Sec. 3.19 from the principle of transmissibility (Sec. 3.3). The present derivation is independent of that principle, however, and will make possible its elimination from the axioms of mechanics (Sec. 16.5).

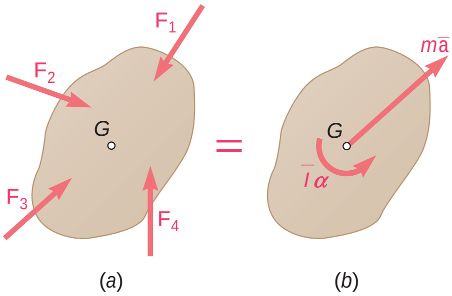


Fig. 16.7 (repeated)

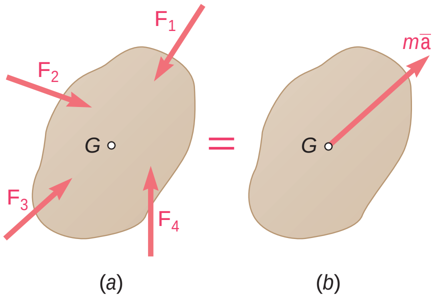


Fig. 16.8 Translation.

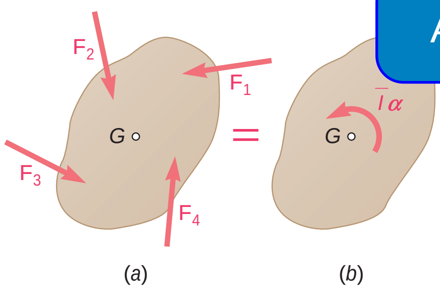


Fig. 16.9 Centroidal rotation.

Translation. In the case of a body in translation, the angular acceleration of the body is identically equal to zero and its effective forces reduce to the vector $m\bar{\mathbf{a}}$ attached at G (Fig. 16.8). Thus, the resultant of the external forces acting on a rigid body in translation passes through the mass center of the body and is equal to $m\bar{\mathbf{a}}$.

Centroidal Rotation. When a slab, or, more generally, a body symmetrical with respect to the reference plane, rotates about a fixed axis perpendicular to the reference plane and passing through its mass center G, we say that the body is in *centroidal rotation*. Since the acceleration $\bar{\mathbf{a}}$ is identically equal to zero, the effective forces of the body reduce to the couple $\bar{I}\alpha$ (Fig. 16.9). Thus, the external forces acting on a body in centroidal rotation are equivalent to a couple of moment $\bar{I}\alpha$.

General Plane Motion. Comparing Fig. 16.7 with Figs. 16.8 and 16.9, we observe that from the point of view of *kinetics*, the most general plane motion of a rigid body symmetrical with respect to the reference plane can be replaced by the sum of a translation and a centroidal rotation. We should note that this statement is more restrictive than the similar statement made earlier from the point of view of *kinematics* (Sec. 15.5), since we now require that the mass center of the body be selected as the reference point.

Referring to Eqs. (16.6), we observe that the first two equations are identical with the equations of motion of a particle of mass m acted

. We thus check that *the mass* *n moves as if the entire mass of* *t, and as if all the external forces acted on it*. We recall that this result has already been obtained in Sec. 14.4 in the general case of a system of particles, the particles being not necessarily rigidly connected. We also note, as we did in Sec. 14.4, that the system of the external forces does not, in general, reduce to a single vector $m\bar{\mathbf{a}}$ attached at G. Therefore, in the general case of the plane motion of a rigid body, *the resultant of the external forces acting on the body does not pass through the mass center of the body*.

Finally, it should be observed that the last of Eqs. (16.6) would still be valid if the rigid body, while subjected to the same applied forces, were constrained to rotate about a fixed axis through G. Thus, *a rigid body in plane motion rotates about its mass center as if this point were fixed*.

*16.5 A REMARK ON THE AXIOMS OF THE MECHANICS OF RIGID BODIES

The fact that two equipollent systems of external forces acting on a rigid body are also equivalent, i.e., have the same effect on that rigid body, has already been established in Sec. 3.19. But there it was derived from the *principle of transmissibility*, one of the axioms used in our study of the statics of rigid bodies. It should be noted that this axiom has not been used in the present chapter because Newton's second and third laws of motion make its use unnecessary in the study of the dynamics of rigid bodies.

In fact, the principle of transmissibility may now be *derived* from the other axioms used in the study of mechanics. This principle stated, without proof (Sec. 3.3), that the conditions of equilibrium or motion of a rigid body remain unchanged if a force $\mathbf{F}$ acting at a given point of the rigid body is replaced by a force $\mathbf{F}'$ of the same magnitude and same direction, but acting at a different point, provided that the two forces have the same line of action. But since $\mathbf{F}$ and $\mathbf{F}'$ have the same moment about any given point, it is clear that they form two equipollent systems of external forces. Thus, we may now *prove,* as a result of what we established in the preceding section, that $\mathbf{F}$ and $\mathbf{F}'$ have the same effect on the rigid body (Fig. 3.3).

The principle of transmissibility can therefore be removed from the list of axioms required for the study of the mechanics of rigid bodies. These axioms are reduced to the parallelogram law of addition of vectors and to Newton's laws of motion.

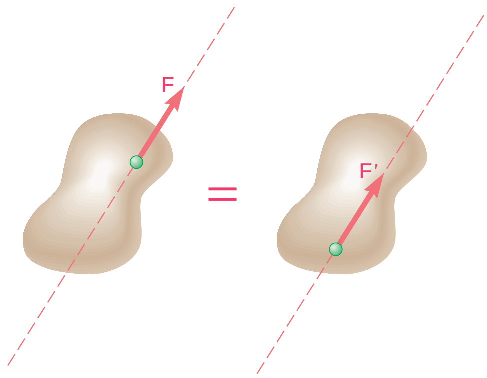


Fig. 3.3 (repeated)

16.6 SOLUTION OF PROBLEMS INVOLVING THE MOTION OF A RIGID BODY

We saw in Sec. 16.4 that when a rigid body is in plane motion, there exists a fundamental relation between the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, . . . , acting on the body, the acceleration $\bar{\mathbf{a}}$ of its mass center, and the angular acceleration A of the body. This relation, which is represented in Fig. 16.7 in the form of a *free-body-diagram equation,* can be used to determine the acceleration $\bar{\mathbf{a}}$ and the angular acceleration A produced by a given system of forces acting on a rigid body or, conversely, to determine the forces which produce a given motion of the rigid body.

The three algebraic equa[illegible] lems of plane motion.† Howe[illegible] that the solution of many pro[illegible] simplified by an appropriate choice of the point about which the moments of the forces are computed. It is therefore preferable to remember the relation existing between the forces and the accelerations in the pictorial form shown in Fig. 16.7 and to derive from this fundamental relation the component or moment equations which fit best the solution of the problem under consideration.

The fundamental relation shown in Fig. 16.7 can be presented in an alternative form if we add to the external forces an inertia vector $-m\bar{\mathbf{a}}$ of sense opposite to that of $\bar{\mathbf{a}}$, attached at G, and an inertia couple $-\bar{I}\mathsf{A}$ of moment equal in magnitude to $\bar{I}\mathsf{a}$ and of sense opposite to that of A (Fig. 16.10). The system obtained is equivalent to zero, and the rigid body is said to be in *dynamic equilibrium.*

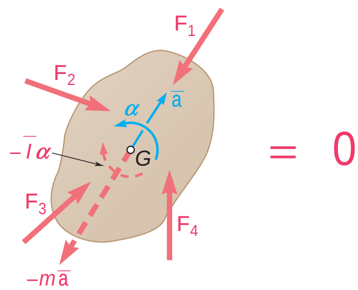


Fig. 16.10

Whether the principle of equivalence of external and effective forces is directly applied, as in Fig. 16.7, or whether the concept of dynamic equilibrium is introduced, as in Fig. 16.10, the use of free-body-diagram equations showing vectorially the relationship existing between the forces applied on the rigid body and the resulting linear and angular accelerations presents considerable advantages over the blind application of formulas (16.6). These advantages can be summarized as follows:

1. The use of a pictorial representation provides a much clearer understanding of the effect of the forces on the motion of the body.

†We recall that the last of Eqs. (16.6) is valid only in the case of the plane motion of a rigid body symmetrical with respect to the reference plane. In all other cases, the methods of Chap. 18 should be used.

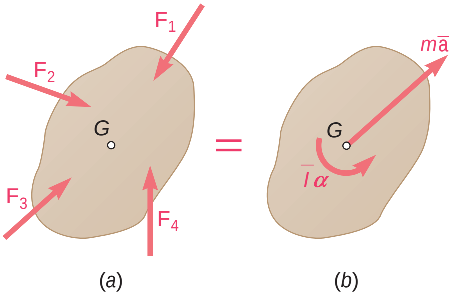


Fig. 16.7 (repeated)

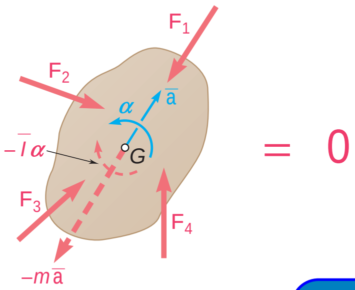


Fig. 16.10 (repeated)

Photo 16.3 The forklift and moving load can be analyzed as a system of two connected rigid bodies in plane motion.

2. This approach makes it possible to divide the solution of a dynamics problem into two parts: In the first part, the analysis of the kinematic and kinetic characteristics of the problem leads to the free-body diagrams of Fig. 16.7 or 16.10; in the second part, the diagram obtained is used to analyze the various forces and vectors involved by the methods of Chap. 3.

3. A unified approach is provided for the analysis of the plane motion of a rigid body, regardless of the particular type of motion involved. While the kinematics of the various motions considered may vary from one case to the other, the approach to the kinetics of the motion is consistently the same. In every case a diagram will be drawn showing the external forces, the vector $m\bar{\mathbf{a}}$ associated with the motion of G, and the couple $\bar{I}\mathbf{\alpha}$ associated with the rotation of the body about G.

4. The resolution of the plane motion of a rigid body into a translation and a centroidal rotation, which is used here, is a basic concept which can be applied effectively throughout the study of mechanics. It will be used again in Chap. 17 with the method of work and energy and the method of impulse and momentum.

5. As you will see in Chap. 18, this approach can be extended to the study of the general three-dimensional motion of a rigid body. The motion of the body will again be resolved into a translation and a rotation about the mass center, and free-body-diagram equations will be used to indicate the relationship existing between the external forces and the rates of change of [illegible]m of the body.

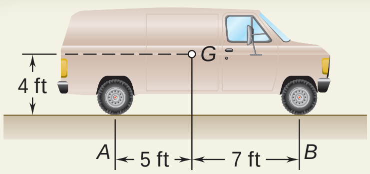
16.7 SYSTEMS OF RIGID BODIES

The method described in the preceding section can also be used in problems involving the plane motion of several connected rigid bodies. For each part of the system, a diagram similar to Fig. 16.7 or Fig. 16.10 can be drawn. The equations of motion obtained from these diagrams are solved simultaneously.

In some cases, as in Sample Prob. 16.3, a single diagram can be drawn for the entire system. This diagram should include all the external forces, as well as the vectors $m\bar{\mathbf{a}}$ and the couples $\bar{I}\mathbf{\alpha}$ associated with the various parts of the system. However, internal forces such as the forces exerted by connecting cables, can be omitted since they occur in pairs of equal and opposite forces and are thus equipollent to zero. The equations obtained by expressing that the system of the external forces is equipollent to the system of the effective forces can be solved for the remaining unknowns.†

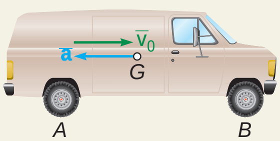
It is not possible to use this second approach in problems involving more than three unknowns, since only three equations of motion are available when a single diagram is used. We need not elaborate upon this point, since the discussion involved would be completely similar to that given in Sec. 6.11 in the case of the equilibrium of a system of rigid bodies.

†Note that we cannot speak of *equivalent* systems since we are not dealing with a single rigid body.



SAMPLE PROBLEM 16.1

When the forward speed of the truck shown was 30 ft/s, the brakes were suddenly applied, causing all four wheels to stop rotating. It was observed that the truck skidded to rest in 20 ft. Determine the magnitude of the normal reaction and of the friction force at each wheel as the truck skidded to rest.



SOLUTION

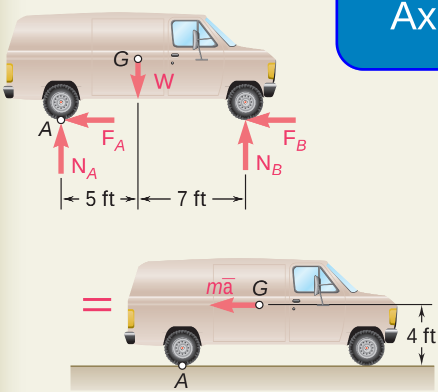
Kinematics of Motion. Choosing the positive sense to the right and using the equations of uniformly accelerated motion, we write

$$\overline{v}_0 = +30\ \text{ft/s} \qquad \overline{v}^2 = \overline{v}_0^2 + 2\overline{a}\,\overline{x} \qquad 0 = (30)^2 + 2\overline{a}(20)$$
$$\overline{a} = -22.5\ \text{ft/s}^2 \qquad \overline{\mathbf{a}} = 22.5\ \text{ft/s}^2 \leftarrow$$

Equations of Motion. The external forces consist of the weight $\mathbf{W}$ of the truck and of the normal reactions and friction forces at the wheels. (The vectors $\mathbf{N}_A$ and $\mathbf{F}_A$ represent the sum of the reactions at the rear wheels, while $\mathbf{N}_B$ and $\mathbf{F}_B$ represent the sum of the reactions at the front wheels.) Since the truck is in translation, the effective forces reduce to the vector $m\overline{\mathbf{a}}$ attached at G. Three equations of motion are obtained by expressing that the system of the external forces is equivalent to the system of the

$= 0$

..., where μ_k is the coefficient of kinetic friction, we find that



$$F_A + F_B = \mu_k(N_A + N_B) = \mu_k W$$

$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $\quad -(F_A + F_B) = -m\overline{a}$

$$-\mu_k W = -\frac{W}{32.2\,\text{ft/s}^2}(22.5\,\text{ft/s}^2)$$
$$\mu_k = 0.699$$

$+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}$: $\quad -W(5\text{ ft}) + N_B(12\text{ ft}) = m\overline{a}(4\text{ ft})$

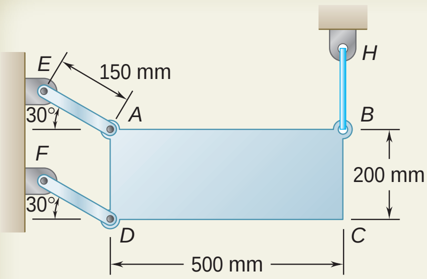
$$-W(5\,\text{ft}) + N_B(12\,\text{ft}) = \frac{W}{32.2\,\text{ft/s}^2}(22.5\,\text{ft/s}^2)(4\,\text{ft})$$
$$N_B = 0.650W$$
$$F_B = \mu_k N_B = (0.699)(0.650W) \qquad F_B = 0.454W$$

$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}$: $\quad N_A + N_B - W = 0$

$$N_A + 0.650W - W = 0$$
$$N_A = 0.350W$$
$$F_A = \mu_k N_A = (0.699)(0.350W) \qquad F_A = 0.245W$$

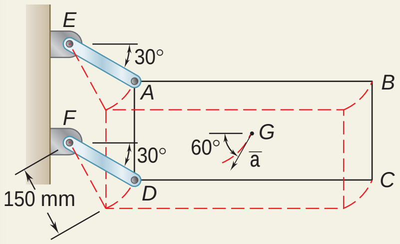
Reactions at Each Wheel. Recalling that the values computed above represent the sum of the reactions at the two front wheels or the two rear wheels, we obtain the magnitude of the reactions at each wheel by writing

$$N_{\text{front}} = \tfrac{1}{2}N_B = 0.325W \qquad N_{\text{rear}} = \tfrac{1}{2}N_A = 0.175W \quad \blacktriangleleft$$
$$F_{\text{front}} = \tfrac{1}{2}F_B = 0.227W \qquad F_{\text{rear}} = \tfrac{1}{2}F_A = 0.122W \quad \blacktriangleleft$$



SAMPLE PROBLEM 16.2

The thin plate *ABCD* of mass 8 kg is held in the position shown by the wire *BH* and two links *AE* and *DF*. Neglecting the mass of the links, determine immediately after wire *BH* has been cut (*a*) the acceleration of the plate, (*b*) the force in each link.

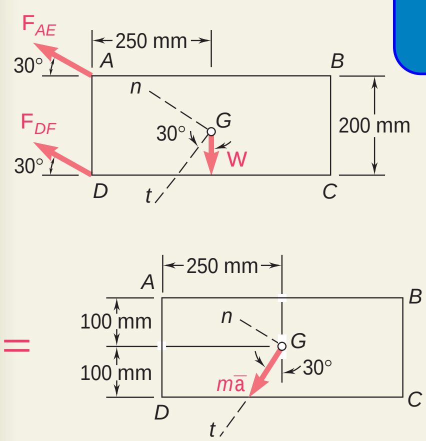


SOLUTION

Kinematics of Motion. After wire *BH* has been cut, we observe that corners *A* and *D* move along parallel circles of radius 150 mm centered, respectively, at *E* and *F*. The motion of the plate is thus a curvilinear translation; the particles forming the plate move along parallel circles of radius 150 mm.

At the instant wire *BH* is cut, the velocity of the plate is zero. Thus the acceleration $\bar{\mathbf{a}}$ of the mass center *G* of the plate is tangent to the circular path which will be described by *G*.

Equations of Motion. The external forces consist of the weight **W** and the forces $\mathbf{F}_{AE}$ and $\mathbf{F}_{DF}$ exerted by the links. Since the plate is in translation, the effective forces reduce to the vector $m\bar{\mathbf{a}}$ attached at *G* and directed along the *t* axis. A free-body-diagram equation is drawn to show that the system of the external forces is equivalent to the system of the effective



$+\swarrow \Sigma F_t = \Sigma(F_t)_{\text{eff}}$:

$$W \cos 30° = m\bar{a}$$
$$mg \cos 30° = m\bar{a}$$
$$\bar{a} = g \cos 30° = (9.81 \text{ m/s}^2) \cos 30° \quad (1)$$
$$\bar{\mathbf{a}} = 8.50 \text{ m/s}^2 \text{ d } 60° \quad \blacktriangleleft$$

b. Forces in Links *AE* and *DF*.

$+\nwarrow \Sigma F_n = \Sigma(F_n)_{\text{eff}}$: $\quad F_{AE} + F_{DF} - W \sin 30° = 0 \quad (2)$

$+\text{i} \Sigma M_G = \Sigma(M_G)_{\text{eff}}$:

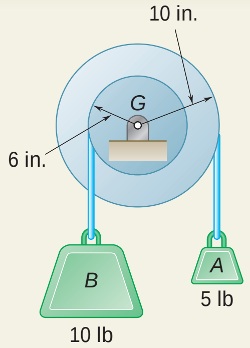
$$(F_{AE} \sin 30°)(250 \text{ mm}) - (F_{AE} \cos 30°)(100 \text{ mm})$$
$$+ (F_{DF} \sin 30°)(250 \text{ mm}) + (F_{DF} \cos 30°)(100 \text{ mm}) = 0$$
$$38.4F_{AE} + 211.6F_{DF} = 0$$
$$F_{DF} = -0.1815F_{AE} \quad (3)$$

Substituting for F_{DF} from (3) into (2), we write

$$F_{AE} - 0.1815F_{AE} - W \sin 30° = 0$$
$$F_{AE} = 0.6109W$$
$$F_{DF} = -0.1815(0.6109W) = -0.1109W$$

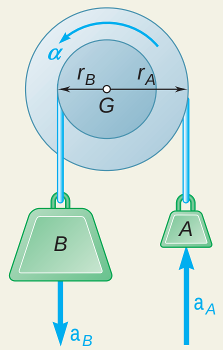
Noting that $W = mg = (8 \text{ kg})(9.81 \text{ m/s}^2) = 78.48$ N, we have

$$F_{AE} = 0.6109(78.48 \text{ N}) \qquad F_{AE} = 47.9 \text{ N } T \quad \blacktriangleleft$$
$$F_{DF} = -0.1109(78.48 \text{ N}) \qquad F_{DF} = 8.70 \text{ N } C \quad \blacktriangleleft$$



SAMPLE PROBLEM 16.3

A pulley weighing 12 lb and having a radius of gyration of 8 in. is connected to two blocks as shown. Assuming no axle friction, determine the angular acceleration of the pulley and the acceleration of each block.



SOLUTION

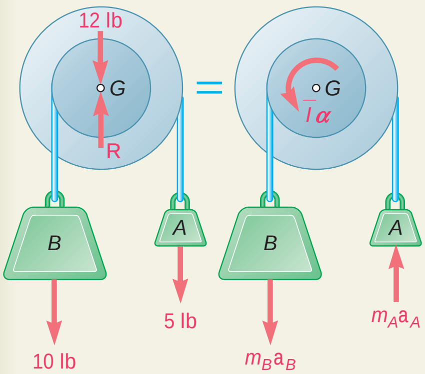
Sense of Motion. Although an arbitrary sense of motion can be assumed (since no friction forces are involved) and later checked by the sign of the answer, we may prefer to determine the actual sense of rotation of the pulley first. The weight of block B required to maintain the equilibrium of the pulley when it is acted upon by the 5-lb block A is first determined. We write

$$+\circlearrowleft \Sigma M_G = 0: \qquad W_B(6 \text{ in.}) - (5 \text{ lb})(10 \text{ in.}) = 0 \qquad W_B = 8.33 \text{ lb}$$

... e pulley will rotate counterclockwise.

... counterclockwise and noting that $a_A = r_A\alpha$ and $a_B = r_B\alpha$, we obtain

$$\mathbf{a}_A = (\tfrac{10}{12} \text{ ft})\alpha \uparrow \qquad \mathbf{a}_B = (\tfrac{6}{12} \text{ ft})\alpha \downarrow$$



Equations of Motion. A single system consisting of the pulley and the two blocks is considered. Forces external to this system consist of the weights of the pulley and the two blocks and of the reaction at G. (The forces exerted by the cables on the pulley and on the blocks are internal to the system considered and cancel out.) Since the motion of the pulley is a centroidal rotation and the motion of each block is a translation, the effective forces reduce to the couple $\bar{I}\boldsymbol{\alpha}$ and the two vectors $m\mathbf{a}_A$ and $m\mathbf{a}_B$. The centroidal moment of inertia of the pulley is

$$\bar{I} = m\bar{k}^2 = \frac{W}{g}\bar{k}^2 = \frac{12 \text{ lb}}{32.2 \text{ ft/s}^2}(\tfrac{8}{12} \text{ ft})^2 = 0.1656 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

Since the system of the external forces is equipollent to the system of the effective forces, we write

$$+\circlearrowleft \Sigma M_G = \Sigma (M_G)_{\text{eff}}:$$

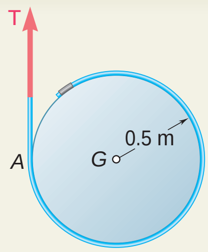
$$(10 \text{ lb})(\tfrac{6}{12} \text{ ft}) - (5 \text{ lb})(\tfrac{10}{12} \text{ ft}) = +\bar{I}\alpha + m_B a_B(\tfrac{6}{12} \text{ ft}) + m_A a_A(\tfrac{10}{12} \text{ ft})$$

$$(10)(\tfrac{6}{12}) - (5)(\tfrac{10}{12}) = 0.1656\alpha + \tfrac{10}{32.2}(\tfrac{6}{12}\alpha)(\tfrac{6}{12}) + \tfrac{5}{32.2}(\tfrac{10}{12}\alpha)(\tfrac{10}{12})$$

$$\alpha = +2.374 \text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 2.37 \text{ rad/s}^2 \circlearrowleft \blacktriangleleft$$

$$a_A = r_A\alpha = (\tfrac{10}{12} \text{ ft})(2.374 \text{ rad/s}^2) \qquad \mathbf{a}_A = 1.978 \text{ ft/s}^2 \uparrow \blacktriangleleft$$

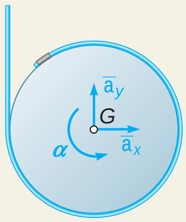
$$a_B = r_B\alpha = (\tfrac{6}{12} \text{ ft})(2.374 \text{ rad/s}^2) \qquad \mathbf{a}_B = 1.187 \text{ ft/s}^2 \downarrow \blacktriangleleft$$



SAMPLE PROBLEM 16.4

A cord is wrapped around a homogeneous disk of radius $r = 0.5$ m and mass $m = 15$ kg. If the cord is pulled upward with a force **T** of magnitude 180 N, determine (*a*) the acceleration of the center of the disk, (*b*) the angular acceleration of the disk, (*c*) the acceleration of the cord.

SOLUTION



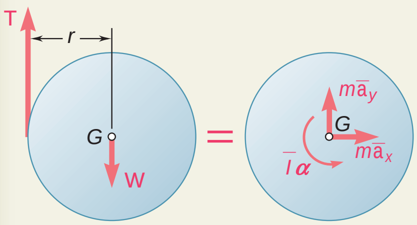
Equations of Motion. We assume that the components $\bar{\mathbf{a}}_x$ and $\bar{\mathbf{a}}_y$ of the acceleration of the center are directed, respectively, to the right and upward and that the angular acceleration of the disk is counterclockwise. The external forces acting on the disk consist of the weight **W** and the force **T** exerted by the cord. This system is equivalent to the system of the effective forces, which consists of a vector of components $m\bar{\mathbf{a}}_x$ and $m\bar{\mathbf{a}}_y$ attached at G and a couple $\bar{I}$A. We write

$\overset{+}{y}\ \Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $\qquad 0 = m\bar{a}_x \qquad\qquad \bar{\mathbf{a}}_x = 0$ ◀

$+\text{x}\Sigma F_y = \Sigma(F_y)_{\text{eff}}$: $\qquad T - W = m\bar{a}_y$

$$\frac{T - W}{}$$

kg)(9.81 m/s^2) = 147.1 N, we



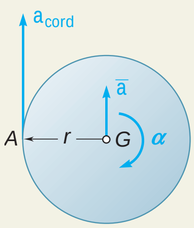
$$\bar{a}_y = \frac{180\text{ N} - 147.1\text{ N}}{15\text{ kg}} = +2.19\text{ m/s}^2 \qquad \bar{\mathbf{a}}_y = 2.19\text{ m/s}^2\text{ x} \text{ ◀}$$

$+1\Sigma M_G = \Sigma(M_G)_{\text{eff}}$: $\qquad -Tr = \bar{I}\text{a}$

$$-Tr = (\tfrac{1}{2}mr^2)\text{a}$$

$$\text{a} = -\frac{2T}{mr} = -\frac{2(180\text{ N})}{(15\text{ kg})(0.5\text{ m})} = -48.0\text{ rad/s}^2$$

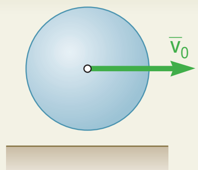
$$\text{A} = 48.0\text{ rad/s}^2\text{ i} \text{ ◀}$$



Acceleration of Cord. Since the acceleration of the cord is equal to the tangential component of the acceleration of point A on the disk, we write

$$\mathbf{a}_{\text{cord}} = (\mathbf{a}_A)_t = \bar{\mathbf{a}} + (\mathbf{a}_{A/G})_t$$
$$= [2.19\text{ m/s}^2\text{ x}] + [(0.5\text{ m})(48\text{ rad/s}^2)\text{x}]$$

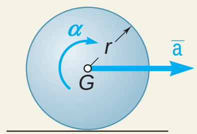
$$\mathbf{a}_{\text{cord}} = 26.2\text{ m/s}^2\text{ x} \text{ ◀}$$



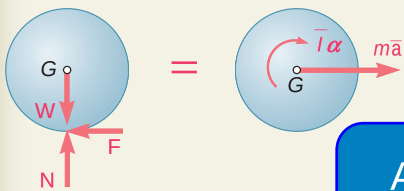
SAMPLE PROBLEM 16.5

A uniform sphere of mass m and radius r is projected along a rough horizontal surface with a linear velocity $\bar{\mathbf{v}}_0$ and no angular velocity. Denoting by μ_k the coefficient of kinetic friction between the sphere and the floor, determine (*a*) the time t_1 at which the sphere will start rolling without sliding, (*b*) the linear velocity and angular velocity of the sphere at time t_1.

SOLUTION



Equations of Motion. The positive sense is chosen to the right for $\bar{\mathbf{a}}$ and clockwise for α. The external forces acting on the sphere consist of the weight $\mathbf{W}$, the normal reaction $\mathbf{N}$, and the friction force $\mathbf{F}$. Since the point of the sphere in contact with the surface is sliding to the right, the friction force $\mathbf{F}$ is directed to the left. While the sphere is sliding, the magnitude of the friction force is $F = \mu_k N$. The effective forces consist of the vector $m\bar{\mathbf{a}}$ attached at G and the couple $\bar{I}\alpha$. Expressing that the system of the external forces is equivalent to the system of the effective forces, we write



$+\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}$: $\quad N - W = 0$

$$N = W = mg \qquad F = \mu_k N = \mu_k mg$$

$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}$: $\quad -F = m\bar{a} \qquad -\mu_k mg = m\bar{a} \qquad \bar{a} = -\mu_k g$

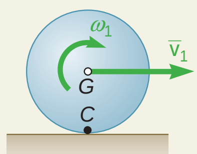
the value obtained for F, we write

$$(\mu_k mg)r = \tfrac{2}{5}mr^2\alpha \qquad \alpha = \frac{5}{2}\frac{\mu_k g}{r}$$

Kinematics of Motion. As long as the sphere both rotates and slides, its linear and angular motions are uniformly accelerated.

$$t = 0, \bar{v} = \bar{v}_0 \qquad \bar{v} = \bar{v}_0 + \bar{a}t = \bar{v}_0 - \mu_k g t \qquad (1)$$

$$t = 0, \omega_0 = 0 \qquad \omega = \omega_0 + \alpha t = 0 + \left(\frac{5}{2}\frac{\mu_k g}{r}\right)t \qquad (2)$$



The sphere will start rolling without sliding when the velocity $\mathbf{v}_C$ of the point of contact C is zero. At that time, $t = t_1$, point C becomes the instantaneous center of rotation, and we have

$$\bar{v}_1 = r\omega_1 \qquad (3)$$

Substituting in (3) the values obtained for $\bar{v}_1$ and ω_1 by making $t = t_1$ in (1) and (2), respectively, we write

$$\bar{v}_0 - \mu_k g t_1 = r\left(\frac{5}{2}\frac{\mu_k g}{r}t_1\right) \qquad t_1 = \frac{2}{7}\frac{\bar{v}_0}{\mu_k g} \quad ◀$$

Substituting for t_1 into (2), we have

$$\omega_1 = \frac{5}{2}\frac{\mu_k g}{r}t_1 = \frac{5}{2}\frac{\mu_k g}{r}\left(\frac{2}{7}\frac{\bar{v}_0}{\mu_k g}\right) \qquad \omega_1 = \frac{5}{7}\frac{\bar{v}_0}{r} \qquad \boldsymbol{\omega}_1 = \frac{5}{7}\frac{\bar{v}_0}{r} \;↻ \quad ◀$$

$$\bar{v}_1 = r\omega_1 = r\left(\frac{5}{7}\frac{\bar{v}_0}{r}\right) \qquad \bar{v}_1 = \tfrac{5}{7}\bar{v}_0 \qquad \bar{\mathbf{v}}_1 = \tfrac{5}{7}\bar{v}_0 \rightarrow \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

This chapter deals with the *plane motion* of rigid bodies, and in this first lesson we considered rigid bodies that are free to move under the action of applied forces.

1. Effective forces. We first recalled that a rigid body consists of a large number of particles. The effective forces of the particles forming the body were found to be equivalent to a vector $m\bar{\mathbf{a}}$ attached at the mass center G of the body and a couple of moment $\bar{I}\boldsymbol{\alpha}$ [Fig. 16.7]. Noting that the applied forces are equivalent to the effective forces, we wrote

$$\Sigma F_x = m\bar{a}_x \qquad \Sigma F_y = m\bar{a}_y \qquad \Sigma M_G = \bar{I}\alpha \tag{16.5}$$

where $\bar{a}_x$ and $\bar{a}_y$ are the x and y components of the acceleration of the mass center G of the body and $\boldsymbol{\alpha}$ is the angular acceleration of the body. It is important to note that when these equations are used, *the moments of the applied forces must be computed with respect to the mass center of the body*. However, you learned a more efficient method of solution based on the use of a free-body-diagram equation.

2. Free-body-diagram equation. Your first step in the solution of a problem should be to draw a

a. A free-body- ...epresenting two equivalent systems o... show *the forces exerted on the body*, including the applied forces, the reactions at the supports, and the weight of the body. *In the second diagram* you should show the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}\boldsymbol{\alpha}$ representing *the effective forces*.

b. Using a free-body-diagram equation allows you to *sum components in any direction and to sum moments about any point.* When writing the three equations of motion needed to solve a given problem, you can therefore select one or more equations involving a single unknown. Solving these equations first and substituting the values obtained for the unknowns into the remaining equation(s) will yield a simpler solution.

3. Plane motion of a rigid body. The problems that you will be asked to solve will fall into one of the following categories.

a. Rigid body in translation. For a body in translation, the angular acceleration is zero. The effective forces reduce to *the vector* $m\bar{\mathbf{a}}$ applied at the mass center [Sample Probs. 16.1 and 16.2].

b. Rigid body in centroidal rotation. For a body in centroidal rotation, the acceleration of the mass center is zero. The effective forces reduce to *the couple* $\bar{I}\alpha$ [Sample Prob. 16.3].

c. Rigid body in general plane motion. You can consider the general plane motion of a rigid body as the sum of a translation and a centroidal rotation. The effective forces are equivalent to the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}\alpha$ [Sample Probs. 16.4 and 16.5].

4. Plane motion of a system of rigid bodies. You first should draw a free-body-diagram equation that includes all the rigid bodies of the system. A vector $m\bar{\mathbf{a}}$ and a couple $\bar{I}\alpha$ are attached to each body. However, the forces exerted on each other by the various bodies of the system can be omitted, since they occur in pairs of equal and opposite forces.

a. If no mo[illegible]ou can use this free-body-diagram e[illegible]n and sum moments about any poin[illegible]he desired unknowns [Sample Prob. [illegible]].

b. If more than three unknowns are involved, you must draw a separate free-body-diagram equation for each of the rigid bodies of the system. Both internal forces and external forces should be included in each of the free-body-diagram equations, and care should be taken to represent with equal and opposite vectors the forces that two bodies exert on each other.

PROBLEMS

CONCEPT QUESTIONS

16.CQ1 Two pendulums, A and B, with the masses and lengths shown are released from rest. Which system has a larger mass moment of inertia about its pivot point?

a. A

b. B

c. They are the same.

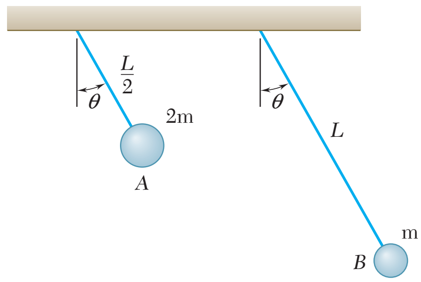


Fig. P16.CQ1 and P16.CQ2

the masses and lengths shown are
m has a larger angular acceleration

a. A

b. B

c. They are the same.

16.CQ3 Two solid cylinders, A and B, have the same mass m and the radii $2r$ and r, respectively. Each is accelerated from rest with a force applied as shown. In order to impart identical angular accelerations to both cylinders, what is the relationship between F_1 and F_2?

a. $F_1 = 0.5F_2$

b. $F_1 = F_2$

c. $F_1 = 2F_2$

d. $F_1 = 4F_2$

e. $F_1 = 8F_2$

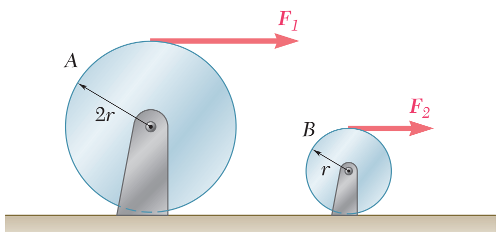


Fig. P16.CQ3

FREE BODY PRACTICE PROBLEMS

16.F1 A 6-ft board is placed in a truck with one end resting against a block secured to the floor and the other leaning against a vertical partition. Draw the FBD and KD necessary to determine the maximum allowable acceleration of the truck if the board is to remain in the position shown.

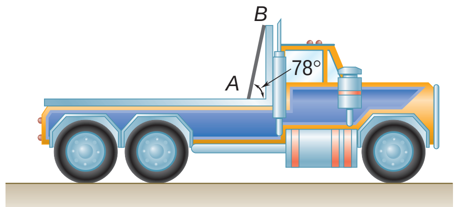


Fig. P16.F1

16.F2 A uniform circular plate of mass 3 kg is attached to two links *AC* and *BD* of the same length. Knowing that the plate is released from rest in the position shown, in which lines joining *G* to *A* and *B* are, respectively, horizontal and vertical, draw the FBD and KD for the plate.

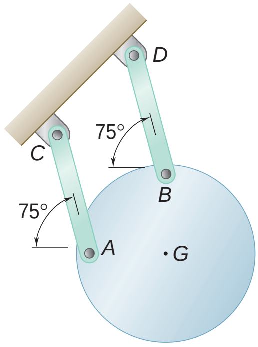


Fig. P16.F2

16.F3 Two uniform disks and
Disk *A* weighs 20 lb a
system is released from
system.

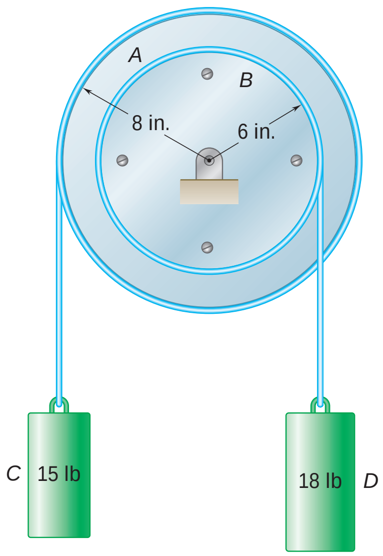


Fig. P16.F3

16.F4 The 400-lb crate shown is lowered by means of two overhead cranes. Knowing the tension in each cable, draw the FBD and KD that can be used to determine the angular acceleration of the crate and the acceleration of the center of gravity.

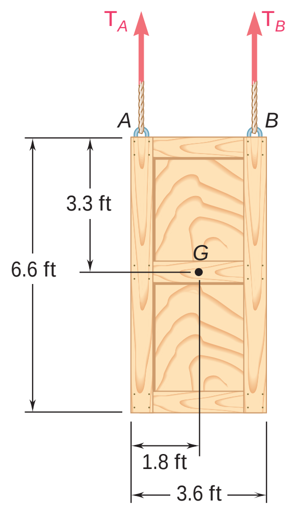


Fig. P16.F4

END-OF-SECTION PROBLEMS

16.1 A conveyor system is fitted with vertical panels, and a 15-in. rod AB weighing 5 lb is lodged between two panels as shown. If the rod is to remain in the position shown, determine the maximum allowable acceleration of the system.

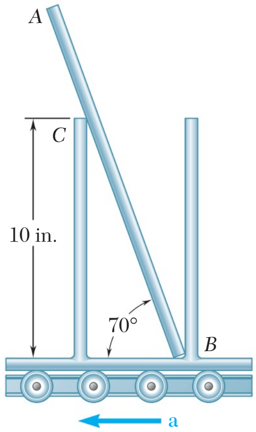


Fig. P16.1 and P16.2

16.2 A conveyor system is fitted with vertical panels, and a 15-in. rod AB weighing 5 lb is lodged between two panels as shown. Knowing that the acceleration of the system is 3 ft/s^2 to the left, determine ... t C, (b) the reaction at B.

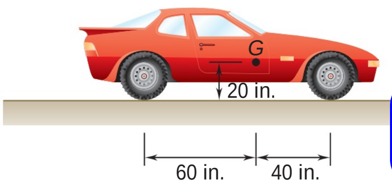


Fig. P16.3

... atic friction between the tires and the road is 0.80 for the automobile shown, determine the maximum possible acceleration on a level road, assuming (a) four-wheel drive, (b) rear-wheel drive, (c) front-wheel drive.

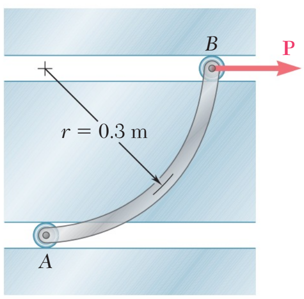


Fig. P16.4

16.4 The motion of the 2.5-kg rod AB is guided by two small wheels which roll freely in horizontal slots. If a force $\mathbf{P}$ of magnitude 8 N is applied at B, determine (a) the acceleration of the rod, (b) the reactions at A and B.

16.5 A uniform rod BC of mass 4 kg is connected to a collar A by a 250-mm cord AB. Neglecting the mass of the collar and cord, determine (a) the smallest constant acceleration $\mathbf{a}_A$ for which the cord and the rod will lie in a straight line, (b) the corresponding tension in the cord.

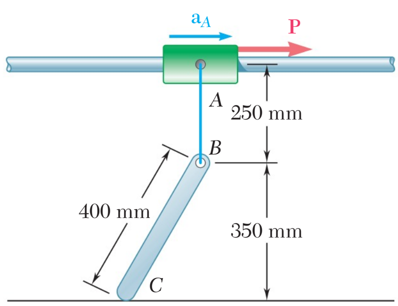


Fig. P16.5

16.6 A 2000-kg truck is being used to lift a 400-kg boulder *B* that is on a 50-kg pallet *A*. Knowing the acceleration of the rear-wheel-drive truck is 1 m/s^2, determine (*a*) the reaction at each of the front wheels, (*b*) the force between the boulder and the pallet.

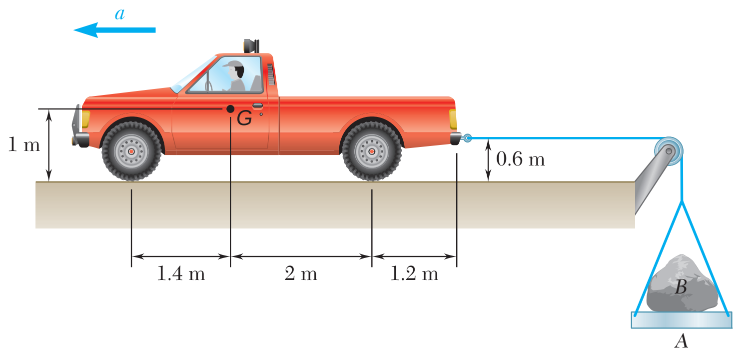


Fig. P16.6

16.7 The support bracket shown is used to transport a cylindrical can from one elevation to another. Knowing that $\mu_s = 0.25$ between the can and the bracket, determine (*a*) the magnitude of the upward acceleration **a** for which the can will slide on the bracket, (*b*) the smallest ratio *h*/*d* for which the can will tip before it slides.

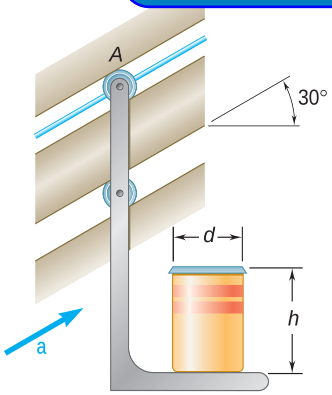


Fig. *P16.7*

16.8 Solve Prob. 16.7, assuming that the acceleration **a** of the bracket is directed downward.

16.9 A 20-kg cabinet is mounted on casters that allow it to move freely ($\mu = 0$) on the floor. If a 100-N force is applied as shown, determine (*a*) the acceleration of the cabinet, (*b*) the range of values of *h* for which the cabinet will not tip.

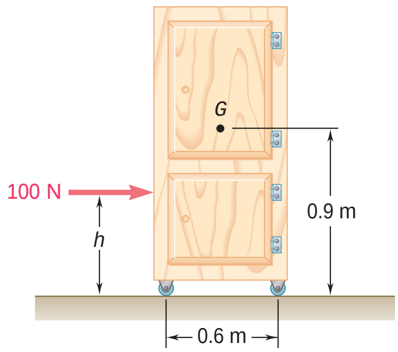


Fig. P16.9

16.10 Solve Prob. 16.9, assuming that the casters are locked and slide on the rough floor ($\mu_k = 0.25$).

16.11 A completely filled barrel and its contents have a combined mass of 90 kg. A cylinder *C* is connected to the barrel at a height $h = 550$ mm as shown. Knowing $\mu_s = 0.40$ and $\mu_k = 0.35$, determine the maximum mass of *C* so the barrel will not tip.

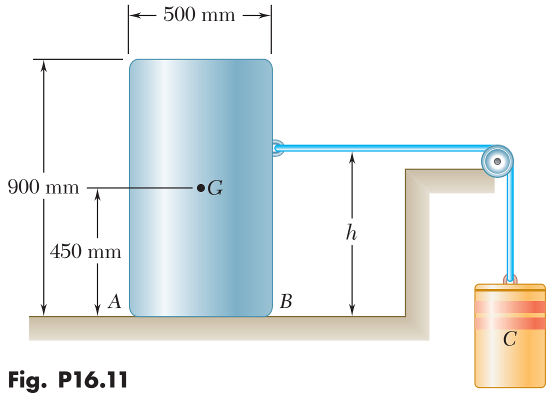


Fig. P16.11

16.12 A 40-kg vase has a 200-mm-diameter base and is being moved using a 100-kg utility cart as shown. The cart moves freely ($\mu = 0$) on the ground. Knowing the coefficient of static friction between the vase and the cart is $\mu_s = 0.4$, determine the maximum force **F** that can be applied if the vase is not to slide or tip.

Fig. ***P16.12***

16.13 The retractable shelf shown is supported by two identical linkage-and-spring systems; only one of the systems is shown. A 20-kg machine is placed on the shelf so that half of its weight is supported by the system shown. If the springs are removed and the system is released from rest, determine (*a*) the acceleration of the machine, (*b*) the tension in link *AB*. Neglect the weight of the shelf and links.

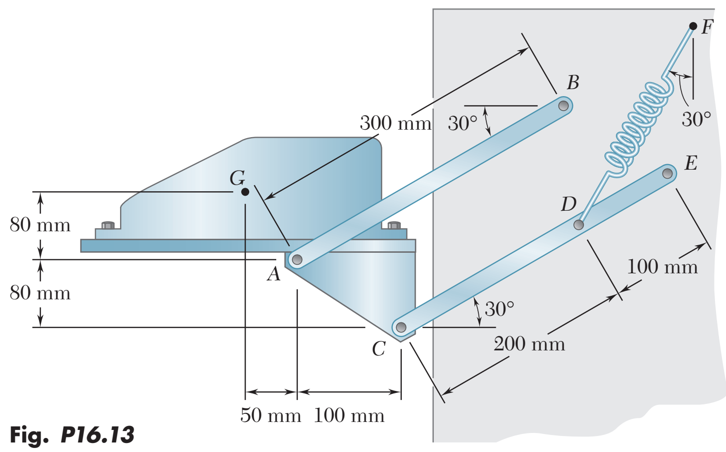


Fig. ***P16.13***

16.14 A uniform rectangular plate has a mass of 5 kg and is held in position by three ropes as shown. Knowing that u = 30°, determine, immediately after rope *CF* has been cut, (*a*) the acceleration of the plate, (*b*) the tension in ropes *AD* and *BE*.

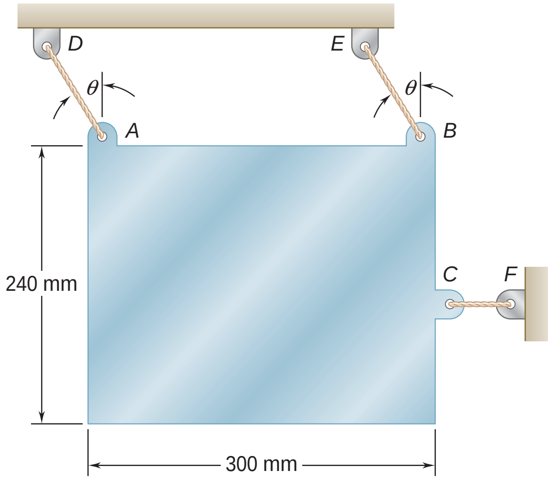


Fig. P16.14 and P16.15

16.15 A uniform rectangular plate has a mass of 5 kg and is held in position by three ropes as sh[illegible] which both ropes *AD* an[illegible] *CF* has been cut.

16.16 Three bars, each of mass 3 kg, are welded together and pin-connected to two links *BE* and *CF*. Neglecting the weight of the links, determine the force in each link immediately after the system is released from rest.

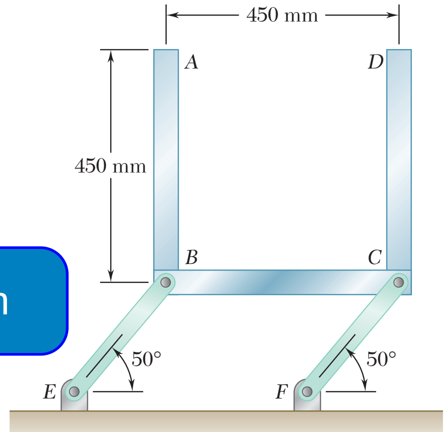


Fig. *P16.16*

16.17 Members *ACE* and *DCB* are each 600 mm long and are connected by a pin at *C*. The mass center of the 10-kg member *AB* is located at *G*. Determine (*a*) the acceleration of *AB* immediately after the system has been released from rest in the position shown, (*b*) the corresponding force exerted by roller *A* on member *AB*. Neglect the weight of members *ACE* and *DCB*.

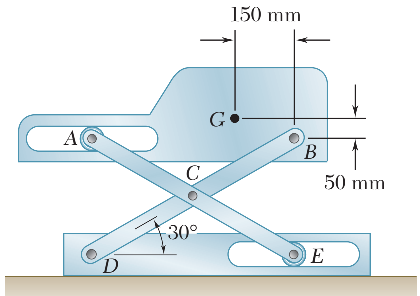


Fig. *P16.17*

16.18 The 15-lb rod *BC* connects a disk centered at *A* to crank *CD*. Knowing that the disk is made to rotate at the constant speed of 180 rpm, determine for the position shown the vertical components of the forces exerted on rod *BC* by pins at *B* and *C*.

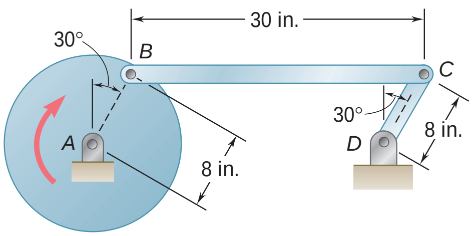


Fig. P16.18

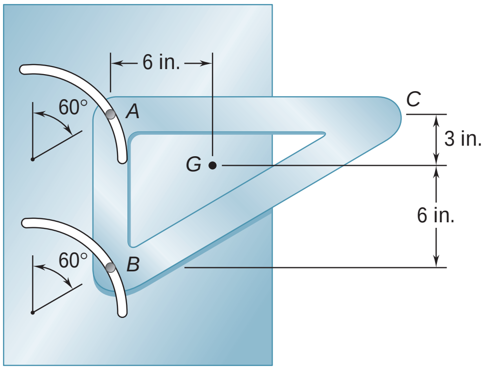


Fig. P16.19

16.19 The triangular weldment ABC is guided by two pins that slide freely in parallel curved slots of radius 6 in. cut in a vertical plate. The weldment weighs 16 lb and its mass center is located at point G. Knowing that at the instant shown the velocity of each pin is 30 in./s downward along the slots, determine (*a*) the acceleration of the weldment, (*b*) the reactions at A and B.

16.20 The coefficients of friction between the 30-lb block and the 5-lb platform BD are $\mu_s = 0.50$ and $\mu_k = 0.40$. Determine the accelerations of the block and of the platform immediately after wire AB has been cut.

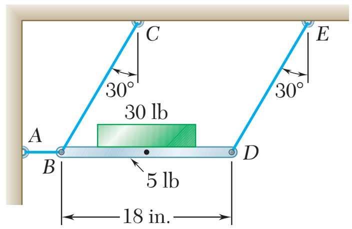


Fig. P16.20

16.21 Draw the shear and bending-moment diagrams for the vertical rod AB of Prob. 16.16.

ment diagrams for the connecting

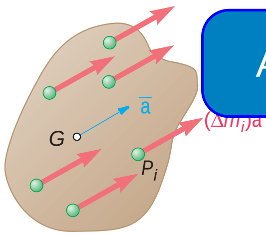


Fig. P16.23

16.23 For a rigid slab in translation, show that the system of the effective forces consists of vectors $(\Delta m_i)\bar{\mathbf{a}}$ attached to the various particles of the slab, where $\bar{\mathbf{a}}$ is the acceleration of the mass center G of the slab. Further show, by computing their sum and the sum of their moments about G, that the effective forces reduce to a single vector $m\bar{\mathbf{a}}$ attached at G.

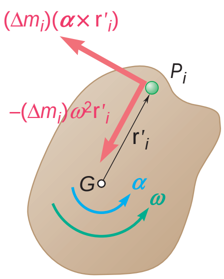


Fig. P16.24

16.24 For a rigid slab in centroidal rotation, show that the system of the effective forces consists of vectors $-(\Delta m_i)\omega^2\mathbf{r}'_i$ and $(\Delta m_i)(\boldsymbol{\alpha} \times \mathbf{r}'_i)$ attached to the various particles P_i of the slab, where $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are the angular velocity and angular acceleration of the slab, and where $\mathbf{r}'_i$ denotes the position vector of the particle P_i relative to the mass center G of the slab. Further show, by computing their sum and the sum of their moments about G, that the effective forces reduce to a couple $\bar{I}\boldsymbol{\alpha}$.

16.25 The rotor of an electric motor has an angular velocity of 3600 rpm when the load and power are cut off. The 50-kg rotor, which has a centroidal radius of gyration of 180 mm, then coasts to rest. Knowing that kinetic friction results in a couple of magnitude 3.5 N · m exerted on the rotor, determine the number of revolutions that the rotor executes before coming to rest.

16.26 It takes 10 min for a 6000-lb flywheel to coast to rest from an angular velocity of 300 rpm. Knowing that the radius of gyration of the flywheel is 36 in., determine the average magnitude of the couple due to kinetic friction in the bearings.

16.27 The 8-in.-radius brake drum is attached to a larger flywheel that is not shown. The total mass moment of inertia of the drum and the flywheel is 14 lb · ft · s^2 and the coefficient of kinetic friction between the drum and the brake shoe is 0.35. Knowing that the angular velocity of the flywheel is 360 rpm counterclockwise when a force **P** of magnitude 75 lb is applied to the pedal C, determine the number of revolutions executed by the flywheel before it comes to rest.

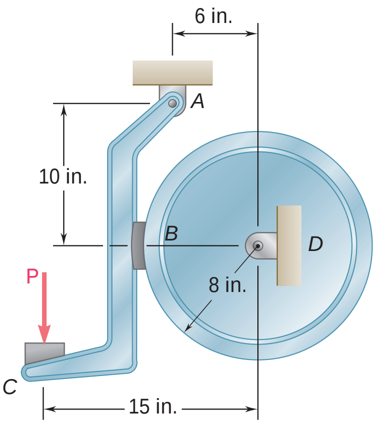


Fig. P16.27

16.28 Solve Prob. 16.27, assuming that the initial angular velocity of the flywheel is 360 rpm clockwise.

16.29 The 100-mm-radius brake drum is attached to a flywheel which is not shown. The drum and flywheel together have a mass of 300 kg and a radius of gyration of 600 mm. The coefficient of kinetic friction between the brake band and the drum is 0.30. Knowing that a force **P** of magnitude 50 N is applied at A when the angular velocity is 180 rpm counterclockwise, determine the time required to stop the flywheel when a = 200 mm and b = 160 mm.

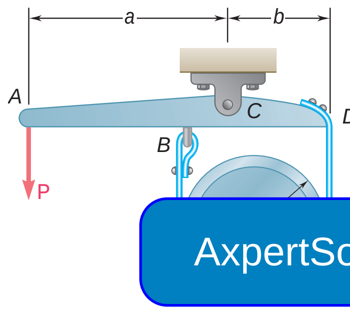


Fig. P16.29

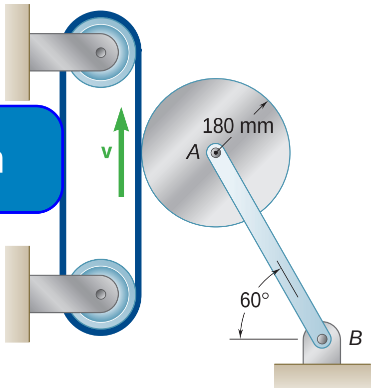


Fig. P16.30

16.30 The 180-mm-radius disk is at rest when it is placed in contact with a belt moving at a constant speed. Neglecting the weight of the link AB and knowing that the coefficient of kinetic friction between the disk and the belt is 0.40, determine the angular acceleration of the disk while slipping occurs.

16.31 Solve Prob. 16.30, assuming that the direction of motion of the belt is reversed.

16.32 In order to determine the mass moment of inertia of a flywheel of radius 600 mm, a 12-kg block is attached to a wire that is wrapped around the flywheel. The block is released and is observed to fall 3 m in 4.6 s. To eliminate bearing friction from the computation, a second block of mass 24 kg is used and is observed to fall 3 m in 3.1 s. Assuming that the moment of the couple due to friction remains constant, determine the mass moment of inertia of the flywheel.

16.33 The flywheel shown has a radius of 20 in., a weight of 250 lb, and a radius of gyration of 15 in. A 30-lb block A is attached to a wire that is wrapped around the flywheel, and the system is released from rest. Neglecting the effect of friction, determine (*a*) the acceleration of block A, (*b*) the speed of block A after it has moved 5 ft.

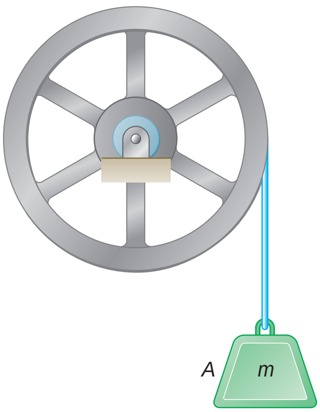


Fig. *P16.32* and P16.33

16.34 Each of the double pulleys shown has a mass moment of inertia of 15 lb · ft · s^2 and is initially at rest. The outside radius is 18 in., and the inner radius is 9 in. Determine (*a*) the angular acceleration of each pulley, (*b*) the angular velocity of each pulley after point *A* on the cord has moved 10 ft.

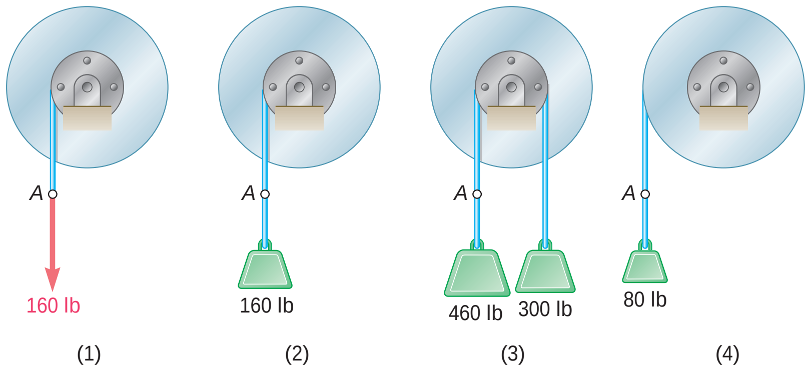


Fig. P16.34

16.35 Each of the gears *A* and *B* has a mass of 9 kg and has a radius of gyration of 200 mm; gear *C* has a mass of 3 kg and has a radius of gyration of 75 mm. If a couple **M** of constant magnitude 5 N-m is applied to gear *C*, determine (*a*) the angular acceleration of gear *A*, (*b*) the tangential force which gear *C* exerts on gear *A*.

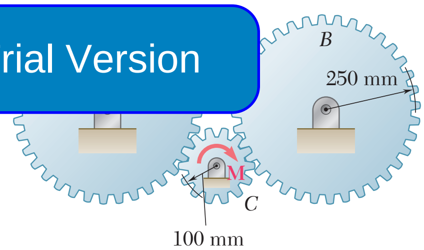


Fig. *P16.35*

16.36 Solve Prob. 16.35, assuming that the couple **M** is applied to disk *A*.

16.37 Gear *A* weighs 1 lb and has a radius of gyration of 1.3 in; gear *B* weighs 6 lb and has a radius of gyration of 3 in.; gear *C* weighs 9 lb and has a radius of gyration of 4.3 in. Knowing a couple **M** of constant magnitude of 40 lb · in is applied to gear *A*, determine (*a*) the angular acceleration of gear *C*, (*b*) the tangential force which gear *B* exerts on gear *C*.

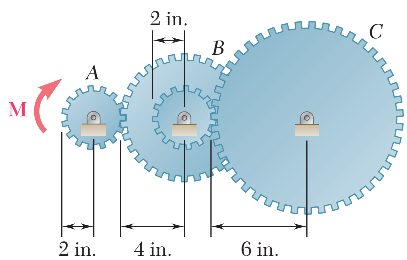


Fig. *P16.37*

16.38 Disks A and B are bolted together, and cylinders D and E are attached to separate cords wrapped on the disks. A single cord passes over disks B and C. Disk A weighs 20 lb and disks B and C each weigh 12 lb. Knowing that the system is released from rest and that no slipping occurs between the cords and the disks, determine the acceleration (*a*) of cylinder D, (*b*) of cylinder E.

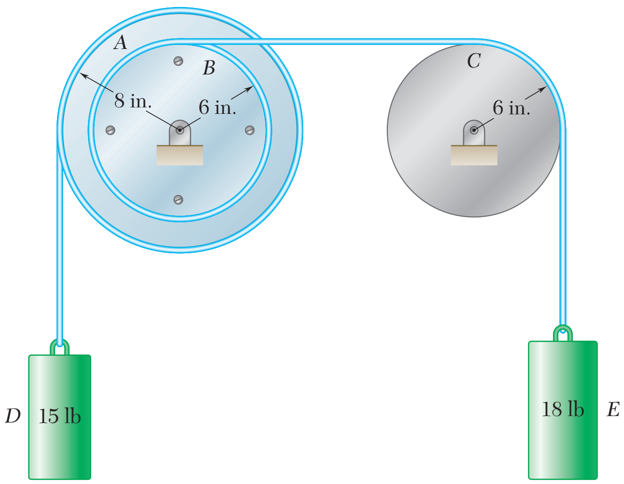


Fig. *P16.38*

16.39 A belt of negligible mass [illegible] pulled to the right with a [illegible] tively, 5 and 20 lb. The sh[illegible] slot and the coefficients o[illegible] cylinders are $\mu_s = 0.50$ and $\mu_k = 0.40$. For $P = 3.6$ lb, determine (*a*) whether slipping occurs between the belt and either cylinder, (*b*) the angular acceleration of each cylinder.

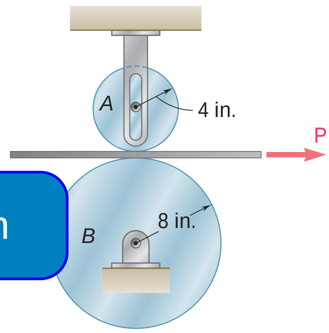


Fig. P16.39

16.40 Solve Prob. 16.39 for $P = 2.00$ lb.

16.41 Disk A has a mass of 6 kg and an initial angular velocity of 360 rpm clockwise; disk B has a mass of 3 kg and is initially at rest. The disks are brought together by applying a horizontal force of magnitude 20 N to the axle of disk A. Knowing that $\mu_k = 0.15$ between the disks and neglecting bearing friction, determine (*a*) the angular acceleration of each disk, (*b*) the final angular velocity of each disk.

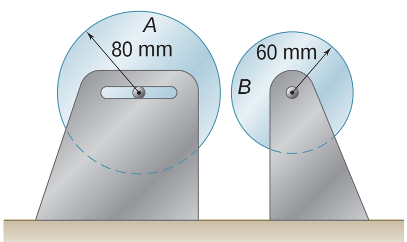


Fig. P16.41

16.42 Solve Prob. 16.41, assuming that initially disk A is at rest and disk B has an angular velocity of 360 rpm clockwise.

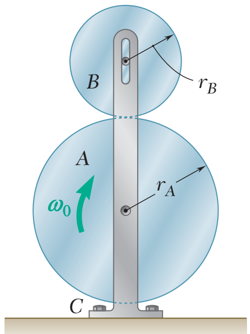


Fig. P16.43 and P16.44

16.43 Disk A has a mass $m_A = 4$ kg, a radius $r_A = 300$ mm, and an initial angular velocity $V_0 = 300$ rpm clockwise. Disk B has a mass $m_B = 1.6$ kg, a radius $r_B = 180$ mm, and is at rest when it is brought into contact with disk A. Knowing that $m_k = 0.35$ between the disks and neglecting bearing friction, determine (a) the angular acceleration of each disk, (b) the reaction at the support C.

16.44 Disk B is at rest when it is brought into contact with disk A, which has an initial angular velocity V_0. (a) Show that the final angular velocities of the disks are independent of the coefficient of friction m_k between the disks as long as $m_k \neq 0$. (b) Express the final angular velocity of disk A in terms of V_0 and the ratio of the masses of the two disks m_A/m_B.

16.45 Cylinder A has an initial angular velocity of 720 rpm clockwise, and cylinders B and C are initially at rest. Disks A and B each weigh 5 lb and have radius $r = 4$ in. Disk C weighs 20 lb and has a radius of 8 in. The disks are brought together when C is placed gently onto A and B. Knowing that $m_k = 0.25$ between A and C and no slipping occurs between B and C, determine (a) the angular acceleration of each disk, (b) the final angular velocity of each disk.

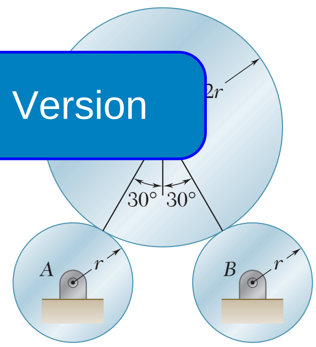


Fig. *P16.45*

16.46 Show that the system of the effective forces for a rigid slab in plane motion reduces to a single vector, and express the distance from the mass center G of the slab to the line of action of this vector in terms of the centroidal radius of gyration $\bar{k}$ of the slab, the magnitude $\bar{a}$ of the acceleration of G, and the angular acceleration a.

16.47 For a rigid slab in plane motion, show that the system of the effective forces consists of vectors $(\Delta m_i)\bar{\mathbf{a}}$, $-(\Delta m_i)V^2\mathbf{r}'_i$, and $(\Delta m_i)(A \times \mathbf{r}'_i)$ attached to the various particles P_i of the slab, where $\bar{\mathbf{a}}$ is the acceleration of the mass center G of the slab, V is the angular velocity of the slab, A is its angular acceleration, and $\mathbf{r}'_i$ denotes the position vector of the particle P_i, relative to G. Further show, by computing their sum and the sum of their moments about G, that the effective forces reduce to a vector $m\bar{\mathbf{a}}$ attached at G and a couple $\bar{I}$A.

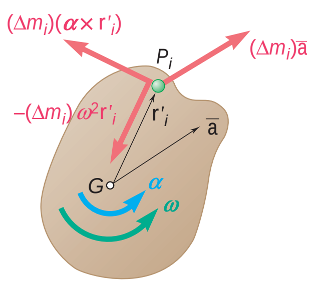


Fig. P16.47

16.48 A uniform slender rod AB rests on a frictionless horizontal surface, and a force **P** of magnitude 0.25 lb is applied at A in a direction perpendicular to the rod. Knowing that the rod weighs 1.75 lb, determine (a) the acceleration of point A, (b) the acceleration of point B, (c) the location of the point on the bar that has zero acceleration.

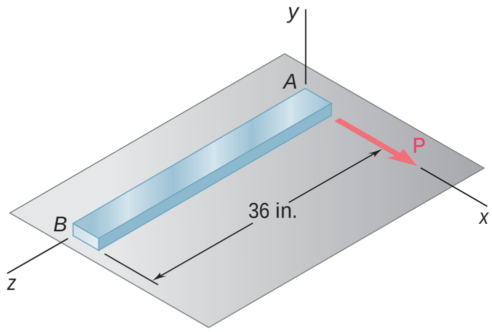


Fig. P16.48

16.49 (a) In Prob. 16.48, determine the point of the rod AB at which the force **P** should be applied if the acceleration of point B is to be zero. (b) Knowing that $P = 0.25$ lb, determine the corresponding acceleration of point A.

16.50 A force **P** of magnitude 3 N is applied to a tape wrapped around a thin hoop of mass 2.4 kg. Knowing that the body rests on a frictionless horizontal surface, determine the acceleration of (a) point A, (b) point B.

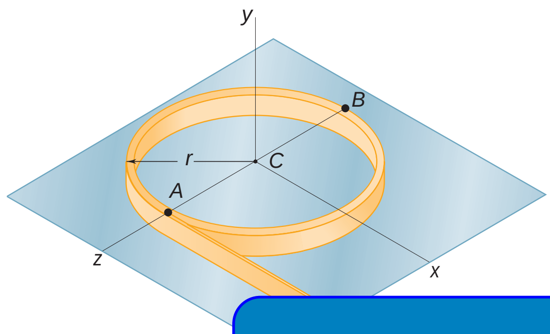


Fig. P16.50

16.51 A force **P** is applied to a tape wrapped around a uniform disk that rests on a frictionless horizontal surface. Show that for each 360° rotation of the disk the center of the disk will move a distance pr.

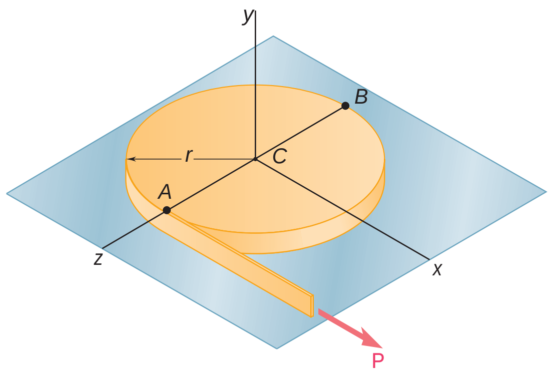


Fig. P16.51

16.52 A 250-lb satellite has a radius of gyration of 24 in. with respect to the y axis and is symmetrical with respect to the zx plane. Its orientation is changed by firing four small rockets A, B, C, and D, each of which produces a 4-lb thrust **T** directed as shown. Determine the angular acceleration of the satellite and the acceleration of its mass center G (a) when all four rockets are fired, (b) when all rockets except D are fired.

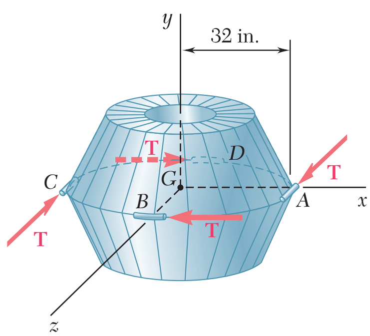


Fig. P16.52

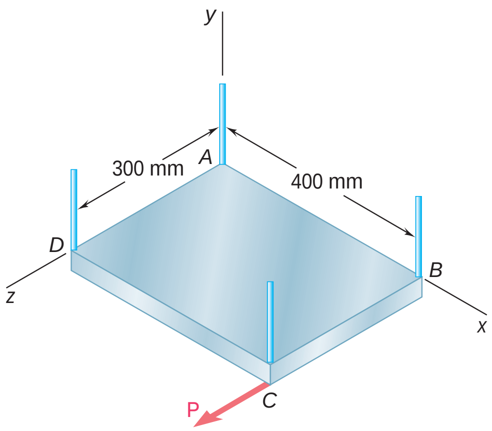


Fig. *P16.53*

16.53 A rectangular plate of mass 5 kg is suspended from four vertical wires, and a force **P** of magnitude 6 N is applied to corner C as shown. Immediately after **P** is applied, determine the acceleration of (a) the midpoint of edge BC, (b) corner B.

16.54 A uniform slender L-shaped bar ABC is at rest on a horizontal surface when a force **P** of magnitude 4 N is applied at point A. Neglecting friction between the bar and the surface and knowing that the mass of the bar is 2 kg, determine (a) the initial angular acceleration of the bar, (b) the initial acceleration of point B.

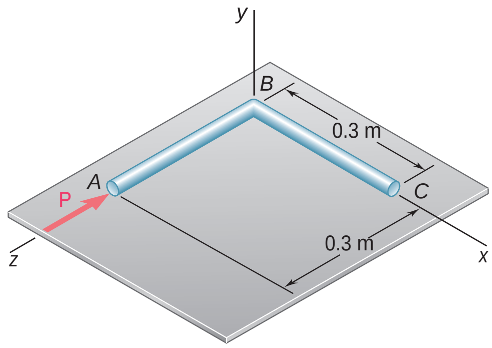


Fig. *P16.54*

16.55 By pulling on the string of a yo-yo, a person manages to make the yo-yo spin, while remaining at the same elevation above the floor. [illegible] by m, the radius of the inner drum [illegible] r, and the centroidal radius of [illegible] rmine the angular acceleration of the yo-yo.

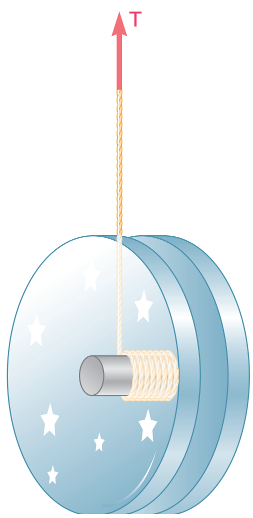


Fig. P16.55 and P16.56

16.56 The 80-g yo-yo shown has a centroidal radius of gyration of 30 mm. The radius of the inner drum on which a string is wound is 6 mm. Knowing that at the instant shown the acceleration of the center of the yo-yo is 1 m/s^2 upward, determine (a) the required tension **T** in the string, (b) the corresponding angular acceleration of the yo-yo.

16.57 A 6-lb sprocket wheel has a centroidal radius of gyration of 2.75 in. and is suspended from a chain as shown. Determine the acceleration of points A and B of the chain, knowing that $T_A = 3$ lb and $T_B = 4$ lb.

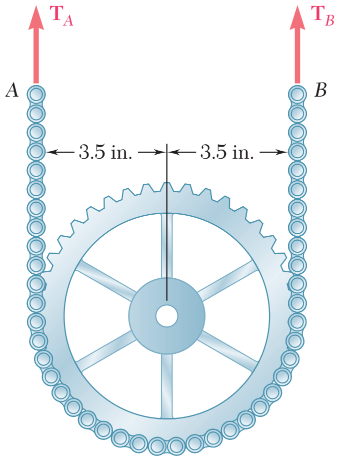


Fig. P16.57

16.58 The steel roll shown has a mass of 1200 kg, a centroidal radius of gyration of 150 mm, and is lifted by two cables looped around its shaft. Knowing that for each cable $T_A = 3100$ N and $T_B = 3300$ N, determine (*a*) the angular acceleration of the roll, (*b*) the acceleration of its mass center.

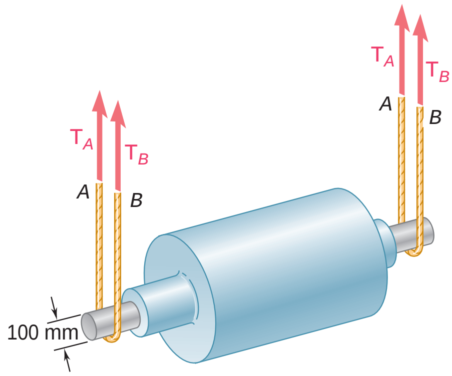


Fig. P16.58 and P16.59

16.59 The steel roll shown has [illegible] of gyration of 150 mm, and is lifted by two cables looped around its shaft. Knowing that at the instant shown the acceleration of the roll is 150 mm/s^2 downward and that for each cable $T_A = 3000$ N, determine (*a*) the corresponding tension T_B, (*b*) the angular acceleration of the roll.

16.60 and *16.61* A 15-ft beam weighing 500 lb is lowered by means of two cables unwinding from overhead cranes. As the beam approaches the ground, the crane operators apply brakes to slow the unwinding motion. Knowing that the deceleration of cable A is 20 ft/s^2 and the deceleration of cable B is 2 ft/s^2, determine the tension in each cable.

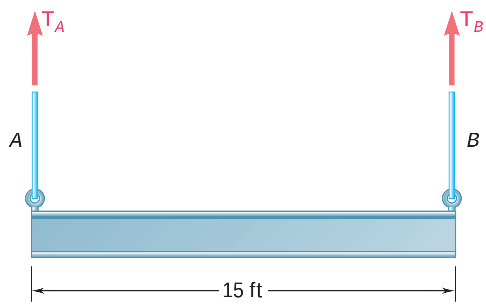


Fig. P16.60

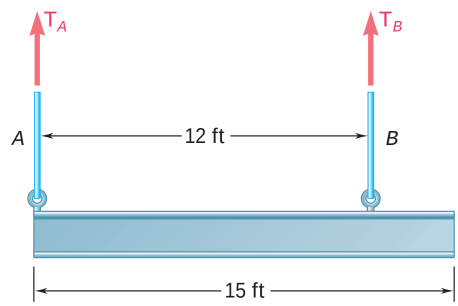


Fig. *P16.61*

16.62 Two uniform cylinders, each of weight $W = 14$ lb and radius $r = 5$ in., are connected by a belt as shown. If the system is released from rest, determine (*a*) the angular acceleration of each cylinder, (*b*) the tension in the portion of belt connecting the two cylinders, (*c*) the velocity of the center of the cylinder *A* after it has moved through 3 ft.

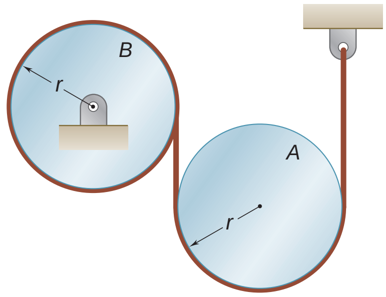


Fig. P16.62

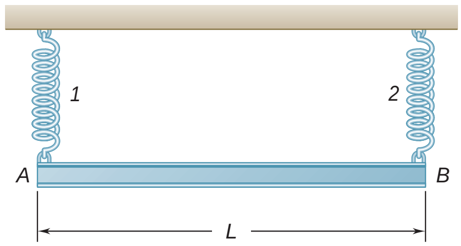


Fig. P16.63

16.63 through 16.65 A beam *AB* of mass *m* and of uniform cross section is suspended from two springs as shown. If spring 2 breaks, determine at that instant (*a*) the angular acceleration of the bar, (*b*) the acceleration of point *A*, (*c*) the acceleration of point *B*.

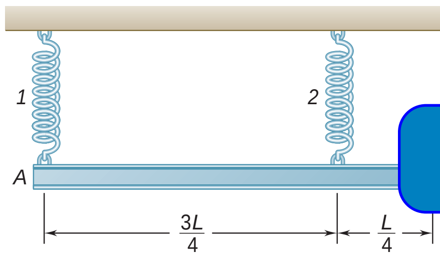


Fig. P16.64

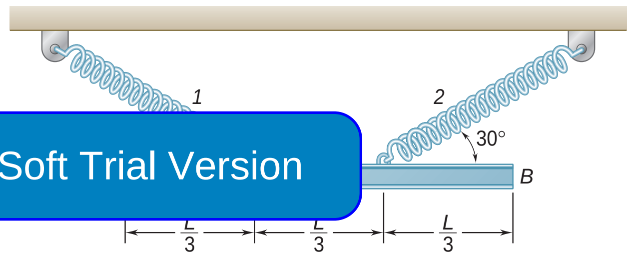


Fig. P16.65

16.66 through 16.68 A thin plate of the shape indicated and of mass *m* is suspended from two springs as shown. If spring 2 breaks, determine the acceleration at that instant (*a*) of point *A*, (*b*) of point *B*.

16.66 A square plate of side *b*
16.67 A circular plate of diameter *b*
16.68 A rectangular plate of height *b* and width *a*

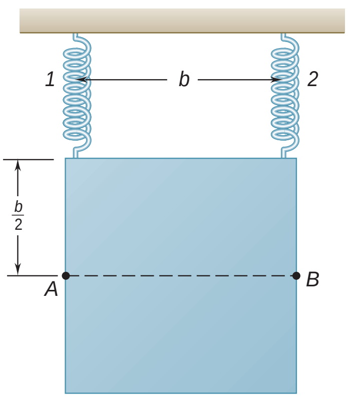


Fig. P16.66

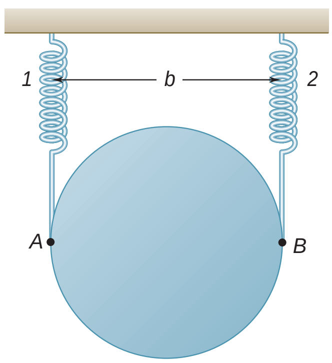


Fig. P16.67

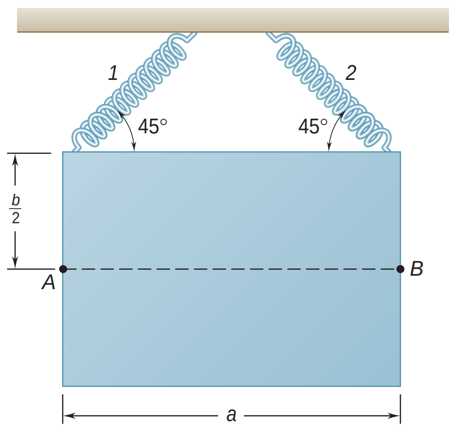


Fig. P16.68

16.69 A sphere of radius r and mass m is projected along a rough horizontal surface with the initial velocities indicated. If the final velocity of the sphere is to be zero, express, in terms of v_0, r, and μ_k, (*a*) the required magnitude of ω_0, (*b*) the time t_1 required for the sphere to come to rest, (*c*) the distance the sphere will move before coming to rest.

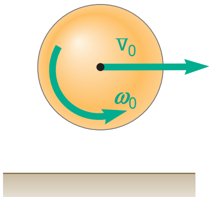


Fig. P16.69

16.70 Solve Prob. 16.69, assuming that the sphere is replaced by a uniform thin hoop of radius r and mass m.

16.71 A bowler projects an 8-in.-diameter ball weighing 12 lb along an alley with a forward velocity $\mathbf{v}_0$ of 15 ft/s and a backspin $\boldsymbol{\omega}_0$ of 9 rad/s. Knowing that the coefficient of kinetic friction between the ball and the alley is 0.10, determine (*a*) the time t_1 at which the ball will start rolling without sliding, (*b*) the speed of the ball at time t_1, (*c*) the distance the ball will have traveled at time t_1.

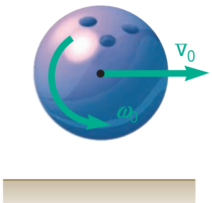


Fig. P16.71

16.72 Solve Prob. 16.71, assuming that the bowler projects the ball with the same forward velocity but with a backspin of 18 rad/s.

16.73 A uniform sphere of rad[illegible] velocity on a belt that m[illegible] $\mathbf{v}_1$. Denoting by μ_k the [illegible] sphere and the belt, determine (*a*) the time t_1 at which the sphere will start rolling without sliding, (*b*) the linear and angular velocities of the sphere at time t_1.

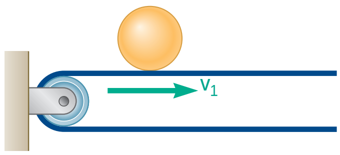


Fig. *P16.73*

16.74 A sphere of radius r and mass m has a linear velocity $\mathbf{v}_0$ directed to the left and no angular velocity as it is placed on a belt moving to the right with a constant velocity $\mathbf{v}_1$. If after first sliding on the belt the sphere is to have no linear velocity relative to the ground as it starts rolling on the belt without sliding, determine in terms of v_1 and the coefficient of kinetic friction μ_k between the sphere and the belt (*a*) the required value of v_0, (*b*) the time t_1 at which the sphere will start rolling on the belt, (*c*) the distance the sphere will have moved relative to the ground at time t_1.

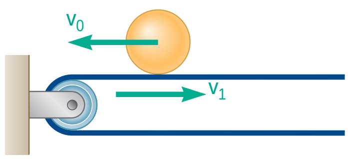


Fig. *P16.74*

16.8 CONSTRAINED PLANE MOTION

Most engineering applications deal with rigid bodies which are moving under given constraints. For example, cranks must rotate about a fixed axis, wheels must roll without sliding, and connecting rods must describe certain prescribed motions. In all such cases, definite relations exist between the components of the acceleration $\bar{\mathbf{a}}$ of the mass center G of the body considered and its angular acceleration A; the corresponding motion is said to be a *constrained motion*.

The solution of a problem involving a constrained plane motion calls first for a *kinematic analysis* of the problem. Consider, for example, a slender rod AB of length l and mass m whose extremities are connected to blocks of negligible mass which slide along horizontal and vertical frictionless tracks. The rod is pulled by a force $\mathbf{P}$ applied at A (Fig. 16.11). We know from Sec. 15.8 that the acceleration $\bar{\mathbf{a}}$ of the mass center G of the rod can be determined at any given instant from the position of the rod, its angular velocity, and its angular acceleration at that instant. Suppose, for example, that the values of u, v, and a are known at a given instant and that we wish to determine the corresponding value of the force $\mathbf{P}$, as well as the reactions at A and B. We should first *determine the components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of the mass center G* by the method of Sec. 15.8. We next apply d'Alembert's principle (Fig. 16.12), using the expressions obtained for $\bar{a}_x$ and $\bar{a}_y$. The unknown forces $\mathbf{P}$, $\mathbf{N}_A$, and $\mathbf{N}_B$ can then be determined by writing and solving the appropriate equations.

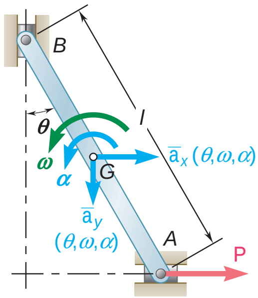


Fig. 16.11

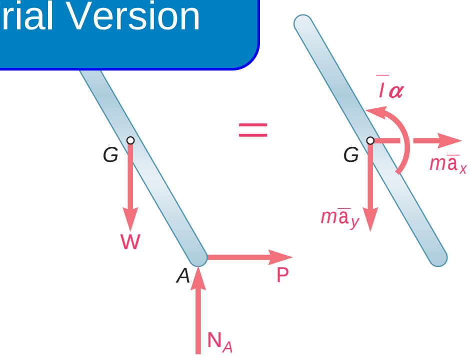


Fig. 16.12

Suppose now that the applied force $\mathbf{P}$, the angle u, and the angular velocity v of the rod are known at a given instant and that we wish to find the angular acceleration a of the rod and the components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of its mass center at that instant, as well as the reactions at A and B. The preliminary kinematic study of the problem will have for its object *to express the components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of G in terms of the angular acceleration* a *of the rod*. This will be done by first expressing the acceleration of a suitable reference point such as A in terms of the angular acceleration a. The components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of G can then be determined in terms of a, and the expressions obtained carried into Fig. 16.12. Three equations can then be derived in terms of a, N_A, and N_B and solved for the three unknowns (see Sample

Prob. 16.10). Note that the method of dynamic equilibrium can also be used to carry out the solution of the two types of problems we have considered (Fig. 16.13).

When a mechanism consists of *several moving parts,* the approach just described can be used with each part of the mechanism. The procedure required to determine the various unknowns is then similar to the procedure followed in the case of the equilibrium of a system of connected rigid bodies (Sec. 6.11).

Earlier, we analyzed two particular cases of constrained plane motion: the translation of a rigid body, in which the angular acceleration of the body is constrained to be zero, and the centroidal rotation, in which the acceleration $\bar{\mathbf{a}}$ of the mass center of the body is constrained to be zero. Two other particular cases of constrained plane motion are of special interest: the *noncentroidal rotation* of a rigid body and the *rolling motion* of a disk or wheel. These two cases can be analyzed by one of the general methods described above. However, in view of the range of their applications, they deserve a few special comments.

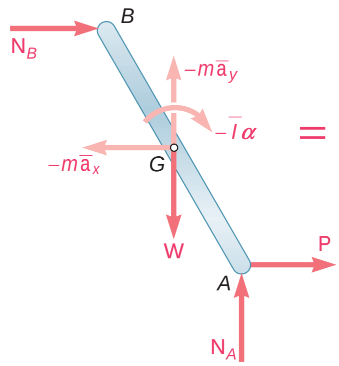


Fig. 16.13

Noncentroidal Rotation. The motion of a rigid body constrained to rotate about a fixed axis which does not pass through its mass center is called *noncentroidal rotation*. The mass center G of the body moves along a circle of radius $\bar{r}$ centered at the point O, where the axis of rotation intersects the plane of reference (Fig. 16.14). Denoting, respectively, by V and A the angular velocity and the angular acceleration of the line [illegible] sions for the tangential and no[illegible] of G:

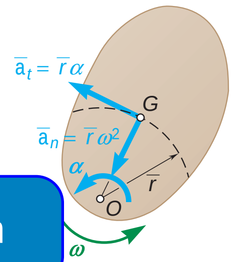


Fig. 16.14

$$\bar{a}_t = \bar{r}\text{a} \qquad \bar{a}_n = \bar{r}\text{v}^2 \tag{16.7}$$

Since line OG belongs to the body, its angular velocity V and its angular acceleration A also represent the angular velocity and the angular acceleration of the body in its motion relative to G. Equations (16.7) define, therefore, the kinematic relation existing between the motion of the mass center G and the motion of the body about G. They should be used to eliminate $\bar{a}_t$ and $\bar{a}_n$ from the equations obtained by applying d'Alembert's principle (Fig. 16.15) or the method of dynamic equilibrium (Fig. 16.16).

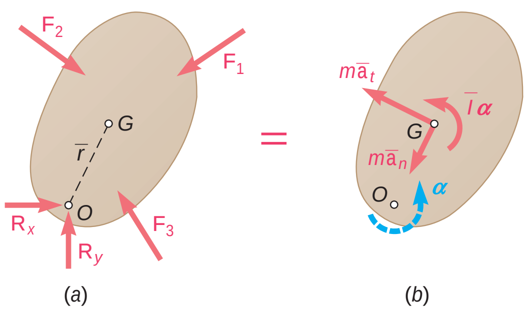


Fig. 16.15

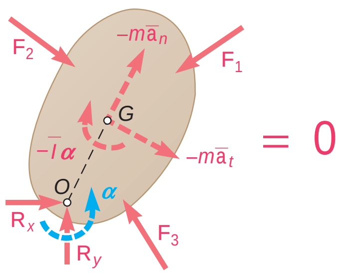


Fig. 16.16

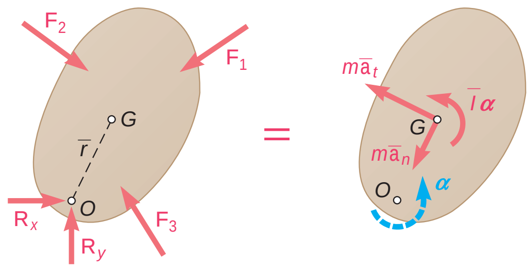


Fig. 16.15 (repeated)

An interesting relation is obtained by equating the moments about the fixed point O of the forces and vectors shown, respectively, in parts a and b of Fig. 16.15. We write

$$+\circlearrowleft \Sigma M_O = \bar{I}\alpha + (m\bar{r}\alpha)\bar{r} = (\bar{I} + m\bar{r}^2)\alpha$$

But according to the parallel-axis theorem, we have $\bar{I} + m\bar{r}^2 = I_O$, where I_O denotes the moment of inertia of the rigid body about the fixed axis. We therefore write

$$\Sigma M_O = I_O\alpha \qquad (16.8)$$

… mportant relation between the … forces about the fixed point O … early understood that this formula does not mean that the system of the external forces is equivalent to a couple of moment $I_O\alpha$. The system of the effective forces, and thus the system of the external forces, reduces to a couple only when O coincides with G—that is, *only when the rotation is centroidal* (Sec. 16.4). In the more general case of noncentroidal rotation, the system of the external forces does not reduce to a couple.

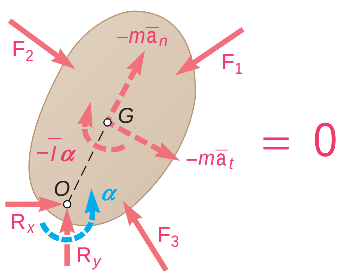


Fig. 16.16 (repeated)

A particular case of noncentroidal rotation is of special interest—the case of *uniform rotation*, in which the angular velocity ω is constant. Since α is zero, the inertia couple in Fig. 16.16 vanishes and the inertia vector reduces to its normal component. This component (also called *centrifugal force*) represents the tendency of the rigid body to break away from the axis of rotation.

Rolling Motion. Another important case of plane motion is the motion of a disk or wheel rolling on a plane surface. If the disk is constrained to roll without sliding, the acceleration $\bar{\mathbf{a}}$ of its mass center G and its angular acceleration α are not independent. Assuming that the disk is balanced, so that its mass center and its geometric center coincide, we first write that the distance $\bar{x}$ traveled by G during a rotation θ of the disk is $\bar{x} = r\theta$, where r is the radius of the disk. Differentiating this relation twice, we write

$$\bar{a} = r\alpha \qquad (16.9)$$

Recalling that the system of the effective forces in plane motion reduces to a vector $m\bar{\mathbf{a}}$ and a couple $\bar{I}\alpha$, we find that in the particular case of the rolling motion of a balanced disk, the effective forces reduce to a vector of magnitude $mr\alpha$ attached at G and to a couple of magnitude $\bar{I}\alpha$. We may thus express that the external forces are equivalent to the vector and couple shown in Fig. 16.17.

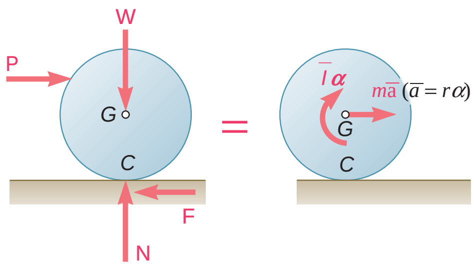


Fig. 16.17

When a disk *rolls without sliding*, there is no relative motion between the point of the disk in contact with the ground and the ground itself. Thus, as far as the computation of the friction force **F** is concerned, a rolling disk can be compared with a block at rest on a surface. The magnitude F of the friction force can have any value, as long as this value does not exceed the maximum value $F_m = \mu_s N$, where μ_s is the coefficient of static friction and N is the magnitude of the normal force. In the case of a rolling disk, the magnitude F of the friction force should therefore be determined independently of N by solving the equation obtained from Fig. 16.17.

When *sliding is impending*, the friction force reaches its maximum value $F_m = \mu_s N$ and can be obtained from N.

When the disk *rotates and slides* at the same time, a relative motion exists between the point of the disk which is in contact with the ground and the ground itself, and the force of friction has the magnitude $F_k = \mu_k N$, where μ_k is the coefficient of kinetic friction. In this case, however, the motion of the mass center G of the disk and the rotation of the disk about G are independent, and $\bar{a}$ is not equal to $r\alpha$.

These three different cas

Rolling, no sliding:		
Rolling, sliding impending:	$F = \mu_s N$	$\bar{a} = r\alpha$
Rotating and sliding:	$F = \mu_k N$	$\bar{a}$ *and* α *independent*

When it is not known whether or not a disk slides, it should first be assumed that the disk rolls without sliding. If F is found smaller than or equal to $\mu_s N$, the assumption is proved correct. If F is found larger than $\mu_s N$, the assumption is incorrect and the problem should be started again, assuming rotating and sliding.


Photo 16.4 As the ball hits the bowling alley, it first spins and slides, then rolls without sliding.

When a disk is *unbalanced*, i.e., when its mass center G does not coincide with its geometric center O, the relation (16.9) does not hold between $\bar{a}$ and α. However, a similar relation holds between the magnitude a_O of the acceleration of the geometric center and the angular acceleration α of an unbalanced disk which rolls without sliding. We have

$$a_O = r\alpha \tag{16.10}$$

To determine $\bar{a}$ in terms of the angular acceleration α and the angular velocity ω of the disk, we can use the relative-acceleration formula

$$\begin{aligned}\bar{\mathbf{a}} = \bar{\mathbf{a}}_G &= \mathbf{a}_O + \mathbf{a}_{G/O} \\ &= \mathbf{a}_O + (\mathbf{a}_{G/O})_t + (\mathbf{a}_{G/O})_n \end{aligned} \tag{16.11}$$

where the three component accelerations obtained have the directions indicated in Fig. 16.18 and the magnitudes $a_O = r\alpha$, $(a_{G/O})_t = (OG)\alpha$, and $(a_{G/O})_n = (OG)\omega^2$.

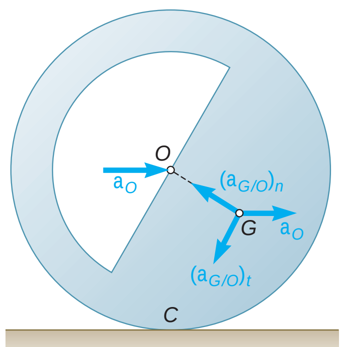
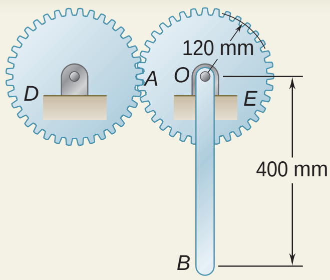


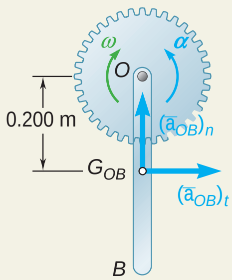
Fig. 16.18



SAMPLE PROBLEM 16.6

The portion AOB of a mechanism consists of a 400-mm steel rod OB welded to a gear E of radius 120 mm which can rotate about a horizontal shaft O. It is actuated by a gear D and, at the instant shown, has a clockwise angular velocity of 8 rad/s and a counterclockwise angular acceleration of 40 rad/s^2. Knowing that rod OB has a mass of 3 kg and gear E a mass of 4 kg and a radius of gyration of 85 mm, determine (*a*) the tangential force exerted by gear D on gear E, (*b*) the components of the reaction at shaft O.

SOLUTION



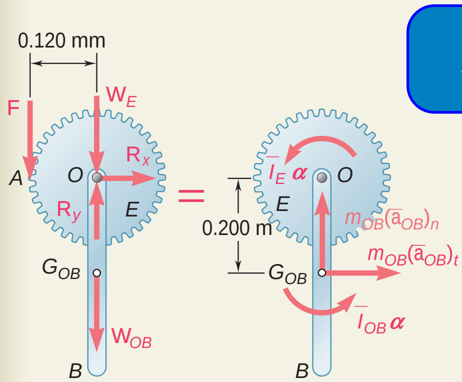
In determining the effective forces of the rigid body AOB, gear E and rod OB will be considered separately. Therefore, the components of the acceleration of the mass center G_{OB} of the rod will be determined first:

$$(\bar{a}_{OB})_t = \bar{r}\alpha = (0.200 \text{ m})(40 \text{ rad/s}^2) = 8 \text{ m/s}^2$$
$$(\bar{a}_{OB})_n = \bar{r}\omega^2 = (0.200 \text{ m})(8 \text{ rad/s})^2 = 12.8 \text{ m/s}^2$$

Equations of Motion. Two sketches of the rigid body AOB have been drawn. The first shows the external forces consisting of the weight $\mathbf{W}_E$ of gear E, the weight $\mathbf{W}_{OB}$ of the rod OB, the force $\mathbf{F}$ exerted by gear D, and ... at O. The magnitudes of the

... $/\text{s}^2) = 39.2$ N

$$W_{OB} = m_{OB}g = (3 \text{ kg})(9.81 \text{ m/s}^2) = 29.4 \text{ N}$$



The second sketch shows the effective forces, which consist of a couple $\bar{I}_E\alpha$ (since gear E is in centroidal rotation) and of a couple and two vector components at the mass center of OB. Since the accelerations are known, we compute the magnitudes of these components and couples:

$$\bar{I}_E\alpha = m_E\bar{k}_E^2\alpha = (4 \text{ kg})(0.085 \text{ m})^2(40 \text{ rad/s}^2) = 1.156 \text{ N}\cdot\text{m}$$
$$m_{OB}(\bar{a}_{OB})_t = (3 \text{ kg})(8 \text{ m/s}^2) = 24.0 \text{ N}$$
$$m_{OB}(\bar{a}_{OB})_n = (3 \text{ kg})(12.8 \text{ m/s}^2) = 38.4 \text{ N}$$
$$\bar{I}_{OB}\alpha = (\tfrac{1}{12}m_{OB}L^2)\alpha = \tfrac{1}{12}(3 \text{ kg})(0.400 \text{ m})^2(40 \text{ rad/s}^2) = 1.600 \text{ N}\cdot\text{m}$$

Expressing that the system of the external forces is equivalent to the system of the effective forces, we write the following equations:

$+\circlearrowleft \Sigma M_O = \Sigma(M_O)_{\text{eff}}$:

$$F(0.120 \text{ m}) = \bar{I}_E\alpha + m_{OB}(\bar{a}_{OB})_t(0.200 \text{ m}) + \bar{I}_{OB}\alpha$$
$$F(0.120 \text{ m}) = 1.156 \text{ N}\cdot\text{m} + (24.0 \text{ N})(0.200 \text{ m}) + 1.600 \text{ N}\cdot\text{m}$$

$F = 63.0$ N $\mathbf{F} = 63.0 \text{ N}\downarrow$ ◀

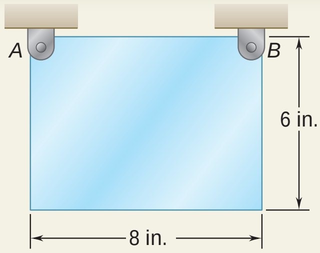
$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $R_x = m_{OB}(\bar{a}_{OB})_t$

$R_x = 24.0$ N $\mathbf{R}_x = 24.0 \text{ N} \rightarrow$ ◀

$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}$: $R_y - F - W_E - W_{OB} = m_{OB}(\bar{a}_{OB})_n$

$R_y - 63.0 \text{ N} - 39.2 \text{ N} - 29.4 \text{ N} = 38.4 \text{ N}$

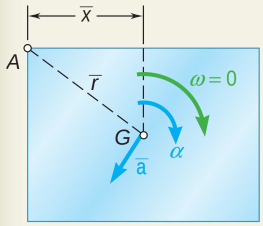
$R_y = 170.0$ N $\mathbf{R}_y = 170.0 \text{ N}\uparrow$ ◀

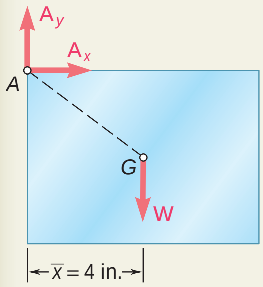


SAMPLE PROBLEM 16.7

A 6 × 8 in. rectangular plate weighing 60 lb is suspended from two pins A and B. If pin B is suddenly removed, determine (a) the angular acceleration of the plate, (b) the components of the reaction at pin A, immediately after pin B has been removed.

SOLUTION





a. Angular Acceleration. We observe that as the plate rotates about point A, its mass center G describes a circle of radius $\bar{r}$ with center at A.

Since the plate is released from rest ($\omega = 0$), the normal component of the acceleration of G is zero. The magnitude of the acceleration $\bar{\mathbf{a}}$ of the mass center G is thus $\bar{a} = \bar{r}\alpha$. We draw the diagram shown to express that the external forces are equivalent to the effective forces:

$$+\text{i}\Sigma M_A = \Sigma(M_A)_{\text{eff}}: \qquad W\bar{x} = (m\bar{a})\bar{r} + \bar{I}\alpha$$

Since $\bar{a} = \bar{r}\alpha$, we have

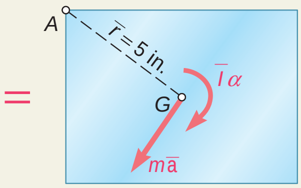
$$W\bar{x} = m(\bar{r}\alpha)\bar{r} + \bar{I}\alpha \qquad \alpha = \frac{W\bar{x}}{\frac{W}{g}\bar{r}^2 + \bar{I}} \qquad (1)$$

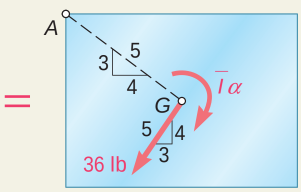
...he plate is

$$\bar{I} = \frac{m}{12}(a^2 + b^2) = \frac{60 \text{ lb}}{12(32.2 \text{ ft/s}^2)}[(\tfrac{8}{12} \text{ ft})^2 + (\tfrac{6}{12} \text{ ft})^2]$$
$$= 0.1078 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

Substituting this value of $\bar{I}$ together with $W = 60$ lb, $\bar{r} = \frac{5}{12}$ ft, and $\bar{x} = \frac{4}{12}$ ft into Eq. (1), we obtain

$$\alpha = +46.4 \text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 46.4 \text{ rad/s}^2 \text{ i} \quad ◀$$





b. Reaction at A. Using the computed value of α, we determine the magnitude of the vector $m\bar{\mathbf{a}}$ attached at G.

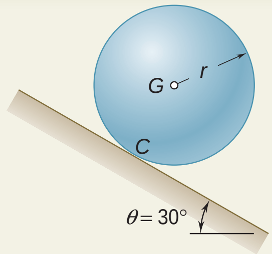
$$m\bar{a} = m\bar{r}\alpha = \frac{60 \text{ lb}}{32.2 \text{ ft/s}^2}(\tfrac{5}{12} \text{ ft})(46.4 \text{ rad/s}^2) = 36.0 \text{ lb}$$

Showing this result on the diagram, we write the equations of motion

$$\overset{+}{\text{y}}\ \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad A_x = -\tfrac{3}{5}(36 \text{ lb})$$
$$= -21.6 \text{ lb} \qquad \mathbf{A}_x = 21.6 \text{ lb z} \quad ◀$$

$$+\text{x}\Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad A_y - 60 \text{ lb} = -\tfrac{4}{5}(36 \text{ lb})$$
$$A_y = +31.2 \text{ lb} \qquad \mathbf{A}_y = 31.2 \text{ lb x} \quad ◀$$

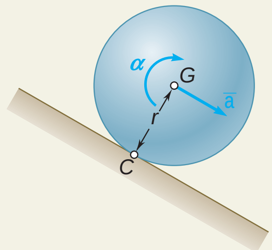
The couple $\bar{I}\alpha$ is not involved in the last two equations; nevertheless, it should be indicated on the diagram.



SAMPLE PROBLEM 16.8

A sphere of radius r and weight W is released with no initial velocity on the incline and rolls without slipping. Determine (*a*) the minimum value of the coefficient of static friction compatible with the rolling motion, (*b*) the velocity of the center G of the sphere after the sphere has rolled 10 ft, (*c*) the velocity of G if the sphere were to move 10 ft down a frictionless 30° incline.

SOLUTION



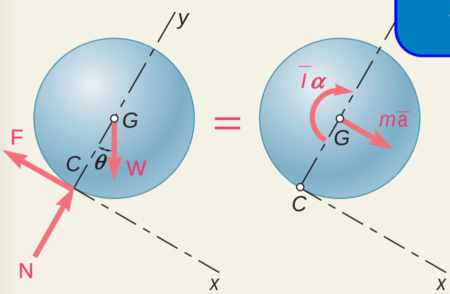
a. Minimum M$_s$ for Rolling Motion. The external forces **W**, **N**, and **F** form a system equivalent to the system of effective forces represented by the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}$A. Since the sphere rolls without sliding, we have $\bar{a} = r$a.

$$+\text{i}\Sigma M_C = \Sigma(M_C)_{\text{eff}}: \qquad (W \sin u)r = (m\bar{a})r + \bar{I}\text{a}$$
$$(W \sin u)r = (mr\text{a})r + \bar{I}\text{a}$$

Noting that $m = W/g$ and $\bar{I} = \frac{2}{5}mr^2$, we write

$$(W \sin u)r = \left(\frac{W}{g}r\text{a}\right)r + \frac{2}{5}\frac{W}{g}r^2\text{a} \qquad \text{a} = +\frac{5g \sin u}{7r}$$

$$\frac{5g \sin u}{} \quad \frac{5(32.2 \text{ ft/s}^2) \sin 30°}{} = 11.50 \text{ ft/s}^2$$



$$W \sin u - F = \frac{}{g} \frac{g \sin u}{7}$$

$$F = +\tfrac{2}{7}W \sin u = \tfrac{2}{7}W \sin 30° \qquad \mathbf{F} = 0.143W \text{ b } 30°$$

$$+\nearrow\Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad N - W \cos u = 0$$
$$N = W \cos u = 0.866W \qquad \mathbf{N} = 0.866W \text{ a } 60°$$
$$\text{m}_s = \frac{F}{N} = \frac{0.143W}{0.866W} \qquad \text{m}_s = 0.165 \quad ◀$$

b. Velocity of Rolling Sphere. We have uniformly accelerated motion:

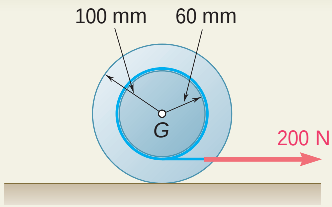
$$\bar{v}_0 = 0 \qquad \bar{a} = 11.50 \text{ ft/s}^2 \qquad \bar{x} = 10 \text{ ft} \qquad \bar{x}_0 = 0$$
$$\bar{v}^2 = \bar{v}_0^2 + 2\bar{a}(\bar{x} - \bar{x}_0) \qquad \bar{v}^2 = 0 + 2(11.50 \text{ ft/s}^2)(10 \text{ ft})$$
$$\bar{v} = 15.17 \text{ ft/s} \qquad \bar{\mathbf{v}} = 15.17 \text{ ft/s c } 30° \quad ◀$$

c. Velocity of Sliding Sphere. Assuming now no friction, we have $F = 0$ and obtain

$$+\text{i}\Sigma M_G = \Sigma(M_G)_{\text{eff}}: \qquad 0 = \bar{I}\text{a} \qquad \text{a} = 0$$
$$+\searrow\Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad W \sin 30° = m\bar{a} \qquad 0.50W = \frac{W}{g}\bar{a}$$
$$\bar{a} = +16.1 \text{ ft/s}^2 \qquad \bar{\mathbf{a}} = 16.1 \text{ ft/s}^2 \text{ c } 30°$$

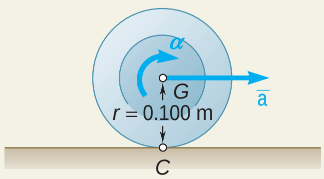
Substituting $\bar{a} = 16.1 \text{ ft/s}^2$ into the equations for uniformly accelerated motion, we obtain

$$\bar{v}^2 = \bar{v}_0^2 + 2\bar{a}(\bar{x} - \bar{x}_0) \qquad \bar{v}^2 = 0 + 2(16.1 \text{ ft/s}^2)(10 \text{ ft})$$
$$\bar{v} = 17.94 \text{ ft/s} \qquad \bar{\mathbf{v}} = 17.94 \text{ ft/s c } 30° \quad ◀$$



SAMPLE PROBLEM 16.9

A cord is wrapped around the inner drum of a wheel and pulled horizontally with a force of 200 N. The wheel has a mass of 50 kg and a radius of gyration of 70 mm. Knowing that $m_s = 0.20$ and $m_k = 0.15$, determine the acceleration of G and the angular acceleration of the wheel.



SOLUTION

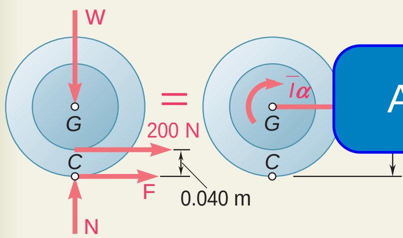
a. Assume Rolling without Sliding. In this case, we have

$$\bar{a} = r\mathrm{a} = (0.100\text{ m})\mathrm{a}$$

We can determine whether this assumption is justified by comparing the friction force obtained with the maximum available friction force. The moment of inertia of the wheel is

$$\bar{I} = m\bar{k}^2 = (50\text{ kg})(0.070\text{ m})^2 = 0.245\text{ kg}\cdot\text{m}^2$$

Equations of Motion



$+\mathrm{i}\,\Sigma M_C = \Sigma(M_C)_{\text{eff}}$: $\quad (200\text{ N})(0.040\text{ m}) = m\bar{a}(0.100\text{ m}) + \bar{I}\mathrm{a}$

$$8.00\text{ N}\cdot\text{m} = (50\text{ kg})(0.100\text{ m})\mathrm{a}(0.100\text{ m}) + (0.245\text{ kg}\cdot\text{m}^2)\mathrm{a}$$

$$\mathrm{a} = +10.74\text{ rad/s}^2$$

$\ldots 0.74\text{ rad/s}^2) = 1.074\text{ m/s}^2$

$\ldots \bar{a}$

$\ldots 50\text{ kg})(1.074\text{ m/s}^2)$

$$F = -146.3\text{ N} \qquad \mathbf{F} = 146.3\text{ N z}$$

$+\mathrm{x}\Sigma F_y = \Sigma(F_y)_{\text{eff}}$:

$$N - W = 0 \qquad N - W = mg = (50\text{ kg})(9.81\text{ m/s}^2) = 490.5\text{ N}$$

$$\mathbf{N} = 490.5\text{ N x}$$

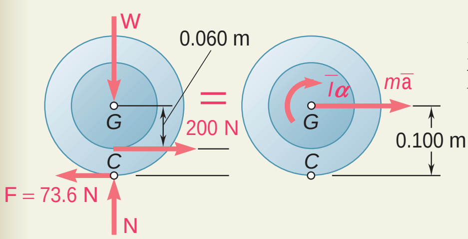
Maximum Available Friction Force

$$F_{\max} = m_s N = 0.20(490.5\text{ N}) = 98.1\text{ N}$$

Since $F > F_{\max}$, the assumed motion is impossible.

b. Rotating and Sliding. Since the wheel must rotate and slide at the same time, we draw a new diagram, where $\bar{\mathbf{a}}$ and A are independent and where

$$F = F_k = m_k N = 0.15(490.5\text{ N}) = 73.6\text{ N}$$



From the computation of part a, it appears that **F** should be directed to the left. We write the following equations of motion:

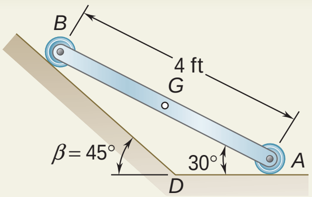
$\overset{+}{\mathrm{y}}\ \Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $\quad 200\text{ N} - 73.6\text{ N} = (50\text{ kg})\bar{a}$

$$\bar{a} = +2.53\text{ m/s}^2 \qquad \bar{\mathrm{a}} = 2.53\text{ m/s}^2\text{ y} \quad ◀$$

$+\mathrm{i}\,\Sigma M_G = \Sigma(M_G)_{\text{eff}}$:

$$(73.6\text{ N})(0.100\text{ m}) - (200\text{ N})(0.060\text{ m}) = (0.245\text{ kg}\cdot\text{m}^2)\mathrm{a}$$

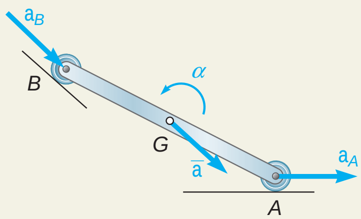
$$\mathrm{a} = -18.94\text{ rad/s}^2 \qquad \mathrm{A} = 18.94\text{ rad/s}^2\text{ l} \quad ◀$$



SAMPLE PROBLEM 16.10

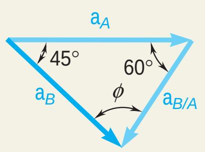
The extremities of a 4-ft rod weighing 50 lb can move freely and with no friction along two straight tracks as shown. If the rod is released with no velocity from the position shown, determine (*a*) the angular acceleration of the rod, (*b*) the reactions at *A* and *B*.

SOLUTION



Kinematics of Motion. Since the motion is constrained, the acceleration of *G* must be related to the angular acceleration A. To obtain this relation, we first determine the magnitude of the acceleration $\mathbf{a}_A$ of point *A* in terms of a. Assuming that A is directed counterclockwise and noting that $a_{B/A} =$ 4a, we write

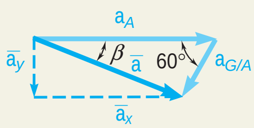
$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$
$$[a_B \text{ c } 45°] = [a_A \text{ y }] + [4\text{a d } 60°]$$



Noting that f = 75° and using the law of sines, we obtain

$$a_A = 5.46\text{a} \qquad a_B = 4.90\text{a}$$

The acceleration of *G* is now obtained by writing



d 60°]

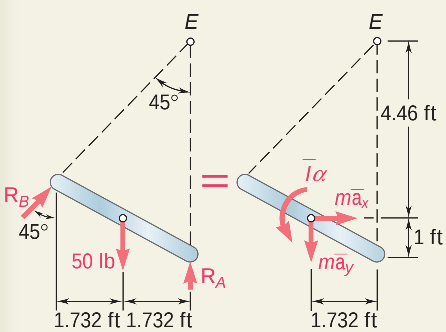
Resolving **a** into *x* and *y* components, we obtain

$$\bar{a}_x = 5.46\text{a} - 2\text{a} \cos 60° = 4.46\text{a} \qquad \bar{\mathbf{a}}_x = 4.46\text{a y}$$
$$\bar{a}_y = -2\text{a} \sin 60° = -1.732\text{a} \qquad \bar{\mathbf{a}}_y = 1.732\text{a w}$$

Kinetics of Motion. We draw a free-body-diagram equation expressing that the system of the external forces is equivalent to the system of the effective forces represented by the vector of components $m\bar{\mathbf{a}}_x$ and $m\bar{\mathbf{a}}_y$ attached at *G* and the couple $\bar{I}$A. We compute the following magnitudes:

$$\bar{I} = \tfrac{1}{12}ml^2 = \frac{1}{12}\frac{50 \text{ lb}}{32.2 \text{ ft/s}^2}(4 \text{ ft})^2 = 2.07 \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \qquad \bar{I}\text{a} = 2.07\text{a}$$

$$m\bar{a}_x = \frac{50}{32.2}(4.46\text{a}) = 6.93\text{a} \qquad m\bar{a}_y = -\frac{50}{32.2}(1.732\text{a}) = -2.69\text{a}$$



Equations of Motion

$+\text{l}\Sigma M_E = \Sigma(M_E)_{\text{eff}}$:
$$(50)(1.732) = (6.93\text{a})(4.46) + (2.69\text{a})(1.732) + 2.07\text{a}$$
$$\text{a} = +2.30 \text{ rad/s}^2 \qquad \text{A} = 2.30 \text{ rad/s}^2 \text{ l} \quad ◀$$

$\overset{+}{\text{y}}\ \Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $\quad R_B \sin 45° = (6.93)(2.30) = 15.94$
$$R_B = 22.5 \text{ lb} \qquad \mathbf{R}_B = 22.5 \text{ lb a } 45° \quad ◀$$

$+\text{x}\Sigma F_y = \Sigma(F_y)_{\text{eff}}$: $\quad R_A + R_B \cos 45° - 50 = -(2.69)(2.30)$
$$R_A = -6.19 - 15.94 + 50 = 27.9 \text{ lb} \qquad \mathbf{R}_A = 27.9 \text{ lb x} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we considered the *plane motion of rigid bodies under constraints*. We found that the types of constraints involved in engineering problems vary widely. For example, a rigid body may be constrained to rotate about a fixed axis or to roll on a given surface, or it may be pin-connected to collars or to other bodies.

1. Your solution of a problem involving the constrained motion of a rigid body, will, in general, consist of two steps. First, you will consider the *kinematics of the motion*, and then you will solve the *kinetics portion of the problem*.

2. The kinematic analysis of the motion is done by using the methods you learned in Chap. 15. Due to the constraints, linear and angular accelerations will be related. (They will *not* be independent, as they were in the last lesson.) You should establish *relationships among the accelerations* (angular as well as linear), and your goal should be to express all accelerations in terms of a *single unknown acceleration*. This is the first step taken in the solution of each of the sample problems in this lesson.

a. For a body in noncentroidal rotation, the components of the acceleration of the mass center are $\bar{a}_t = \bar{r}\alpha$ and $\bar{a}_n = \bar{r}\omega^2$, where ω will generally be known [Sample Probs. 16.6 and 16.7].

b. For a roass center is $\bar{a} = r\alpha$ [Sample Prob.

c. For a bose of action, if neither $\bar{a}$ nor α is known or readily obtainable, is to express $\bar{a}$ in terms of α [Sample Prob. 16.10].

3. The kinetic analysis of the motion is carried out as follows.

a. Start by drawing a free-body-diagram equation. This was done in all the sample problems of this lesson. In each case the left-hand diagram shows the external forces, including the applied forces, the reactions, and the weight of the body. The right-hand diagram shows the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}\alpha$.

b. Next, reduce the number of unknowns in the free-body-diagram equation by using the relationships among the accelerations that you found in your kinematic analysis. You will then be ready to consider equations that can be written by summing components or moments. Choose first an equation that involves a single unknown. After solving for that unknown, substitute the value obtained into the other equations, which you will then solve for the remaining unknowns.

(continued)

4. When solving problems involving rolling disks or wheels, keep in mind the following.

a. If sliding is impending, the friction force exerted on the rolling body has reached its maximum value, $F_m = \mu_s N$, where N is the normal force exerted on the body and μ_s is the coefficient of *static friction* between the surfaces of contact.

b. If sliding is not impending, the friction force F can have *any value* smaller than F_m and should, therefore, be considered as an independent unknown. After you have determined F, be sure to check that it is smaller than F_m; if it is not, *the body does not roll*, but rotates and slides as described in the next paragraph.

c. If the body rotates and slides at the same time, then the body is *not rolling* and the acceleration $\bar{a}$ of the mass center is *independent* of the angular acceleration α of the body: $\bar{a} \neq r\alpha$. On the other hand, the friction force has a well-defined value, $F = \mu_k N$, where μ_k is the coefficient of *kinetic friction* between the surfaces of contact.

d. For an unbalanced rolling disk or wheel, the relation $\bar{a} = r\alpha$ between the acceleration $\bar{a}$ of the mass center G and the angular acceleration α of the disk or wheel *does not hold anymore*. However, a similar relation holds between the acceleration a_O o[illegible]cceleration α of the disk or wheel: a[illegible]s $\bar{a}$ in terms of α and ω (Fig. 16.18)[illegible]

5. For a system of connected rigid bodies, the goal of your *kinematic analysis* should be to determine all the accelerations from the given data, or to express them all in terms of a single unknown. (For systems with several degrees of freedom, you will need to use as many unknowns as there are degrees of freedom.)

Your *kinetic analysis* will generally be carried out by drawing a free-body-diagram equation for the entire system, as well as for one or several of the rigid bodies involved. In the latter case, both internal and external forces should be included, and care should be taken to represent with equal and opposite vectors the forces that two bodies exert on each other.

PROBLEMS

CONCEPT QUESTIONS

16.CQ4 A cord is attached to a spool when a force **P** is applied to the cord as shown. Assuming the spool rolls without slipping, what direction does the spool move for each case?

Case 1: **a.** left **b.** right **c.** It would not move.
Case 2: **a.** left **b.** right **c.** It would not move.
Case 3: **a.** left **b.** right **c.** It would not move.

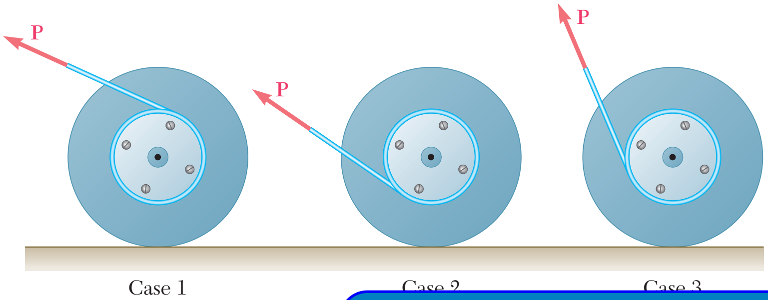


Fig. P16.CQ4 and P16.CQ5

16.CQ5 A cord is attached to a spool when a force **P** is applied to the cord as shown. Assuming the spool rolls without slipping, in what direction does the friction force act for each case?

Case 1: **a.** left **b.** right **c.** The friction force would be zero.
Case 2: **a.** left **b.** right **c.** The friction force would be zero.
Case 3: **a.** left **b.** right **c.** The friction force would be zero.

16.CQ6 A front-wheel-drive car starts from rest and accelerates to the right. Knowing that the tires do not slip on the road, what is the direction of the friction force the road applies to the front tires?

a. left
b. right
c. The friction force is zero.

16.CQ7 A front-wheel-drive car starts from rest and accelerates to the right. Knowing that the tires do not slip on the road, what is the direction of the friction force the road applies to the rear tires?

a. left
b. right
c. The friction force is zero.

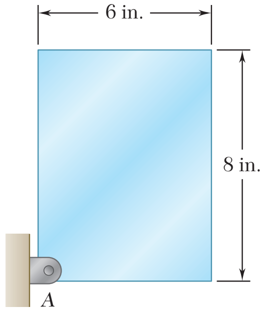


Fig. P16.F5

FREE BODY PRACTICE PROBLEMS

16.F5 A uniform 6 × 8-in. rectangular plate of mass m is pinned at A. Knowing the angular velocity of the plate at the instant shown is V, draw the FBD and KD.

16.F6 Two identical 4-lb slender rods AB and BC are connected by a pin at B and by the cord AC. The assembly rotates in a vertical plane under the combined effect of gravity and a couple **M** applied to rod AB. Knowing that in the position shown the angular velocity of the assembly is V, draw the FBD and KD that can be used to determine the angular acceleration of the assembly.

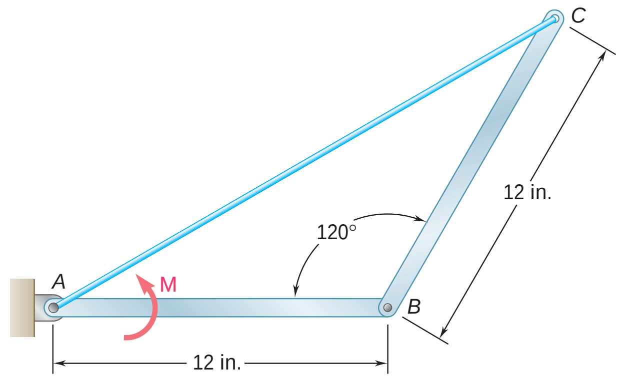


Fig. P16.F6

16.F7 The 4-lb uniform rod AB is attached to collars of negligible mass … the fixed rods shown. Rod AB is at … n a horizontal force **P** is applied to … ng to the left. Draw the FBD and … KD for the rod.

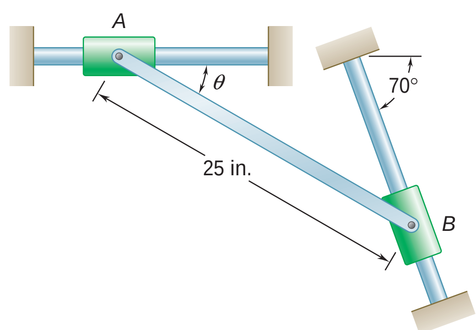


Fig. P16.F7

16.F8 A uniform disk of mass $m = 4$ kg and radius $r = 150$ mm is supported by a belt $ABCD$ that is bolted to the disk at B and C. If the belt suddenly breaks at a point located between A and B, draw the FBD and KD for the disk immediately after the break.

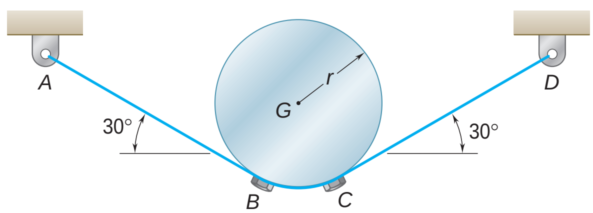


Fig. P16.F8

END-OF-SECTION PROBLEMS

16.75 Show that the couple $\bar{I}\alpha$ of Fig. 16.15 can be eliminated by attaching the vectors $m\bar{\mathbf{a}}_t$ and $m\bar{\mathbf{a}}_n$ at a point P called the *center of percussion*, located on line OG at a distance $GP = \bar{k}^2/\bar{r}$ from the mass center of the body.

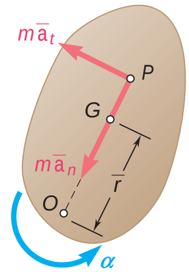


Fig. P16.75

16.76 A uniform slender rod of length $L = 900$ mm and mass $m = 4$ kg is suspended from a hinge at C. A horizontal force $\mathbf{P}$ of magnitude 75 N is applied at end B. Knowing that $\bar{r} = 225$ mm, determine (*a*) the angular acceleration of the rod, (*b*) the components of the reaction at C.

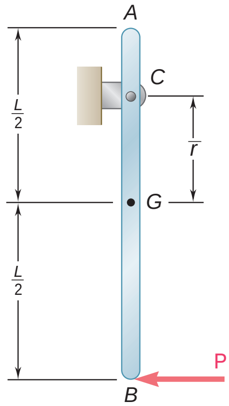


Fig. P16.76

16.77 In Prob. 16.76, determine (*a*) the distance $\bar{r}$ for which the horizontal component of the reaction at C is zero, (*b*) the corresponding angular acceleration [illegible]

16.78 A uniform slender rod of l[illegible] freely from a hinge at A. If a force $\mathbf{P}$ of magnitude 1.5 lb is applied at B horizontally to the left ($h = L$), determine (*a*) the angular acceleration of the rod, (*b*) the components of the reaction at A.

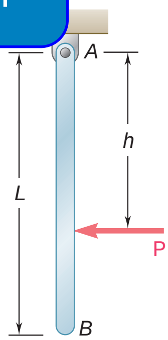


Fig. P16.78

16.79 In Prob. 16.78, determine (*a*) the distance h for which the horizontal component of the reaction at A is zero, (*b*) the corresponding angular acceleration of the rod.

16.80 The uniform slender rod AB is welded to the hub D, and the system rotates about the vertical axis DE with a constant angular velocity V. (*a*) Denoting by w the mass per unit length of the rod, express the tension in the rod at a distance z from end A in terms of w, l, z, and V, (*b*) Determine the tension in the rod for $w = 0.3$ kg/m, $l = 400$ mm, $z = 250$ mm, and v = 150 rpm.

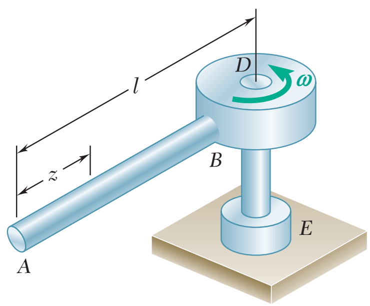


Fig. P16.80

16.81 The shutter shown was formed by removing one quarter of a disk of 0.75-in. radius and is used to interrupt a beam of light emanating from a lens at C. Knowing that the shutter weighs 0.125 lb and rotates at the constant rate of 24 cycles per second, determine the magnitude of the force exerted by the shutter on the shaft at A.

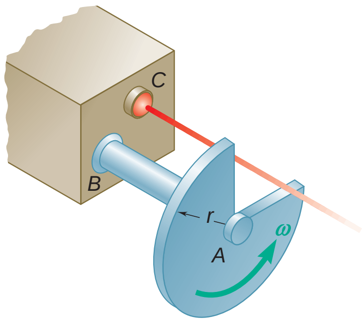


Fig. *P16.81*

16.82 A 6-in.-diameter hole is cut as shown in a thin disk of 15-in. diameter. The disk rotates in a horizontal plane about its geometric center A at the constant rate of 480 rpm. Knowing that the disk has a mass of 60 lb after the hole has been cut, determine the horizontal component of the force exerted by the shaft on the disk at A.

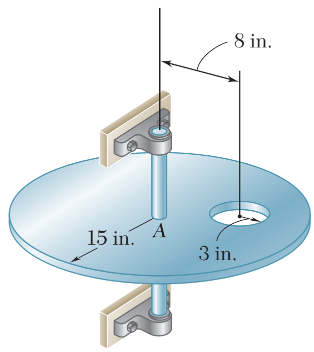


Fig. *P16.82*

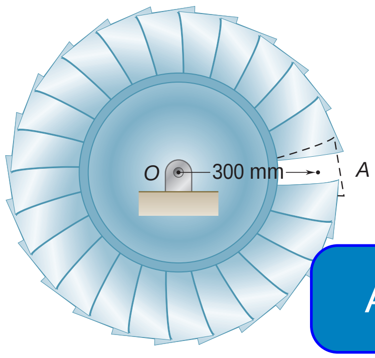


Fig. P16.83

…tes at a constant rate of 9600 rpm. …the disk coincides with the center of rotation O, determine the reaction at O immediately after a single blade at A, of mass 45 g, becomes loose and is thrown off.

16.84 and 16.85 A uniform rod of length L and mass m is supported as shown. If the cable attached at end B suddenly breaks, determine (a) the acceleration of end B, (b) the reaction at the pin support.

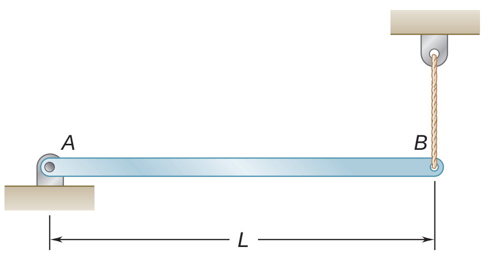


Fig. P16.84

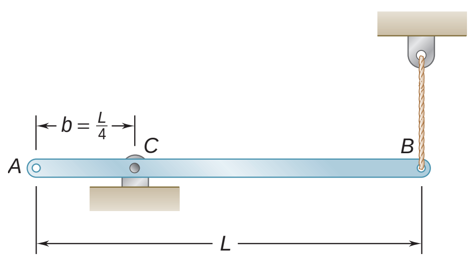


Fig. P16.85

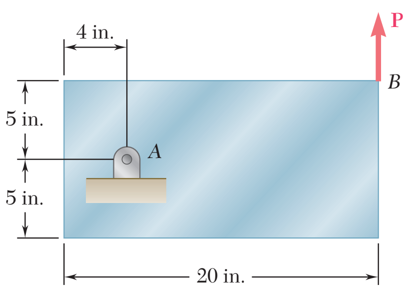


Fig. P16.86

16.86 A 12-lb uniform plate rotates about A in a vertical plane under the combined effect of gravity and of the vertical force **P**. Knowing that at the instant shown the plate has an angular velocity of 20 rad/s and an angular acceleration of 30 rad/s^2 both counterclockwise, determine (a) the force **P**, (b) the components of the reaction at A.

16.87 A 1.5-kg slender rod is welded to a 5-kg uniform disk as shown. The assembly swings freely about *C* in a vertical plane. Knowing that in the position shown the assembly has an angular velocity of 10 rad/s clockwise, determine (*a*) the angular acceleration of the assembly, (*b*) the components of the reaction at *C*.

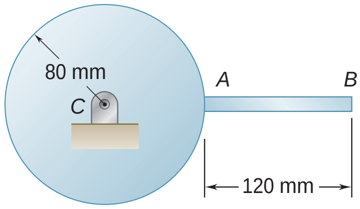


Fig. P16.87

16.88 Two uniform rods, *ABC* of weight 6 lb and *DCE* of weight 8 lb, are connected by a pin at *C* and by two cords *BD* and *BE*. The T-shaped assembly rotates in a vertical plane under the combined effect of gravity and of a couple **M** which is applied to rod *ABC*. Knowing that at the instant shown the tension in cord *BE* is 2 lb and the tension in cord *BD* is 0.5 lb, determine (*a*) the angular acceleration of the assembly, (*b*) the couple **M**.

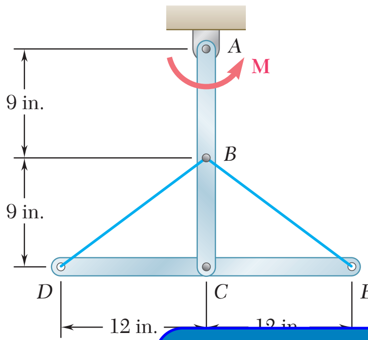


Fig. P16.88

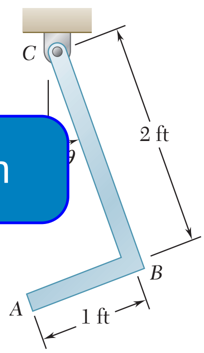


Fig. *P16.89*

16.89 The object *ABC* consists of two slender rods welded together at point *B*. Rod *AB* has a weight of 2 lb and bar *BC* has a weight of 4 lb. Knowing the magnitude of the angular velocity of *ABC* is 10 rad/s when u = 0°, determine the components of the reaction at point *C* when u = 0°.

16.90 A 3.5-kg slender rod *AB* and a 2-kg slender rod *BC* are connected by a pin at *B* and by the cord *AC*. The assembly can rotate in a vertical plane under the combined effect of gravity and a couple **M** applied to rod *BC*. Knowing that in the position shown the angular velocity of the assembly is zero and the tension in cord *AC* is equal to 25 N, determine (*a*) the angular acceleration of the assembly, (*b*) the magnitude of the couple **M**.

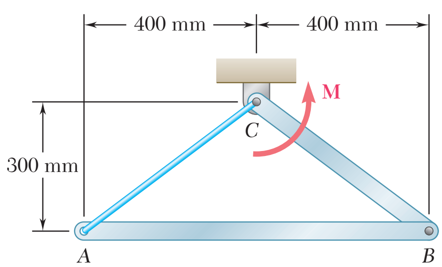


Fig. *P16.90*

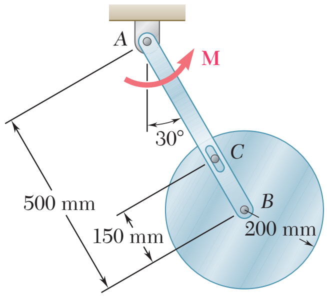


Fig. ***P16.91***

16.91 A 9-kg uniform disk is attached to the 5-kg slender rod *AB* by means of frictionless pins at *B* and *C*. The assembly rotates in a vertical plane under the combined effect of gravity and of a couple **M** which is applied to rod *AB*. Knowing that at the instant shown the assembly has an angular velocity of 6 rad/s and an angular acceleration of 25 rad/s^2, both counterclockwise, determine (*a*) the couple **M**, (*b*) the force exerted by pin *C* on member *AB*.

16.92 Derive the equation $\Sigma M_C = I_C$a for the rolling disk of Fig. 16.17, where ΣM_C represents the sum of the moments of the external forces about the instantaneous center *C*, and I_C is the moment of inertia of the disk about *C*.

16.93 Show that in the case of an unbalanced disk, the equation derived in Prob. 16.92 is valid only when the mass center *G*, the geometric center *O*, and the instantaneous center *C* happen to lie in a straight line.

16.94 A wheel of radius *r* and centroidal radius of gyration $\bar{k}$ is released from rest on the incline and rolls without sliding. Derive an expression for the acceleration of the center of the wheel in terms of *r*, $\bar{k}$, b, and *g*.

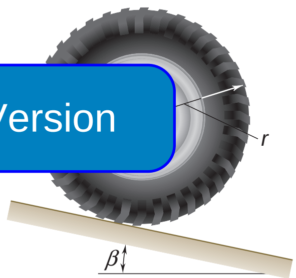


Fig. P16.94

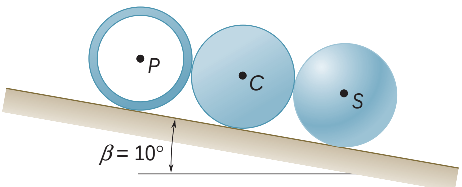


Fig. P16.95

16.95 A homogeneous sphere *S*, a uniform cylinder *C*, and a thin pipe *P* are in contact when they are released from rest on the incline shown. Knowing that all three objects roll without slipping, determine, after 4 s of motion, the clear distance between (*a*) the pipe and the cylinder, (*b*) the cylinder and the sphere.

16.96 A 40-kg flywheel of radius $R = 0.5$ m is rigidly attached to a shaft of radius $r = 0.05$ m that can roll along parallel rails. A cord is attached as shown and pulled with a force **P** of magnitude 150 N. Knowing the centroidal radius of gyration is $\bar{k} = 0.4$ m, determine (*a*) the angular acceleration of the flywheel, (*b*) the velocity of the center of gravity after 5 s.

16.97 A 40-kg flywheel of radius $R = 0.5$ m is rigidly attached to a shaft of radius $r = 0.05$ m that can roll along parallel rails. A cord is attached as shown and pulled with a force **P**. Knowing the centroidal radius of gyration is $\bar{k} = 0.4$ m and the coefficient of static friction is $m_s = 0.4$, determine the largest magnitude of force **P** for which no slipping will occur.

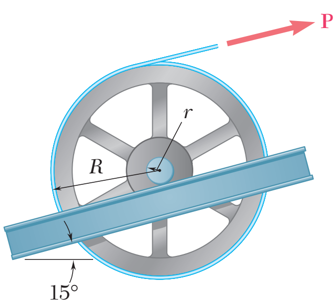


Fig. ***P16.96*** **and** ***P16.97***

16.98 through *16.101* A drum of 60-mm radius is attached to a disk of 120-mm radius. The disk and drum have a total mass of 6 kg and a combined radius of gyration of 90 mm. A cord is attached as shown and pulled with a force **P** of magnitude 20 N. Knowing that the disk rolls without sliding, determine (*a*) the angular acceleration of the disk and the acceleration of *G*, (*b*) the minimum value of the coefficient of static friction compatible with this motion.

16.102 through 16.105 A drum of 4-in. radius is attached to a disk of 8-in. radius. The disk and drum have a combined weight of 10 lb and a combined radius of gyration of 6 in. A cord is attached as shown and pulled with a force **P** of magnitude 5 lb. Knowing that the coefficients of static and kinetic friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, respectively, determine (*a*) whether or not the disk slides, (*b*) the angular acceleration of the disk and the acceleration of *G*.

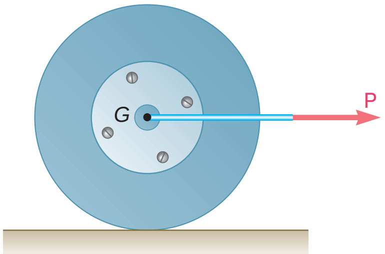


Fig. P16.98 and P16.102

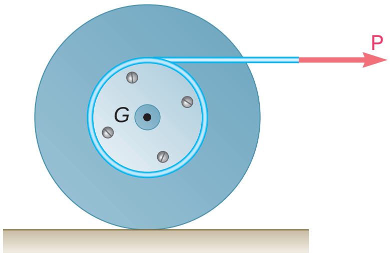


Fig. P16.99 and P16.103

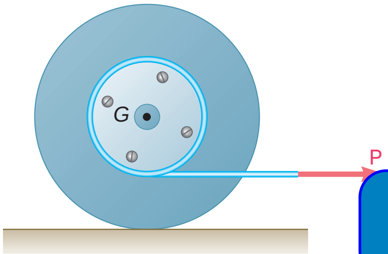


Fig. *P16.100* and P16.104

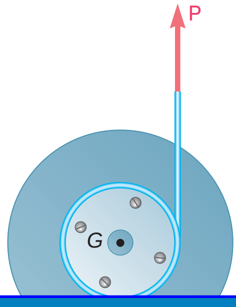


Fig. *P16.101* and P16.105

***16.106* and 16.107** A 12-in.-radius cylinder of weight 16 lb rests on a 6-lb carriage. The system is at rest when a force **P** of magnitude 4 lb is applied. Knowing that the cylinder rolls without sliding on the carriage and neglecting the mass of the wheels of the carriage, determine (*a*) the acceleration of the carriage, (*b*) the acceleration of point *A*, (*c*) the distance the cylinder has rolled with respect to the carriage after 0.5 s.

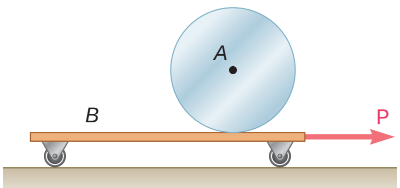


Fig. *P16.106*

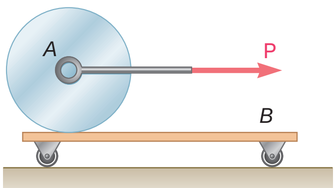


Fig. P16.107

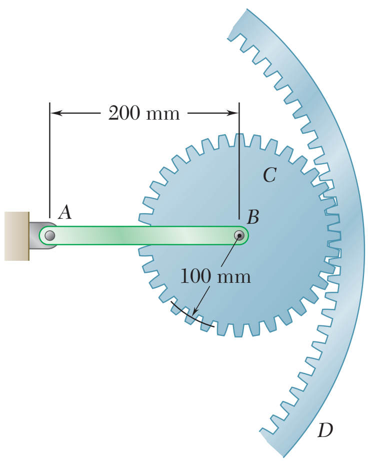


Fig. P16.108

16.108 Gear *C* has a mass of 5 kg and a centroidal radius of gyration of 75 mm. The uniform bar *AB* has a mass of 3 kg and gear *D* is stationary. If the system is released from rest in the position shown, determine (*a*) the angular acceleration of gear *C*, (*b*) the acceleration of point *B*.

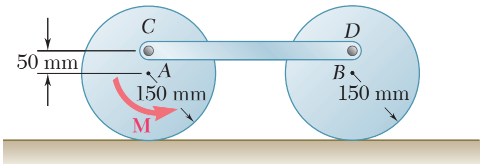


Fig. P16.109

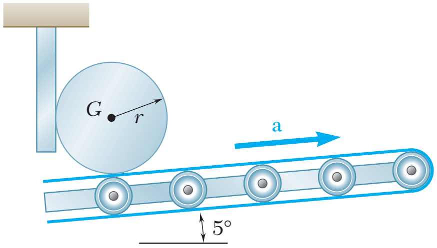


Fig. *P16.110*

16.109 Two uniform disks A and B, each of mass 2 kg, are connected by a 1.5-kg rod CD as shown. A counterclockwise couple **M** of moment 2.5 N-m is applied to disk A. Knowing that the disks roll without sliding, determine (*a*) the acceleration of the center of each disk, (*b*) the horizontal component of the force exerted on disk B by pin D.

16.110 A 10-lb cylinder of radius $r = 4$ in. is resting on a conveyor belt when the belt is suddenly turned on and it experiences an acceleration of magnitude $a = 6$ ft/s^2. The smooth vertical bar holds the cylinder in place when the belt is not moving. Knowing the cylinder rolls without slipping and the friction between the vertical bar and the cylinder is negligible, determine (*a*) the angular acceleration of the cylinder, (*b*) the components of the force the conveyor belt applies to the cylinder.

16.111 A hemisphere of weight W and radius r is released from rest in the position shown. Determine (*a*) the minimum value of m_s for which the hemisphere starts to roll without sliding, (*b*) the corresponding acceleration of point B [*Hint:* Note that $OG = \frac{3}{8}r$ and that, by the parallel-axis theorem, $\bar{I} = \frac{2}{5}mr^2 - m(OG)^2$.]

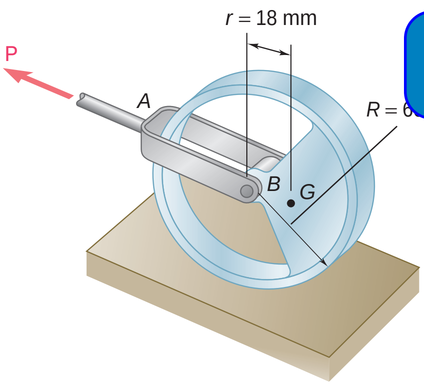


Fig. P16.113

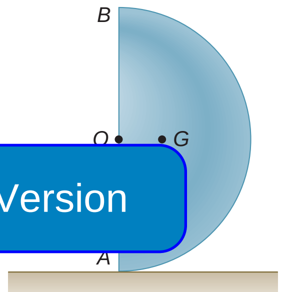


Fig. P16.111

16.112 Solve Prob. 16.111, considering a half cylinder instead of a hemisphere. [*Hint:* Note that $OG = 4r/3p$ and that, by the parallel-axis theorem, $\bar{I} = \frac{1}{2}mr^2 - m(OG)^2$.]

16.113 The center of gravity G of a 1.5-kg unbalanced tracking wheel is located at a distance $r = 18$ mm from its geometric center B. The radius of the wheel is $R = 60$ mm and its centroidal radius of gyration is 44 mm. At the instant shown the center B of the wheel has a velocity of 0.35 m/s and an acceleration of 1.2 m/s^2, both directed to the left. Knowing that the wheel rolls without sliding and neglecting the mass of the driving yoke AB, determine the horizontal force **P** applied to the yoke.

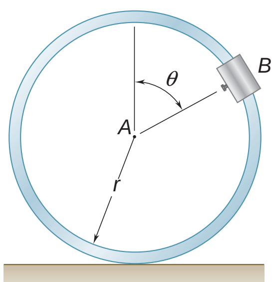


Fig. P16.114 and P16.115

16.114 A small clamp of mass m_B is attached at B to a hoop of mass m_h. The system is released from rest when $u = 90°$ and rolls without sliding. Knowing that $m_h = 3m_B$, determine (*a*) the angular acceleration of the hoop, (*b*) the horizontal and vertical components of the acceleration of B.

16.115 A small clamp of mass m_B is attached at B to a hoop of mass m_h. Knowing that the system is released from rest and rolls without sliding, derive an expression for the angular acceleration of the hoop in terms of m_B, m_h, r, and u.

16.116 A 4-lb bar is attached to a 10-lb uniform cylinder by a square pin, P, as shown. Knowing that $r = 16$ in., $h = 8$ in., $\theta = 20°$, $L = 20$ in., and $\omega = 2$ rad/s at the instant shown, determine the reactions at P at this instant assuming that the cylinder rolls without sliding down the incline.

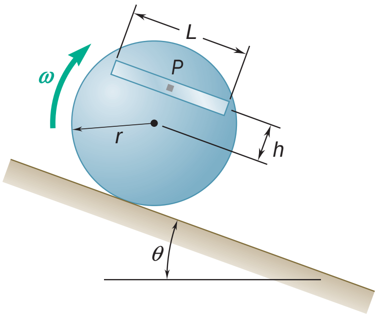


Fig. P16.116

16.117 The ends of the 20-lb uniform rod AB are attached to collars of negligible mass that slide without friction along fixed rods. If the rod is released from rest when $\theta = 25°$, determine immediately after release (a) the angular acceleration of the rod, (b) the reaction at A, (c) the reaction at B.

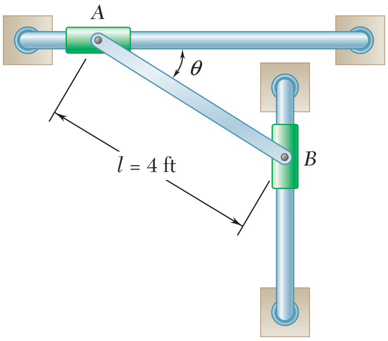


Fig. P16.117 and P16.118

16.118 The ends of the 20-lb u
negligible mass that slide
cal force **P** is applied to
to start from rest with an upward acceleration of 40 ft/s². Deter-
mine (a) the force **P**, (b) the reaction at A.

16.119 The motion of the 3-kg uniform rod AB is guided by small wheels of negligible weight that roll along without friction in the slots shown. If the rod is released from rest in the position shown, determine immediately after release (a) the angular acceleration of the rod, (b) the reaction at B.

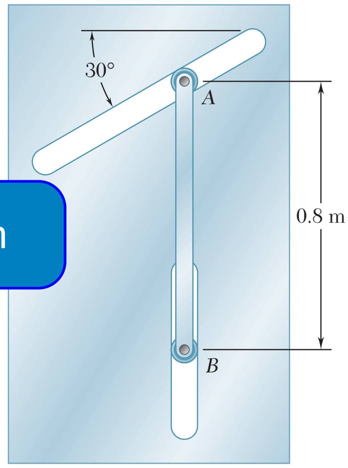


Fig. P16.119

16.120 A beam AB of length L and mass m is supported by two cables as shown. If cable BD breaks, determine at that instant the tension in the remaining cable as a function of its initial angular orientation θ.

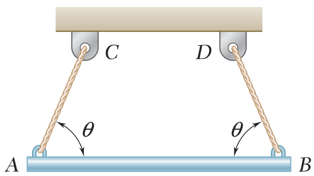


Fig. P16.120

16.121 End A of a uniform 10-kg bar is attached to a horizontal rope and end B contacts a floor with negligible friction. Knowing that the bar is released from rest in the position shown, determine immediately after release (a) the angular acceleration of the bar, (b) the tension in the rope, (c) the reaction at B.

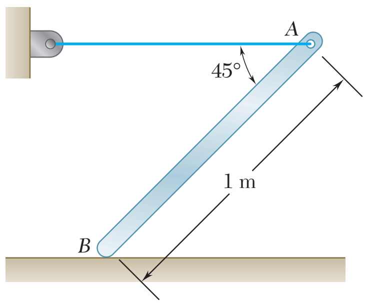


Fig. P16.121

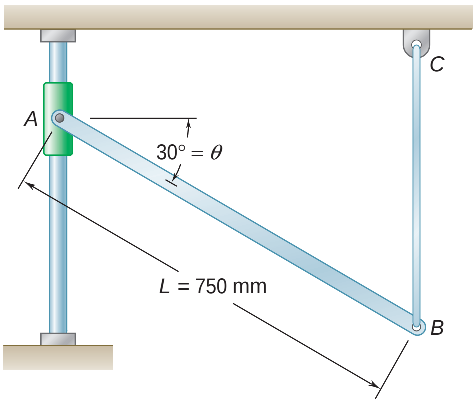
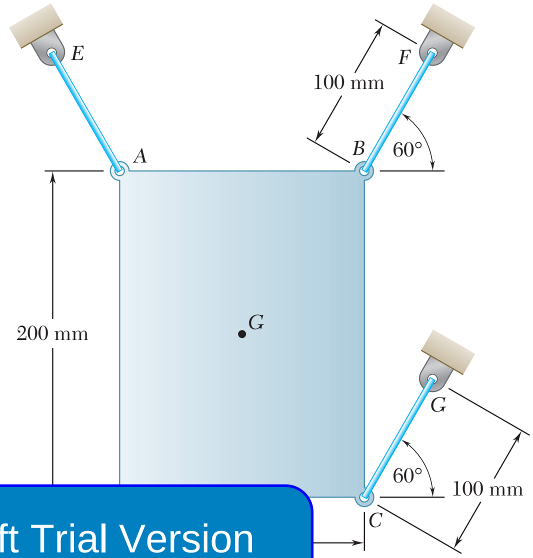


Fig. *P16.122*

16.122 End A of the 8-kg uniform rod AB is attached to a collar that can slide without friction on a vertical rod. End B of the rod is attached to a vertical cable BC. If the rod is released from rest in the position shown, determine immediately after release (*a*) the angular acceleration of the rod, (*b*) the reaction at A.

16.123 A uniform thin plate $ABCD$ has a mass of 8 kg and is held in position by three inextensible cords AE, BF, and CG. If cord AE is cut, determine at that instant (*a*) if the plate is undergoing translation or general plane motion, (*b*) the tension in cords BF and CG.



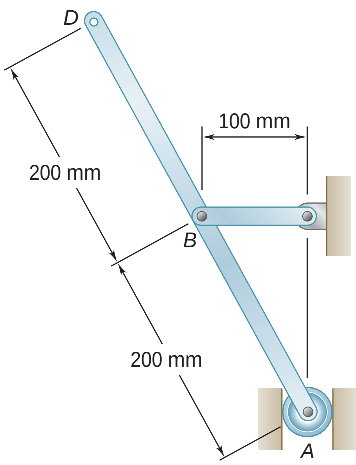


Fig. P16.124

16.124 The 4-kg uniform rod ABD is attached to the crank BC and is fitted with a small wheel that can roll without friction along a vertical slot. Knowing that at the instant shown crank BC rotates with an angular velocity of 6 rad/s clockwise and an angular acceleration of 15 rad/s^2 counterclockwise, determine the reaction at A.

16.125 The 7-lb uniform rod AB is connected to crank BD and to a collar of negligible weight, which can slide freely along rod EF. Knowing that in the position shown crank BD rotates with an angular velocity of 15 rad/s and an angular acceleration of 60 rad/s^2, both clockwise, determine the reaction at A.

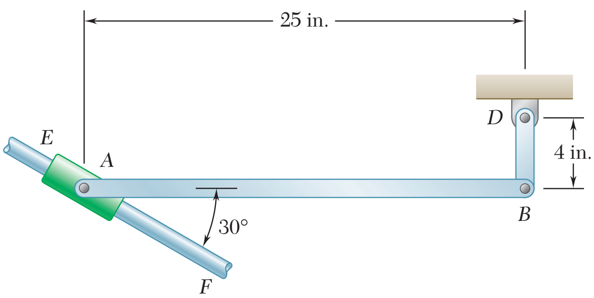


Fig. P16.125

16.126 In Prob. 16.125, determine the reaction at A, knowing that in the position shown crank BD rotates with an angular velocity of 15 rad/s clockwise and an angular acceleration of 60 rad/s^2 counterclockwise.

16.127 The 250-mm uniform rod *BD*, of mass 5 kg, is connected as shown to disk *A* and to a collar of negligible mass, that may slide freely along a vertical rod. Knowing that disk *A* rotates counterclockwise at a constant rate of 500 rpm, determine the reactions at *D* when u = 0.

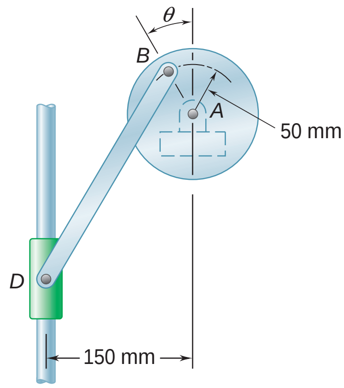
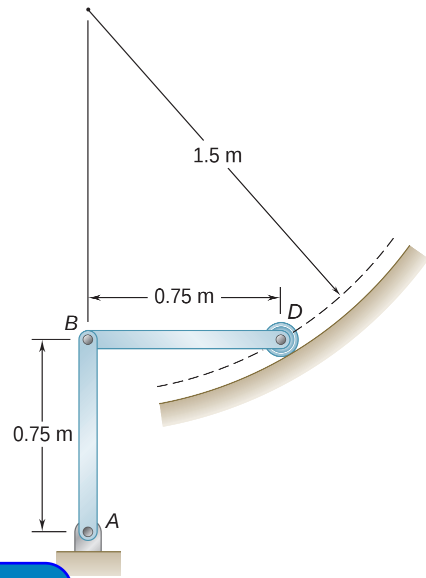


Fig. P16.127

16.128 Solve Prob. 16.127 when u = 90°.

16.129 The 4-kg uniform slender bar *BD* is attached to bar *AB* and a wheel of negligible mass that rolls on a circular surface. Knowing that at the instant shown … and no angular accelerat…



…16.129

16.130 The motion of the unifo… mass $m = 3$ kg is guided by pins at *A* and *B* that slide freely in frictionless slots, circular and horizontal, cut into a vertical plate as shown. Knowing that at the instant shown the rod has an angular velocity of 3 rad/s counterclockwise and u = 30°, determine the reactions at points *A* and *B*.

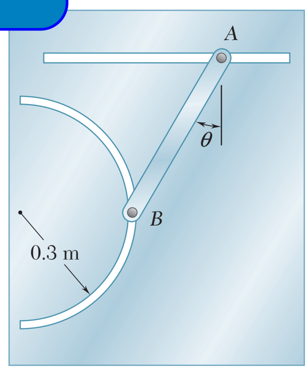


Fig. *P16.130*

16.131 At the instant shown, the 20-ft-long, uniform 100-lb pole *ABC* has an angular velocity of 1 rad/s counterclockwise and point *C* is sliding to the right. A 120-lb horizontal force **P** acts at *B*. Knowing the coefficient of kinetic friction between the pole and the ground is 0.3, determine at this instant (*a*) the acceleration of the center of gravity, (*b*) the normal force between the pole and the ground.

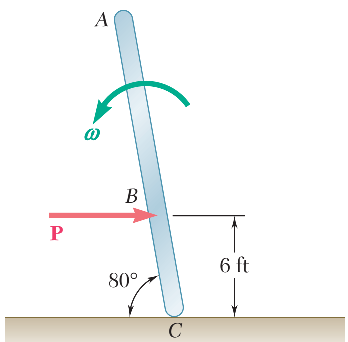


Fig. P16.131

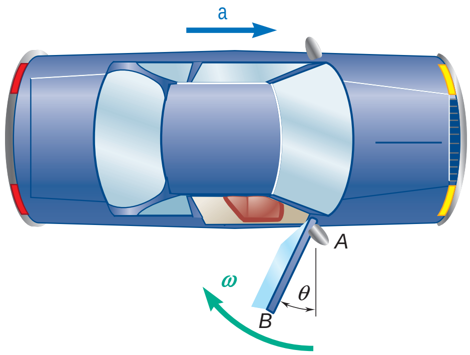


Fig. *P16.132*

16.132 A driver starts his car with the door on the passenger's side wide open (u = 0). The 80-lb door has a centroidal radius of gyration $\bar{k}$ = 12.5 in., and its mass center is located at a distance r = 22 in. from its vertical axis of rotation. Knowing that the driver maintains a constant acceleration of 6 ft/s^2, determine the angular velocity of the door as it slams shut (u = 90°).

16.133 For the car of Prob. 16.132, determine the smallest constant acceleration that the driver can maintain if the door is to close and latch, knowing that as the door hits the frame its angular velocity must be at least 2 rad/s for the latching mechanism to operate.

16.134 Two 8-lb uniform bars are connected to form the linkage shown. Neglecting the effect of friction, determine the reaction at D immediately after the linkage is released from rest in the position shown.

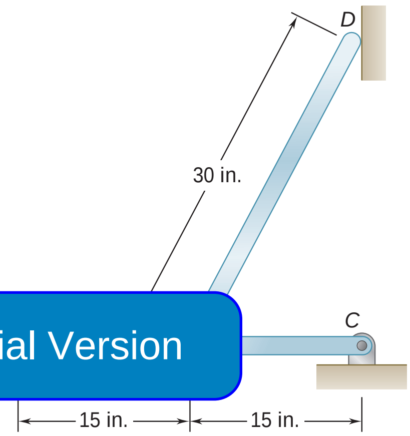


Fig. P16.134

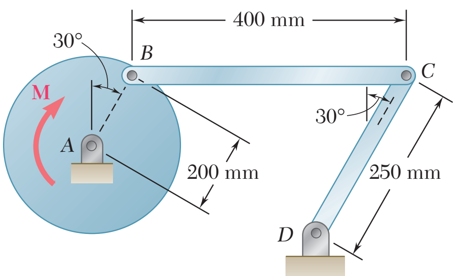


Fig. P16.135 and P16.136

***16.135** The 6-kg rod BC connects a 10-kg disk centered at A to a 5-kg rod CD. The motion of the system is controlled by the couple **M** applied to disk A. Knowing that at the instant shown disk A has an angular velocity of 36 rad/s clockwise and no angular acceleration, determine (*a*) the couple **M**, (*b*) the components of the force exerted at C on rod BC.

***16.136** The 6-kg rod BC connects a 10-kg disk centered at A to a 5-kg rod CD. The motion of the system is controlled by the couple **M** applied to disk A. Knowing that at the instant shown disk A has an angular velocity of 36 rad/s clockwise and an angular acceleration of 150 rad/s^2 counterclockwise, determine (*a*) the couple **M**, (*b*) the components of the force exerted at C on rod BC.

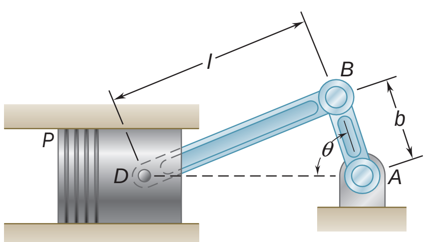


Fig. P16.137

16.137 In the engine system shown l = 250 mm and b = 100 mm. The connecting rod BD is assumed to be a 1.2-kg uniform slender rod and is attached to the 1.8-kg piston P. During a test of the system, crank AB is made to rotate with a constant angular velocity of 600 rpm clockwise with no force applied to the face of the piston. Determine the forces exerted on the connecting rod at B and D when u = 180°. (Neglect the effect of the weight of the rod.)

16.138 Solve Prob. 16.137 when u = 90°.

16.139 The 4-lb rod AB and the 6-lb rod BC are connected as shown to a disk that is made to rotate in a vertical plane at a constant angular velocity of 6 rad/s clockwise. For the position shown, determine the forces exerted at A and B on rod AB.

16.140 The 4-lb rod AB and the 6-lb rod BC are connected as shown to a disk that is made to rotate in a vertical plane. Knowing that at the instant shown the disk has an angular acceleration of 18 rad/s^2 clockwise and no angular velocity, determine the components of the forces exerted at A and B on rod AB.

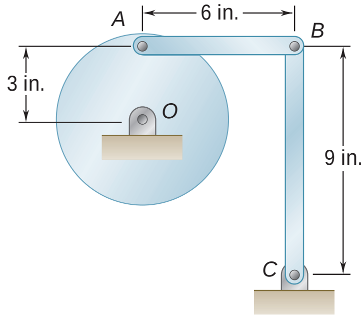
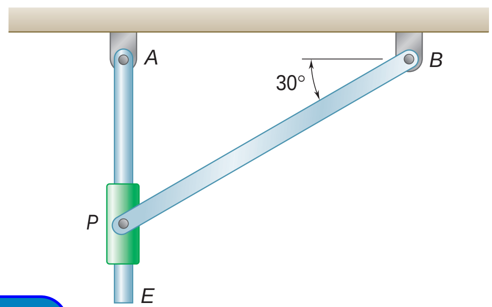


Fig. P16.139 and P16.140

16.141 Two rotating rods in the vertical plane are connected by a slider block P of negligible mass. The rod attached at A has a weight of 1.6 lb and a length of 8 in. Rod BP weighs 2 lb and is 10 in. long and the friction between block P and AE is negligible. The motion of the system is controlled by a couple $\mathbf{M}$ applied to rod BP. Knowing that rod BP has a constant angular velocity of 20 rad/s clockwise, determine (a) the couple $\mathbf{M}$, (b) the components of the force exerted on AE by block P.

16.142 Two rotating rods in the vertical plane are connected by a slider block P of negligible mass. The rod attached at A has a mass of 0.8 kg and a length of 160 mm. Rod BP has a mass of 1 kg and is 200 mm long and the friction between block P and AE is negligible. The motion of the system is controlled by a couple $\mathbf{M}$ applied to bar BP. Knowing that at the instant shown rod BP has an angular velocity of 20 rad/s clockwise and an angular acceleration of 80 rad/s^2 clockwise, determine (a) the c[illegible] exerted on AE by block [illegible]



[illegible]***16.141*** **and** ***P16.142***

***16.143** Draw the shear and bending-moment diagrams for the rod of Prob. 16.77 immediately after the cable at B breaks.

***16.144** A uniform slender bar AB of mass m is suspended as shown from a uniform disk of the same mass m. Neglecting the effect of friction, determine the accelerations of points A and B immediately after a horizontal force $\mathbf{P}$ has been applied at B.

16.145 A uniform rod AB, of mass 15 kg and length 1 m, is attached to the 20-kg cart C. Neglecting friction, determine immediately after the system has been released from rest, (a) the acceleration of the cart, (b) the angular acceleration of the rod.

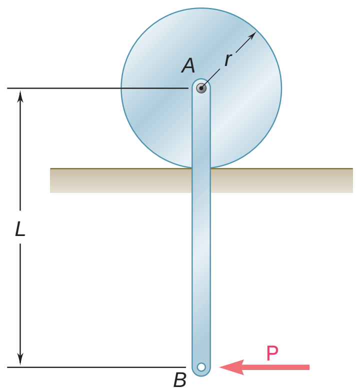


Fig. P16.144

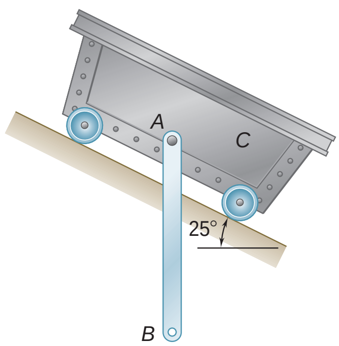


Fig. P16.145

***16.146** The 5-kg slender rod AB is pin-connected to an 8-kg uniform disk as shown. Immediately after the system is released from rest, determine the acceleration of (*a*) point A, (*b*) point B.

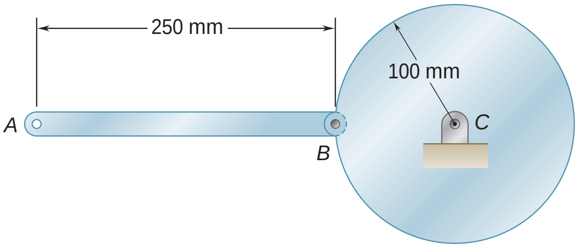


Fig. P16.146

***16.147 and *16.148** The 6-lb cylinder B and the 4-lb wedge A are held at rest in the position shown by cord C. Assuming that the cylinder rolls without sliding on the wedge and neglecting friction between the wedge and the ground, determine, immediately after cord C has been cut, (*a*) the acceleration of the wedge, (*b*) the angular acceleration of the cylinder.

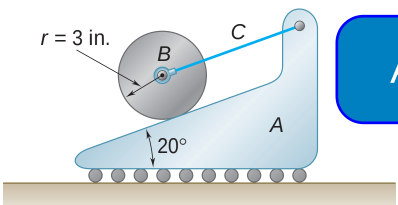


Fig. P16.147

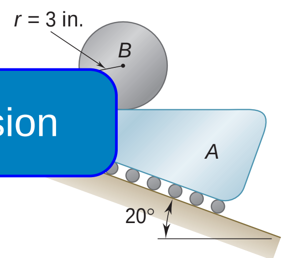


Fig. P16.148

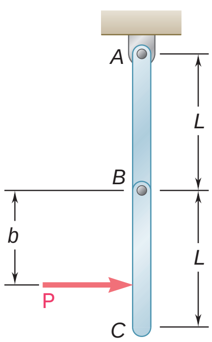


Fig. *P16.149* and *P16.150*

***16.149** Each of the 3-kg bars AB and BC is of length $L = 500$ mm. A horizontal force **P** of magnitude 20 N is applied to bar BC as shown. Knowing that $b = L$ (**P** is applied at C), determine the angular acceleration of each bar.

***16.150** Each of the 3-kg bars AB and BC is of length $L = 500$ mm. A horizontal force **P** of magnitude 20 N is applied to bar BC. For the position shown, determine (*a*) the distance b for which the bars move as if they formed a single rigid body, (*b*) the corresponding angular acceleration of the bars.

***16.151** (*a*) Determine the magnitude and the location of the maximum bending moment in the rod of Prob. 16.78. (*b*) Show that the answer to part *a* is independent of the weight of the rod.

***16.152** Draw the shear and bending-moment diagrams for the rod of Prob. 16.84 immediately after the cable at B breaks.

REVIEW AND SUMMARY

In this chapter, we studied the *kinetics of rigid bodies,* i.e., the relations existing between the forces acting on a rigid body, the shape and mass of the body, and the motion produced. Except for the first two sections, which apply to the most general case of the motion of a rigid body, our analysis was restricted to the *plane motion of rigid slabs* and rigid bodies symmetrical with respect to the reference plane. The study of the plane motion of nonsymmetrical rigid bodies and of the motion of rigid bodies in three-dimensional space will be considered in Chap. 18.

Fundamental equations of motion for a rigid body

We first recalled [Sec. 16.2] the two fundamental equations derived in Chap. 14 for the motion of a system of particles and observed that they apply in the most general case of the motion of a rigid body. The first equation defines the motion of the mass center G of the body; we have

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \tag{16.1}$$

where m is the mass of the b second is related to the motio frame of reference; we wrote

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{16.2}$$

where $\dot{\mathbf{H}}_G$ is the rate of change of the angular momentum $\mathbf{H}_G$ of the body about its mass center G. Together, Eqs. (16.1) and (16.2) express that *the system of the external forces is equipollent to the system consisting of the vector* $m\bar{\mathbf{a}}$ *attached at* G *and the couple of moment* $\dot{\mathbf{H}}_G$ (Fig. 16.19).

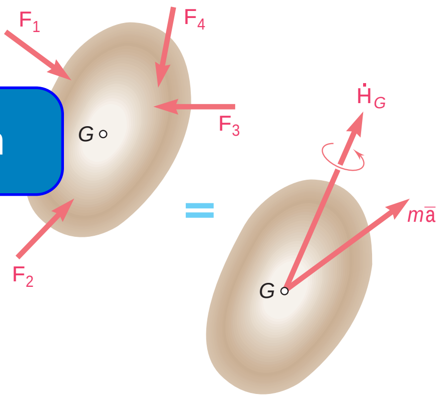


Fig. 16.19

Angular momentum in plane motion

Restricting our analysis at this point and for the rest of the chapter to the plane motion of rigid slabs and rigid bodies symmetrical with respect to the reference plane, we showed [Sec. 16.3] that the angular momentum of the body could be expressed as

$$\mathbf{H}_G = \bar{I}\mathrm{V} \tag{16.4}$$

where $\bar{I}$ is the moment of inertia of the body about a centroidal axis perpendicular to the reference plane and V is the angular velocity of the body. Differentiating both members of Eq. (16.4), we obtained

$$\dot{\mathbf{H}}_G = \bar{I}\dot{\mathrm{V}} = \bar{I}\mathrm{A} \tag{16.5}$$

which shows that in the restricted case considered here, the rate of change of the angular momentum of the rigid body can be represented by a vector of the same direction as $\boldsymbol{\alpha}$ (i.e., perpendicular to the plane of reference) and of magnitude $\bar{I}\alpha$.

Equations for the plane motion of a rigid body

It follows from [Sec. 16.4] that the plane motion of a rigid slab or of a rigid body symmetrical with respect to the reference plane is defined by the three scalar equations

$$\Sigma F_x = m\bar{a}_x \qquad \Sigma F_y = m\bar{a}_y \qquad \Sigma M_G = \bar{I}\alpha \qquad (16.6)$$

D'Alembert's principle

It further follows that *the external forces acting on the rigid body are actually* ***equivalent*** *to the effective forces of the various particles forming the body*. This statement, known as *d'Alembert's principle*, can be expressed in the form of the vector diagram shown in Fig. 16.20, where the effective forces have been represented by a vector $m\bar{\mathbf{a}}$ attached at G and a couple $\bar{I}\boldsymbol{\alpha}$. In the particular case of a slab in *translation*, the effective forces shown in part b of this figure reduce to the single vector $m\bar{\mathbf{a}}$, while in the particular case of a slab in *centroidal rotation*, they reduce to the single couple $\bar{I}\boldsymbol{\alpha}$; in any other case of plane motion, both the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}\boldsymbol{\alpha}$ should be included.

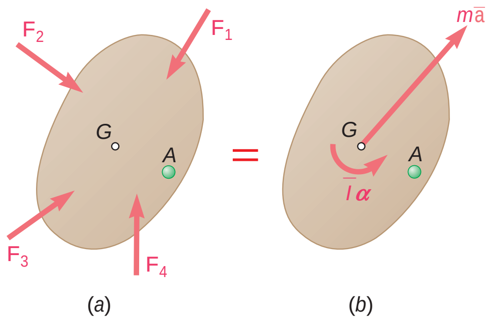


Fig. 16.20

Free-body-diagram equation

Any problem involving the plane motion of a rigid slab may be solved by drawing a *free-body-diagram equation* similar to that of Fig. 16.20 [Sec. 16.6]. Three equations of motion can then be obtained by ...nents, and moments about an ...ectors involved [Sample Probs. ...ive solution can be obtained by adding to the external forces an *inertia vector* $-m\bar{\mathbf{a}}$ of sense opposite to that of $\bar{\mathbf{a}}$, attached at G, and an *inertia couple* $-\bar{I}\boldsymbol{\alpha}$ of sense opposite to that of $\boldsymbol{\alpha}$. The system obtained in this way is equivalent to zero, and the slab is said to be in *dynamic equilibrium*.

Connected rigid bodies

The method described above can also be used to solve problems involving the plane motion of several connected rigid bodies [Sec. 16.7]. A free-body-diagram equation is drawn for each part of the system and the equations of motion obtained are solved simultaneously. In some cases, however, a single diagram can be drawn for the entire system, including all the external forces as well as the vectors $m\bar{\mathbf{a}}$ and the couples $\bar{I}\boldsymbol{\alpha}$ associated with the various parts of the system [Sample Prob. 16.3].

Constrained plane motion

In the second part of the chapter, we were concerned with rigid bodies *moving under given constraints* [Sec. 16.8]. While the kinetic analysis of the constrained plane motion of a rigid slab is the same as above, it must be supplemented by a *kinematic analysis* which has for its object to express the components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of the mass center G of the slab in terms of its angular acceleration α. Problems solved in this way included the *noncentroidal rotation* of rods and plates [Sample Probs. 16.6 and 16.7], the *rolling motion* of spheres and wheels [Sample Probs. 16.8 and 16.9], and the plane motion of *various types of linkages* [Sample Prob. 16.10].

REVIEW PROBLEMS

16.153 A cyclist is riding a bicycle at a speed of 20 mph on a horizontal road. The distance between the axles is 42 in., and the mass center of the cyclist and the bicycle is located 26 in. behind the front axle and 40 in. above the ground. If the cyclist applies the brakes only on the front wheel, determine the shortest distance in which he can stop without being thrown over the front wheel.

16.154 The forklift truck shown weighs 2250 lb and is used to lift a crate of weight $W = 2500$ lb. The truck is moving to the left at a speed of 10 ft/s when the brakes are applied on all four wheels. Knowing that the coefficient of static friction between the crate and the fork lift is 0.30, determine the smallest distance in which the truck can be brought to a stop if the crate is not to slide and if the truck is not to tip forward.

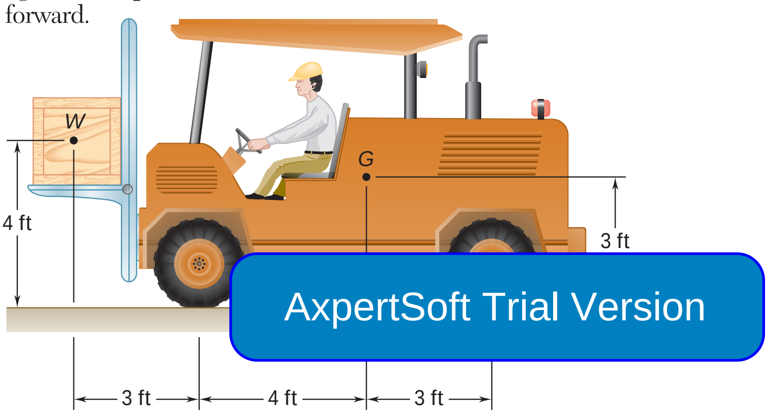


Fig. P16.154

16.155 A 5-kg uniform disk is attached to the 3-kg uniform rod BC by means of a frictionless pin AB. An elastic cord is wound around the edge of the disk and is attached to a ring at E. Both ring E and rod BC can rotate freely about the vertical shaft. Knowing that the system is released from rest when the tension in the elastic cord is 15 N, determine (*a*) the angular acceleration of the disk, (*b*) the acceleration of the center of the disk.

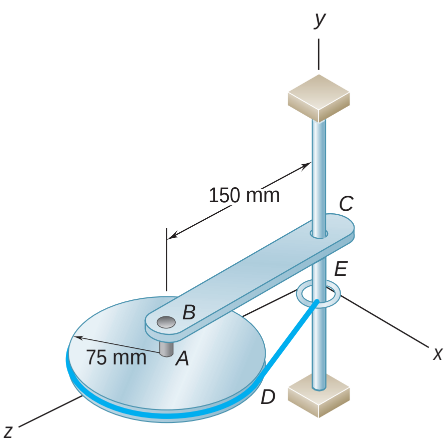


Fig. *P16.155*

16.156 Identical cylinders of mass m and radius r are pushed by a series of moving arms. Assuming the coefficient of friction between all surfaces to be m < 1 and denoting by a the magnitude of the acceleration of the arms, derive an expression for (*a*) the maximum allowable value of a if each cylinder is to roll without sliding, (*b*) the minimum allowable value of a if each cylinder is to move to the right without rotating.

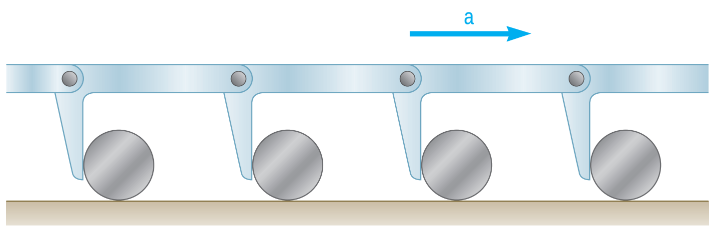


Fig. P16.156

16.157 The uniform rod AB of weight W is released from rest when b = 70°. Assuming that the friction force between end A and the surface is large enough to prevent sliding, determine immediately after release (*a*) the angular acceleration of the rod, (*b*) the normal reaction at A, (*c*) the friction force at A.

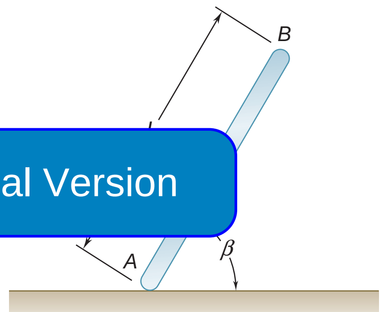


Fig. P16.157 and P16.158

16.158 The uniform rod AB of weight W is released from rest when b = 70°. Assuming that the friction force is zero between end A and the surface, determine immediately after release (*a*) the angular acceleration of the rod, (*b*) the acceleration of the mass center of the rod, (*c*) the reaction at A.

16.159 A bar of mass $m = 5$ kg is held as shown between four disks, each of mass $m' = 2$ kg and radius $r = 75$ mm. Knowing that the normal forces on the disks are sufficient to prevent any slipping, for each of the cases shown determine the acceleration of the bar immediately after it has been released from rest.

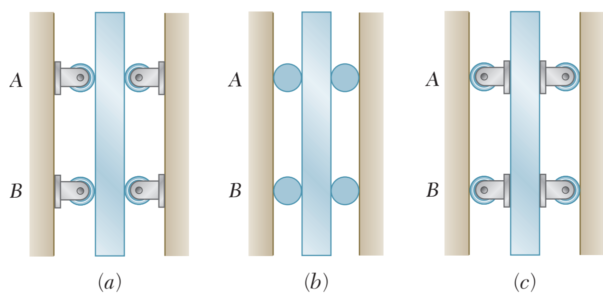


Fig. *P16.159*

16.160 A uniform plate of mass m is suspended in each of the ways shown. For each case determine immediately after the connection B has been released (*a*) the angular acceleration of the plate, (*b*) the acceleration of its mass center.

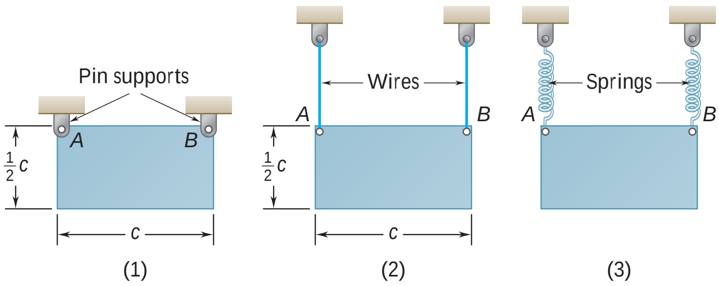


Fig. P16.160

16.161 A cylinder with a circular hole is rolling without slipping on a fixed curved surface as shown. The cylinder would have a weight of 16 lb without the hole, but with the hole it has a weight of 15 lb. Knowing that at the instant shown the disk has an angular velocity of 5 rad/s clockwise, determine (*a*) the angular acceleration of the disk, (*b*) the components
and the ground at this i

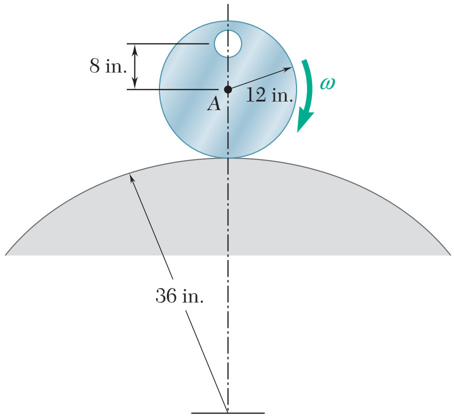


Fig. *P16.161*

16.162 The motion of a square plate of side 150 mm and mass 2.5 kg is guided by pins at corners A and B that slide in slots cut in a vertical wall. Immediately after the plate is released from rest in the position shown, determine (*a*) the angular acceleration of the plate, (*b*) the reaction at corner A.

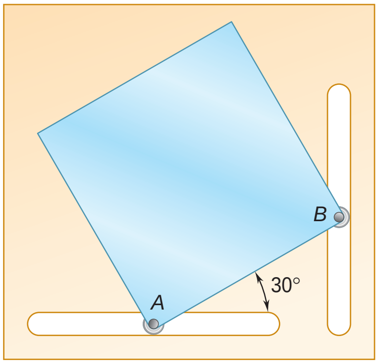


Fig. P16.162

16.163 The motion of a square plate of side 150 mm and mass 2.5 kg is guided by a pin at corner *A* that slides in a horizontal slot cut in a vertical wall. Immediately after the plate is released from rest in the position shown, determine (*a*) the angular acceleration of the plate, (*b*) the reaction at corner *A*.

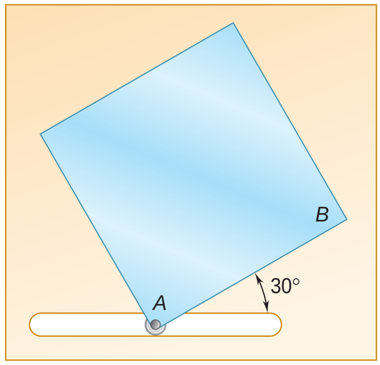


Fig. P16.163

16.164 The Geneva mechanism shown is used to provide an intermittent rotary motion of disk *S*. Disk *D* weighs 2 lb and has a radius of gyration of 0.9 in., and disk *S* weighs 6 lb and has a radius of gyration of 1.5 in. The motion of the system is controlled by a couple [illegible] attached to disk *D* and can slide [illegible] slots cut in disk *S*. It is desirable [illegible] *S* be zero as the pin enters and [illegible] will occur if the distance between the centers of the disks and the radii of the disks are related as shown. Knowing disk *D* rotates with a constant counterclockwise angular velocity of 8 rad/s and the friction between the slot and pin *P* is negligible, determine when f = 150° (*a*) the couple **M**, (*b*) the magnitude of the force pin *P* applies to disk *S*.

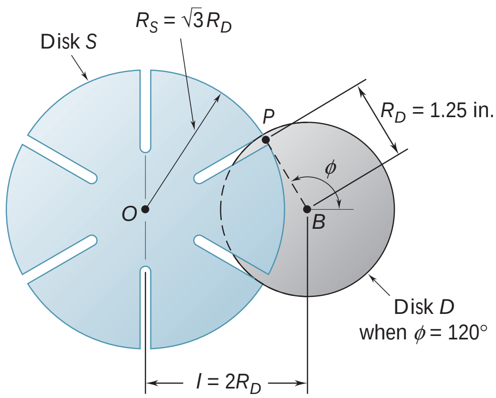


Fig. P16.164

COMPUTER PROBLEMS

16.C1 The 5-lb rod AB is released from rest in the position shown. (a) Assuming that the friction force between end A and the surface is large enough to prevent sliding, using software calculate the normal reaction and the friction force at A immediately after release for values of b from 0 to 85°. (b) Knowing that the coefficient of static friction between the rod and the floor is actually equal to 0.50, determine the range of values of b for which the rod will slip immediately after being released from rest.

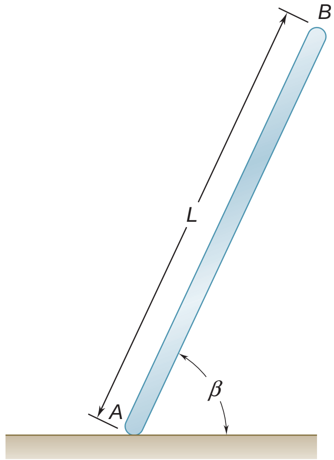


Fig. P16.C1

16.C2 End A of the 5-kg rod AB is moved to the left at a constant speed v_A = 1.5 m/s. Using computational software calculate and plot the normal reactions at ends A and B of the rod for values of u from 0 to 50°. Determine the value of u at which end B of the rod loses contact with the wall.

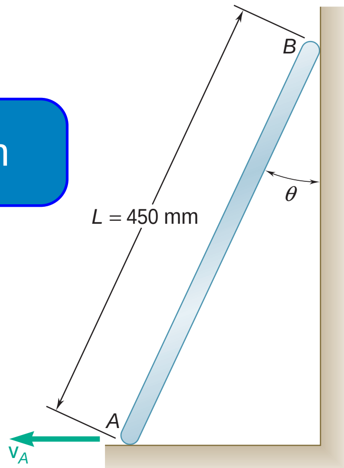


Fig. P16.C2

16.C3 A 30-lb cylinder of diameter b = 8 in. and height h = 6 in. is placed on a 10-lb platform CD that is held in the position shown by three cables. It is desired to determine the minimum value of m_s between the cylinder and the platform for which the cylinder does not slip on the platform, immediately after cable AB is cut. Using computational software calculate and plot the minimum allowable value of m_s for values of u from 0 to 30°. Knowing that the actual value of m_s is 0.60, determine the value of u at which slipping impends.

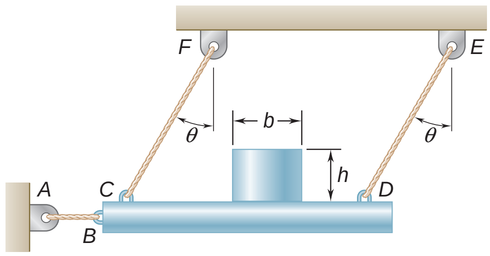


Fig. P16.C3

16.C4 For the engine system of Prob. 15.C3 of Chap. 15, the masses of piston P and the connecting rod BD are 2.5 kg and 3 kg, respectively. Knowing that during a test of the system no force is applied to the face of the piston, use computational software to calculate and plot the horizontal and vertical components of the dynamic reactions exerted on the connecting rod at B and D for values of u from 0 to 180°.

16.C5 A uniform slender bar AB of mass m is suspended from springs AC and BD as shown. Using computational software calculate and plot the accelerations of ends A and B, immediately after spring AC has broken, for values of u from 0 to 90°.

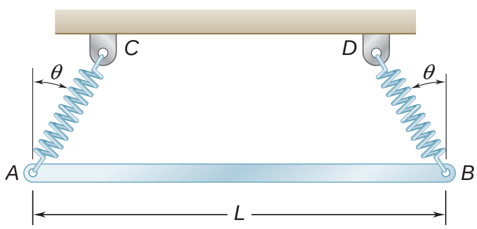
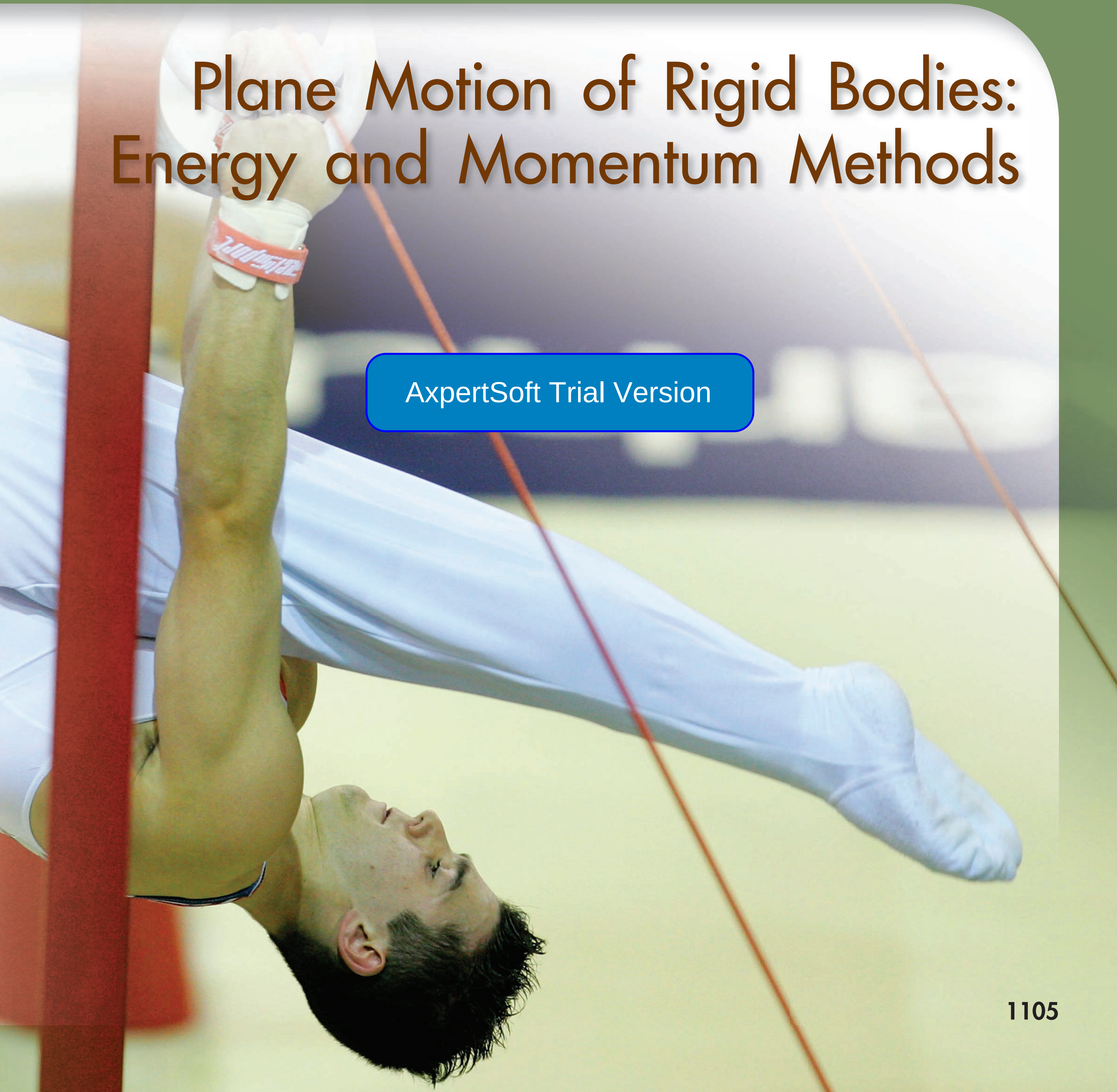


Fig. P16.C5

In this chapter the energy and momentum methods will be added to the tools available for your study of the motion of rigid bodies. For example, by using the principle of conservation of energy and direct application of Newton's secon [illegible] forces exerted on the hands of this gymnast can be determined as he swings from one stationary hold to another.

CHAPTER 17

Plane Motion of Rigid Bodies: Energy and Momentum Methods



17.1 INTRODUCTION

In this chapter the method of work and energy and the method of impulse and momentum will be used to analyze the plane motion of rigid bodies and of systems of rigid bodies.

The method of work and energy will be considered first. In Secs. 17.2 through 17.5, the work of a force and of a couple will be defined, and an expression for the kinetic energy of a rigid body in plane motion will be obtained. The principle of work and energy will then be used to solve problems involving displacements and velocities. In Sec. 17.6, the principle of conservation of energy will be applied to the solution of a variety of engineering problems.

In the second part of the chapter, the principle of impulse and momentum will be applied to the solution of problems involving velocities and time (Secs. 17.8 and 17.9) and the concept of conservation of angular momentum will be introduced and discussed (Sec. 17.10).

In the last part of the chapter (Secs. 17.11 and 17.12), problems involving the eccentric impact of rigid bodies will be considered. As was done in Chap. 13, where we analyzed the impact of particles, the coefficient of restitution between the colliding bodies will be used together with the principle of impulse and momentum in the solution of impact problems. It will also be shown that the method used is applicable not only when the colliding bodies move freely after the impact but also when the bodies are partially constrained in their motion.

ND ENERGY

The principle of work and energy will now be used to analyze the plane motion of rigid bodies. As was pointed out in Chap. 13, the method of work and energy is particularly well adapted to the solution of problems involving velocities and displacements. Its main advantage resides in the fact that the work of forces and the kinetic energy of particles are scalar quantities.

In order to apply the principle of work and energy to the analysis of the motion of a rigid body, it will again be assumed that the rigid body is made of a large number n of particles of mass Δm_i. Recalling Eq. (14.30) of Sec. 14.8, we write

$$T_1 + U_{1\rightarrow 2} = T_2 \tag{17.1}$$

where T_1, T_2 = initial and final values of total kinetic energy of particles forming the rigid body

$U_{1\rightarrow 2}$ = work of all forces acting on various particles of the body

The total kinetic energy

$$T = \frac{1}{2}\sum_{i=1}^{n} \Delta m_i\, v_i^2 \tag{17.2}$$

is obtained by adding positive scalar quantities and is itself a positive scalar quantity. You will see later how T can be determined for various types of motion of a rigid body.

Photo 17.1 The work done by friction reduces the kinetic energy of the wheel.

The expression $U_{1 \rightarrow 2}$ in (17.1) represents the work of all the forces acting on the various particles of the body, whether these forces are internal or external. However, as you will see presently, the total work of the internal forces holding together the particles of a rigid body is zero. Consider two particles A and B of a rigid body and the two equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ they exert on each other (Fig. 17.1). While, in general, small displacements $d\mathbf{r}$ and $d\mathbf{r}'$ of the two particles are different, the components of these displacements along AB must be equal; otherwise, the particles would not remain at the same distance from each other and the body would not be rigid. Therefore, the work of $\mathbf{F}$ is equal in magnitude and opposite in sign to the work of $-\mathbf{F}$, and their sum is zero. Thus, the total work of the internal forces acting on the particles of a rigid body is zero, and *the expression $U_{1 \rightarrow 2}$ in Eq. (17.1) reduces to the work of the external forces* acting on the body during the displacement considered.

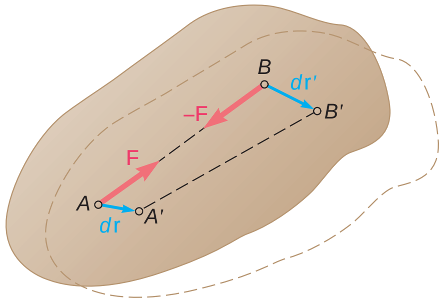


Fig. 17.1

17.3 WORK OF FORCES ACTING ON A RIGID BODY

We saw in Sec. 13.2 that the work of a force $\mathbf{F}$ during a displacement of its point of application from A_1 to A_2 is

$$U_{1 \rightarrow 2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \tag{17.3}$$

or

$U_{1 \rightarrow 2} =$

where F is the magnitude of the force, α is the angle it forms with the direction of motion of its point of application A, and s is the variable of integration which measures the distance traveled by A along its path.

In computing the work of the external forces acting on a rigid body, it is often convenient to determine the work of a couple without considering separately the work of each of the two forces forming the couple. Consider the two forces $\mathbf{F}$ and $-\mathbf{F}$ forming a couple of moment $\mathbf{M}$ and acting on a rigid body (Fig. 17.2). Any small displacement of the rigid body bringing A and B, respectively, into A' and B'' can be divided into two parts: in one part points A and B undergo equal displacements $d\mathbf{r}_1$; in the other part A' remains fixed while B' moves into B'' through a displacement $d\mathbf{r}_2$ of magnitude $ds_2 = r\,d\theta$. In the first part of the motion, the work of $\mathbf{F}$ is equal in magnitude and opposite in sign to the work of $-\mathbf{F}$ and their sum is zero. In the second part of the motion, only force $\mathbf{F}$ works, and its work is $dU = F\,ds_2 = Fr\,d\theta$. But the product Fr is equal to the magnitude M of the moment of the couple. Thus, the work of a couple of moment $\mathbf{M}$ acting on a rigid body is

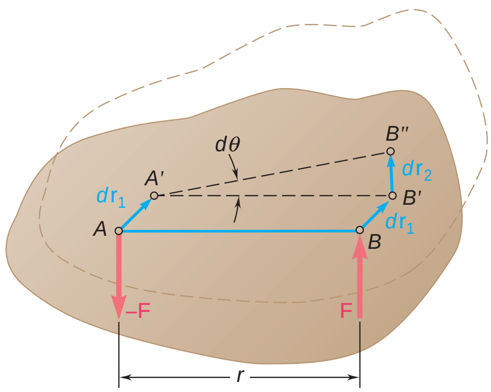


Fig. 17.2

$$dU = M\,d\theta \tag{17.4}$$

where $d\theta$ is the small angle expressed in radians through which the body rotates. We again note that work should be expressed in units obtained by multiplying units of force by units of length. The work of the couple during a finite rotation of the rigid body is obtained by integrating both members of (17.4) from the initial value u_1 of the angle u to its final value u_2. We write

$$U_{1\,y\,2} = \int_{\mathsf{u}_1}^{\mathsf{u}_2} M\, d\mathsf{u} \tag{17.5}$$

When the moment **M** *of the couple is constant,* formula (17.5) reduces to

$$U_{1\,y\,2} = M(\mathsf{u}_2 - \mathsf{u}_1) \tag{17.6}$$

It was pointed out in Sec. 13.2 that a number of forces encountered in problems of kinetics *do no work.* They are forces applied to fixed points or acting in a direction perpendicular to the displacement of their point of application. Among the forces which do no work the following have been listed: the reaction at a frictionless pin when the body supported rotates about the pin, the reaction at a frictionless surface when the body in contact moves along the surface, and the weight of a body when its center of gravity moves horizontally. We can add now that *when a rigid body rolls without sliding on a fixed surface, the friction force* **F** *at the point of contact C does no work.* The velocity $\mathbf{v}_C$ of the point of contact C is zero, and the work of the friction force **F** during a small displacement of the rigid body is

$$\ldots v_C\, dt) = 0$$

… RIGID BODY IN PLANE MOTION

Consider a rigid body of mass m in plane motion. We recall from Sec. 14.7 that, if the absolute velocity $\mathbf{v}_i$ of each particle P_i of the body is expressed as the sum of the velocity $\overline{\mathbf{v}}$ of the mass center G of the body and of the velocity $\mathbf{v}'_i$ of the particle relative to a frame $Gx'y'$ attached to G and of fixed orientation (Fig. 17.3), the kinetic energy of the system of particles forming the rigid body can be written in the form

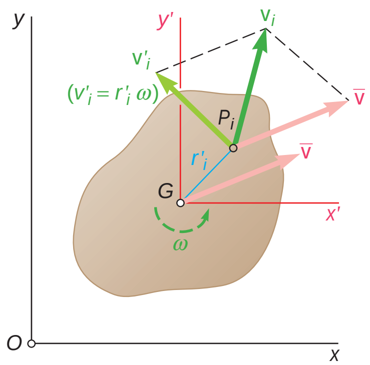


Fig. 17.3

$$T = \tfrac{1}{2}m\overline{v}^2 + \frac{1}{2}\sum_{i=1}^{n} \Delta m_i v_i'^2 \tag{17.7}$$

But the magnitude v'_i of the relative velocity of P_i is equal to the product $r'_i\mathsf{v}$ of the distance r'_i of P_i from the axis through G perpendicular to the plane of motion and of the magnitude v of the angular velocity of the body at the instant considered. Substituting into (17.7), we have

$$T = \tfrac{1}{2}m\overline{v}^2 + \frac{1}{2}\left(\sum_{i=1}^{n} r_i'^2\, \Delta m_i\right) w^2 \tag{17.8}$$

or, since the sum represents the moment of inertia $\overline{I}$ of the body about the axis through G,

$$T = \tfrac{1}{2}m\overline{v}^2 + \tfrac{1}{2}\overline{I}\mathsf{v}^2 \tag{17.9}$$

We note that in the particular case of a body in translation ($\omega = 0$), the expression obtained reduces to $\frac{1}{2}m\bar{v}^2$, while in the case of a centroidal rotation ($\bar{v} = 0$), it reduces to $\frac{1}{2}\bar{I}\omega^2$. We conclude that the kinetic energy of a rigid body in plane motion can be separated into two parts: (1) the kinetic energy $\frac{1}{2}m\bar{v}^2$ associated with the motion of the mass center G of the body, and (2) the kinetic energy $\frac{1}{2}\bar{I}\omega^2$ associated with the rotation of the body about G.

Noncentroidal Rotation. The relation (17.9) is valid for any type of plane motion and can therefore be used to express the kinetic energy of a rigid body rotating with an angular velocity ω about a fixed axis through O (Fig. 17.4). In that case, however, the kinetic energy of the body can be expressed more directly by noting that the speed v_i of the particle P_i is equal to the product $r_i\omega$ of the distance r_i of P_i from the fixed axis and the magnitude ω of the angular velocity of the body at the instant considered. Substituting into (17.2), we write

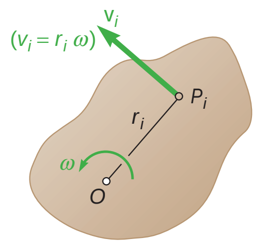


Fig. 17.4

$$T = \frac{1}{2}\sum_{i=1}^{n} \Delta m_i (r_i\omega)^2 = \frac{1}{2}\left(\sum_{i=1}^{n} r_i^2\,\Delta m_i\right)\omega^2$$

or, since the last sum represents the moment of inertia I_O of the body about the fixed axis through O,

$$T = \tfrac{1}{2}I_O\omega^2 \qquad (17.10)$$

We note that the results o[illegible] of plane slabs or to the motion [illegible] respect to the reference plane, and can be applied to the study of the plane motion of any rigid body, regardless of its shape. However, since Eq. (17.9) is applicable to any plane motion while Eq. (17.10) is applicable only in cases involving noncentroidal rotation, Eq. (17.9) will be used in the solution of all the sample problems.

17.5 SYSTEMS OF RIGID BODIES

When a problem involves several rigid bodies, each rigid body can be considered separately and the principle of work and energy can be applied to each body. Adding the kinetic energies of all the particles and considering the work of all the forces involved, we can also write the equation of work and energy for the entire system. We have

$$T_1 + U_{1\rightarrow 2} = T_2 \qquad (17.11)$$

where T represents the arithmetic sum of the kinetic energies of the rigid bodies forming the system (all terms are positive) and $U_{1\rightarrow 2}$ represents the work of all the forces acting on the various bodies, whether these forces are *internal* or *external* from the point of view of the system as a whole.

The method of work and energy is particularly useful in solving problems involving pin-connected members, blocks and pulleys connected by inextensible cords, and meshed gears. In all these cases, the internal forces occur by pairs of equal and opposite forces, and the points of application of the forces in each pair *move through equal distances* during a small displacement of the system. As a result, the work of the internal forces is zero and $U_{1\rightarrow 2}$ reduces to the work of the *forces external to the system.*

17.6 CONSERVATION OF ENERGY

We saw in Sec. 13.6 that the work of conservative forces, such as the weight of a body or the force exerted by a spring, can be expressed as a change in potential energy. When a rigid body, or a system of rigid bodies, moves under the action of conservative forces, the principle of work and energy stated in Sec. 17.2 can be expressed in a modified form. Substituting for $U_{1\rightarrow 2}$ from (13.19′) into (17.1), we write

$$\boxed{T_1 + V_1 = T_2 + V_2} \tag{17.12}$$

Formula (17.12) indicates that when a rigid body, or a system of rigid bodies, moves under the action of conservative forces, *the sum of the kinetic energy and of the potential energy of the system remains constant.* It should be noted that in the case of the plane motion of a rigid body, the kinetic energy of the body should include both the *translational* term $\frac{1}{2}m\bar{v}^2$ and the *rotational* term $\frac{1}{2}\bar{I}\omega^2$.

[illegible] f the principle of conservation [illegible] d AB, of length l and mass m, [illegible] ocks of negligible mass sliding [illegible] e assume that the rod is released with no initial velocity from a horizontal position (Fig. 17.5*a*), and we wish to determine its angular velocity after it has rotated through an angle θ (Fig. 17.5*b*).

Since the initial velocity is zero, we have $T_1 = 0$. Measuring the potential energy from the level of the horizontal track, we write $V_1 = 0$. After the rod has rotated through θ, the center of gravity G of the rod is at a distance $\frac{1}{2}l \sin\theta$ below the reference level and we have

$$V_2 = -\tfrac{1}{2}Wl\sin\theta = -\tfrac{1}{2}mgl\sin\theta$$

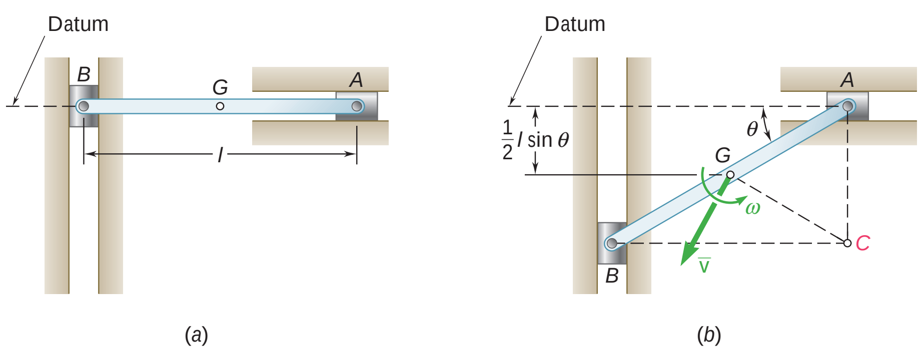


Fig. 17.5

Observing that in this position the instantaneous center of the rod is located at C and that $CG = \frac{1}{2}l$, we write $\bar{v}_2 = \frac{1}{2}l\omega$ and obtain

$$T_2 = \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\omega_2^2 = \tfrac{1}{2}m(\tfrac{1}{2}l\omega)^2 + \tfrac{1}{2}(\tfrac{1}{12}ml^2)\omega^2$$
$$= \frac{1}{2}\frac{ml^2}{3}\omega^2$$

Applying the principle of conservation of energy, we write

$$T_1 + V_1 = T_2 + V_2$$
$$0 = \frac{1}{2}\frac{ml^2}{3}\omega^2 - \tfrac{1}{2}mgl\sin\theta$$
$$\omega = \left(\frac{3g}{l}\sin\theta\right)^{1/2}$$

The advantages of the method of work and energy, as well as its shortcomings, were indicated in Sec. 13.4. Here we should add that the method of work and energy must be supplemented by the application of d'Alembert's principle when reactions at fixed axles, rollers, or sliding blocks are to be determined. For example, in order to compute the reactions at the extremities A and B of the rod of Fig. 17.5*b*, a diagram should be drawn to express that the system of the external forces applied to the rod is equivalent to the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}\boldsymbol{\alpha}$. The angular velocity $\boldsymbol{\omega}$ of the rod, however, is determined by the method of work and energy before the equations of motion are solved for the reactions. The complete analysis of the motion of the rod an
requires, therefore, the comb
energy and of the principle of e
tive forces.

17.7 POWER

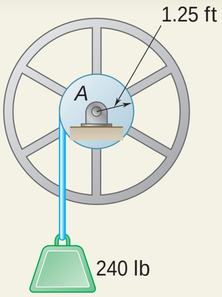
Power was defined in Sec. 13.5 as the time rate at which work is done. In the case of a body acted upon by a force $\mathbf{F}$, and moving with a velocity $\mathbf{v}$, the power was expressed as follows:

$$\text{Power} = \frac{dU}{dt} = \mathbf{F} \cdot \mathbf{v} \tag{13.13}$$

In the case of a rigid body rotating with an angular velocity $\boldsymbol{\omega}$ and acted upon by a couple of moment $\mathbf{M}$ parallel to the axis of rotation, we have, by (17.4),

$$\text{Power} = \frac{dU}{dt} = \frac{M\,d\theta}{dt} = M\omega \tag{17.13}$$

The various units used to measure power, such as the watt and the horsepower, were defined in Sec. 13.5.

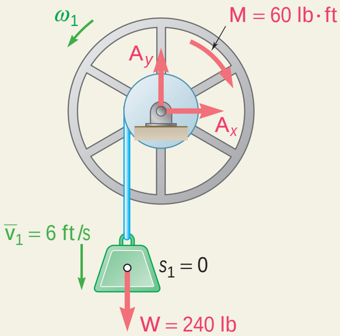


SAMPLE PROBLEM 17.1

A 240-lb block is suspended from an inextensible cable which is wrapped around a drum of 1.25-ft radius rigidly attached to a flywheel. The drum and flywheel have a combined centroidal moment of inertia $\bar{I} = 10.5$ lb · ft · s^2. At the instant shown, the velocity of the block is 6 ft/s directed downward. Knowing that the bearing at A is poorly lubricated and that the bearing friction is equivalent to a couple **M** of magnitude 60 lb · ft, determine the velocity of the block after it has moved 4 ft downward.

SOLUTION

We consider the system formed by the flywheel and the block. Since the cable is inextensible, the work done by the internal forces exerted by the cable cancels. The initial and final positions of the system and the external forces acting on the system are as shown.



Kinetic Energy. ***Position 1.***

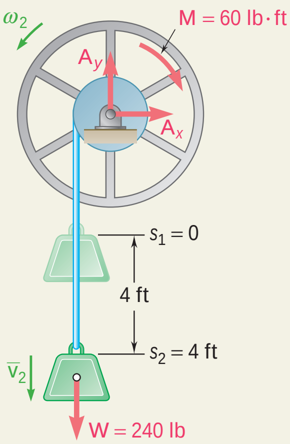
Block: $\bar{v}_1 = 6$ ft/s

Flywheel: $\omega_1 = \dfrac{\bar{v}_1}{\ } = \dfrac{6 \text{ ft/s}}{\ } = 4.80$ rad/s

$$= \frac{1}{2}\frac{\ }{32.2 \text{ ft/s}^2}(6 \text{ ft/s})^2 + \tfrac{1}{2}(10.5 \text{ lb} \cdot \text{ft} \cdot \text{s}^2)(4.80 \text{ rad/s})^2$$
$$= 255 \text{ ft} \cdot \text{lb}$$

Position 2. Noting that $\omega_2 = \bar{v}_2/1.25$, we write

$$T_2 = \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\omega_2^2$$
$$= \frac{1}{2}\frac{240}{32.2}(\bar{v}_2)^2 + (\tfrac{1}{2})(10.5)\left(\frac{\bar{v}_2}{1.25}\right)^2 = 7.09\bar{v}_2^2$$



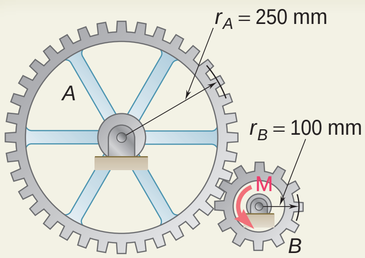
Work. During the motion, only the weight **W** of the block and the friction couple **M** do work. Noting that **W** does positive work and that the friction couple **M** does negative work, we write

$$s_1 = 0 \qquad s_2 = 4 \text{ ft}$$
$$\theta_1 = 0 \qquad \theta_2 = \frac{s_2}{r} = \frac{4 \text{ ft}}{1.25 \text{ ft}} = 3.20 \text{ rad}$$

$$U_{1\to 2} = W(s_2 - s_1) - M(\theta_2 - \theta_1)$$
$$= (240 \text{ lb})(4 \text{ ft}) - (60 \text{ lb} \cdot \text{ft})(3.20 \text{ rad})$$
$$= 768 \text{ ft} \cdot \text{lb}$$

Principle of Work and Energy

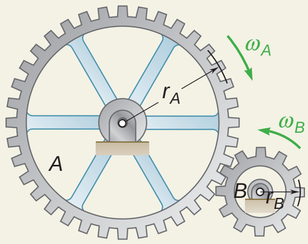
$$T_1 + U_{1\to 2} = T_2$$
$$255 \text{ ft} \cdot \text{lb} + 768 \text{ ft} \cdot \text{lb} = 7.09\bar{v}_2^2$$
$$\bar{v}_2 = 12.01 \text{ ft/s} \qquad \bar{\mathbf{v}}_2 = 12.01 \text{ ft/s} \downarrow$$



SAMPLE PROBLEM 17.2

Gear A has a mass of 10 kg and a radius of gyration of 200 mm; gear B has a mass of 3 kg and a radius of gyration of 80 mm. The system is at rest when a couple $\mathbf{M}$ of magnitude 6 N · m is applied to gear B. Neglecting friction, determine (a) the number of revolutions executed by gear B before its angular velocity reaches 600 rpm, (b) the tangential force which gear B exerts on gear A.

SOLUTION



Motion of Entire System. Noting that the peripheral speeds of the gears are equal, we write

$$r_A \mathrm{v}_A = r_B \mathrm{v}_B \qquad \mathrm{v}_A = \mathrm{v}_B \frac{r_B}{r_A} = \mathrm{v}_B \frac{100 \text{ mm}}{250 \text{ mm}} = 0.40\mathrm{v}_B$$

For $\mathrm{v}_B = 600$ rpm, we have

$$\mathrm{v}_B = 62.8 \text{ rad/s} \qquad \mathrm{v}_A = 0.40\mathrm{v}_B = 25.1 \text{ rad/s}$$
$$\bar{I}_A = m_A \bar{k}_A^2 = (10 \text{ kg})(0.200 \text{ m})^2 = 0.400 \text{ kg} \cdot \text{m}^2$$
$$\bar{I}_B = m_B \bar{k}_B^2 = (3 \text{ kg})(0.080 \text{ m})^2 = 0.0192 \text{ kg} \cdot \text{m}^2$$

Ki... ...is initially at rest, $T_1 = 0$. Adding the ... $\mathrm{v}_B = 600$ rpm, we obtain

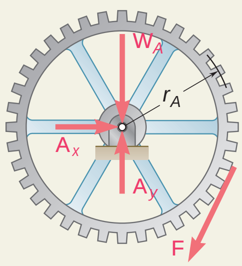
$$= \tfrac{1}{2}(0.400 \text{ kg} \cdot \text{m}^2)(25.1 \text{ rad/s})^2 + \tfrac{1}{2}(0.0192 \text{ kg} \cdot \text{m}^2)(62.8 \text{ rad/s})^2$$
$$= 163.9 \text{ J}$$

Work. Denoting by u_B the angular displacement of gear B, we have

$$U_{1\,y\,2} = M\mathrm{u}_B = (6 \text{ N} \cdot \text{m})(\mathrm{u}_B \text{ rad}) = (6\mathrm{u}_B) \text{ J}$$

Principle of Work and Energy

$$T_1 + U_{1\,y\,2} = T_2$$
$$0 + (6\theta_B) \text{ J} = 163.9 \text{ J}$$
$$\theta_B = 27.32 \text{ rad} \qquad \theta_B = 4.35 \text{ rev} \quad ◀$$



Motion of Gear A. ***Kinetic Energy.*** Initially, gear A is at rest, so $T_1 = 0$. When $\mathrm{v}_B = 600$ rpm, the kinetic energy of gear A is

$$T_2 = \tfrac{1}{2}\bar{I}_A \mathrm{v}_A^2 = \tfrac{1}{2}(0.400 \text{ kg} \cdot \text{m}^2)(25.1 \text{ rad/s})^2 = 126.0 \text{ J}$$

Work. The forces acting on gear A are as shown. The tangential force $\mathbf{F}$ does work equal to the product of its magnitude and of the length $\mathrm{u}_A r_A$ of the arc described by the point of contact. Since $\mathrm{u}_A r_A = \mathrm{u}_B r_B$, we have

$$U_{1\,y\,2} = F(\theta_B r_B) = F(27.3 \text{ rad})(0.100 \text{ m}) = F(2.73 \text{ m})$$

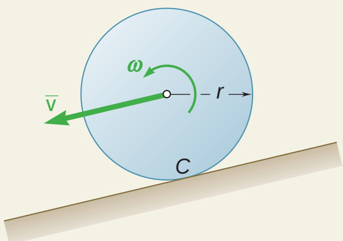
Principle of Work and Energy

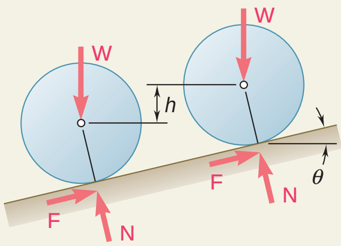
$$T_1 + U_{1\,y\,2} = T_2$$
$$0 + F(2.73 \text{ m}) = 126.0 \text{ J}$$
$$F = +46.2 \text{ N} \qquad \mathbf{F} = 46.2 \text{ N} \swarrow \quad ◀$$

SAMPLE PROBLEM 17.3

A sphere, a cylinder, and a hoop, each having the same mass and the same radius, are released from rest on an incline. Determine the velocity of each body after it has rolled through a distance corresponding to a change in elevation h.

SOLUTION





The problem will first be solved in general terms, and then results for each body will be found. We denote the mass by m, the centroidal moment of inertia by $\bar{I}$, the weight by W, and the radius by r.

Kinematics. Since each body rolls, the instantaneous center of rotation is located at C and we write

$$\omega = \frac{\bar{v}}{r}$$

Kinetic Energy

$$T_1 = 0$$
$$T_2 = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2$$
$$= \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2}\bar{I}\left(\frac{\bar{v}}{r}\right)^2 = \tfrac{1}{2}\left(m + \frac{\bar{I}}{r^2}\right)\bar{v}^2$$

motion does no work,

Principle of Work and Energy

$$T_1 + U_{1\rightarrow 2} = T_2$$
$$0 + Wh = \tfrac{1}{2}\left(m + \frac{\bar{I}}{r^2}\right)\bar{v}^2 \qquad \bar{v}^2 = \frac{2Wh}{m + \bar{I}/r^2}$$

Noting that $W = mg$, we rearrange the result and obtain

$$\bar{v}^2 = \frac{2gh}{1 + \bar{I}/mr^2}$$

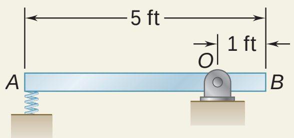
Velocities of Sphere, Cylinder, and Hoop. Introducing successively the particular expression for $\bar{I}$, we obtain

Sphere:	$\bar{I} = \frac{2}{5} mr^2$	$\bar{v} = 0.845\sqrt{2gh}$ ◀
Cylinder:	$\bar{I} = \frac{1}{2} mr^2$	$\bar{v} = 0.816\sqrt{2gh}$ ◀
Hoop:	$\bar{I} = mr^2$	$\bar{v} = 0.707\sqrt{2gh}$ ◀

Remark. Let us compare the results with the velocity attained by a frictionless block sliding through the same distance. The solution is identical to the above solution except that $\omega = 0$; we find $\bar{v} = \sqrt{2gh}$.

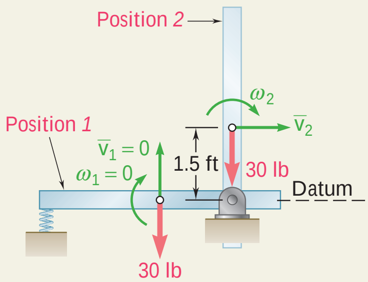
Comparing the results, we note that the velocity of the body is independent of both its mass and radius. However, the velocity does depend upon the quotient $\bar{I}/mr^2 = \bar{k}^2/r^2$, which measures the ratio of the rotational kinetic energy to the translational kinetic energy. Thus the hoop, which has the largest $\bar{k}$ for a given radius r, attains the smallest velocity, while the sliding block, which does not rotate, attains the largest velocity.

SAMPLE PROBLEM 17.4



A 30-lb slender rod AB is 5 ft long and is pivoted about a point O which is 1 ft from end B. The other end is pressed against a spring of constant k = 1800 lb/in. until the spring is compressed 1 in. The rod is then in a horizontal position. If the rod is released from this position, determine its angular velocity and the reaction at the pivot O as the rod passes through a vertical position.

SOLUTION



Position 1. ***Potential Energy.*** Since the spring is compressed 1 in., we have x_1 = 1 in.

$$V_e = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}(1800 \text{ lb/in.})(1 \text{ in.})^2 = 900 \text{ in} \cdot \text{lb}$$

Choosing the datum as shown, we have $V_g = 0$; therefore,

$$V_1 = V_e + V_g = 900 \text{ in} \cdot \text{lb} = 75 \text{ ft} \cdot \text{lb}$$

Kinetic Energy. Since the velocity in position *1* is zero, we have $T_1 = 0$.

Position 2. ***Potential Energy.*** The elongation of the spring is zero, and we have $V_e = 0$. Since the center of gravity of the rod is now 1.5 ft above the datum,

$$V_g = (30 \text{ lb})(+1.5 \text{ ft}) = 45 \text{ ft} \cdot \text{lb}$$
$$45 \text{ ft} \cdot \text{lb}$$

e angular velocity of the rod in position bout O and write $\bar{v}_2 = \bar{r}\mathrm{v}_2 = 1.5\mathrm{v}_2$.

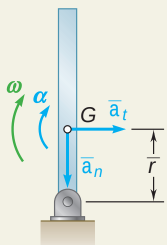
$$\bar{I} = \tfrac{1}{12}ml^2 = \frac{1}{12}\frac{30 \text{ lb}}{32.2 \text{ ft/s}^2}(5 \text{ ft})^2 = 1.941 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$T_2 = \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\mathrm{v}_2^2 = \frac{1}{2}\frac{30}{32.2}(1.5\mathrm{v}_2)^2 + \tfrac{1}{2}(1.941)\mathrm{v}_2^2 = 2.019\mathrm{v}_2^2$$

Conservation of Energy

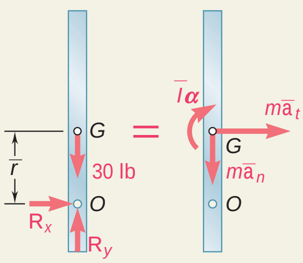
$$T_1 + V_1 = T_2 + V_2$$
$$0 + 75 \text{ ft} \cdot \text{lb} = 2.019\mathrm{v}_2^2 + 45 \text{ ft} \cdot \text{lb}$$

$$\mathrm{V}_2 = 3.86 \text{ rad/si} \quad ◀$$



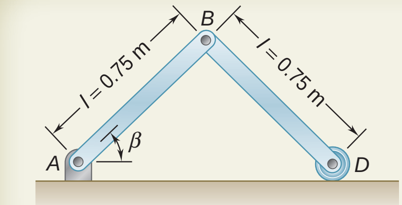
Reaction in Position 2. Since v_2 = 3.86 rad/s, the components of the acceleration of G as the rod passes through position *2* are

$$\bar{a}_n = \bar{r}\mathrm{v}_2^2 = (1.5 \text{ ft})(3.86 \text{ rad/s})^2 = 22.3 \text{ ft/s}^2 \qquad \bar{\mathbf{a}}_n = 22.3 \text{ ft/s}^2 \text{w}$$
$$\bar{a}_t = \bar{r}\mathrm{a} \qquad \bar{\mathbf{a}}_t = \bar{r}\mathrm{a} \text{ y}$$



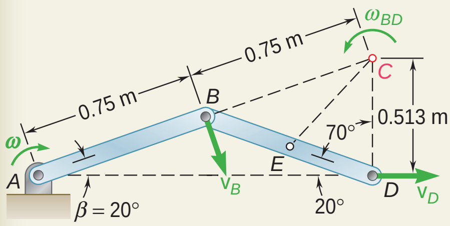
We express that the system of external forces is equivalent to the system of effective forces represented by the vector of components $m\bar{\mathbf{a}}_t$ and $m\bar{\mathbf{a}}_n$ attached at G and the couple $\bar{I}$A.

$$+\text{i}\Sigma M_O = \Sigma(M_O)_{\text{eff}}: \qquad 0 = \bar{I}\mathrm{a} + m(\bar{r}\mathrm{a})\bar{r} \qquad \mathrm{a} = 0$$
$$\overset{+}{\text{y}}\ \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad R_x = m(\bar{r}\mathrm{a}) \qquad R_x = 0$$
$$+\text{x}\ \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad R_y - 30 \text{ lb} = -m\bar{a}_n$$
$$R_y - 30 \text{ lb} = -\frac{30 \text{ lb}}{32.2 \text{ ft/s}^2}(22.3 \text{ ft/s}^2)$$
$$R_y = +9.22 \text{ lb} \qquad \mathbf{R} = 9.22 \text{ lb x} \quad ◀$$



SAMPLE PROBLEM 17.5

Each of the two slender rods shown is 0.75 m long and has a mass of 6 kg. If the system is released from rest with $\beta = 60°$, determine (*a*) the angular velocity of rod *AB* when $\beta = 20°$, (*b*) the velocity of point *D* at the same instant.



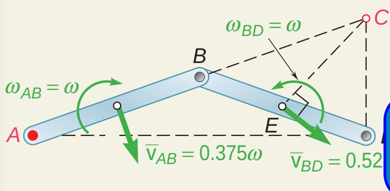
SOLUTION

Kinematics of Motion When $\beta = 20°$. Since $\mathbf{v}_B$ is perpendicular to the rod *AB* and $\mathbf{v}_D$ is horizontal, the instantaneous center of rotation of rod *BD* is located at *C*. Considering the geometry of the figure, we obtain

$$BC = 0.75 \text{ m} \qquad CD = 2(0.75 \text{ m}) \sin 20° = 0.513 \text{ m}$$

Applying the law of cosines to triangle *CDE*, where *E* is located at the mass center of rod *BD*, we find $EC = 0.522$ m. Denoting by ω the angular velocity of rod *AB*, we have

$$\bar{v}_{AB} = (0.375 \text{ m})\omega \qquad \bar{\mathbf{v}}_{AB} = 0.375\omega \searrow$$
$$v_B = (0.75 \text{ m})\omega \qquad \mathbf{v}_B = 0.75\omega \searrow$$



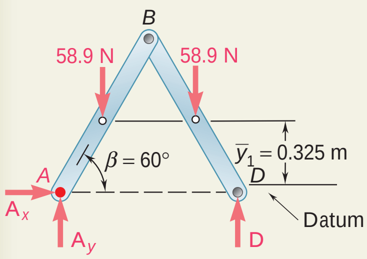
Since rod *BD* seems to rotate about point *C*, we write

$$\ldots \text{ m})\omega_{BD} \qquad \omega_{BD} = \omega \text{ l}$$
$$\bar{\mathbf{v}}_{BD} = 0.522\omega \searrow$$

Position 1. ***Potential Energy.*** Choosing the datum as shown, and observing that $W = (6 \text{ kg})(9.81 \text{ m/s}^2) = 58.86$ N, we have

$$V_1 = 2W\bar{y}_1 = 2(58.86 \text{ N})(0.325 \text{ m}) = 38.26 \text{ J}$$

Kinetic Energy. Since the system is at rest, $T_1 = 0$.



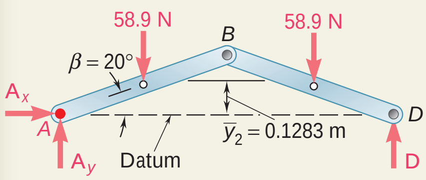
Position 1

Position 2. ***Potential Energy***

$$V_2 = 2W\bar{y}_2 = 2(58.86 \text{ N})(0.1283 \text{ m}) = 15.10 \text{ J}$$

Kinetic Energy

$$I_{AB} = \bar{I}_{BD} = \tfrac{1}{12}ml^2 = \tfrac{1}{12}(6 \text{ kg})(0.75 \text{ m})^2 = 0.281 \text{ kg} \cdot \text{m}^2$$
$$T_2 = \tfrac{1}{2}m\bar{v}_{AB}^2 + \tfrac{1}{2}\bar{I}_{AB}\omega_{AB}^2 + \tfrac{1}{2}m\bar{v}_{BD}^2 + \tfrac{1}{2}\bar{I}_{BD}\omega_{BD}^2$$
$$= \tfrac{1}{2}(6)(0.375\omega)^2 + \tfrac{1}{2}(0.281)\omega^2 + \tfrac{1}{2}(6)(0.522\omega)^2 + \tfrac{1}{2}(0.281)\omega^2$$
$$= 1.520\omega^2$$



Position 2

Conservation of Energy

$$T_1 + V_1 = T_2 + V_2$$
$$0 + 38.26 \text{ J} = 1.520\omega^2 + 15.10 \text{ J}$$
$$\omega = 3.90 \text{ rad/s} \qquad \omega_{AB} = 3.90 \text{ rad/s i} \quad ◀$$

Velocity of Point *D*

$$v_D = (CD)\omega = (0.513 \text{ m})(3.90 \text{ rad/s}) = 2.00 \text{ m/s}$$
$$\mathbf{v}_D = 2.00 \text{ m/s y} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson we introduced energy methods to determine the velocity of rigid bodies for various positions during their motion. As you found out previously in Chap. 13, energy methods should be considered for problems involving displacements and velocities.

1. The method of work and energy, when applied to all of the particles forming a rigid body, yields the equation

$$T_1 + U_{1 y 2} = T_2 \qquad (17.1)$$

where T_1 and T_2 are, respectively, the initial and final values of the total kinetic energy of the particles forming the body and $U_{1 y 2}$ is the *work done by the external forces* exerted on the rigid body.

a. Work of forces and couples. To the expression for the work of a force (Chap. 13), we added the expression for the work of a couple and wrote

$$U_{1 y 2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \qquad U_{1 y 2} = \int_{\mathsf{u}_1}^{\mathsf{u}_2} M\, d\mathsf{u} \qquad (17.3,\ 17.5)$$

When the moment of a couple is constant, the work of the couple is

(17.6)

where u_1 and u … 1 and 17.2].

b. The kinetic energy of a rigid body in plane motion was found by considering the motion of the body as the sum of a translation with its mass center and a rotation about the mass center.

$$T = \tfrac{1}{2} m\bar{v}^2 + \tfrac{1}{2} \bar{I}\mathsf{v}^2 \qquad (17.9)$$

where $\bar{v}$ is the velocity of the mass center and v is the angular velocity of the body [Sample Probs. 17.3 and 17.4].

2. For a system of rigid bodies we again used the equation

$$T_1 + U_{1 y 2} = T_2 \qquad (17.1)$$

where T is the sum of the kinetic energies of the bodies forming the system and U is the work done by *all the forces acting on the bodies,* internal as well as external. Your computations will be simplified if you keep the following in mind.

a. The forces exerted on each other by pin-connected members or by meshed gears are equal and opposite, and, since they have the same point of application, they undergo equal small displacements. Therefore, *their total work is zero* and can be omitted from your calculations [Sample Prob. 17.2].

(*continued*)

b. The forces exerted by an inextensible cord on the two bodies it connects have the same magnitude and their points of application move through equal distances, but the work of one force is positive and the work of the other is negative. Therefore, *their total work is zero* and can again be omitted from your calculations [Sample Prob. 17.1].

c. The forces exerted by a spring on the two bodies it connects also have the same magnitude, but their points of application will generally move through different distances. Therefore, *their total work is usually not zero* and should be taken into account in your calculations.

3. The principle of conservation of energy can be expressed as

$$T_1 + V_1 = T_2 + V_2 \tag{17.12}$$

where V represents the potential energy of the system. This principle can be used when a body or a system of bodies is acted upon by conservative forces, such as the force exerted by a spring or the force of gravity [Sample Probs. 17.4 and 17.5].

4. The last section of this lesson was devoted to power, which is the time rate at which work is done. For a body acted upon by a couple of moment **M**, the power can be expres

(17.13)

where ω is the angular velocity of the body expressed in rad/s. As you did in Chap. 13, you should express power either in watts or in horsepower (1 hp = 550 ft · lb/s).

PROBLEMS

CONCEPT QUESTIONS

17.CQ1 A round object of mass m and radius r is released from rest at the top of a curved surface and rolls without slipping until it leaves the surface with a horizontal velocity as shown. Will a solid sphere, a solid cylinder, or a hoop travel the greatest distance x?

a. Solid sphere
b. Solid cylinder
c. Hoop
d. They will all travel the same distance.

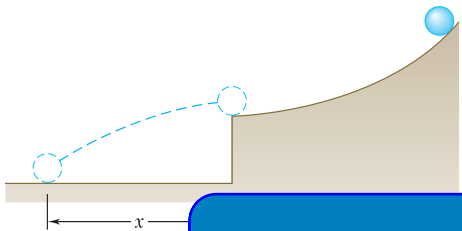


Fig. P17.CQ1

17.CQ2 A solid steel sphere A of radius r and mass m is released from rest and rolls without slipping down an incline as shown. After traveling a distance d, the sphere has a speed v. If a solid steel sphere of radius $2r$ is released from rest on the same incline, what will its speed be after rolling a distance d?

a. $0.25\,v$
b. $0.5\,v$
c. v
d. $2v$
e. $4v$

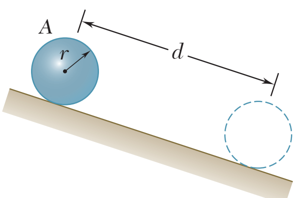


Fig. P17.CQ2

17.CQ3 Slender bar *A* is rigidly connected to a massless rod *BC* in Case 1 and two massless cords in Case 2 as shown. The vertical thickness of bar *A* is negligible compared to *L*. In both cases *A* is released from rest at an angle $u = u_0$. When $u = 0°$, which system will have the larger kinetic energy?

a. Case 1

b. Case 2

c. The kinetic energy will be the same.

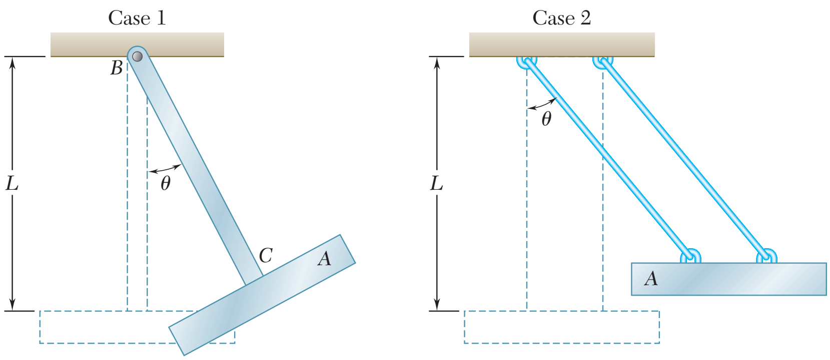


Fig. P17.CQ3 and P17.CQ5

e speeds of the centers of gravity n $u = 0°$?

b. Case 2 will be larger.

c. The speeds will be the same.

17.CQ5 Slender bar *A* is rigidly connected to a massless rod *BC* in Case 1 and two massless cords in Case 2 as shown. The vertical thickness of bar *A* is not negligible compared to *L*. In both cases *A* is released from rest at an angle $u = u_0$. When $u = 0°$, which system will have the largest kinetic energy?

a. Case 1

b. Case 2

c. The kinetic energy will be the same.

END-OF-SECTION PROBLEMS

17.1 The rotor of an electric motor has an angular velocity of 3600 rpm when the load and power are cut off. The 50-kg rotor then coasts to rest after 5000 revolutions. Knowing that the kinetic friction of the rotor produces a couple of magnitude 4 N · m, determine the centroidal radius of gyration of the rotor.

17.2 It is known that 1500 revolutions are required for the 6000-lb flywheel to coast to rest from an angular velocity of 300 rpm. Knowing that the centroidal radius of gyration of the flywheel is 36 in., determine the average magnitude of the couple due to kinetic friction in the bearings.

17.3 Two disks of the same material are attached to a shaft as shown. Disk A has a weight of 30 lb and a radius $r = 5$ in. Disk B is three times as thick as disk A. Knowing that a couple **M** of magnitude 15 lb · ft is to be applied to disk A when the system is at rest, determine the radius nr of disk B if the angular velocity of the system is to be 600 rpm after four revolutions.

17.4 Two disks of the same material are attached to a shaft as shown. Disk A is of radius r and has a thickness b, while disk B is of radius nr and thickness $3b$. A couple **M** of constant magnitude is applied when the system is at rest and is removed after the system has executed two revolutions. Determine the value of n which results in the largest final speed for a point on the rim of disk B.

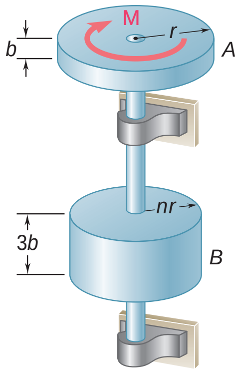


Fig. P17.3 and P17.4

17.5 The flywheel of a small punch rotates at 300 rpm. It is known that 1800 ft · lb of work must be done each time a hole is punched. It is desired that the speed of the flywheel after one punching be not less than 90 percent of the original speed of 300 rpm. (*a*) Determine the required moment of inertia of the flywheel. (*b*) If a constant 25-lb · ft couple is applied to the shaft of the flywheel, determine the number of revolutions which must occur between each punching, knowing that the initial velocity is to be 300 rpm at the start of each punching.

17.6 The flywheel of a punching machine has a mass of 300 kg and a radius of gyration of 600 mm. Each punching operation requires 2500 J of work. (*a*) Kn [illegible] 300 rpm just before a p [illegible] ately after the punchin [illegible] applied to the shaft of the flywheel, determine the number of revolutions executed before the speed is again 300 rpm.

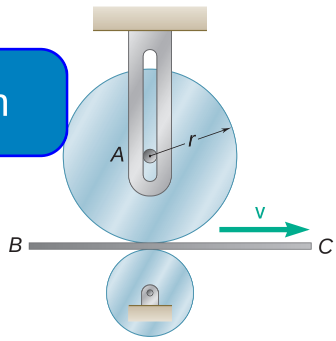


Fig. P17.7 and *P17.8*

17.7 Disk A, of weight 10 lb and radius $r = 6$ in., is at rest when it is placed in contact with belt BC, which moves to the right with a constant speed $v = 40$ ft/s. Knowing that $m_k = 0.20$ between the disk and the belt, determine the number of revolutions executed by the disk before it attains a constant angular velocity.

17.8 Disk A is of constant thickness and is at rest when it is placed in contact with belt BC, which moves with a constant velocity **v**. Denoting by m_k the coefficient of kinetic friction between the disk and the belt, derive an expression for the number of revolutions executed by the disk before it attains a constant angular velocity.

17.9 The 10-in.-radius brake drum is attached to a larger flywheel which is not shown. The total mass moment of inertia of the flywheel and drum is 16 lb · ft · s^2 and the coefficient of kinetic friction between the drum and the brake shoe is 0.40. Knowing that the initial angular velocity is 240 rpm clockwise, determine the force which must be exerted by the hydraulic cylinder if the system is to stop in 75 revolutions.

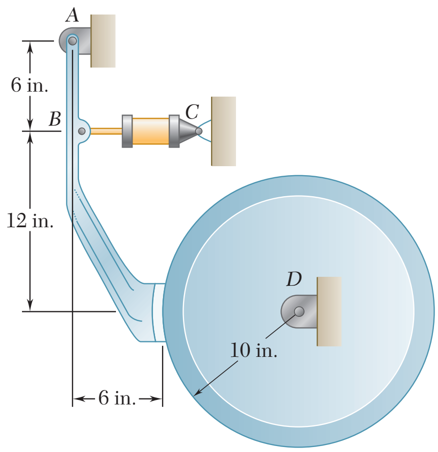


Fig. *P17.9*

17.10 Solve Prob. 17.9, assuming that the initial angular velocity of the flywheel is 240 rpm counterclockwise.

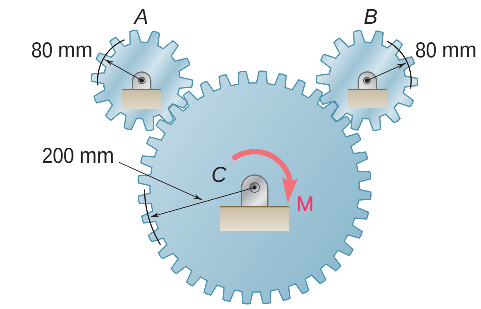


Fig. P17.11

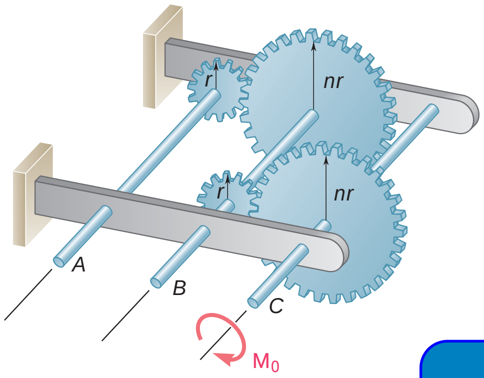


Fig. P17.13

17.11 Each of the gears A and B has a mass of 2.4 kg and a radius of gyration of 60 mm, while gear C has a mass of 12 kg and a radius of gyration of 150 mm. A couple $\mathbf{M}$ of constant magnitude 10 N · m is applied to gear C. Determine (*a*) the number of revolutions of gear C required for its angular velocity to increase from 100 to 450 rpm, (*b*) the corresponding tangential force acting on gear A.

17.12 Solve Prob. 17.11, assuming that the 10-N · m couple is applied to gear B.

17.13 The gear train shown consists of four gears of the same thickness and of the same material; two gears are of radius r, and the other two are of radius nr. The system is at rest when the couple $\mathbf{M}_0$ is applied to shaft C. Denoting by I_0 the moment of inertia of a gear of radius r, determine the angular velocity of shaft A if the couple $\mathbf{M}_0$ is applied for one revolution of shaft C.

17.14 The double pulley shown has a mass of 15 kg and a centroidal radius of gyration of 160 mm. Cylinder A and block B are attached to cords that are wrapped on the pulleys as shown. The coefficient of kinetic friction between block B and the surface is 0.2. Knowing that the system is at rest in the position shown when a constant force $\mathbf{P}$ = 200 N is applied to cylinder A, determine (*a*) the velocity of cylinder A as it strikes the ground, (*b*) the total distance that block B moves before coming to rest.

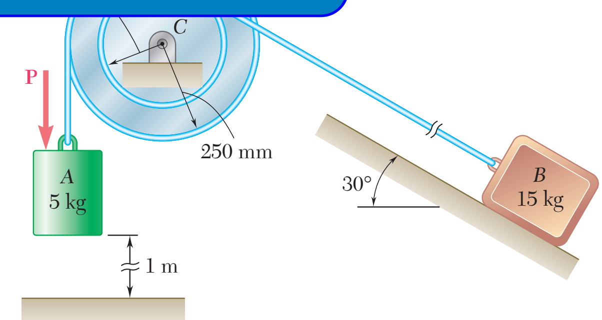


Fig. *P17.14*

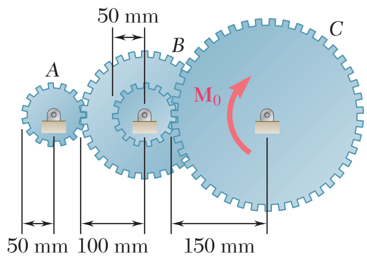


Fig. *P17.15*

17.15 Gear A has a mass of 1 kg and a radius of gyration of 30 mm; gear B has a mass of 4 kg and a radius of gyration of 75 mm; gear C has a mass of 9 kg and a radius of gyration of 100 mm. The system is at rest when a couple $\mathbf{M}_0$ of constant magnitude 4 N · m is applied to gear C. Assuming that no slipping occurs between the gears, determine the number of revolutions required for disk A to reach an angular velocity of 300 rpm.

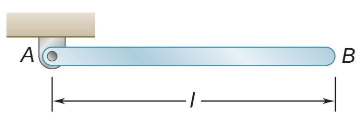


Fig. P17.16

17.16 A slender rod of length l and weight W is pivoted at one end as shown. It is released from rest in a horizontal position and swings freely. (*a*) Determine the angular velocity of the rod as it passes through a vertical position and determine the corresponding reaction at the pivot. (*b*) Solve part *a* for W = 1.8 lb and l = 3 ft.

17.17 A slender rod of length l is pivoted about a point C located at a distance b from its center G. It is released from rest in a horizontal position and swings freely. Determine (*a*) the distance b for which the angular velocity of the rod as it passes through a vertical position is maximum, (*b*) the corresponding values of its angular velocity and of the reaction at C.

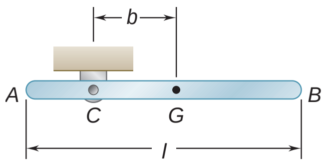


Fig. P17.17

17.18 and 17.19 A slender 9-lb rod can rotate in a vertical plane about a pivot at B. A spring of constant $k = 30$ lb/ft and of unstretched length 6 in. is attached to the rod as shown. Knowing that the rod is released from rest in the position shown, determine its angular velocity after it has rotated through 90°.

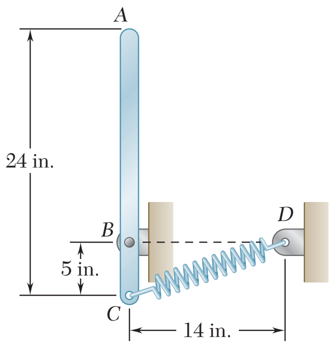


Fig. P17.18

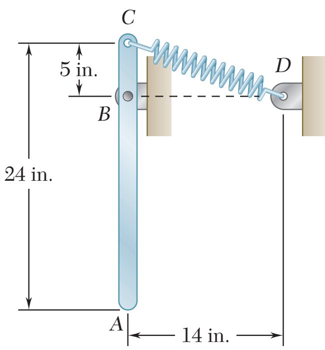


Fig. P17.19

17.20 A 160-lb gymnast is exe horizontal bar. In the pos clockwise angular velocity and will maintain his body straight and rigid as he swings downward. Assuming that during the swing the centroidal radius of gyration of his body is 1.5 ft, determine his angular velocity and the force exerted on his hands after he has rotated through (*a*) 90°, (*b*) 180°.

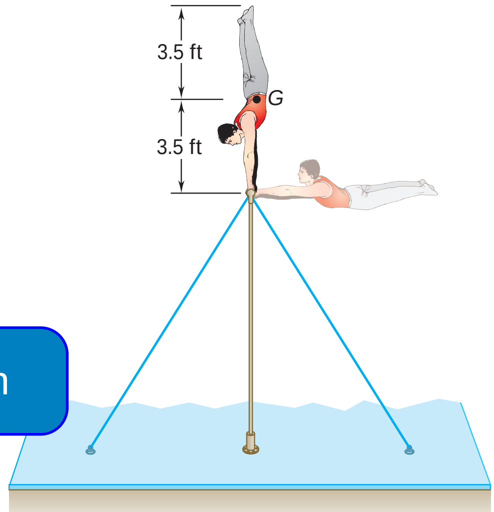


Fig. P17.20

17.21 A collar with a mass of 1 kg is rigidly attached at a distance d = 300 mm from the end of a uniform slender rod AB. The rod has a mass of 3 kg and is of length L = 600 mm. Knowing that the rod is released from rest in the position shown, determine the angular velocity of the rod after it has rotated through 90°.

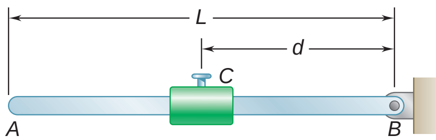


Fig. *P17.21* and *P17.22*

17.22 A collar with a mass of 1 kg is rigidly attached to a slender rod AB of mass 3 kg and length L = 600 mm. The rod is released from rest in the position shown. Determine the distance d for which the angular velocity of the rod is maximum after it has rotated through 90°.

17.23 Two identical slender rods AB and BC are welded together to form an L-shaped assembly. The assembly is pressed against a spring at D and released from the position shown. Knowing that the maximum angle of rotation of the assembly in its subsequent motion is 90° counterclockwise, determine the magnitude of the angular velocity of the assembly as it passes through the position where rod AB forms an angle of 30° with the horizontal.

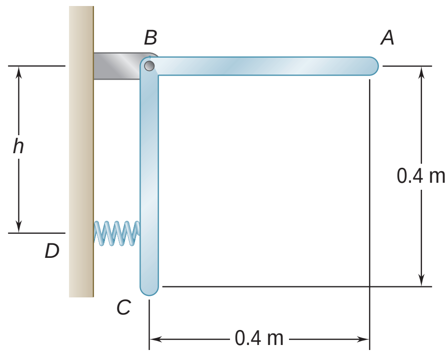


Fig. P17.23

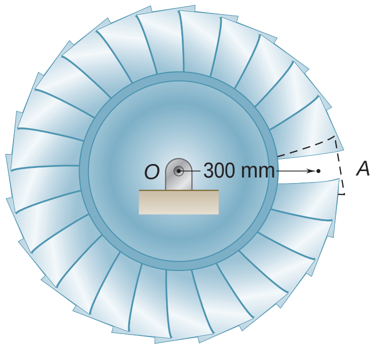


Fig. P17.24

17.24 The 30-kg turbine disk has a centroidal radius of gyration of 175 mm and is rotating clockwise at a constant rate of 60 rpm when a small blade of weight 0.5 N at point A becomes loose and is thrown off. Neglecting friction, determine the change in the angular velocity of the turbine disk after it has rotated through (*a*) 90°, (*b*) 270°.

17.25 A rope is wrapped around a cylinder of radius r and mass m as shown. Knowing that the cylinder is released from rest, determine the velocity of the center of the cylinder after it has moved downward a distance s.

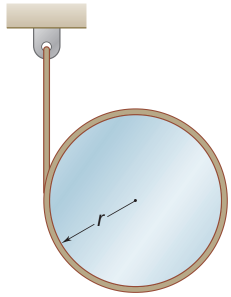


Fig. P17.25

17.26 Solve Prob. 17.25, assuming that the cylinder is replaced by a thin-walled pipe of radius r and mass m.

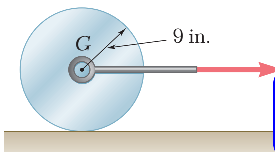


Fig. P17.27

…er, initially at rest, is acted upon …wing that the body rolls without …ocity of its center G after it has moved 5 ft, (*b*) the friction force required to prevent slipping.

17.28 A small sphere of mass m and radius r is released from rest at A and rolls without sliding on the curved surface to point B where it leaves the surface with a horizontal velocity. Knowing that $a = 1.5$ m and $b = 1.2$ m, determine (*a*) the speed of the sphere as it strikes the ground at C, (*b*) the corresponding distance c.

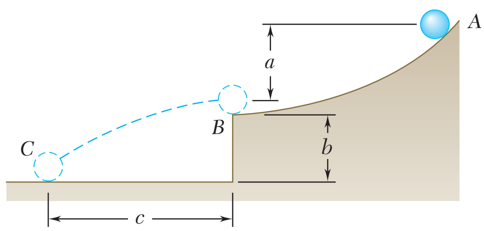


Fig. ***P17.28***

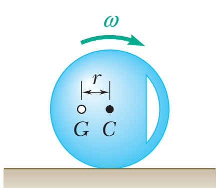


Fig. P17.29

17.29 The mass center G of a 3-kg wheel of radius $R = 180$ mm is located at a distance $r = 60$ mm from its geometric center C. The centroidal radius of gyration of the wheel is $\bar{k} = 90$ mm. As the wheel rolls without sliding, its angular velocity is observed to vary. Knowing that v = 8 rad/s in the position shown, determine (*a*) the angular velocity of the wheel when the mass center G is directly above the geometric center C, (*b*) the reaction at the horizontal surface at the same instant.

17.30 A half section of pipe of mass m and radius r is released from rest in the position shown. Knowing that the pipe rolls without sliding, determine (*a*) its angular velocity after it has rolled through 90°, (*b*) the reaction at the horizontal surface at the same instant. [*Hint:* Note that $GO = 2r/p$ and that, by the parallel-axis theorem, $\bar{I} = mr^2 - m(GO)^2$.]

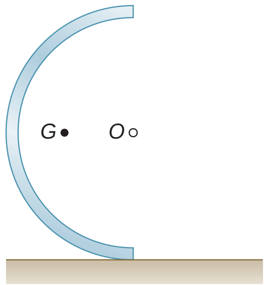


Fig. P17.30

17.31 A sphere of mass m and radius r rolls without slipping inside a curved surface of radius R. Knowing that the sphere is released from rest in the position shown, derive an expression for (*a*) the linear velocity of the sphere as it passes through B, (*b*) the magnitude of the vertical reaction at that instant.

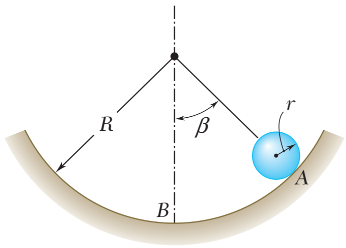


Fig. P17.31

17.32 Two uniform cylinders, each of weight $W = 14$ lb and radius $r = 5$ in., are connected by a [illegible] shown the angular velo[illegible] determine (*a*) the distanc[illegible] the angular velocity of cy[illegible] (*b*) the tension in the portion of belt connecting the two cylinders.

17.33 Two uniform cylinders, each of weight $W = 14$ lb and radius $r = 5$ in., are connected by a belt as shown. If the system is released from rest, determine (*a*) the velocity of the center of cylinder A after it has moved through 3 ft, (*b*) the tension in the portion of belt connecting the two cylinders.

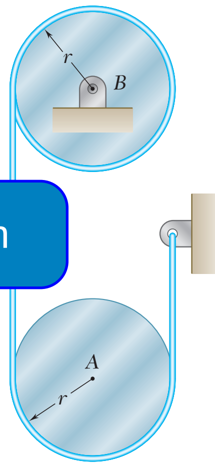


Fig. P17.32 and P17.33

17.34 A bar of mass $m = 5$ kg is held as shown between four disks each of mass $m' = 2$ kg and radius $r = 75$ mm. Knowing that the forces exerted on the disks are sufficient to prevent slipping and that the bar is released from rest, for each of the cases shown determine the velocity of the bar after it has moved through the distance h.

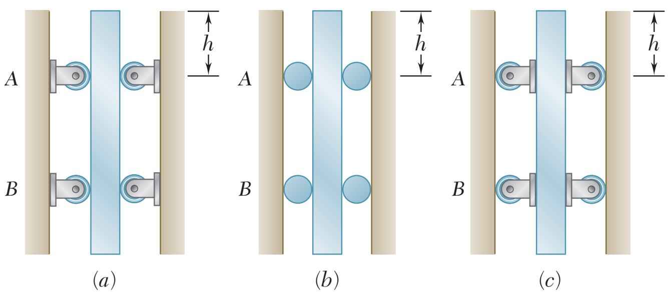


Fig. *P17.34*

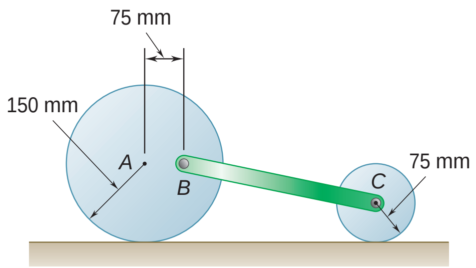


Fig. P17.35

17.35 The 5-kg rod BC is attached by pins to two uniform disks as shown. The mass of the 150-mm-radius disk is 6 kg and that of the 75-mm-radius disk is 1.5 kg. Knowing that the system is released from rest in the position shown, determine the velocity of the rod after disk A has rotated through 90°.

17.36 The motion of the uniform rod AB is guided by small wheels of negligible mass that roll on the surface shown. If the rod is released from rest when $\theta = 0$, determine the velocities of A and B when $\theta = 30°$.

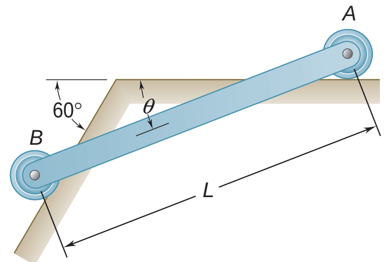


Fig. P17.36

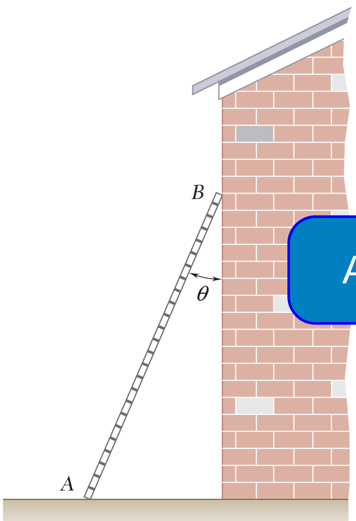


Fig. P17.37 and P17.38

17.37 A 5-m-long ladder has a mass of 15 kg and is placed against a house at an angle $\theta = 20°$. Knowing that the ladder is released from rest, determine the angular velocity of the ladder and the velocity of end A when $\theta = 45°$. Assume the ladder can slide freely on the ... ertical wall.

... , and centroidal mass moment of ... use at an angle $\theta = \theta_0$. Knowing that the ladder is released from rest, determine the angular velocity of the ladder when $\theta = \theta_2$. Assume the ladder can slide freely on the horizontal ground and on the vertical wall.

17.39 The ends of a 9-lb rod AB are constrained to move along slots cut in a vertical plate as shown. A spring of constant $k = 3$ lb/in. is attached to end A in such a way that its tension is zero when $\theta = 0$. If the rod is released from rest when $\theta = 50°$, determine the angular velocity of the rod and the velocity of end B when $\theta = 0$.

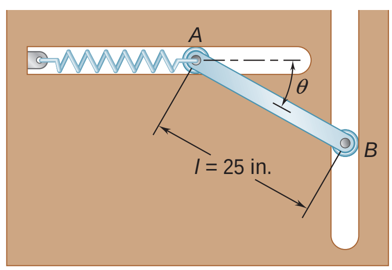


Fig. P17.39 and P17.40

17.40 The ends of a 9-lb rod AB are constrained to move along slots cut in a vertical plate as shown. A spring of constant $k = 3$ lb/in. is attached to end A in such a way that its tension is zero when $\theta = 0$. If the rod is released from rest when $\theta = 0$, determine the angular velocity of the rod and the velocity of end B when $\theta = 30°$.

17.41 The motion of a slender rod of length R is guided by pins at A and B which slide freely in slots cut in a vertical plate as shown. If end B is moved slightly to the left and then released, determine the angular velocity of the rod and the velocity of its mass center (*a*) at the instant when the velocity of end B is zero, (*b*) as end B passes through point D.

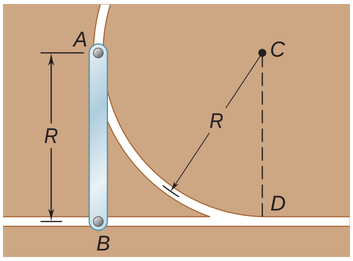


Fig. *P17.41*

17.42 Each of the two rods shown is of length $L = 1$ m and has a mass of 5 kg. Point D is connected to a spring of constant $k = 20$ N/m and is constrained to move along a vertical slot. Knowing that the system is released from rest when rod BD is horizontal and the spring connected to point D is initially unstretched, determine the velocity of point D when it is directly to the right of point A.`

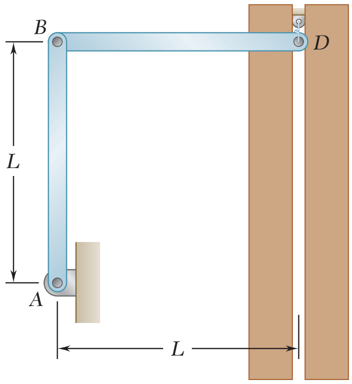


Fig. P17.42

17.43 The 4-kg rod AB is attached to a collar of negligible mass at A and to a flywheel at B. The flywheel has a mass of 16 kg and a radius of gyration of 180 mm.
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the velocity of the flywh

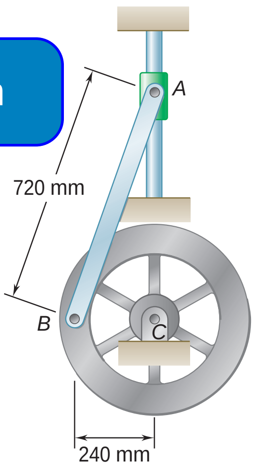


Fig. P17.43 and P17.44

17.44 If in Prob. 17.43 the angular velocity of the flywheel is to be the same in the position shown and when point B is directly above C, determine the required value of its angular velocity in the position shown.

17.45 The uniform rods AB and BC weigh 2.4 kg and 4 kg, respectively, and the small wheel at C is of negligible weight. If the wheel is moved slightly to the right and then released, determine the velocity of pin B after rod AB has rotated through 90°.

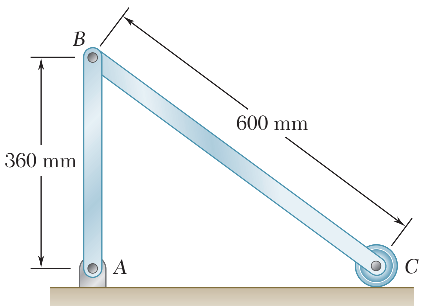


Fig. P17.45 and P17.46

17.46 The uniform rods AB and BC weigh 2.4 kg and 4 kg, respectively, and the small wheel at C is of negligible weight. Knowing that in the position shown the velocity of wheel C is 2 m/s to the right, determine the velocity of pin B after rod AB has rotated through 90°.

17.47 The 80-mm-radius gear shown has a mass of 5 kg and a centroidal radius of gyration of 60 mm. The 4-kg rod AB is attached to the center of the gear and to a pin at B that slides freely in a vertical slot. Knowing that the system is released from rest when $\theta = 60°$, determine the velocity of the center of the gear when $\theta = 20°$.

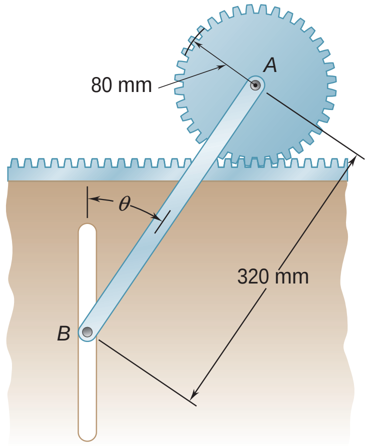


Fig. P17.47

17.48 Knowing that the maximum allowable couple that can be applied to a shaft is 15.5 kip · in., determine the maximum horsepower that can be transmitted by the shaft at (*a*) 180 rpm, (*b*) 480 rpm.

...sed to form a gear train which will ... A to a machine tool at F. (Bearings ... he sketch.) Knowing that the fre-quency of the motor is 30 Hz, determine the magnitude of the couple which is applied to shaft (*a*) AB, (*b*) CD, (*c*) EF.

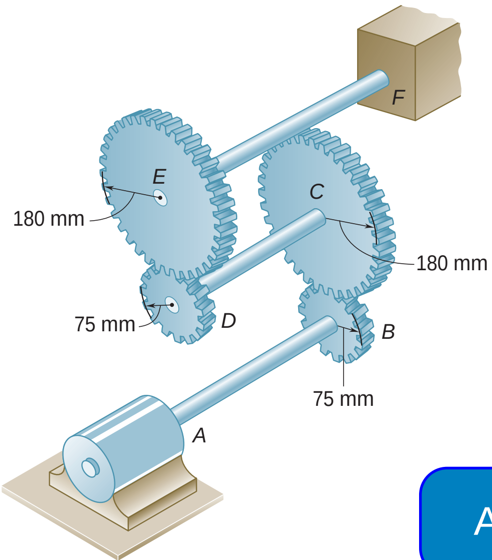


Fig. P17.49

17.50 The shaft-disk-belt arrangement shown is used to transmit 2.4 kW from point A to point D. Knowing that the maximum allowable couples that can be applied to shafts AB and CD are 25 N · m and 80 N · m, respectively, determine the required minimum speed of shaft AB.

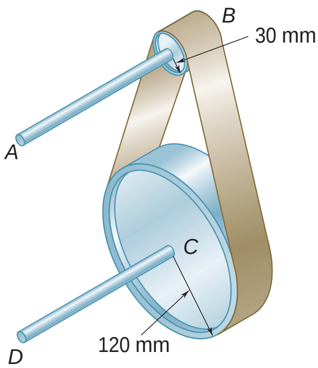


Fig. P17.50

17.51 The experimental setup shown is used to measure the power output of a small turbine. When the turbine is operating at 200 rpm, the readings of the two spring scales are 10 and 22 lb, respectively. Determine the power being developed by the turbine.

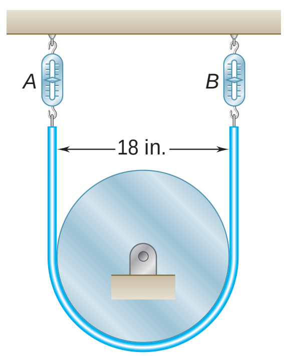


Fig. P17.51

17.8 PRINCIPLE OF IMPULSE AND MOMENTUM FOR THE PLANE MOTION OF A RIGID BODY

The principle of impulse and momentum will now be applied to the analysis of the plane motion of rigid bodies and of systems of rigid bodies. As was pointed out in Chap. 13, the method of impulse and momentum is particularly well adapted to the solution of problems involving time and velocities. Moreover, the principle of impulse and momentum provides the only practicable method for the solution of problems involving impulsive motion or impact (Secs. 17.11 and 17.12).

Considering again a rigid body as made of a large number of particles P_i, we recall from Sec. 14.9 that the system formed by the momenta of the particles at time t_1 and the system of the impulses of the external forces applied from t_1 to t_2 are together equipollent to the system formed by the momenta of the particles at time t_2. Since the vectors associated with a rigid body can be considered as sliding vectors, it follows (Sec. 3.19) that the systems of vectors shown in Fig. 17.6 are not only equipollent but truly *equivalent* in

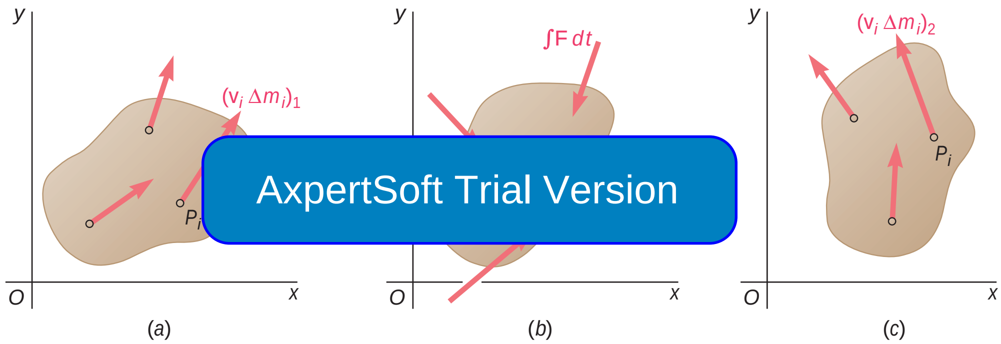


Fig. 17.6

the sense that the vectors on the left-hand side of the equals sign can be transformed into the vectors on the right-hand side through the use of the fundamental operations listed in Sec. 3.13. We therefore write

$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1 \rightarrow 2} = \textbf{Syst Momenta}_2 \qquad (17.14)$$

But the momenta $\mathbf{v}_i\,\Delta m_i$ of the particles can be reduced to a vector attached at G, equal to their sum

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{v}_i\,\Delta m_i$$

and a couple of moment equal to the sum of their moments about G

$$\mathbf{H}_G = \sum_{i=1}^{n} \mathbf{r}'_i \times \mathbf{v}_i\,\Delta m_i$$

We recall from Sec. 14.3 that $\mathbf{L}$ and $\mathbf{H}_G$ define, respectively, the linear momentum and the angular momentum about G of the system

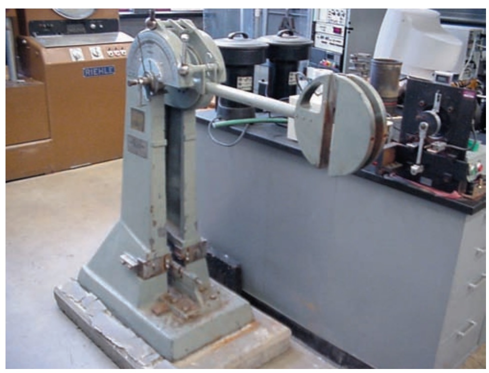


Photo 17.2 A Charpy impact test is used to determine the amount of energy absorbed by a material during impact by subtracting the final gravitation potential energy of the arm from its initial gravitational potential energy.

of particles forming the rigid body. We also note from Eq. (14.14) that $\mathbf{L} = m\overline{\mathbf{v}}$. On the other hand, restricting the present analysis to the plane motion of a rigid slab or of a rigid body symmetrical with respect to the reference plane, we recall from Eq. (16.4) that $\mathbf{H}_G = \bar{I}\boldsymbol{\omega}$. We thus conclude that the system of the momenta $\mathbf{v}_i\,\Delta m_i$ is equivalent to the *linear momentum vector* $m\overline{\mathbf{v}}$ attached at G and to the *angular momentum couple* $\bar{I}\boldsymbol{\omega}$ (Fig. 17.7). Observing that the

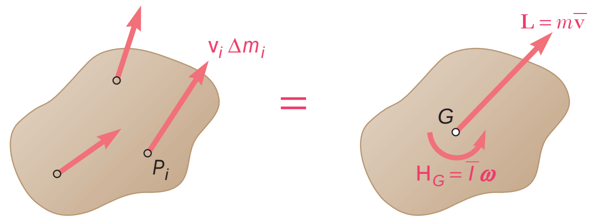


Fig. 17.7

system of momenta reduces to the vector $m\overline{\mathbf{v}}$ in the particular case of a translation ($\boldsymbol{\omega} = 0$) and to the couple $\bar{I}\boldsymbol{\omega}$ in the particular case of a centroidal rotation ($\overline{\mathbf{v}} = \mathbf{0}$), we verify once more that the plane motion of a rigid body symmetrical with respect to the reference plane can be resolved into a translation with the mass center G and a rotation about G.

[illegible]ta in parts a and c of Fig. 17.6 [illegible]ector and angular momentum [illegible] shown in Fig. 17.8. This figure

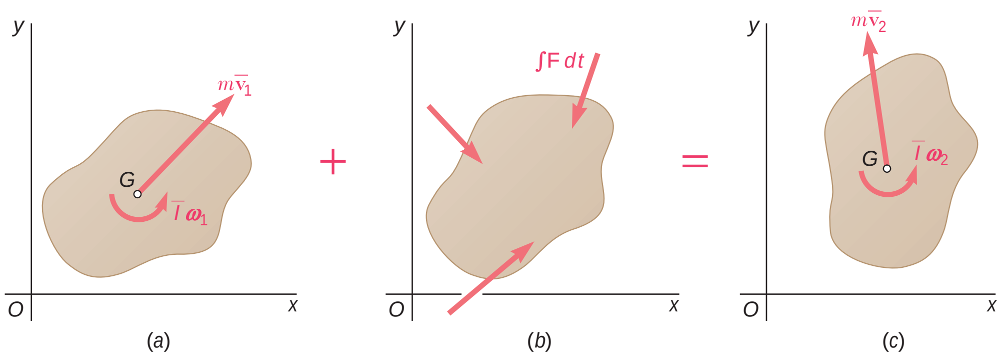


Fig. 17.8

expresses as a free-body-diagram equation the fundamental relation (17.14) in the case of the plane motion of a rigid slab or of a rigid body symmetrical with respect to the reference plane.

Three equations of motion can be derived from Fig. 17.8. Two equations are obtained by summing and equating the x and y *components* of the momenta and impulses, and the third equation is obtained by summing and equating the *moments* of these vectors *about any given point.* The coordinate axes can be chosen fixed in space, or allowed to move with the mass center of the body while maintaining a fixed direction. In either case, the point about which moments are taken should keep the same position relative to the coordinate axes during the interval of time considered.

In deriving the three equations of motion for a rigid body, care should be taken not to add linear and angular momenta indiscriminately. Confusion can be avoided by remembering that $m\bar{v}_x$ and $m\bar{v}_y$ represent the *components of a vector,* namely, the linear momentum vector $m\bar{\mathbf{v}}$, while $\bar{I}\omega$ represents the *magnitude of a couple,* namely, the angular momentum couple $\bar{I}\boldsymbol{\omega}$. Thus the quantity $\bar{I}\omega$ should be added only to the *moment* of the linear momentum $m\bar{\mathbf{v}}$, never to this vector itself nor to its components. All quantities involved will then be expressed in the same units, namely $\text{N} \cdot \text{m} \cdot \text{s}$ or $\text{lb} \cdot \text{ft} \cdot \text{s}$.

Noncentroidal Rotation. In this particular case of plane motion, the magnitude of the velocity of the mass center of the body is $\bar{v} = \bar{r}\omega$, where $\bar{r}$ represents the distance from the mass center to the fixed axis of rotation and ω represents the angular velocity of the body at the instant considered; the magnitude of the momentum vector attached at G is thus $m\bar{v} = m\bar{r}\omega$. Summing the moments about O of the momentum vector and momentum couple (Fig. 17.9)

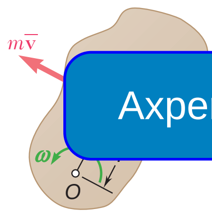


Fig. 17.9

and using the parallel-axis theorem for moments of inertia, we find that the angular momentum $\mathbf{H}_O$ of the body about O has the magnitude†

$$\bar{I}\omega + (m\bar{r}\omega)\bar{r} = (\bar{I} + m\bar{r}^2)\omega = I_O\omega \tag{17.15}$$

Equating the moments about O of the momenta and impulses in (17.14), we write

$$I_O\omega_1 + \sum \int_{t_1}^{t_2} M_O\, dt = I_O\omega_2 \tag{17.16}$$

In the general case of plane motion of a rigid body symmetrical with respect to the reference plane, Eq. (17.16) can be used with respect to the instantaneous axis of rotation under certain conditions. It is recommended, however, that all problems of plane motion be solved by the general method described earlier in this section.

†Note that the sum $\mathbf{H}_A$ of the moments about an arbitrary point A of the momenta of the particles of a rigid slab is, in general, *not* equal to $I_A\omega$. (See Prob. 17.67.)

17.9 SYSTEMS OF RIGID BODIES

The motion of several rigid bodies can be analyzed by applying the principle of impulse and momentum to each body separately (Sample Prob. 17.6). However, in solving problems involving no more than three unknowns (including the impulses of unknown reactions), it is often convenient to apply the principle of impulse and momentum to the system as a whole. The momentum and impulse diagrams are drawn for the entire system of bodies. For each moving part of the system, the diagrams of momenta should include a momentum vector, a momentum couple, or both. Impulses of forces internal to the system can be omitted from the impulse diagram, since they occur in pairs of equal and opposite vectors. Summing and equating successively the x components, y components, and moments of all vectors involved, one obtains three relations which express that the momenta at time t_1 and the impulses of the external forces form a system equipollent to the system of the momenta at time t_2.† Again, care should be taken not to add linear and angular momenta indiscriminately; each equation should be checked to make sure that consistent units have been used. This approach has been used in Sample Prob. 17.8 and, further on, in Sample Probs. 17.9 and 17.10.

17.10 CONSERVATION OF ANGULAR MOMENTUM

When no external force acts on a rigid body or a system of rigid

orces are zero and the system
t to the system of the momenta
uccessively the x components,
y components, and moments of the momenta at times t_1 and t_2, we conclude that the total linear momentum of the system is conserved in any direction and that its total angular momentum is conserved about any point.

There are many engineering applications, however, in which *the linear momentum is not conserved* yet *the angular momentum* $\mathbf{H}_O$ of the system about a given point O *is conserved* that is, in which

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2 \qquad (17.17)$$

Such cases occur when the lines of action of all external forces pass through O or, more generally, when the sum of the angular impulses of the external forces about O is zero.

Problems involving *conservation of angular momentum* about a point O can be solved by the general method of impulse and momentum, i.e., by drawing momentum and impulse diagrams as described in Secs. 17.8 and 17.9. Equation (17.17) is then obtained by summing and equating moments about O (Sample Prob. 17.8). As you will see later in Sample Prob. 17.9, two additional equations can be written by summing and equating x and y components and these equations can be used to determine two unknown linear impulses, such as the impulses of the reaction components at a fixed point.

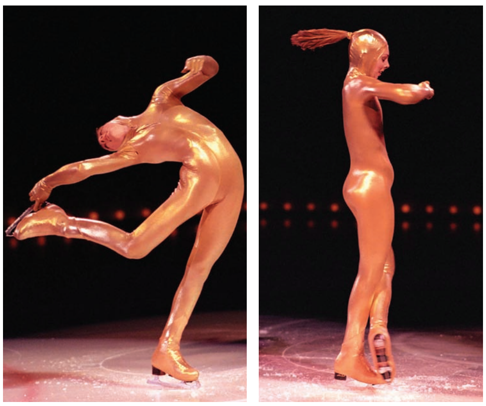
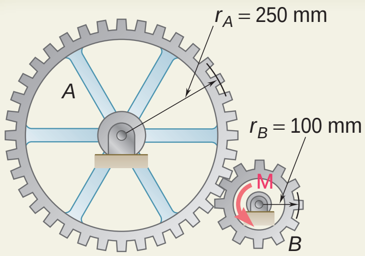


Photo 17.3 A figure skater at the beginning and at the end of a spin. By using the principle of conservation of angular momentum you will find that her angular velocity is much higher at the end of the spin.

†Note that as in Sec. 16.7, we cannot speak of *equivalent* systems since we are not dealing with a single rigid body.



SAMPLE PROBLEM 17.6

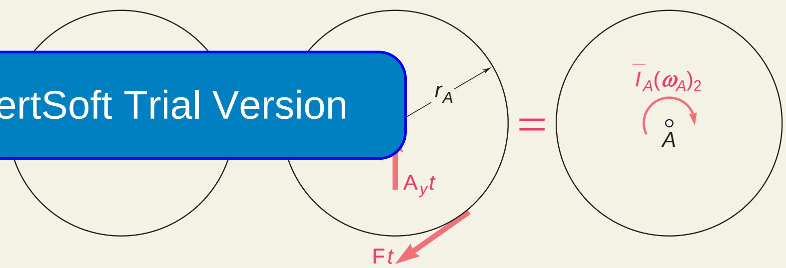
Gear A has a mass of 10 kg and a radius of gyration of 200 mm, and gear B has a mass of 3 kg and a radius of gyration of 80 mm. The system is at rest when a couple **M** of magnitude 6 N · m is applied to gear B. (These gears were considered in Sample Prob. 17.2.) Neglecting friction, determine (*a*) the time required for the angular velocity of gear B to reach 600 rpm, (*b*) the tangential force which gear B exerts on gear A.

SOLUTION

We apply the principle of impulse and momentum to each gear separately. Since all forces and the couple are constant, their impulses are obtained by multiplying them by the unknown time t. We recall from Sample Prob. 17.2 that the centroidal moments of inertia and the final angular velocities are

$$\bar{I}_A = 0.400 \text{ kg}\cdot\text{m}^2 \qquad \bar{I}_B = 0.0192 \text{ kg}\cdot\text{m}^2$$
$$(\mathrm{v}_A)_2 = 25.1 \text{ rad/s} \qquad (\mathrm{v}_B)_2 = 62.8 \text{ rad/s}$$

Principle of Impulse and Momentum for Gear *A*. The systems of initial momenta, impulses, and final momenta are shown in three separate sketches.

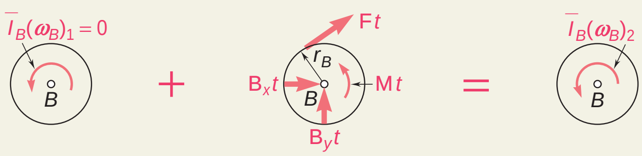


$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\,y\,2} = \textbf{Syst Momenta}_2$$

+1moments about A: $\qquad 0 - Ftr_A = -\bar{I}_A(\mathrm{v}_A)_2$

$$Ft(0.250 \text{ m}) = (0.400 \text{ kg}\cdot\text{m}^2)(25.1 \text{ rad/s})$$
$$Ft = 40.2 \text{ N}\cdot\text{s}$$

Principle of Impulse and Momentum for Gear *B*.



$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\,y\,2} = \textbf{Syst Momenta}_2$$

+1moments about B: $\qquad 0 + Mt - Ftr_B = \bar{I}_B(\mathrm{v}_B)_2$

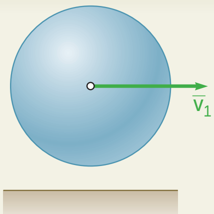
$$+(6 \text{ N}\cdot\text{m})t - (40.2 \text{ N}\cdot\text{s})(0.100 \text{ m}) = (0.0192 \text{ kg}\cdot\text{m}^2)(62.8 \text{ rad/s})$$

$$t = 0.871 \text{ s} \quad ◀$$

Recalling that $Ft = 40.2$ N · s, we write

$$F(0.871 \text{ s}) = 40.2 \text{ N}\cdot\text{s} \qquad F = +46.2 \text{ N}$$

Thus, the force exerted by gear B on gear A is $\qquad \mathbf{F} = 46.2 \text{ N} \swarrow \quad ◀$



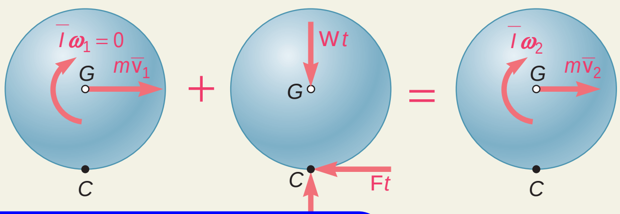
SAMPLE PROBLEM 17.7

A uniform sphere of mass m and radius r is projected along a rough horizontal surface with a linear velocity $\bar{\mathbf{v}}_1$ and no angular velocity. Denoting by m_k the coefficient of kinetic friction between the sphere and the surface, determine (*a*) the time t_2 at which the sphere will start rolling without sliding, (*b*) the linear and angular velocities of the sphere at time t_2.

SOLUTION

While the sphere is sliding relative to the surface, it is acted upon by the normal force **N**, the friction force **F**, and its weight **W** of magnitude $W = mg$.

Principle of Impulse and Momentum. We apply the principle of impulse and momentum to the sphere from the time $t_1 = 0$ when it is placed on the surface until the time $t_2 = t$ when it starts rolling without sliding.



$_{\to 2}$ = **Syst Momenta**$_2$

$Wt = 0$ (1)

+y x components: $m\bar{v}_1 - Ft = m\bar{v}_2$ (2)

+i moments about G: $Ftr = \bar{I}\,\mathrm{v}_2$ (3)

From (1) we obtain $N = W = mg$. During the entire time interval considered, sliding occurs at point C and we have $F = \mathrm{m}_k N = \mathrm{m}_k mg$. Substituting CS for F into (2), we write

$$m\bar{v}_1 - \mathrm{m}_k mgt = m\bar{v}_2 \qquad \bar{v}_2 = \bar{v}_1 - \mathrm{m}_k gt \qquad (4)$$

Substituting $F = \mathrm{m}_k mg$ and $\bar{I} = \frac{2}{5}mr^2$ into (3),

$$\mathrm{m}_k mgtr = \tfrac{2}{5}mr^2\mathrm{v}_2 \qquad \mathrm{v}_2 = \frac{5}{2}\frac{\mathrm{m}_k g}{r}t \qquad (5)$$

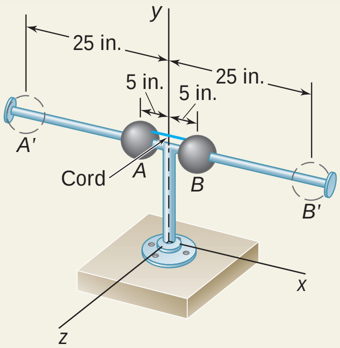
The sphere will start rolling without sliding when the velocity $\mathbf{v}_C$ of the point of contact is zero. At that time, point C becomes the instantaneous center of rotation, and we have $\bar{v}_2 = r\mathrm{v}_2$. Substituting from (4) and (5), we write

$$\bar{v}_2 = r\mathrm{v}_2 \qquad \bar{v}_1 - \mathrm{m}_k gt = r\left(\frac{5}{2}\frac{\mathrm{m}_k g}{r}t\right) \qquad t = \frac{2}{7}\frac{\bar{v}_1}{\mathrm{m}_k g} \quad ◀$$

Substituting this expression for t into (5),

$$\mathrm{v}_2 = \frac{5}{2}\frac{\mathrm{m}_k g}{r}\left(\frac{2}{7}\frac{\bar{v}_1}{\mathrm{m}_k g}\right) \qquad \mathrm{v}_2 = \frac{5}{7}\frac{\bar{v}_1}{r} \qquad \mathrm{V}_2 = \frac{5}{7}\frac{\bar{v}_1}{r}\ \mathrm{i} \quad ◀$$

$$\bar{v}_2 = r\mathrm{v}_2 \qquad \bar{v}_2 = r\left(\frac{5}{7}\frac{\bar{v}_1}{r}\right) \qquad \bar{\mathbf{v}}_2 = \tfrac{5}{7}\,\bar{v}_1\ \mathrm{y} \quad ◀$$

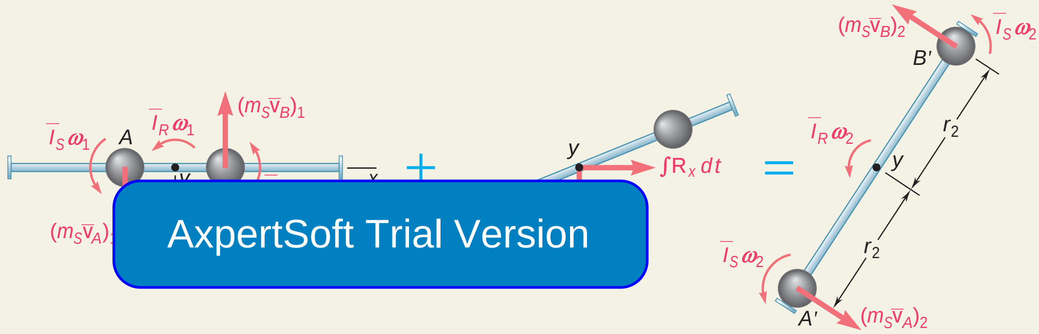


SAMPLE PROBLEM 17.8

Two solid spheres of radius 3 in., weighing 2 lb each, are mounted at A and B on the horizontal rod $A'B'$, which rotates freely about the vertical with a counterclockwise angular velocity of 6 rad/s. The spheres are held in position by a cord which is suddenly cut. Knowing that the centroidal moment of inertia of the rod and pivot is $\bar{I}_R = 0.25 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$, determine (*a*) the angular velocity of the rod after the spheres have moved to positions A' and B', (*b*) the energy lost due to the plastic impact of the spheres and the stops at A' and B'.

SOLUTION

a. Principle of Impulse and Momentum. In order to determine the final angular velocity of the rod, we will express that the initial momenta of the various parts of the system and the impulses of the external forces are together equipollent to the final momenta of the system.



$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\,y\,2} = \textbf{Syst Momenta}_2$$

Observing that the external forces consist of the weights and the reaction at the pivot, which have no moment about the y axis, and noting that $\bar{v}_A = \bar{v}_B = \bar{r}\mathsf{v}$, we equate moments about the y axis:

$$2(m_S\bar{r}_1\mathsf{v}_1)\bar{r}_1 + 2\bar{I}_S\mathsf{v}_1 + \bar{I}_R\mathsf{v}_1 = 2(m_S\bar{r}_2\mathsf{v}_2)\bar{r}_2 + 2\bar{I}_S\mathsf{v}_2 + \bar{I}_R\mathsf{v}_2$$

$$(2m_S\bar{r}_1^2 + 2\bar{I}_S + \bar{I}_R)\mathsf{v}_1 = (2m_S\bar{r}_2^2 + 2\bar{I}_S + \bar{I}_R)\mathsf{v}_2 \qquad (1)$$

which expresses that *the angular momentum of the system about the y axis is conserved.* We now compute

$$\bar{I}_S = \tfrac{2}{5}m_Sa^2 = \tfrac{2}{5}(2\text{ lb}/32.2\text{ ft/s}^2)(\tfrac{3}{12}\text{ ft})^2 = 0.00155\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$m_S\bar{r}_1^2 = (2/32.2)(\tfrac{5}{12})^2 = 0.0108 \qquad m_S\bar{r}_2^2 = (2/32.2)(\tfrac{25}{12})^2 = 0.2696$$

Substituting these values, and $\bar{I}_R = 0.25$ and $\mathsf{v}_1 = 6$ rad/s into (1):

$$0.275(6\text{ rad/s}) = 0.792\mathsf{v}_2 \qquad \mathsf{V}_2 = 2.08\text{ rad/s l} \quad ◀$$

b. Energy Lost. The kinetic energy of the system at any instant is

$$T = 2(\tfrac{1}{2}m_S\bar{v}^2 + \tfrac{1}{2}\bar{I}_S\mathsf{v}^2) + \tfrac{1}{2}\bar{I}_R\mathsf{v}^2 = \tfrac{1}{2}(2m_S\bar{r}^2 + 2\bar{I}_S + \bar{I}_R)\mathsf{v}^2$$

Recalling the numerical values found above, we have

$$T_1 = \tfrac{1}{2}(0.275)(6)^2 = 4.95\text{ ft}\cdot\text{lb} \qquad T_2 = \tfrac{1}{2}(0.792)(2.08)^2 = 1.713\text{ ft}\cdot\text{lb}$$

$$\Delta T = T_2 - T_1 = 1.71 - 4.95 \qquad \Delta T = -3.24\text{ ft}\cdot\text{lb} \quad ◀$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to use the method of impulse and momentum to solve problems involving the plane motion of rigid bodies. As you found out previously in Chap. 13, this method is most effective when used in the solution of problems involving velocities and time.

1. The principle of impulse and momentum for the plane motion of a rigid body is expressed by the following vector equation:

$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1 \rightarrow 2} = \textbf{Syst Momenta}_2 \qquad (17.14)$$

where **Syst Momenta** represents the system of the momenta of the particles forming the rigid body, and **Syst Ext Imp** represents the system of all the external impulses exerted during the motion.

a. The system of the momenta of a rigid body is equivalent to a linear momentum vector $m\bar{\mathbf{v}}$ attached at the mass center of the body and an angular momentum couple $\bar{I}\omega$ (Fig. 17.7).

b. You should draw a free-body-diagram equation for the rigid body to express graphically the above vector equation. Your diagram equation will consist of three sketches of the body, representing respectively the initial momenta, the impulses of the external forces, and the final momenta. It will show that the system of the initial momen[illegible]ernal forces are together equivalent t[illegible]).

c. By using the [illegible] components in any direction and sum moments about any point. When summing moments about a point, remember to include the *angular momentum* $\bar{I}\omega$ of the body, as well as the *moments* of the components of its *linear momentum.* In most cases you will be able to select and solve an equation that involves only one unknown. This was done in all the sample problems of this lesson.

2. In problems involving a system of rigid bodies, you can apply the principle of impulse and momentum to the system as a whole. Since internal forces occur in equal and opposite pairs, they will not be part of your solution [Sample Prob. 17.8].

3. Conservation of angular momentum about a given axis occurs when, for a system of rigid bodies, *the sum of the moments of the external impulses about that axis is zero.* You can indeed easily observe from the free-body-diagram equation that the initial and final angular momenta of the system about that axis are equal and, thus, that *the angular momentum of the system about the given axis is conserved.* You can then sum the angular momenta of the various bodies of the system and the moments of their linear momenta about that axis to obtain an equation which can be solved for one unknown [Sample Prob. 17.8].

CONCEPT QUESTIONS

17.CQ6 Slender bar A is rigidly connected to a massless rod BC in Case 1 and two massless cords in Case 2 as shown. The vertical thickness of bar A is negligible compared to L. If bullet D strikes A with a speed v_0 and becomes embedded in it, how will the speeds of the center of gravity of A immediately after the impact compare for the two cases?

a. Case 1 will be larger.

b. Case 2 will be larger.

c. The speeds will be the same.

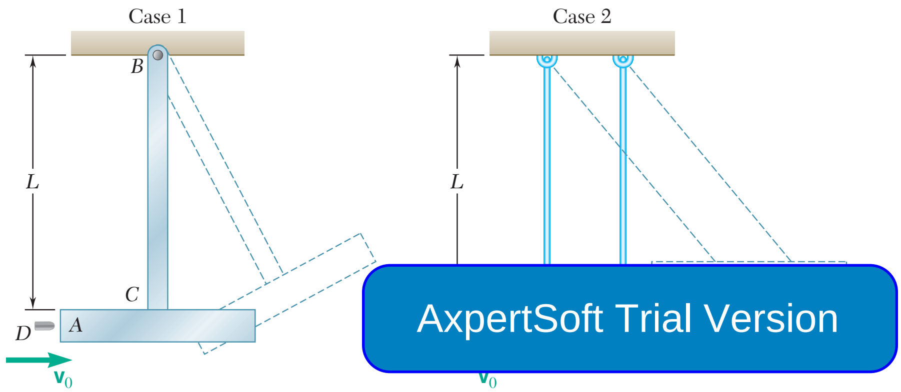


Fig. P17.CQ6

17.CQ7 A 1-m-long uniform slender bar AB has an angular velocity of 12 rad/s and its center of gravity has a velocity of 2 m/s as shown. About which point is the angular momentum of A smallest at this instant?

a. P_1

b. P_2

c. P_3

d. P_4

e. It is the same about all the points.

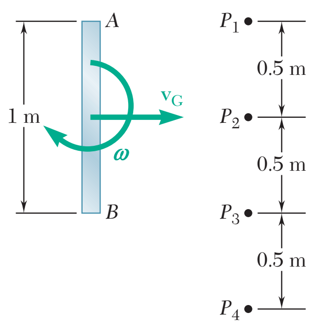


Fig. P17.CQ7

IMPULSE-MOMENTUM PRACTICE PROBLEMS

17.F1 The 350-kg flywheel of a small hoisting engine has a radius of gyration of 600 mm. If the power is cut off when the angular velocity of the flywheel is 100 rpm clockwise, draw an impulse-momentum diagram that can be used to determine the time required for the system to come to rest.

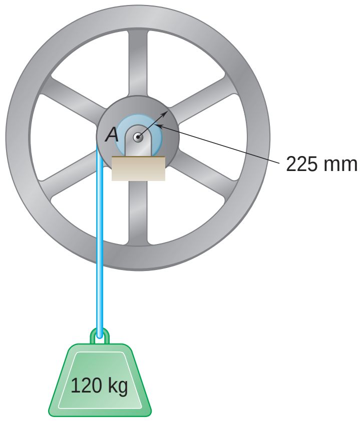


Fig. P17.F1

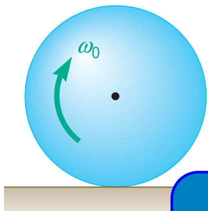


Fig. P17.F2

17.F2 A sphere of radius r and mass m is placed on a horizontal floor with no linear velocity but with a clockwise angular velocity ω_0. Denoting by μ_k the coefficient of kinetic friction between the sphere [illegible]e-momentum diagram that can be [illegible]t which the sphere will start rolling [illegible]

17.F3 Two panels A and B are attached with hinges to a rectangular plate and held by a wire as shown. The plate and the panels are made of the same material and have the same thickness. The entire assembly is rotating with an angular velocity ω_0 when the wire breaks. Draw the impulse-momentum diagram that is needed to determine the angular velocity of the assembly after the panels have come to rest against the plate.

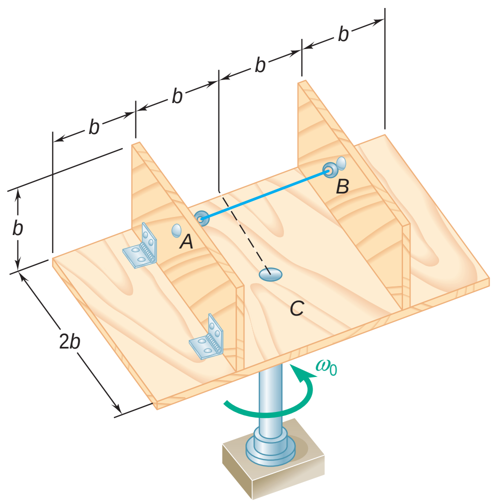


Fig. P17.F3

END-OF-SECTION PROBLEMS

17.52 The rotor of an electric motor has a mass of 25 kg, and it is observed that 4.2 min is required for the rotor to coast to rest from an angular velocity of 3600 rpm. Knowing that kinetic friction produces a couple of magnitude 1.2 N · m, determine the centroidal radius of gyration for the rotor.

17.53 A small grinding wheel is attached to the shaft of an electric motor which has a rated speed of 3600 rpm. When the power is turned off, the unit coasts to rest in 70 s. The grinding wheel and rotor have a combined weight of 6 lb and a combined radius of gyration of 2 in. Determine the average magnitude of the couple due to kinetic friction in the bearings of the motor.

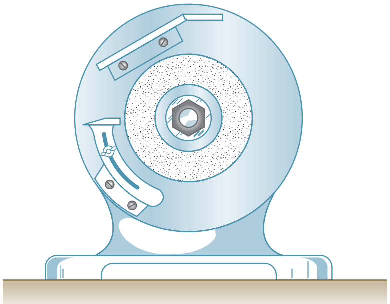


Fig. P17.53

17.54 A bolt located 50 mm from the center of an automobile wheel is tightened by applying the couple shown for 0.10 s. Assuming that the wheel is free to rotate and is initially at rest, determine the resulting angular velocity of the wheel. The wheel has a mass of 19 kg and has a radius of gyration of 250 mm.

Fig. P17.54

17.55 Two disks of the same thickness and same material are attached to a shaft as shown. The 8-lb disk A has a radius $r_A = 3$ in., and disk B has a radius $r_B = 4.5$ in. Knowing that a couple $\mathbf{M}$ of magnitude 20 lb · in. is applied to disk A when the system is at rest, determine the time required for the angular velocity of the system to reach 960 rpm.

17.56 Two disks of the same thickness and same material are attached to a shaft as shown. The 3-kg disk A has a radius $r_A = 100$ mm, and disk B has a radius $r_B = 125$ mm. Knowing that the angular velocity of the system is to be increased from 200 rpm to 800 rpm during a 3-s interval, determine the magnitude of the couple $\mathbf{M}$ that must be applied to disk A.

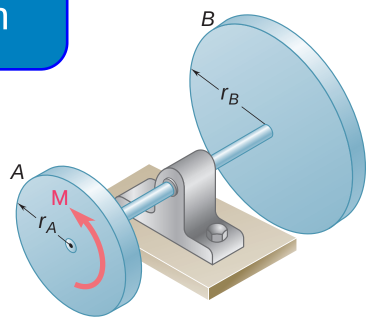


Fig. P17.55 and *P17.56*

17.57 A disk of constant thickness, initially at rest, is placed in contact with a belt that moves with a constant velocity $\mathbf{v}$. Denoting by μ_k the coefficient of kinetic friction between the disk and the belt, derive an expression for the time required for the disk to reach a constant angular velocity.

17.58 Disk A, of weight 5 lb and radius $r = 3$ in., is at rest when it is placed in contact with a belt which moves at a constant speed $v = 50$ ft/s. Knowing that $\mu_k = 0.20$ between the disk and the belt, determine the time required for the disk to reach a constant angular velocity.

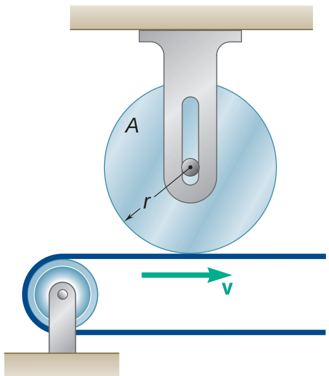


Fig. *P17.57* and P17.58

17.59 A cylinder of radius r and weight W with an initial counterclockwise angular velocity V_0 is placed in the corner formed by the floor and a vertical wall. Denoting by m_k the coefficient of kinetic friction between the cylinder and the wall and the floor, derive an expression for the time required for the cylinder to come to rest.

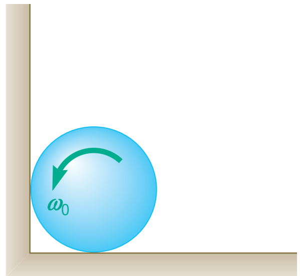


Fig. P17.59

17.60 and 17.61 Two uniform disks and two cylinders are assembled as indicated. Disk A has a mass of 10 kg and disk B has a mass of 6 kg. Knowing that the system is released from rest, determine the time required for cylinder C to have a speed of 0.5 m/s.

17.60 Disks A and B are bolted together and the cylinders are attached to separate cords wrapped on the disks.

17.61 The cylinders are attached to a single cord that passes over the disks. Assume that no slipping occurs between

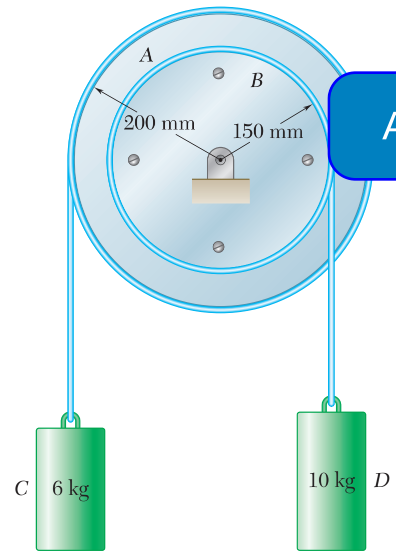


Fig. P17.60

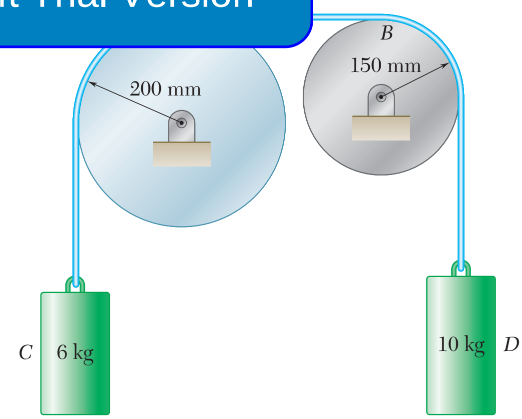


Fig. P17.61

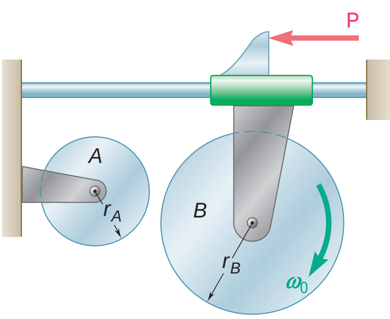


Fig. P17.62 and P17.63

17.62 Disk B has an initial angular velocity V_0 when it is brought into contact with disk A which is at rest. Show that the final angular velocity of disk B depends only on V_0 and the ratio of the masses m_A and m_B of the two disks.

17.63 The 7.5-lb disk A has a radius $r_A = 6$ in. and is initially at rest. The 10-lb disk B has a radius $r_B = 8$ in. and an angular velocity V_0 of 900 rpm when it is brought into contact with disk A. Neglecting friction in the bearings, determine (*a*) the final angular velocity of each disk, (*b*) the total impulse of the friction force exerted on disk A.

17.64 A tape moves over the two drums shown. Drum A weighs 1.4 lb and has a radius of gyration of 0.75 in., while drum B weighs 3.5 lb and has a radius of gyration of 1.25 in. In the lower portion of the tape the tension is constant and equal to $T_A = 0.75$ lb. Knowing that the tape is initially at rest, determine (*a*) the required constant tension T_B if the velocity of the tape is to be $v = 10$ ft/s after 0.24 s, (*b*) the corresponding tension in the portion of the tape between the drums.

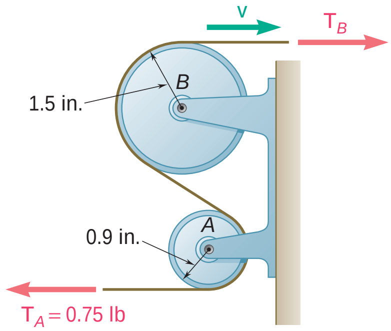
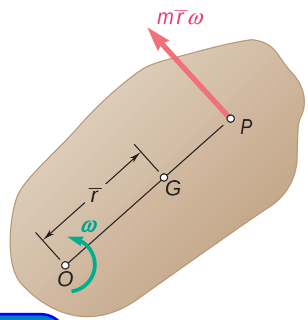


Fig. P17.64

17.65 Show that the system of momenta for a rigid slab in plane motion reduces to a single vector, and express the distance from the mass center G to the line of action of this vector in terms of the centroidal radius of gyration $\bar{k}$ of the slab, the magnitude $\bar{v}$ of the velocity of G, and the angular velocity ω.

17.66 Show that, when a rigid slab rotates about a fixed axis through O perpendicular to the slab, the system of the momenta of its particles is equivalent to a single vector of magnitude $m\bar{r}\omega$, perpendicular to the line OG, and applied to a point P on this line, called the *center of percussion*, at a distance $GP = \bar{k}^2/\bar{r}$ from the mass center of the slab.



P17.66

17.67 Show that the sum $\mathbf{H}_A$ of the moments about a point A of the momenta of the particles of a rigid slab in plane motion is equal to $I_A\omega$, where ω is the angular velocity of the slab at the instant considered and I_A the moment of inertia of the slab about A, if and only if one of the followi
center of the slab, (*b*) A
(*c*) the velocity of A is d
the mass center G.

17.68 Consider a rigid slab initially at rest and subjected to an impulsive force $\mathbf{F}$ contained in the plane of the slab. We define the *center of percussion* P as the point of intersection of the line of action of $\mathbf{F}$ with the perpendicular drawn from G. (*a*) Show that the instantaneous center of rotation C of the slab is located on line GP at a distance $GC = \bar{k}^2/GP$ on the opposite side of G. (*b*) Show that if the center of percussion were located at C the instantaneous center of rotation would be located at P.

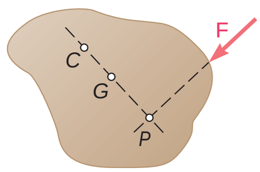


Fig. P17.68

17.69 A flywheel is rigidly attached to a 1.5-in.-radius shaft that rolls without sliding along parallel rails. Knowing that after being released from rest the system attains a speed of 6 in./s in 30 s, determine the centroidal radius of gyration of the system.

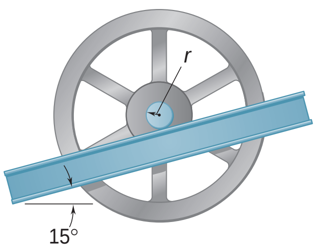


Fig. P17.69

17.70 A wheel of radius r and centroidal radius of gyration $\bar{k}$ is released from rest on the incline shown at time $t = 0$. Assuming that the wheel rolls without sliding, determine (*a*) the velocity of its center at time t, (*b*) the coefficient of static friction required to prevent slipping.

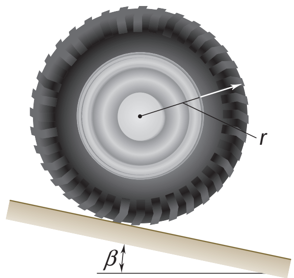


Fig. P17.70

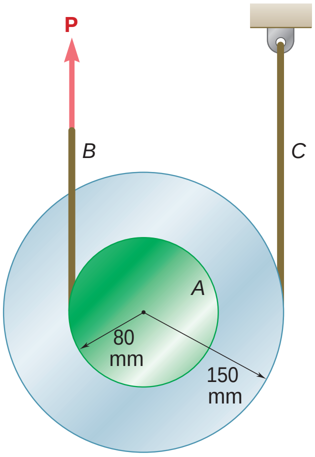


Fig. P17.71

17.71 The double pulley shown has a mass of 3 kg and a radius of gyration of 100 mm. Knowing that when the pulley is at rest, a force **P** of magnitude 24 N is applied to cord B, determine (*a*) the velocity of the center of the pulley after 1.5 s, (*b*) the tension in cord C.

17.72 and *17.73* A 9-in.-radius cylinder of weight 18 lb rests on a 6-lb carriage. The system is at rest when a force **P** of magnitude 2.5 lb is applied as shown for 1.2 s. Knowing that the cylinder rolls without sliding on the carriage and neglecting the mass of the wheels of the carriage, determine the resulting velocity of (*a*) the carriage,

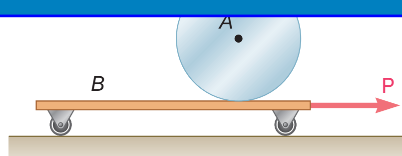


Fig. P17.72

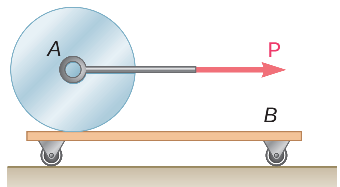


Fig. *P17.73*

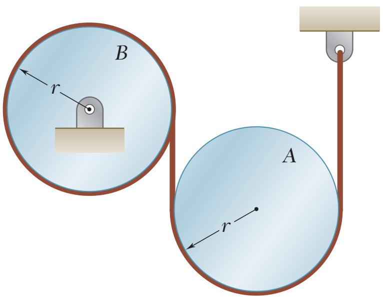


Fig. P17.74 and P17.75

17.74 Two uniform cylinders, each of mass $m = 6$ kg and radius $r = 125$ mm, are connected by a belt as shown. If the system is released from rest when $t = 0$, determine (*a*) the velocity of the center of cylinder B at $t = 3$ s, (*b*) the tension in the portion of belt connecting the two cylinders.

17.75 Two uniform cylinders, each of mass $m = 6$ kg and radius $r = 125$ mm, are connected by a belt as shown. Knowing that at the instant shown the angular velocity of cylinder A is 30 rad/s counterclockwise, determine (*a*) the time required for the angular velocity of cylinder A to be reduced to 5 rad/s, (*b*) the tension in the portion of belt connecting the two cylinders.

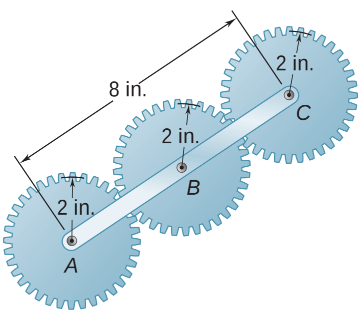


Fig. *P17.76*

17.76 In the gear arrangement shown, gears A and C are attached to rod ABC, which is free to rotate about B, while the inner gear B is fixed. Knowing that the system is at rest, determine the magnitude of the couple **M** which must be applied to rod ABC, if 2.5 s later the angular velocity of the rod is to be 240 rpm clockwise. Gears A and C weigh 2.5 lb each and may be considered as disks of radius 2 in.; rod ABC weighs 4 lb.

17.77 A sphere of radius r and mass m is projected along a rough horizontal surface with the initial velocities shown. If the final velocity of the sphere is to be zero, express (*a*) the required magnitude of V_0 in terms of v_0 and r, (*b*) the time required for the sphere to come to rest in terms of v_0 and the coefficient of kinetic friction m_k.

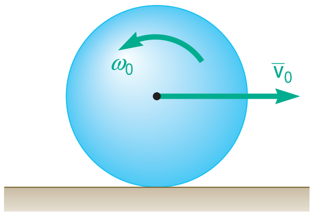


Fig. P17.77

17.78 A bowler projects an 8.5-in.-diameter ball weighing 16 lb along an alley with a forward velocity $\mathbf{v}_0$ of 25 ft/s and a backspin V_0 of 9 rad/s. Knowing that the coefficient of kinetic friction between the ball and the alley is 0.10, determine (*a*) the time t_1 at which the ball will start rolling without sliding, (*b*) the speed of the ball at time t_1.

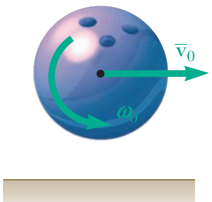


Fig. P17.78

17.79 Four rectangular panels, each of length b and height $\frac{1}{2}b$, are attached with hinges to a circular plate of diameter $1\,\bar{2}\,b$ and held by a wire loop in the position shown. The plate and the panels are made of the same material and have the same thickness. The entire assembly is rotatin
wire breaks. Determine
the panels have come to

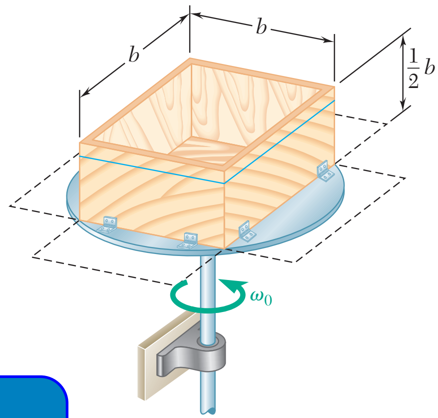


Fig. P17.79

17.80 A 2.5-lb disk of radius 4 in. is attached to the yoke *BCD* by means of short shafts fitted in bearings at *B* and *D*. The 1.5-lb yoke has a radius of gyration of 3 in. about the x axis. Initially the assembly is rotating at 120 rpm with the disk in the plane of the yoke ($u = 0$). If the disk is slightly disturbed and rotates with respect to the yoke until $u = 90°$, where it is stopped by a small bar at *D*, determine the final angular velocity of the assembly.

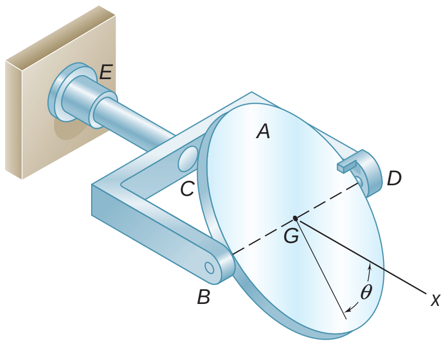


Fig. P17.80

17.81 Two 10-lb disks and a small motor are mounted on a 15-lb rectangular platform which is free to rotate about a central vertical spindle. The normal operating speed of the motor is 180 rpm. If the motor is started when the system is at rest, determine the angular velocity of all elements of the system after the motor has attained its normal operating speed. Neglect the mass of the motor and of the belt.

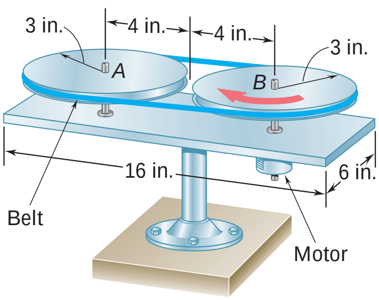


Fig. P17.81

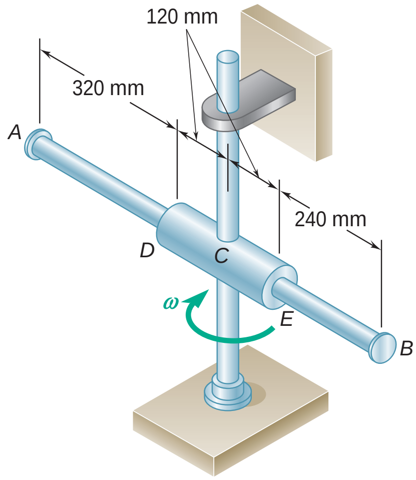


Fig. P17.82

17.82 A 3-kg rod of length 800 mm can slide freely in the 240-mm cylinder *DE*, which in turn can rotate freely in a horizontal plane. In the position shown the assembly is rotating with an angular velocity of magnitude $\omega = 40$ rad/s and end *B* of the rod is moving toward the cylinder at a speed of 75 mm/s relative to the cylinder. Knowing that the centroidal mass moment of inertia of the cylinder about a vertical axis is 0.025 kg · m^2 and neglecting the effect of friction, determine the angular velocity of the assembly as end *B* of the rod strikes end *E* of the cylinder.

17.83 A 1.6-kg tube *AB* can slide freely on rod *DE* which in turn can rotate freely in a horizontal plane. Initially the assembly is rotating with an angular velocity $\omega = 5$ rad/s and the tube is held in position by a cord. The moment of inertia of the rod and bracket about the vertical axis of rotation is 0.30 kg · m^2 and the centroidal moment of inertia of the tube about a vertical axis is 0.0025 kg · m^2. If the cord suddenly breaks, determine (*a*) the angular velocity of the assembly after the tube has moved to end *E*, (*b*) the energy lost during the plastic impact at *E*.

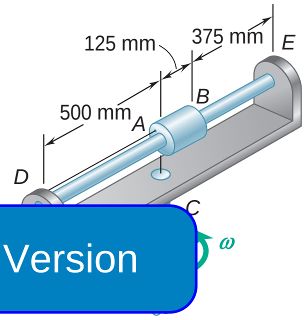


Fig. P17.83

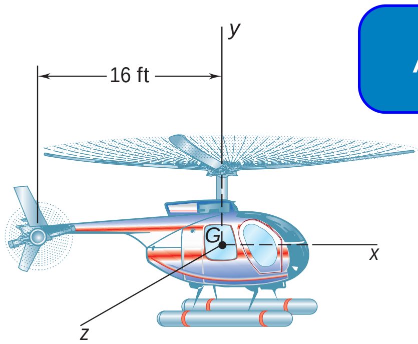


Fig. *P17.84*

17.84 In the helicopter shown, a vertical tail propeller is used to prevent rotation of the cab as the speed of the main blades is changed. Assuming that the tail propeller is not operating, determine the final angular velocity of the cab after the speed of the main blades has been changed from 180 to 240 rpm. (The speed of the main blades is measured relative to the cab, and the cab has a centroidal moment of inertia of 650 lb · ft · s^2. Each of the four main blades is assumed to be a slender 14-ft rod weighing 55 lb.)

17.85 Assuming that the tail propeller in Prob. 17.84 is operating and that the angular velocity of the cab remains zero, determine the final horizontal velocity of the cab when the speed of the main blades is changed from 180 to 240 rpm. The cab weighs 1250 lb and is initially at rest. Also determine the force exerted by the tail propeller if the change in speed takes place uniformly in 12 s.

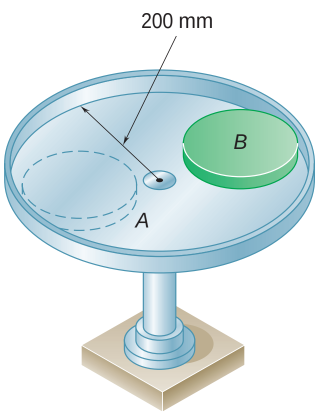


Fig. P17.86

17.86 The circular platform *A* is fitted with a rim of 200-mm inner radius and can rotate freely about the vertical shaft. It is known that the platform-rim unit has a mass of 5 kg and a radius of gyration of 175 mm with respect to the shaft. At a time when the platform is rotating with an angular velocity of 50 rpm, a 3-kg disk *B* of radius 80 mm is placed on the platform with no velocity. Knowing that disk *B* then slides until it comes to rest relative to the platform against the rim, determine the final angular velocity of the platform.

17.87 Two 4-kg disks and a small motor are mounted on a 6-kg rectangular platform which is free to rotate about a central vertical spindle. The normal operating speed of the motor is 240 rpm. If the motor is started when the system is at rest, determine the angular velocity of all elements of the system after the motor has attained its normal operating speed. Neglect the mass of the motor and of the belt.

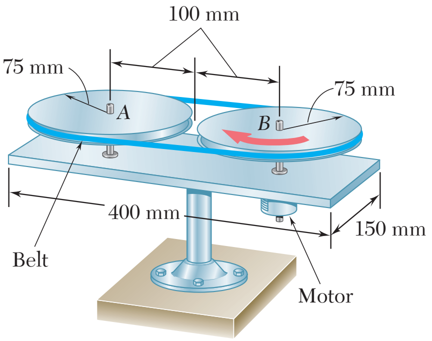
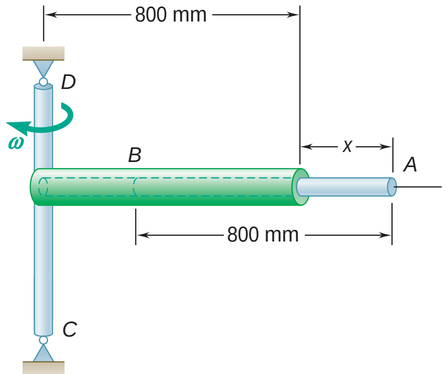


Fig. P17.87

17.88 The 4-kg rod *AB* can slide freely inside the 6-kg tube *CD*. The rod was entirely within the tube ($x = 0$) and released with no initial velocity relative to the tube when the angular velocity of the assembly was 5 rad/s. Neglecting the effect of friction, determine the speed of the rod rel



17.88

17.89 A 1.8-kg collar *A* and a 0
a frame, consisting of the horizontal rod *OE* and the vertical rod *CD*, which is free to rotate about its vertical axis of symmetry. The two collars are connected by a cord running over a pulley that is attached to the frame at *O*. At the instant shown, the velocity v_A of collar *A* has a magnitude of 2.1 m/s and a stop prevents collar *B* from moving. The stop is suddenly removed and collar *A* moves toward *E*. As it reaches a distance of 0.12 m from *O*, the magnitude of its velocity is observed to be 2.5 m/s. Determine at that instant the magnitude of the angular velocity of the frame and the moment of inertia of the frame and pulley system about *CD*.

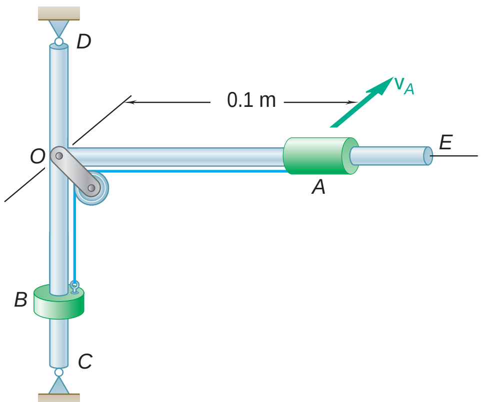


Fig. P17.89

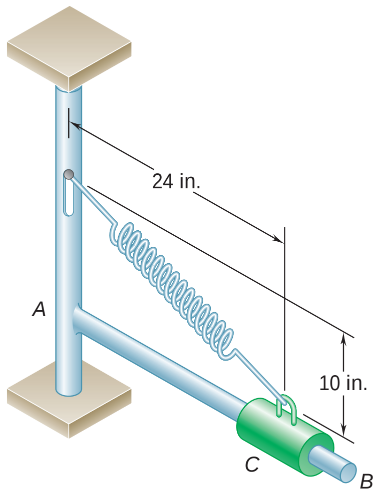


Fig. P17.90

17.90 A 6-lb collar C is attached to a spring and can slide on rod AB, which in turn can rotate in a horizontal plane. The mass moment of inertia of rod AB with respect to end A is $0.35 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$. The spring has a constant $k = 15$ lb/in. and an undeformed length of 10 in. At the instant shown the velocity of the collar relative to the rod is zero and the assembly is rotating with an angular velocity of 12 rad/s. Neglecting the effect of friction, determine (*a*) the angular velocity of the assembly as the collar passes through a point located 7.5 in. from end A of the rod, (*b*) the corresponding velocity of the collar relative to the rod.

17.91 A small 4-lb collar C can slide freely on a thin ring of weight 6 lb and radius 10 in. The ring is welded to a short vertical shaft, which can rotate freely in a fixed bearing. Initially the ring has an angular velocity of 35 rad/s and the collar is at the top of the ring (u = 0) when it is given a slight nudge. Neglecting the effect of friction, determine (*a*) the angular velocity of the ring as the collar passes through the position u = 90°, (*b*) the corresponding velocity of the collar relative to the ring.

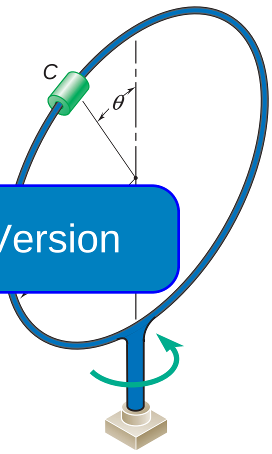


Fig. P17.91

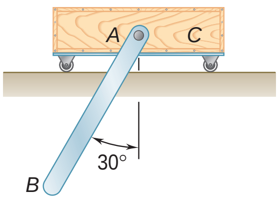


Fig. *P17.92*

17.92 A uniform rod AB, of mass 7 kg and length 1.2 m, is attached to the 11-kg cart C. Knowing that the system is released from rest in the position shown and neglecting friction, determine (*a*) the velocity of point B as rod AB passes through a vertical position, (*b*) the corresponding velocity of cart C.

17.93 In Prob. 17.82, determine the velocity of rod AB relative to cylinder DE as end B of the rod strikes end E of the cylinder.

17.94 In Prob. 17.83, determine the velocity of the tube relative to the rod as the tube strikes end E of the assembly.

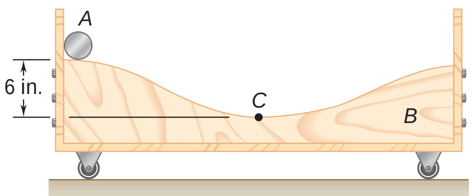


Fig. P17.95

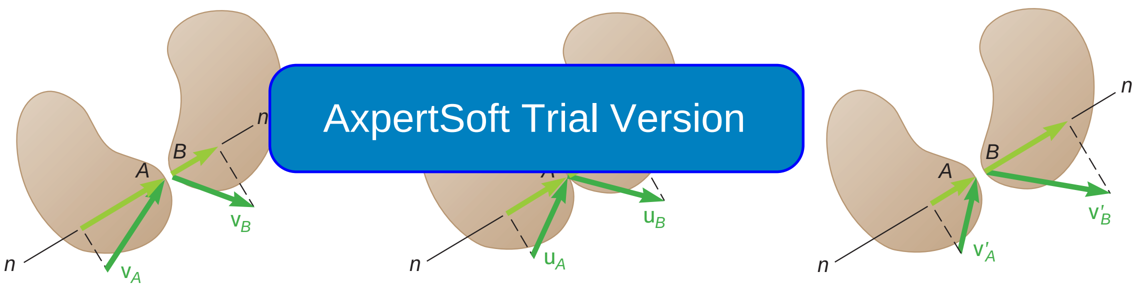
17.95 The 6-lb steel cylinder A and the 10-lb wooden cart B are at rest in the position shown when the cylinder is given a slight nudge, causing it to roll without sliding along the top surface of the cart. Neglecting friction between the cart and the ground, determine the velocity of the cart as the cylinder passes through the lowest point of the surface at C.

17.11 IMPULSIVE MOTION

You saw in Chap. 13 that the method of impulse and momentum is the only practicable method for the solution of problems involving the impulsive motion of a particle. Now you will find that problems involving the impulsive motion of a rigid body are particularly well suited to a solution by the method of impulse and momentum. Since the time interval considered in the computation of linear impulses and angular impulses is very short, the bodies involved can be assumed to occupy the same position during that time interval, making the computation quite simple.

17.12 ECCENTRIC IMPACT

In Secs. 13.13 and 13.14, you learned to solve problems of *central impact,* i.e., problems in which the mass centers of the two colliding bodies are located on the line of impact. You will now analyze the *eccentric impact* of two rigid bodies. Consider two bodies which collide, and denote by $\mathbf{v}_A$ and $\mathbf{v}_B$ the velocities before impact *of the two points of contact* A and B (Fig. 17.10a). Under the impact, the two bodies will *deform,* and at the end of the period of deformation, the velocities $\mathbf{u}_A$ and $\mathbf{u}_B$ of A and B will have equal components along the line of impact nn (Fig. 17.10b). A period of *restitution* will then take place, at the end of which A and B will have velocities $\mathbf{v}'_A$ and $\mathbf{v}'_B$ (Fig. 17.10c). Assuming that the bodies are frictionless, we find that the forces they exert on each other are directed along the line of impact. Denoting the magnitude of the impulse of one of these forces during the period of deformation by $\int P\,dt$ and the magnitude of its impulse during the period of restitution by $\int R\,dt$, we recall that the coefficient of restitution e is defined as the ratio



(a) (b) (c)

Fig. 17.10

$$e = \frac{\int R\,dt}{\int P\,dt} \qquad (17.18)$$

We propose to show that the relation established in Sec. 13.13 between the relative velocities of two particles before and after impact also holds between the components along the line of impact

Photo 17.4 When the rotating bat contacts the ball it applies an impulsive force to the ball requiring the method of impulse and momentum to be used to determine the final velocities of the ball and bat.

of the relative velocities of the two points of contact A and B. We propose to show, therefore, that

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n] \tag{17.19}$$

It will first be assumed that the motion of each of the two colliding bodies of Fig. 17.10 is unconstrained. Thus the only impulsive forces exerted on the bodies during the impact are applied at A and B, respectively. Consider the body to which point A belongs and draw the three momentum and impulse diagrams corresponding to the period of deformation (Fig. 17.11). We denote by $\bar{\mathbf{v}}$ and $\bar{\mathbf{u}}$,

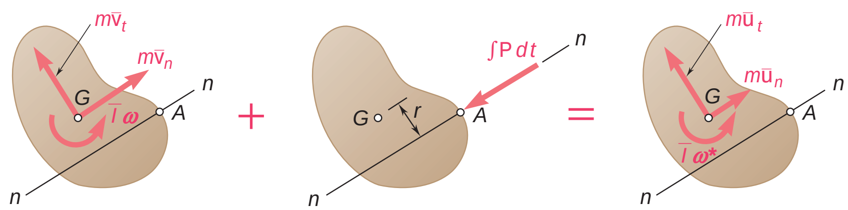


Fig. 17.11

center at the beginning and at , and we denote by ω and ω^* e same instants. Summing and equating the components of the momenta and impulses along the line of impact nn, we write

$$m\bar{v}_n - \int P\,dt = m\bar{u}_n \tag{17.20}$$

Summing and equating the moments about G of the momenta and impulses, we also write

$$\bar{I}\omega - r\int P\,dt = \bar{I}\omega^* \tag{17.21}$$

where r represents the perpendicular distance from G to the line of impact. Considering now the period of restitution, we obtain in a similar way

$$m\bar{u}_n - \int R\,dt = m\bar{v}'_n \tag{17.22}$$

$$\bar{I}\omega^* - r\int R\,dt = \bar{I}\omega' \tag{17.23}$$

where $\bar{\mathbf{v}}'$ and ω' represent, respectively, the velocity of the mass center and the angular velocity of the body after impact. Solving (17.20) and (17.22) for the two impulses and substituting into (17.18), and then solving (17.21) and (17.23) for the same two impulses and substituting again into (17.18), we obtain the following two alternative expressions for the coefficient of restitution:

$$e = \frac{\bar{u}_n - \bar{v}'_n}{\bar{v}_n - \bar{u}_n} \qquad e = \frac{\omega^* - \omega'}{\omega - \omega^*} \tag{17.24}$$

Multiplying by r the numerator and denominator of the second expression obtained for e, and adding respectively to the numerator and denominator of the first expression, we have

$$e = \frac{\bar{u}_n + r\omega^* - (\bar{v}'_n + r\omega')}{\bar{v}_n + r\omega - (\bar{u}_n + r\omega^*)} \tag{17.25}$$

Observing that $\bar{v}_n + r\omega$ represents the component $(v_A)_n$ along nn of the velocity of the point of contact A and that, similarly, $\bar{u}_n + r\omega^*$ and $\bar{v}'_n + r\omega'$ represent, respectively, the components $(u_A)_n$ and $(v'_A)_n$, we write

$$e = \frac{(u_A)_n - (v'_A)_n}{(v_A)_n - (u_A)_n} \tag{17.26}$$

The analysis of the motion of the second body leads to a similar expression for e in terms of the components along nn of the successive velocities of point B. Recalling that $(u_A)_n = (u_B)_n$, and eliminating these two velocity components by a manipulation similar to the one used in Sec. 13.13, we obtain relation (17.19).

If one or both of the colliding bodies is constrained to rotate about a fixed point O, as in the case of a compound pendulum (Fig. 17.12*a*), an impulsive reaction will be exerted at O (Fig. 17.12*b*).

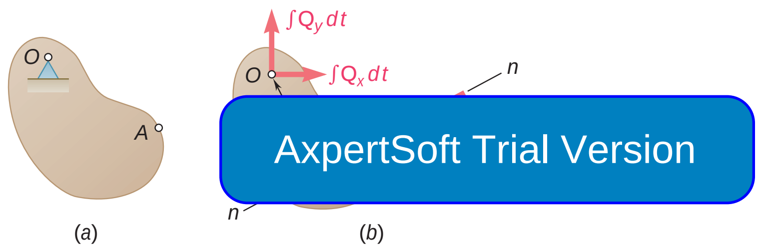


Fig. 17.12

Let us verify that while their derivation must be modified, Eqs. (17.26) and (17.19) remain valid. Applying formula (17.16) to the period of deformation and to the period of restitution, we write

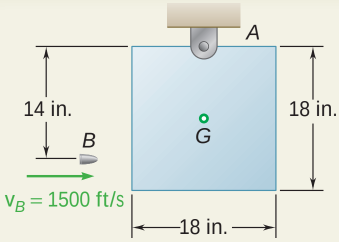
$$I_O\omega - r\int P\,dt = I_O\omega^* \tag{17.27}$$
$$I_O\omega^* - r\int R\,dt = I_O\omega' \tag{17.28}$$

where r represents the perpendicular distance from the fixed point O to the line of impact. Solving (17.27) and (17.28) for the two impulses and substituting into (17.18), and then observing that $r\omega$, $r\omega^*$, and $r\omega'$ represent the components along nn of the successive velocities of point A, we write

$$e = \frac{\omega^* - \omega'}{\omega - \omega^*} = \frac{r\omega^* - r\omega'}{r\omega - r\omega^*} = \frac{(u_A)_n - (v'_A)_n}{(v_A)_n - (u_A)_n}$$

and check that Eq. (17.26) still holds. Thus Eq. (17.19) remains valid when one or both of the colliding bodies is constrained to rotate about a fixed point O.

In order to determine the velocities of the two colliding bodies after impact, relation (17.19) should be used in conjunction with one or several other equations obtained by applying the principle of impulse and momentum (Sample Prob. 17.10).

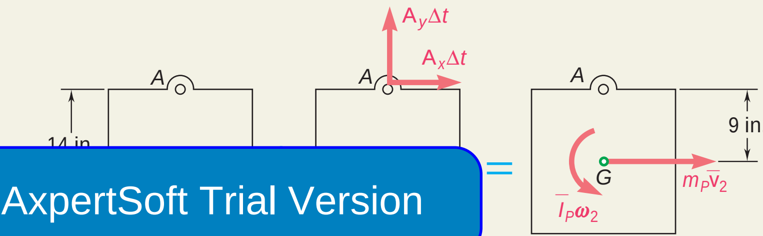


SAMPLE PROBLEM 17.9

A 0.05-lb bullet B is fired with a horizontal velocity of 1500 ft/s into the side of a 20-lb square panel suspended from a hinge at A. Knowing that the panel is initially at rest, determine (a) the angular velocity of the panel immediately after the bullet becomes embedded, (b) the impulsive reaction at A, assuming that the bullet becomes embedded in 0.0006 s.

SOLUTION

Principle of Impulse and Momentum. We consider the bullet and the panel as a single system and express that the initial momenta of the bullet and panel and the impulses of the external forces are together equipollent to the final momenta of the system. Since the time interval $\Delta t = 0.0006$ s is very short, we neglect all nonimpulsive forces and consider only the external impulses $\mathbf{A}_x\,\Delta t$ and $\mathbf{A}_y\,\Delta t$.



$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\to2} = \textbf{Syst Momenta}_2$$

$+\circlearrowleft$ moments about A: $\quad m_Bv_B(\tfrac{14}{12}\text{ ft}) + 0 = m_P\bar{v}_2(\tfrac{9}{12}\text{ ft}) + \bar{I}_P\omega_2 \qquad (1)$

$\xrightarrow{+}$ x components: $\quad m_Bv_B + A_x\,\Delta t = m_P\bar{v}_2 \qquad (2)$

$+\uparrow y$ components: $\quad 0 + A_y\,\Delta t = 0 \qquad (3)$

The centroidal mass moment of inertia of the square panel is

$$\bar{I}_P = \tfrac{1}{6}m_Pb^2 = \frac{1}{6}\left(\frac{20\text{ lb}}{32.2}\right)(\tfrac{18}{12}\text{ ft})^2 = 0.2329\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

Substituting this value as well as the given data into (1) and noting that

$$\bar{v}_2 = (\tfrac{9}{12}\text{ ft})\omega_2$$

we write

$$\left(\frac{0.05}{32.2}\right)(1500)(\tfrac{14}{12}) = 0.2329\omega_2 + \left(\frac{20}{32.2}\right)(\tfrac{9}{12}\omega_2)(\tfrac{9}{12})$$

$$\omega_2 = 4.67\text{ rad/s} \qquad \boldsymbol{\omega}_2 = 4.67\text{ rad/s}\circlearrowleft \blacktriangleleft$$

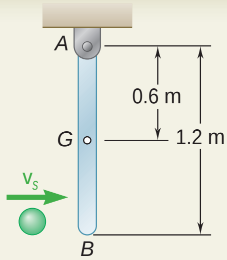
$$\bar{v}_2 = (\tfrac{9}{12}\text{ ft})\omega_2 = (\tfrac{9}{12}\text{ ft})(4.67\text{ rad/s}) = 3.50\text{ ft/s}$$

Substituting $\bar{v}_2 = 3.50$ ft/s, $\Delta t = 0.0006$ s, and the given data into Eq. (2), we have

$$\left(\frac{0.05}{32.2}\right)(1500) + A_x(0.0006) = \left(\frac{20}{32.2}\right)(3.50)$$

$$A_x = -259\text{ lb} \qquad \mathbf{A}_x = 259\text{ lb}\leftarrow \blacktriangleleft$$

From Eq. (3), we find $\quad A_y = 0 \qquad \mathbf{A}_y = 0 \blacktriangleleft$

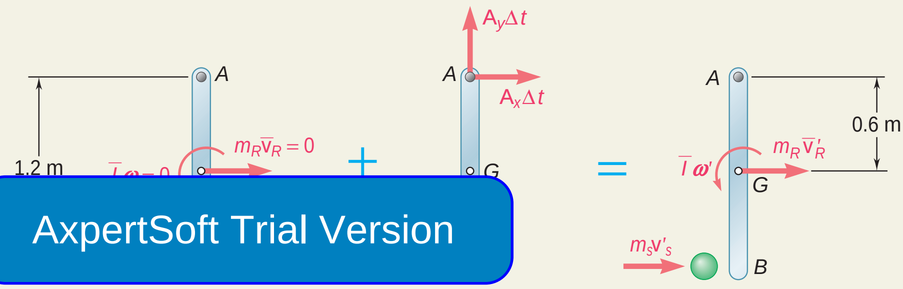


SAMPLE PROBLEM 17.10

A 2-kg sphere moving horizontally to the right with an initial velocity of 5 m/s strikes the lower end of an 8-kg rigid rod AB. The rod is suspended from a hinge at A and is initially at rest. Knowing that the coefficient of restitution between the rod and the sphere is 0.80, determine the angular velocity of the rod and the velocity of the sphere immediately after the impact.

SOLUTION

Principle of Impulse and Momentum. We consider the rod and sphere as a single system and express that the initial momenta of the rod and sphere and the impulses of the external forces are together equipollent to the final momenta of the system. We note that the only impulsive force external to the system is the impulsive reaction at A.



$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\,y\,2} = \textbf{Syst Momenta}_2$$

+l moments about A:

$$m_s v_s(1.2\text{ m}) = m_s v'_s(1.2\text{ m}) + m_R \bar{v}'_R(0.6\text{ m}) + \bar{I}\mathsf{v}' \qquad (1)$$

Since the rod rotates about A, we have $\bar{v}'_R = \bar{r}\mathsf{v}' = (0.6\text{ m})\mathsf{v}'$. Also,

$$\bar{I} = \tfrac{1}{12}mL^2 = \tfrac{1}{12}(8\text{ kg})(1.2\text{ m})^2 = 0.96\text{ kg}\cdot\text{m}^2$$

Substituting these values and the given data into Eq. (1), we have

$$(2\text{ kg})(5\text{ m/s})(1.2\text{ m}) = (2\text{ kg})v'_s(1.2\text{ m}) + (8\text{ kg})(0.6\text{ m})\mathsf{v}'(0.6\text{ m}) + (0.96\text{ kg}\cdot\text{m}^2)\mathsf{v}'$$

$$12 = 2.4v'_s + 3.84\mathsf{v}' \qquad (2)$$

Relative Velocities. Choosing positive to the right, we write

$$v'_B - v'_s = e(v_s - v_B)$$

Substituting $v_s = 5$ m/s, $v_B = 0$, and $e = 0.80$, we obtain

$$v'_B - v'_s = 0.80(5\text{ m/s}) \qquad (3)$$

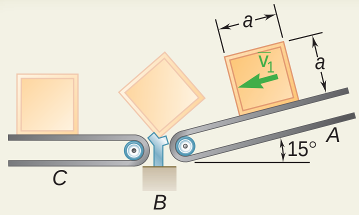
Again noting that the rod rotates about A, we write

$$v'_B = (1.2\text{ m})\mathsf{v}' \qquad (4)$$

Solving Eqs. (2) to (4) simultaneously, we obtain

$\mathsf{v}' = 3.21$ rad/s $\qquad$ $\mathsf{V}' = 3.21$ rad/s l ◀

$v'_s = -0.143$ m/s $\qquad$ $\mathbf{v}'_s = -0.143$ m/s z ◀

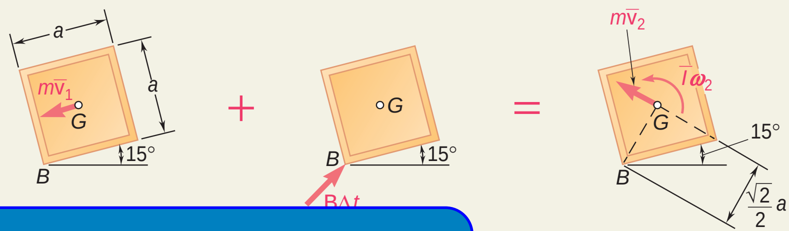


SAMPLE PROBLEM 17.11

A square package of side a and mass m moves down a conveyor belt A with a constant velocity $\bar{\mathbf{v}}_1$. At the end of the conveyor belt, the corner of the package strikes a rigid support at B. Assuming that the impact at B is perfectly plastic, derive an expression for the smallest magnitude of the velocity $\bar{\mathbf{v}}_1$ for which the package will rotate about B and reach conveyor belt C.

SOLUTION

Principle of Impulse and Momentum. Since the impact between the package and the support is perfectly plastic, the package rotates about B during the impact. We apply the principle of impulse and momentum to the package and note that the only impulsive force external to the package is the impulsive reaction at B.



$_2$ = **Syst Momenta**$_2$

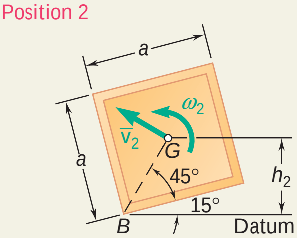
+Moments about B: $(m\bar{v}_1)(\frac{1}{2}a) + 0 = (m\bar{v}_2)(\frac{1}{2}\sqrt{2}a) + \bar{I}\omega_2$ (1)

Since the package rotates about B, we have $\bar{v}_2 = (GB)\omega_2 = \frac{1}{2}\sqrt{2}a\omega_2$. We substitute this expression, together with $\bar{I} = \frac{1}{6}ma^2$, into Eq. (1):

$$(m\bar{v}_1)(\tfrac{1}{2}a) = m(\tfrac{1}{2}\sqrt{2}a\omega_2)(\tfrac{1}{2}\sqrt{2}a) + \tfrac{1}{6}ma^2\omega_2 \qquad \bar{v}_1 = \tfrac{4}{3}a\omega_2 \qquad (2)$$

Principle of Conservation of Energy. We apply the principle of conservation of energy between position 2 and position 3.

Position 2



$GB = \frac{1}{2}\sqrt{2}a = 0.707a$

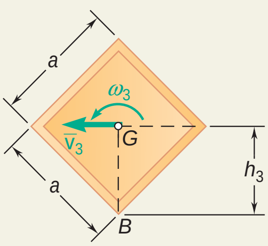
$h_2 = GB\sin(45° + 15°)$

$= 0.612a$

Position 2. $V_2 = Wh_2$. Recalling that $\bar{v}_2 = \frac{1}{2}\sqrt{2}a\omega_2$, we write

$$T_2 = \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\omega_2^2 = \tfrac{1}{2}m(\tfrac{1}{2}\sqrt{2}a\omega_2)^2 + \tfrac{1}{2}(\tfrac{1}{6}ma^2)\omega_2^2 = \tfrac{1}{3}ma^2\omega_2^2$$

Position 3



$h_3 = GB = 0.707a$

Position 3. Since the package must reach conveyor belt C, it must pass through position 3 where G is directly above B. Also, since we wish to determine the smallest velocity for which the package will reach this position, we choose $\bar{v}_3 = \omega_3 = 0$. Therefore $T_3 = 0$ and $V_3 = Wh_3$.

Conservation of Energy

$$T_2 + V_2 = T_3 + V_3$$

$$\tfrac{1}{3}ma^2\omega_2^2 + Wh_2 = 0 + Wh_3$$

$$\omega_2^2 = \frac{3W}{ma^2}(h_3 - h_2) = \frac{3g}{a^2}(h_3 - h_2) \qquad (3)$$

Substituting the computed values of h_2 and h_3 into Eq. (3), we obtain

$$\omega_2^2 = \frac{3g}{a^2}(0.707a - 0.612a) = \frac{3g}{a^2}(0.095a) \qquad \omega_2 = \sqrt{0.285g/a}$$

$$\bar{v}_1 = \tfrac{4}{3}a\omega_2 = \tfrac{4}{3}a\sqrt{0.285g/a} \qquad \bar{v}_1 = 0.712\sqrt{ga}$$

SOLVING PROBLEMS ON YOUR OWN

This lesson was devoted to the *impulsive motion* and to the *eccentric impact of rigid bodies.*

1. Impulsive motion occurs when a rigid body is subjected to a very large force $\mathbf{F}$ for a very short interval of time Δt; the resulting impulse $\mathbf{F}\,\Delta t$ is both finite and different from zero. Such forces are referred to as *impulsive forces* and are encountered whenever there is an impact between two rigid bodies. Forces for which the impulse is zero are referred to as *nonimpulsive forces.* As you saw in Chap. 13, the following forces can be assumed to be nonimpulsive: the *weight* of a body, the force exerted by a *spring,* and any other force which is *known* to be small by comparison with the impulsive forces. Unknown reactions, however, *cannot be assumed* to be nonimpulsive.

2. Eccentric impact of rigid bodies. You saw that when two bodies collide, the velocity components along the line of impact of the *points of contact A and B* before and after impact satisfy the following equation:

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n] \tag{17.19}$$

where the left-hand member is the *relative velocity after the impact,* and the right-hand member is [illegible] nd the *relative velocity before the i*[illegible]

This equat[illegible] velocity components of the points of contact before and after an impact that you used for particles in Chap. 13.

3. To solve a problem involving an impact you should use the *method of impulse and momentum* and take the following steps.

a. Draw a free-body-diagram equation of the body that will express that the system consisting of the momenta immediately before impact and of the impulses of the external forces is equivalent to the system of the momenta immediately after impact.

b. The free-body-diagram equation will relate the velocities before and after impact and the impulsive forces and reactions. In some cases, you will be able to determine the unknown velocities and impulsive reactions by solving equations obtained by summing components and moments [Sample Prob. 17.9].

c. In the case of an impact in which $e > 0$, the number of unknowns will be greater than the number of equations that you can write by summing components and moments, and you should supplement the equations obtained from the free-body-diagram equation with Eq. (17.19), which relates the relative velocities of the points of contact before and after impact [Sample Prob. 17.10].

d. During an impact you must use the method of impulse and momentum. However, *before and after the impact* you can, if necessary, use some of the other methods of solution that you have learned, such as the method of work and energy [Sample Prob. 17.11].

PROBLEMS

IMPULSE-MOMENTUM PRACTICE PROBLEMS

17.F4 A uniform slender rod AB of mass m is at rest on a frictionless horizontal surface when hook C engages a small pin at A. Knowing that the hook is pulled upward with a constant velocity $\mathbf{v}_0$, draw the impulse-momentum diagram that is needed to determine the impulse exerted on the rod at A and B. Assume that the velocity of the hook is unchanged and that the impact is perfectly plastic.

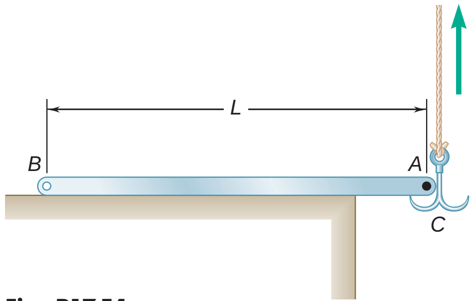


Fig. P17.F4

17.F5 A uniform slender rod AB of length L is falling freely with a velocity $\mathbf{v}_0$ when cord AC suddenly becomes taut. Assuming that the impact [illegible] ...pulse-momentum diagram that is [illegible] ...r velocity of the rod and the veloc- [illegible] ...ly after the cord becomes taut.

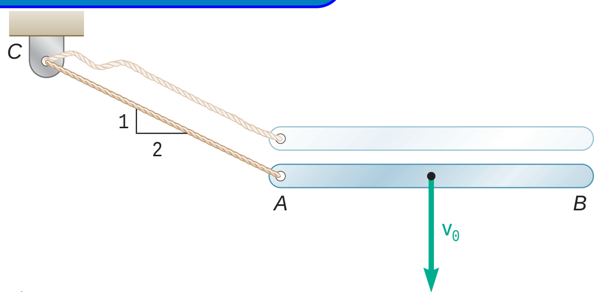


Fig. P17.F5

17.F6 A slender rod CDE of length L and mass m is attached to a pin support at its midpoint D. A second and identical rod AB is rotating about a pin support at A with an angular velocity ω_1 when its end B strikes end C of rod CDE. The coefficient of restitution between the rods is e. Draw the impulse-momentum diagrams that are needed to determine the angular velocity of each rod immediately after the impact.

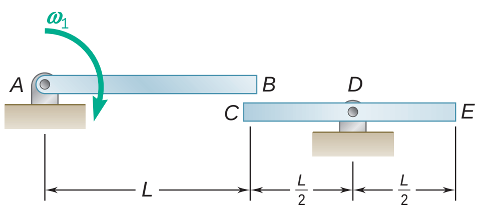


Fig. P17.F6

END-OF-SECTION PROBLEMS

17.96 At what height h above its center G should a billiard ball of radius r be struck horizontally by a cue if the ball is to start rolling without sliding?

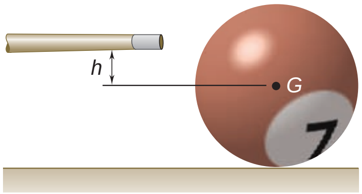


Fig. P17.96

17.97 A bullet weighing 0.08 lb is fired with a horizontal velocity of 1800 ft/s into the lower end of a slender 15-lb bar of length $L = 30$ in. Knowing that $h = 12$ in. and that the bar is initially at rest, determine (*a*) the angular velocity of the bar immediately after the bullet becomes embedded, (*b*) the impulsive reaction at C, assuming that the bullet becomes embedded in 0.001 s.

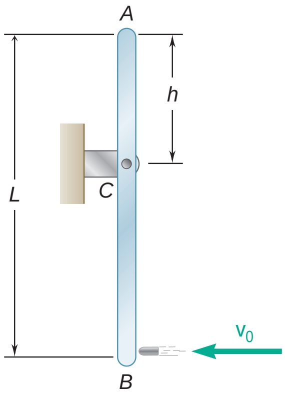


Fig. P17.97

17.98 In Prob. 17.97, determin sive reaction at C is to be ity of the bar immediate

17.99 An 16-lb wooden panel is suspended from a pin support at A and is initially at rest. A 4-lb metal sphere is released from rest at B and falls into a hemispherical cup C attached to the panel at a point located on its top edge. Assuming that the impact is perfectly plastic, determine the velocity of the mass center G of the panel immediately after the impact.

17.100 A 16-lb wooden panel is suspended from a pin support at A and is initially at rest. A 4-lb metal sphere is released from rest at B' and falls into a hemispherical cup C' attached to the panel at the same level as the mass center G. Assuming that the impact is perfectly plastic, determine the velocity of the mass center G of the panel immediately after the impact.

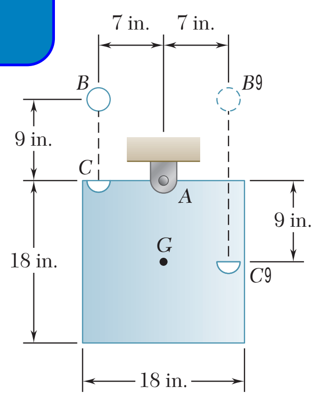


Fig. *P17.99* and *P17.100*

17.101 A 45-g bullet is fired with a velocity of 400 m/s at u = 30° into a 9-kg square panel of side $b = 200$ mm. Knowing that $h = 150$ mm and that the panel is initially at rest, determine (*a*) the velocity of the center of the panel immediately after the bullet becomes embedded, (*b*) the impulsive reaction at A, assuming that the bullet becomes embedded in 2 ms.

17.102 A 45-g bullet is fired with a velocity of 400 m/s at u = 5° into a 9-kg square panel of side $b = 200$ mm. Knowing that the panel is initially at rest, determine (*a*) the required distance h if the horizontal component of the impulsive reaction at A is to be zero, (*b*) the corresponding velocity of the center of the panel immediately after the bullet becomes embedded.

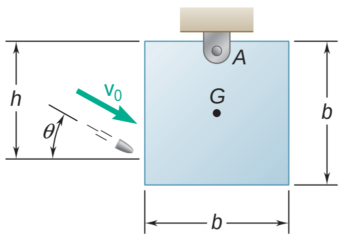


Fig. P17.101 and P17.102

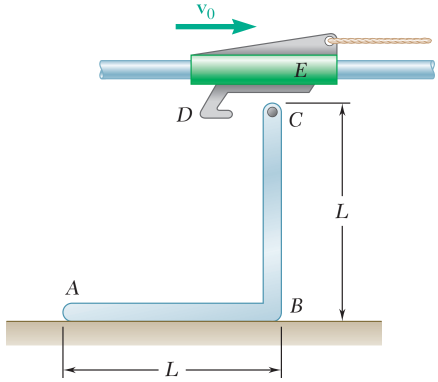


Fig. P17.103

17.103 Two uniform rods, each of mass m, form the L-shaped rigid body ABC which is initially at rest on the frictionless horizontal surface when hook D of the carriage E engages a small pin at C. Knowing that the carriage is pulled to the right with a constant velocity $\mathbf{v}_0$, determine immediately after the impact (*a*) the angular velocity of the body, (*b*) the velocity of corner B. Assume that the velocity of the carriage is unchanged and that the impact is perfectly plastic.

17.104 The uniform slender rod AB of weight 5 lb and length 30 in. forms an angle b = 30° with the vertical as it strikes the smooth corner shown with a vertical velocity $\mathbf{v}_1$ of magnitude 8 ft/s and no angular velocity. Assuming that the impact is perfectly plastic, determine the angular velocity of the rod immediately after the impact.

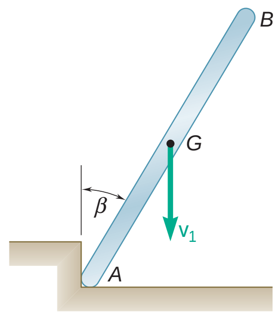


Fig. P17.104

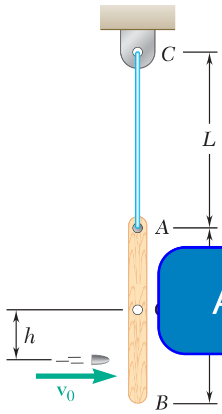


Fig. P17.105

17.105 A bullet weighing 0.08 lb is fired with a horizontal velocity of rod AB of length L = 30 in. The suspended by a cord of length L = for which, immediately after the bullet becomes embedded, the instantaneous center of rotation of the rod is point C.

17.106 A uniform sphere of radius r rolls down the incline shown without slipping. It hits a horizontal surface and, after slipping for a while, it starts rolling again. Assuming that the sphere does not bounce as it hits the horizontal surface, determine its angular velocity and the velocity of its mass center after it has resumed rolling.

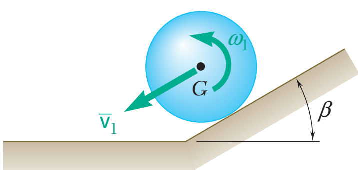


Fig. P17.106

17.107 A uniformly loaded rectangular crate is released from rest in the position shown. Assuming that the floor is sufficiently rough to prevent slipping and that the impact at B is perfectly plastic, determine the smallest value of the ratio a/b for which corner A will remain in contact with the floor.

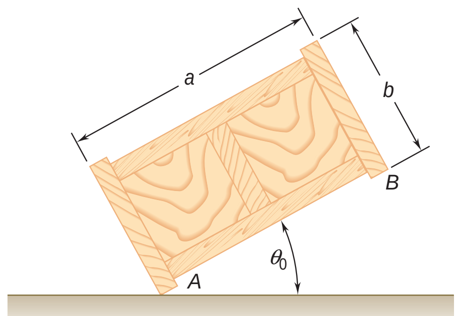


Fig. P17.107

17.108 A bullet of mass m is fired with a horizontal velocity $\mathbf{v}_0$ and at a height $h = \frac{1}{2}R$ into a wooden disk of much larger mass M and radius R. The disk rests on a horizontal plane and the coefficient of friction between the disk and the plane is finite. (a) Determine the linear velocity $\bar{\mathrm{v}}_1$ and the angular velocity v_1 of the disk immediately after the bullet has penetrated the disk. (b) Describe the ensuing motion of the disk and determine its linear velocity after the motion has become uniform.

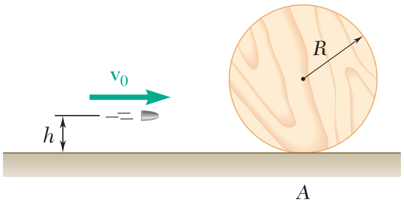
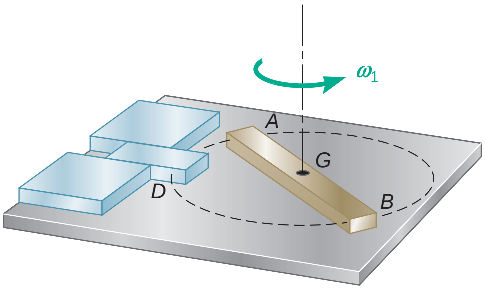


Fig. P17.108 and P17.109

17.109 Determine the height h at which the bullet of Prob. 17.108 should be fired (a) if the disk is to roll without sliding immediately after impact, (b) if the disk is to slide without rolling immediately after impact.

17.110 A uniform slender bar of length L = 200 mm and mass m = 0.5 kg is supported by a frictionless horizontal table. Initially the bar is spinning about its mass center G with a constant angular speed v_1 = 6 rad/s. Suddenly latch D is moved to the right and is struck by end A of the bar. Knowing that the coefficient of restitution between A and D is e = 0.6, determine the angular velocity of the bar and the velocity of its mass center immediately after the impact.



P17.110

17.111 A uniform slender rod of length L is dropped onto rigid supports at A and B. Since support B is slightly lower than support A, the rod strikes A with a velocity $\bar{\mathbf{v}}_1$ before it strikes B. Assuming perfectly elastic impact at both A and B, determine the angular velocity of the rod and the velocit[illegible] rod (a) strikes support [illegible] support A.

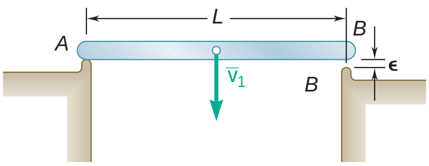


Fig. *P17.111*

17.112 The slender rod AB of length L forms an angle b with the vertical as it strikes the frictionless surface shown with a vertical velocity $\bar{\mathbf{v}}_1$ and no angular velocity. Assuming that the impact is perfectly plastic, derive an expression for the angular velocity of the rod immediately after the impact.

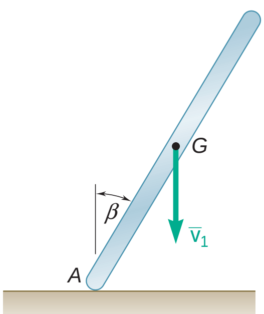


Fig. P17.112

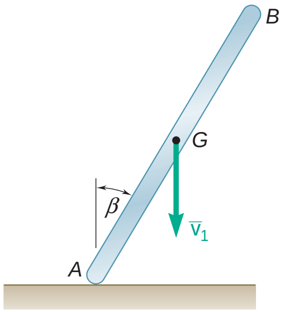


Fig. *P17.113*

17.113 The slender rod AB of length $L = 1$ m forms an angle b $= 30°$ with the vertical as it strikes the frictionless surface shown with a vertical velocity $\bar{\mathbf{v}}_1 = 2$ m/s and no angular velocity. Knowing that the coefficient of restitution between the rod and the ground is $e = 0.8$, determine the angular velocity of the rod immediately after the impact.

17.114 The trapeze/lanyard air drop (t/LAD) launch is a proposed innovative method for airborne launch of a payload-carrying rocket. The release sequence involves several steps as shown in (1) where the payload rocket is shown at various instances during the launch. To investigate the first step of this process, where the rocket body drops freely from the carrier aircraft until the 2-m lanyard stops the vertical motion of B, a trial rocket is tested as shown in (2). The rocket can be considered a uniform 1×7-m rectangle with a mass of 4000 kg. Knowing that the rocket is released from rest and falls vertically 2 m before the lanyard becomes taut, determine the angular velocity of the rocket immediately after the lanyard is taut.

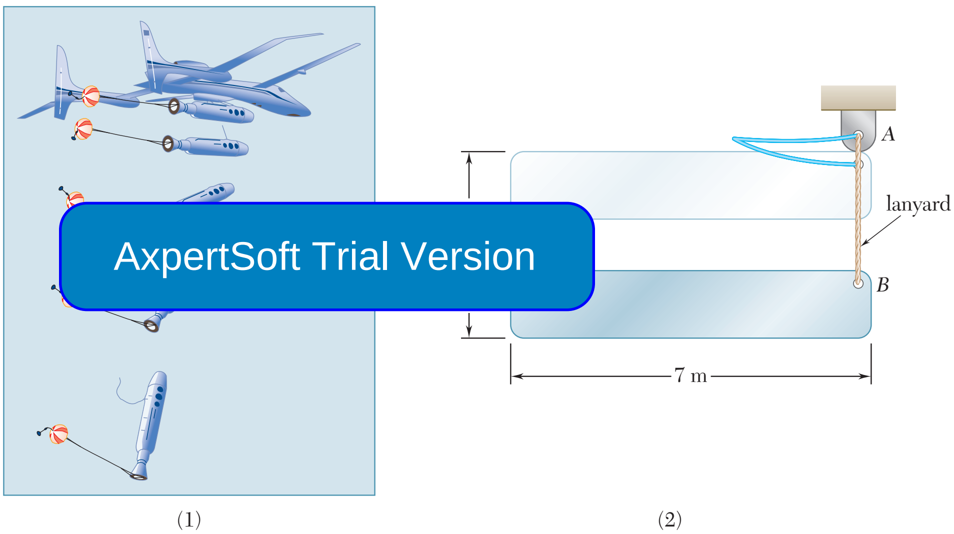


Fig. *P17.114*

17.115 The uniform rectangular block shown is moving along a frictionless surface with a velocity $\bar{\mathbf{v}}_1$ when it strikes a small obstruction at B. Assuming that the impact between corner A and obstruction B is perfectly plastic, determine the magnitude of the velocity $\bar{\mathbf{v}}_1$ for which the maximum angle u through which the block will rotate will be 30°.

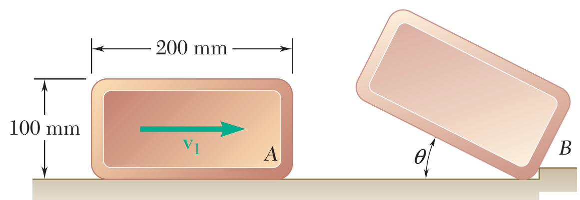


Fig. P17.115

17.116 A slender rod of length L and mass m is released from rest in the position shown. It is observed that after the rod strikes the vertical surface it rebounds to form an angle of 30° with the vertical. (*a*) Determine the coefficient of restitution between knob K and the surface. (*b*) Show that the same rebound can be expected for any position of knob K.

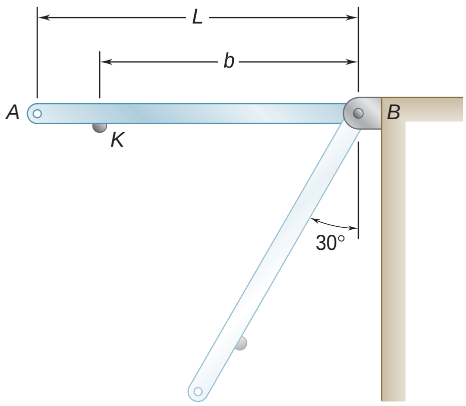


Fig. P17.116

17.117 A slender rod of mass m and length L is released from rest in the position shown and hits edge D. Assuming perfectly plastic impact at D, determine for $b = 0.6L$, (*a*) the angular velocity of the rod immediately after the impact, (*b*) the maximum angle through which the rod will rotate after the impact.

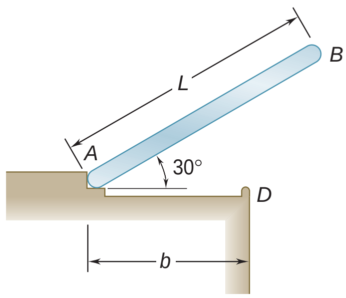


Fig. P17.117

17.118 A uniformly loaded squar [illegible] D directly above A; it ro [illegible] floor, and then rotates a [illegible] prevent slipping and the impact at D is perfectly plastic. Denoting by V_0 the angular velocity of the crate immediately before B strikes the floor, determine (*a*) the angular velocity of the crate immediately after B strikes the floor, (*b*) the fraction of the kinetic energy of the crate lost during the impact, (*c*) the angle u through which the crate will rotate after B strikes the floor.

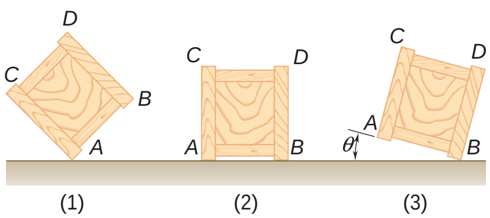


Fig. P17.118

17.119 A 1-oz bullet is fired with a horizontal velocity of 750 mi/h into the 18-lb wooden beam AB. The beam is suspended from a collar of negligible mass that can slide along a horizontal rod. Neglecting friction between the collar and the rod, determine the maximum angle of rotation of the beam during its subsequent motion.

17.120 For the beam of Prob. 17.119, determine the velocity of the 1-oz bullet for which the maximum angle of rotation of the beam will be 90°.

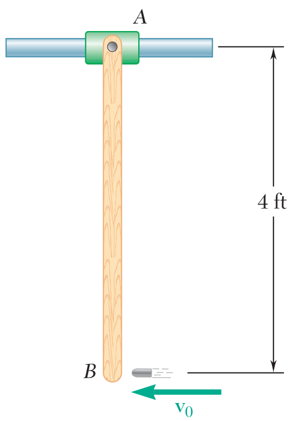


Fig. P17.119

17.121 The plank *CDE* has a mass of 15 kg and rests on a small pivot at *D*. The 55-kg gymnast *A* is standing on the plank at *C* when the 70-kg gymnast *B* jumps from a height of 2.5 m and strikes the plank at *E*. Assuming perfectly plastic impact and that gymnast *A* is standing absolutely straight, determine the height to which gymnast *A* will rise.

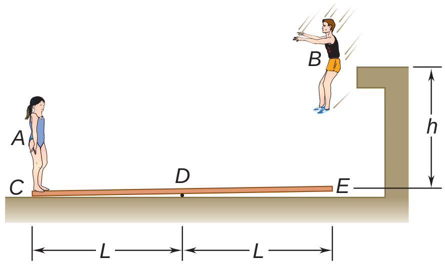


Fig. P17.121

17.122 Solve Prob. 17.121, assuming that the gymnasts change places so that gymnast *A* jumps onto the plank while gymnast *B* stands at *C*.

17.123 A small plate *B* is attached to a cord that is wrapped around a uniform 8-lb disk of radius $R = 9$ in. A 3-lb collar *A* is released from rest and falls through a distance $h = 15$ in. before hitting plate *B*. Assuming that the impact is perfectly plastic and neglecting the weight of the plate, determine immediately after the impact (*a*) the velocity of the collar, (*b*) the angular velocity of the disk.

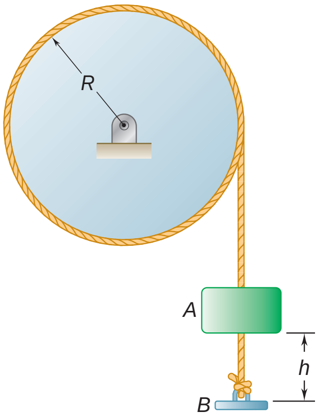


Fig. P17.123

hat the coefficient of restitution

17.125 Two identical slender rods may swing freely from the pivots shown. Rod *A* is released from rest in a horizontal position and swings to a vertical position, at which time the small knob *K* strikes rod *B* which was at rest. If $h = \frac{1}{2}l$ and $e = \frac{1}{2}$, determine (*a*) the angle through which rod *B* will swing, (*b*) the angle through which rod *A* will rebound.

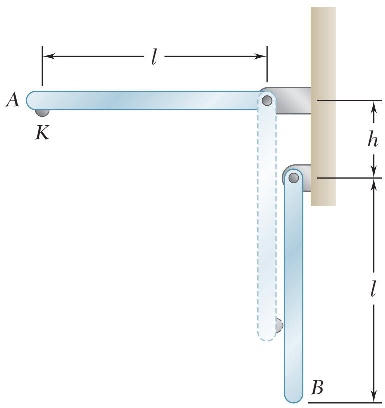


Fig. P17.125

17.126 A 2-kg solid sphere of radius $r = 40$ mm is dropped from a height $h = 200$ mm and lands on a uniform slender plank *AB* of mass 4 kg and length $L = 500$ mm which is held by two inextensible cords. Knowing that the impact is perfectly plastic and that the sphere remains attached to the plank at a distance $a = 40$ mm from the left end, determine the velocity of the sphere immediately after impact. Neglect the thickness of the plank.

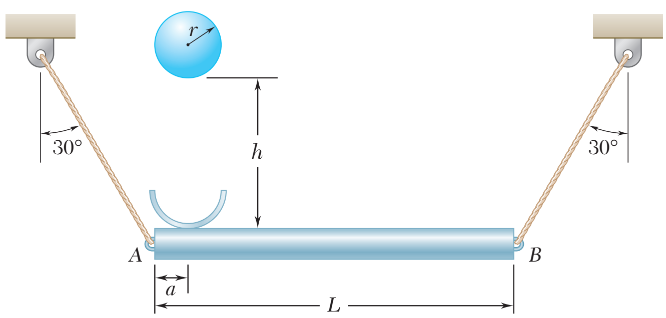


Fig. P17.126

17.127 and 17.128 Member ABC has a mass of 2.4 kg and is attached to a pin support at B. An 800-g sphere D strikes the end of member ABC with a vertical velocity $\mathbf{v}_1$ of 3 m/s. Knowing that $L = 750$ mm and that the coefficient of restitution between the sphere and member ABC is 0.5, determine immediately after the impact (*a*) the angular velocity of member ABC, (*b*) the velocity of the sphere.

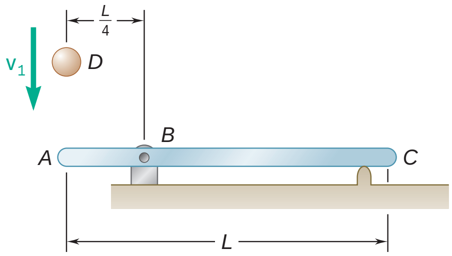


Fig. P17.127

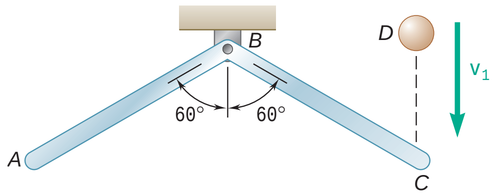


Fig. P17.128

17.129 Sphere A of mass $m_A = 2$ kg and radius $r = 40$ mm rolls without slipping with a velocity $\bar{\mathbf{v}}_1 = 2$ m/s on a horizontal surface when it hits squarely a uniform slender bar B of mass is $m_B = 0.5$ kg and length $L = 100$ mm that is standing on end and is at rest. Denoting by m_k the coefficient of kinetic friction between the sphere and the horizontal surface, negle bar, and knowing the co 0.1, determine the angu immediately after the impact.

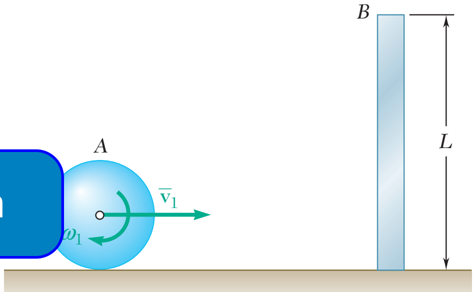


Fig. *P17.129*

17.130 A large 3-lb sphere with a radius $r = 3$ in. is thrown into a light basket at the end of a thin, uniform rod weighing 2 lb and length $L = 10$ in. as shown. Immediately before the impact the angular velocity of the rod is 3 rad/s counterclockwise and the velocity of the sphere is 2 ft/s down. Assume the sphere sticks in the basket. Determine after the impact (*a*) the angular velocity of the bar and sphere, (*b*) the components of the reactions at A.

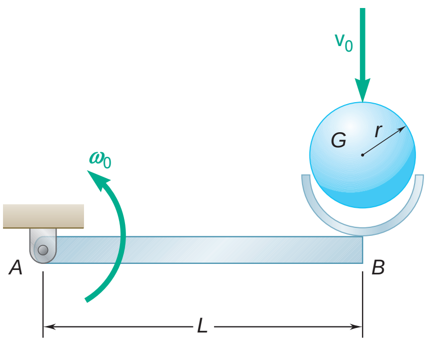


Fig. *P17.130*

17.131 A small rubber ball of radius r is thrown against a rough floor with a velocity $\bar{\mathbf{v}}_A$ of magnitude v_0 and a backspin V_A of magnitude V_0. It is observed that the ball bounces from A to B, then from B to A, then from A to B, etc. Assuming perfectly elastic impact, determine the required magnitude V_0 of the backspin in terms of $\bar{v}_0$ and r.

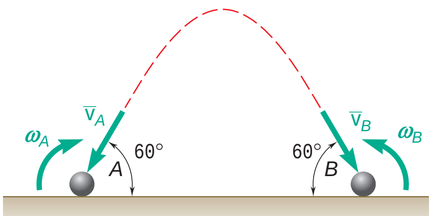


Fig. P17.131

17.132 Sphere A of mass m and radius r rolls without slipping with a velocity $\bar{\mathbf{v}}_1$ on a horizontal surface when it hits squarely an identical sphere B that is at rest. Denoting by m_k the coefficient of kinetic friction between the spheres and the surface, neglecting friction between the spheres, and assuming perfectly elastic impact, determine (*a*) the linear and angular velocities of each sphere immediately after the impact, (*b*) the velocity of each sphere after it has started rolling uniformly.

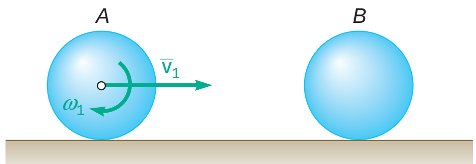


Fig. P17.132

17.133 In a game of pool, ball A is rolling without slipping with a velocity $\bar{\mathbf{v}}_0$ as it hits obliquely ball B, which is at rest. Denoting by r the radius of each ball and by m_k the coefficient of kinetic friction between a ball and the table, and assuming perfectly elastic impact, determine (*a*) the linear and angular velocity of each ball immediately after the impact, (*b*) the velocity of ball B after it has started rolling uniformly.

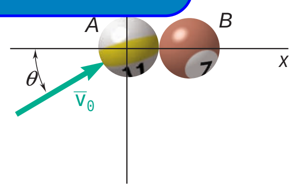


Fig. P17.133

17.134 Each of the bars AB and BC is of length $L = 400$ mm and mass $m = 1.2$ kg. Determine the angular velocity of each bar immediately after the impulse $\mathbf{Q}\Delta t = (1.5 \text{ N} \cdot \text{s})\mathbf{i}$ is applied at C.

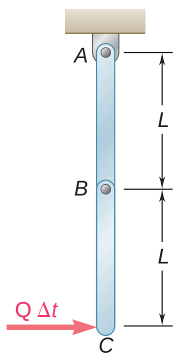


Fig. P17.134

REVIEW AND SUMMARY

In this chapter we again considered the method of work and energy and the method of impulse and momentum. In the first part of the chapter we studied the method of work and energy and its application to the analysis of the motion of rigid bodies and systems of rigid bodies.

Principle of work and energy for a rigid body

In Sec. 17.2, we first expressed the principle of work and energy for a rigid body in the form

$$T_1 + U_{1\to2} = T_2 \qquad (17.1)$$

where T_1 and T_2 represent the initial and final values of the kinetic energy of the rigid body and $U_{1\to2}$ represents the work of the *external forces* acting on the rigid body.

Work of a force or a couple

In Sec. 17.3, we recalled the expression found in Chap. 13 for the work of a force **F** applied at a

$$U_{1\to2} = \int_{s_1} (F\cos\alpha)\,ds \qquad (17.3')$$

where F was the magnitude of the force, α the angle it formed with the direction of motion of A, and s the variable of integration measuring the distance traveled by A along its path. We also derived the expression for the *work of a couple of moment* **M** applied to a rigid body during a rotation in θ of the rigid body:

$$U_{1\to2} = \int_{\theta_1}^{\theta_2} M\,d\theta \qquad (17.5)$$

Kinetic energy in plane motion

We then derived an expression for the kinetic energy of a rigid body in plane motion [Sec. 17.4]. We wrote

$$T = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2 \qquad (17.9)$$

where $\bar{v}$ is the velocity of the mass center G of the body, ω is the angular velocity of the body, and $\bar{I}$ is its moment of inertia about an axis through G perpendicular to the plane of reference (Fig. 17.13) [Sample Prob. 17.3]. We noted that the kinetic energy of a rigid body in plane motion can be separated into two parts: (1) the kinetic energy $\frac{1}{2}m\bar{v}^2$ associated with the motion of the mass center G of the body, and (2) the kinetic energy $\frac{1}{2}\bar{I}\omega^2$ associated with the rotation of the body about G.

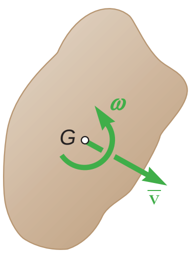


Fig. 17.13

For a rigid body rotating about a fixed axis through O with an angular velocity V, we had

Kinetic energy in rotation

$$T = \tfrac{1}{2} I_O \mathrm{v}^2 \qquad (17.10)$$

where I_O was the moment of inertia of the body about the fixed axis. We noted that the result obtained is not limited to the rotation of plane slabs or of bodies symmetrical with respect to the reference plane, but is valid regardless of the shape of the body or of the location of the axis of rotation.

Systems of rigid bodies

Equation (17.1) can be applied to the motion of systems of rigid bodies [Sec. 17.5] as long as all the forces acting on the various bodies involved—internal as well as external to the system—are included in the computation of $U_{1 \mathrm{y} 2}$. However, in the case of systems consisting of pin-connected members, or blocks and pulleys connected by inextensible cords, or meshed gears, the points of application of the internal forces move through equal distances and the work of these forces cancels out [Sample Probs. 17.1 and 17.2].

Conservation of energy

When a rigid body, or a system of rigid bodies, moves under the action of conservative forces, the principle of work and energy can be expressed in the form

$$T_1 + V_1 = T_2 + V_2 \qquad (17.12)$$

…f conservation of energy [Sec. …lve problems involving conservative forces such as the force of gravity or the force exerted by a spring [Sample Probs. 17.4 and 17.5]. However, when a reaction is to be determined, the principle of conservation of energy must be supplemented by the application of d'Alembert's principle [Sample Prob. 17.4].

Power

In Sec. 17.7, we extended the concept of power to a rotating body subjected to a couple, writing

$$\text{Power} = \frac{dU}{dt} = \frac{M\,d\mathrm{u}}{dt} = M\mathrm{v} \qquad (17.13)$$

where M is the magnitude of the couple and v the angular velocity of the body.

The middle part of the chapter was devoted to the method of impulse and momentum and its application to the solution of various types of problems involving the plane motion of rigid slabs and rigid bodies symmetrical with respect to the reference plane.

Principle of impulse and momentum for a rigid body

We first recalled the *principle of impulse and momentum* as it was derived in Sec. 14.9 for a system of particles and applied it to the *motion of a rigid body* [Sec. 17.8]. We wrote

$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1 \mathrm{y} 2} = \textbf{Syst Momenta}_2 \qquad (17.14)$$

Next we showed that for a rigid slab or a rigid body symmetrical with respect to the reference plane, the system of the momenta of the particles forming the body is equivalent to a vector $m\bar{\mathbf{v}}$ attached at the mass center G of the body and a couple $\bar{I}V$ (Fig. 17.14). The vector

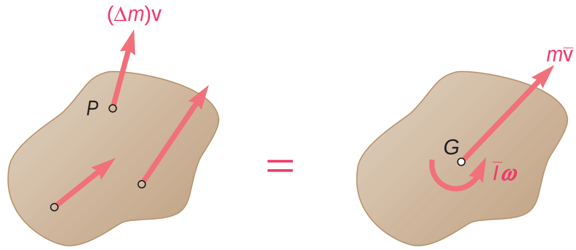


Fig. 17.14

$m\bar{\mathbf{v}}$ is associated with the translation of the body with G and represents the *linear momentum* of the body, while the couple $\bar{I}V$ corresponds to the rotation of the body about G and represents the *angular momentum* of the body about an axis through G.

Equation (17.14) can be expressed graphically as shown in Fig. 17.15 by drawing three diagrams ... of the initial momenta of the b... acting on the body, and the sys...

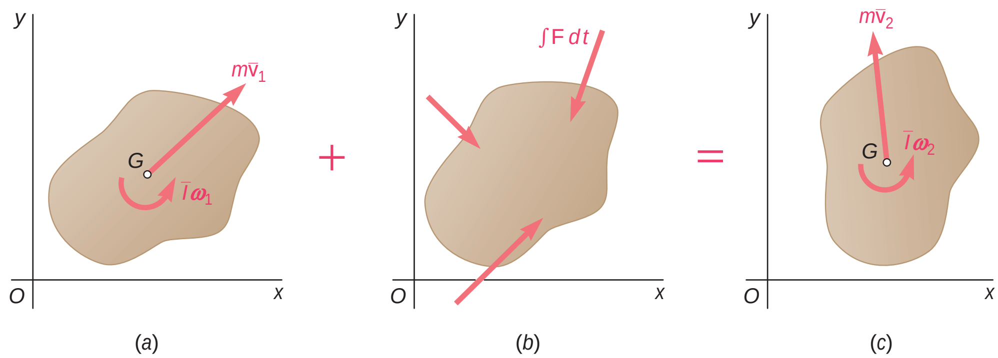


Fig. 17.15

Summing and equating respectively the *x components*, the *y components*, and the *moments about any given point* of the vectors shown in that figure, we obtain three equations of motion which can be solved for the desired unknowns [Sample Probs. 17.6 and 17.7].

In problems dealing with several connected rigid bodies [Sec. 17.9], each body can be considered separately [Sample Prob. 17.6], or, if no more than three unknowns are involved, the principle of impulse