Introduction to

Finite Elements in Engineering

Tirupathi R. Chandrupatla
Rowan University
Glassboro, New Jersey

Ashok D. Belegundu
The Pennsylvania State University
University Park, Pennsylvania

Solutions Manual

Prentice Hall, Upper Saddle River, New Jersey 07458

CONTENTS

PREFACE

This solutions manual serves as an aid to professors in teaching from the book **Introduction to Finite Elements in Engineering, 4th Edition**. *The problems in the book fall into the following categories:*

1. *Simple problems to understand the concepts*
2. *Derivations and direct solutions*
3. *Solutions requiring computer runs*
4. *Solutions requiring program modifications*

Our basic philosophy in the development of this manual is to provide a complete guidance to the teacher in formulating, modeling, and solving the problems. Complete solutions are given for problems in all categories stated. For some larger problems such as those in three dimensional stress analysis, complete formulation and modeling aspects are discussed. The students should be able to proceed from the guidelines provided.

For problems involving distributed and other types of loading, the nodal loads are to be calculated for the input data. The programs do not generate the loads. This calculation and the boundary condition decisions enable the student to develop a physical sense for the problems. The students may be encouraged to modify the programs to calculate the loads automatically.

The students should be introduced to the programs in Chapter 12 right from the point of solving problems in Chapter 6. This will enable the students to solve larger problems with ease. The input data file for each program has been provided. Data for a problem should follow this format. The best strategy is to copy the example file and edit it for the problem under consideration. The data from program MESHGEN will need some editing to complete the information on boundary conditions, loads, and material properties.

We thank you for your enthusiastic response to our first three editions of the book. We look forward to receive your feedback of your experiences, comments, and suggestions for making improvements to the book and this manual.

Tirupathi R. Chandrupatla P.E., CMfgE
Department of Mechanical Engineering
Rowan University, Glassboro, NJ 08028
e-mail: chandrupatla@rowan.edu

Ashok D. Belegundu
Department of Mechanical and Nuclear Engineering
The Pennsylvania State University
University Park, PA 16802
e-mail: adb3@psu.edu

CHAPTER 1
FUNDAMENTAL CONCEPTS

1.1 We use the first three steps of Eq. 1.11

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

Adding the above, we get

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

Adding and subtracting $\nu\frac{\sigma_x}{E}$ from the first equation,

$$\varepsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

Similar expressions can be obtained for ε_y, and ε_z.

From the relationship for γ_{yz} and Eq. 1.12,

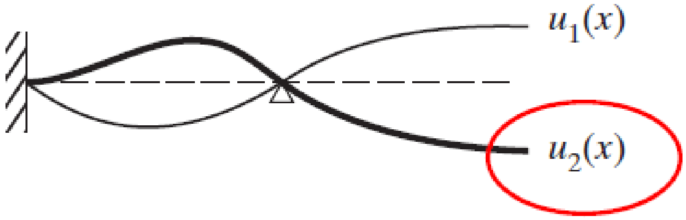
$$\tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz} \qquad \text{etc.}$$

Above relations can be written in the form

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

where **D** is the material property matrix defined in Eq. 1.15. ■

1.2 Note that $u_2(x)$ satisfies the zero slope boundary condition at the support.



■

1.3 Plane strain condition implies that

$$\varepsilon_z = 0 = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

which gives

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

We have, $\sigma_x = 20000$ psi $\quad \sigma_y = -10000$ psi $\quad E = 30\times10^6$ psi $\quad \nu = 0.3$.

On substituting the values,

$$\sigma_z = 3000 \text{ psi} \qquad \blacksquare$$

1.4 Displacement field

$$u = 10^{-4}\left(-x^2 + 2y^2 + 6xy\right)$$
$$v = 10^{-4}\left(3x + 6y - y^2\right)$$
$$\frac{\partial u}{\partial x} = 10^{-4}(-2x + 6y) \quad \frac{\partial u}{\partial y} = 10^{-4}(4y + 6x)$$
$$\frac{\partial v}{\partial x} = 3\times10^{-4} \qquad \frac{\partial v}{\partial y} = 10^{-4}(6 + 2y)$$
$$\varepsilon = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

at $x = 1, y = 0$

$$\varepsilon = 10^{-4}\begin{Bmatrix} -2 \\ 6 \\ 9 \end{Bmatrix}$$

$\blacksquare$

1.5 On inspection, we note that the displacements u and v are given by

$$u = 0.1\,y + 4$$
$$v = 0$$

It is then easy to see that

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.1$$

■

1.6 The displacement field is given as

$$u = 1 + 3x + 4x^3 + 6xy^2$$
$$v = xy - 7x^2$$

(a) The strains are then given by

$$\varepsilon_x = \frac{\partial u}{\partial x} = 3 + 12x^2 + 6y^2$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = x$$

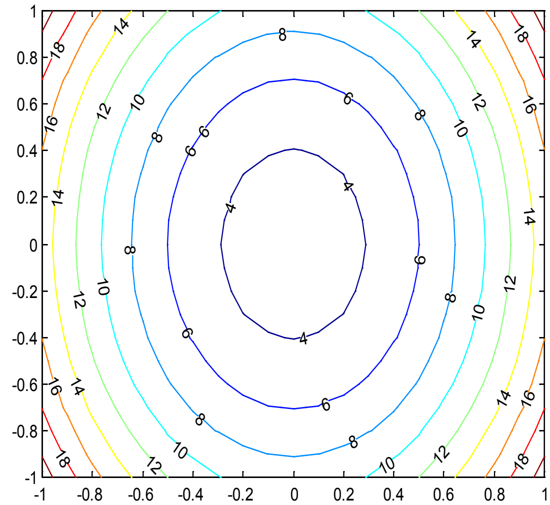
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 12xy + y - 14x$$

(b) In order to draw the contours of the strain field using MATLAB, we need to create a script file, which may be edited as a text file and save with ".m" extension. The file for plotting ε_x is given below

file "prob1p5b.m"

```
[X,Y] = meshgrid(-1:.1:1,-1:.1:1);
Z = 3.+12.*X.^2+6.*Y.^2;
[C,h] = contour(X,Y,Z);
clabel(C,h);
```

On running the program, the contour map is shown as follows:



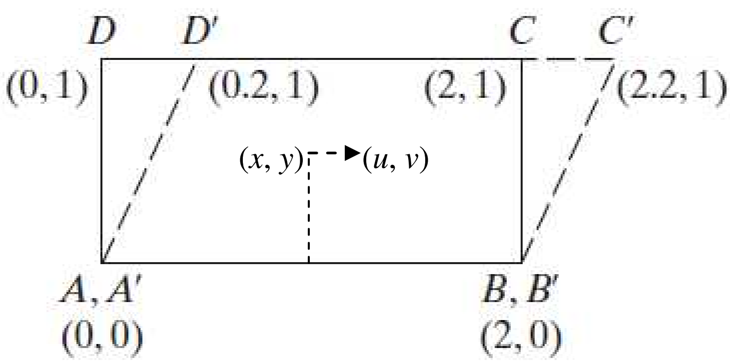
Contours of ε_x

Contours of ε_y and γ_{xy} are obtained by changing Z in the script file. The numbers on the contours show the function values.

(c) The maximum value of ε_x is at any of the corners of the square region. The maximum value is 21.

■

1.7



a) $$u = \frac{0.2}{1} y \quad \Rightarrow \quad u = 0.2y \qquad v = 0$$

b) $$\varepsilon_x = \frac{\partial u}{\partial x} = 0 \quad \varepsilon_y = \frac{\partial v}{\partial y} = 0 \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.2$$

1.8

$$\sigma_x = 40\text{ MPa} \quad \sigma_y = 20\text{ MPa} \quad \sigma_z = 30\text{ MPa}$$
$$\tau_{yz} = -30\text{ MPa} \quad \tau_{xz} = 15\text{ MPa} \quad \tau_{xy} = 10\text{ MPa}$$
$$\mathbf{n} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}^{\mathrm{T}}$$

From Eq.1.8 we get

$$T_x = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z$$
$$= 35.607\text{ MPa}$$
$$T_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z$$
$$= -6.213\text{ MPa}$$
$$T_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z$$
$$= 13.713\text{ MPa}$$
$$\sigma_n = T_x n_x + T_y n_y + T_z n_z$$
$$= 24.393\text{ MPa}$$

■

1.9 From the derivation made in P1.1, we have

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y + \nu\varepsilon_z\right]$$

which can be written in the form

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_x + \nu\varepsilon_v\right]$$

and

$$\tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz}$$

Lame's constants λ and μ are defined in the expressions

$$\sigma_x = \lambda\varepsilon_v + 2\mu\varepsilon_x$$
$$\tau_{yz} = \mu\gamma_{yz}$$

On inspection,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$
$$\mu = \frac{E}{2(1+\nu)}$$

μ is same as the shear modulus G. ■

1.10

$$
\begin{aligned}
\varepsilon &= 1.2\times10^{-5} \\
\Delta T &= 30^0 C \\
E &= 200\,\text{GPa} \\
\alpha &= 12\times10^{-6}\,/^0\,C \\
\varepsilon_0 &= \alpha\Delta T = 3.6\times10^{-4} \\
\sigma &= E(\varepsilon-\varepsilon_0) = -69.6\,\text{MPa}
\end{aligned}
$$

■

1.11

$$
\begin{aligned}
\varepsilon_x &= \frac{du}{dx} = 1+2x^2 \\
\delta &= \int_0^L \frac{du}{dx}dx = \left(x+\frac{2}{3}x^3\right)\Big|_0^L \\
&= L\left(1+\frac{2}{3}L^2\right)
\end{aligned}
$$

■

1.12 Following the steps of Example 1.1, we have

$$
\begin{bmatrix} (80+40+50) & -80 \\ -80 & 80 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 50 \end{Bmatrix}
$$

Above matrix form is same as the set of equations:

$$170\,q_1 - 80\,q_2 = 60$$

$$-80\,q_1 + 80\,q_2 = 50$$

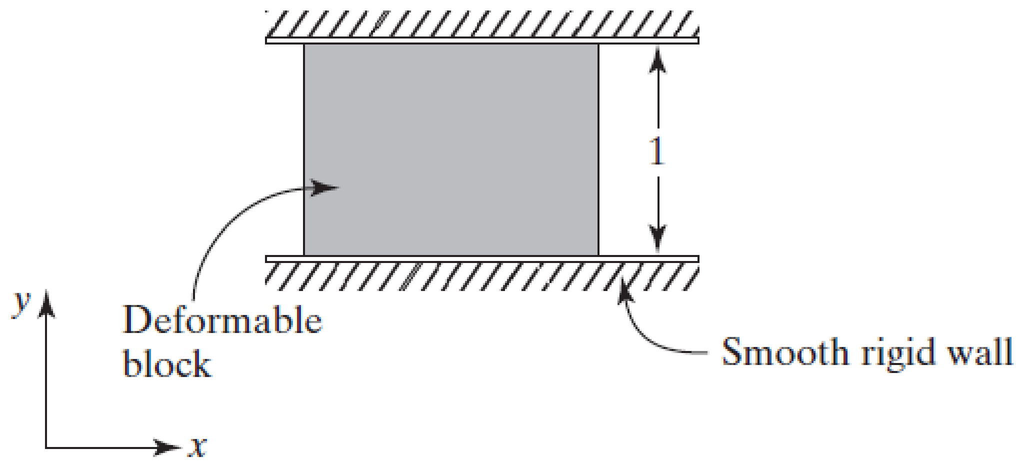
Solving for q_1 and q_2, we get

$$q_1 = 1.222\text{ mm}$$

$$q_2 = 1.847\text{ mm}$$

■

1.13



When the wall is smooth, $\sigma_x = 0$. ΔT is the temperature rise.

a) When the block is thin in the *z* direction, it corresponds to plane stress condition. The rigid walls in the *y* direction require $\varepsilon_y = 0$. The generalized Hooke's law yields the equations

$$\varepsilon_x = -\nu \frac{\sigma_y}{E} + \alpha \Delta T$$

$$\varepsilon_y = \frac{\sigma_y}{E} + \alpha \Delta T$$

From the second equation, setting $\varepsilon_y = 0$, we get $\sigma_y = -E\alpha\Delta T$. ε_x is then calculated using the first equation as $(1-\nu)\alpha\Delta T$.

b) When the block is very thick in the *z* direction, plain strain condition prevails. Now we have $\varepsilon_z = 0$, in addition to $\varepsilon_y = 0$. σ_z is not zero.

$$\varepsilon_x = -\nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} + \alpha \Delta T$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} + \alpha \Delta T = 0$$

$$\varepsilon_z = -\nu \frac{\sigma_y}{E} + \frac{\sigma_z}{E} + \alpha \Delta T = 0$$

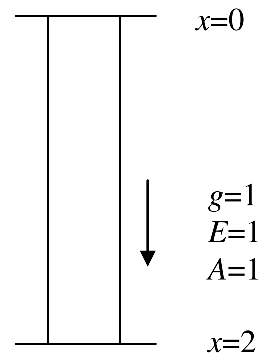
From the last two equations, we get

$$\sigma_y = \frac{-E\alpha\Delta T}{1+\nu} \quad \sigma_z = -\frac{1+2\nu}{1+\nu} E\alpha\Delta T$$

ε_x is now obtained from the first equation. ■

1.14 For thin block, it is plane stress condition. Treating the nominal size as 1, we may set the initial strain $\varepsilon_0 = \alpha \Delta T = \dfrac{0.1}{1}$ in part (a) of problem 1.13. Thus $\sigma_y = -0.1E$. ■

1.15



The potential energy Π is given by

$$\Pi = \frac{1}{2}\int_0^2 EA\left(\frac{du}{dx}\right)^2 dx - \int_0^2 ugAdx$$

Consider the polynomial from Example 1.2,

$$u = a_3\left(-2x + x^2\right)$$

$$\frac{du}{dx} = \left(-2 + 2x\right)a_3 = 2\left(-1 + x\right)a_3$$

On substituting the above expressions and integrating, the first term of becomes

$$2a_3^{\,2}\left(\frac{2}{3}\right)$$

and the second term

$$\int_0^2 ugAdx = \int_0^2 udx = a_3\left(-x^2 + \frac{x^3}{3}\right)\Big|_0^2$$

$$= -\frac{4}{3}a_3$$

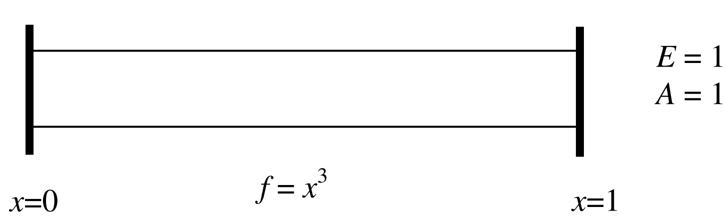
Thus

$$\Pi = \frac{4}{3}\left(a_3^{\,2} + a_3\right)$$

$$\frac{\partial \Pi}{\partial a_3} = 0 \Rightarrow \quad a_3 = -\frac{1}{2}$$

this gives $\quad u_{x=1} = -\dfrac{1}{2}\left(-2 + 1\right) = 0.5$ ■

1.16



We use the displacement field defined by $u = a_0 + a_1x + a_2x^2$.

$$u = 0 \text{ at } x = 0 \Rightarrow a_0 = 0$$

$$u = 0 \text{ at } x = 1 \Rightarrow a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$$

We then have $u = a_1x(1-x)$, and $du/dx = a_1(1-x)$.

The potential energy is now written as

$$\begin{aligned}
\Pi &= \frac{1}{2}\int_0^1 \left(\frac{du}{dx}\right)^2 dx - \int_0^1 fudx \\
&= \frac{1}{2}\int_0^1 a_1^{\,2}(1-2x)^2\,dx - \int_0^1 x^3 a_1 x(1-x)dx \\
&= \frac{1}{2}\int_0^1 a_1^{\,2}\left(1-4x+4x^2\right)dx - \int_0^1 a_1\left(x^4-x^5\right)dx \\
&= \frac{1}{2}a_1^{\,2}\left(1-\frac{4}{2}+\frac{4}{3}\right) - a_1\left(\frac{1}{5}-\frac{1}{6}\right) \\
&= \frac{a_1^{\,2}}{6} - \frac{a_1}{30}
\end{aligned}$$

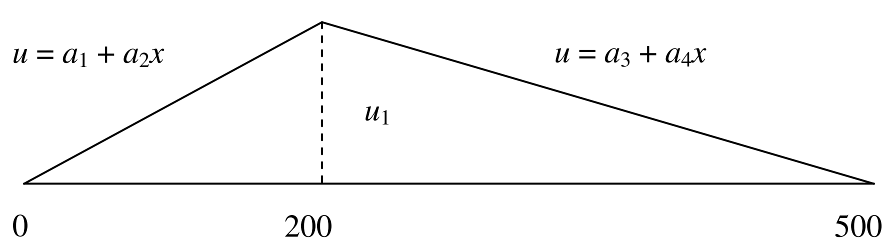
$$\frac{\partial \Pi}{\partial a_1} = 0 \quad \Rightarrow \quad \frac{a_1}{3} - \frac{1}{30} = 0$$

This yields, $a_1 = 0.1$

Displacemen $u = 0.1x(1-x)$

Stress $\sigma = E\,du/dx = 0.1(1-x)$ ■

1.17 Let u_1 be the displacement at $x = 200$ mm. Piecewise linear displacement that is continuous in the interval $0 \le x \le 500$ is represented as shown in the figure.



$0 \le x \le 200$

$u = 0$ at $x = 0 \;\Rightarrow\; a_1 = 0$

$u = u_1$ at $x = 200 \;\Rightarrow\; a_2 = u_1/200$

$\Rightarrow\; u = (u_1/200)x \qquad du/dx = u_1/200$

$200 \le x \le 500$

$u = 0$ at $x = 500 \;\Rightarrow\; a_3 + 500\, a_4 = 0$

$u = u_1$ at $x = 200 \;\Rightarrow\; a_3 + 200\, a_4 = u_1$

$\Rightarrow\; a_4 = -u_1/300 \qquad a_3 = (5/3)u_1$

$\Rightarrow\; u = (5/3)u_1 - (u_1/300)x \qquad du/dx = \; - u_1/200$

$$\Pi = \frac{1}{2}\int_0^{200} E_{al}A_1\left(\frac{du}{dx}\right)^2 dx + \frac{1}{2}\int_{200}^{500} E_{st}A_2\left(\frac{du}{dx}\right)^2 dx - 10000u_1$$

$$\Pi = \frac{1}{2}E_{al}A_1\left(\frac{u_1}{200}\right)^2 200 + \frac{1}{2}E_{st}A_2\left(-\frac{u_1}{300}\right)^2 300 - 10000u_1$$

$$= \frac{1}{2}\left(\frac{E_{al}A_1}{200} + \frac{E_{st}A_2}{300}\right)u_1^{\,2} - 10000u_1$$

$$\frac{\partial \Pi}{\partial u_1} = 0 \;\Rightarrow\; \left(\frac{E_{al}A_1}{200} + \frac{E_{st}A_2}{300}\right)u_1 - 10000 = 0$$

Note that using the units MPa (N/mm^2) for modulus of elasticity and mm^2 for area and mm for length will result in displacement in mm, and stress in MPa.

Thus, $E_{al} = 70000$ MPa, $E_{st} = 200000$, and $A_1 = 900\ mm^2$, $A_2 = 1200\ mm^2$. On substituting these values into the above equation, we get

$$u_1 = 0.009 \text{ mm}$$

This is precisely the solution obtained from strength of materials approach ■

1.18

In the Galerkin method, we start from the equilibrium equation

$$\frac{d}{dx}EA\frac{du}{dx} + g = 0$$

Following the steps of Example 1.3, we get

$$\int_0^2 -EA\frac{du}{dx}\frac{d\phi}{dx}dx + \int_0^2 g\phi dx$$

Introducing

$$u = \left(2x - x^2\right)u_1, \text{ and}$$
$$\phi = \left(2x - x^2\right)\phi_1$$

where u_1 and ϕ_1 are the values of u and ϕ at $x = 1$ respectively,

$$\phi_1\left[-u_1\int_0^2 (1-2x)^2\,dx + \int_0^2 \left(2x - x^2\right)dx\right] = 0$$

On integrating, we get

$$\phi_1\left(-\frac{8}{3}u_1 + \frac{4}{3}\right) = 0$$

This is to be satisfied for every ϕ_1, which gives the solution

$$u_1 = 0.5 \qquad \blacksquare$$

1.19 We use

$$u = a_1 + a_2x + a_3x^2 + a_4x^3$$
$$u = 0 \text{ at } x = 0$$
$$u = 0 \text{ at } x = 2$$

This implies that

$$0 = a_1$$
$$0 = a_1 + 2a_2 + 4_3 + 8a_4$$

and

$$u = a_3\left(x^2 - 2x\right) + a_4\left(x^3 - 4x\right)$$
$$\frac{du}{dx} = 2a_3(x-1) + a_4\left(3x^2 - 4\right)$$

a_3 and a_4 are considered as independent variables in

$$\Pi = \frac{1}{2}\int_0^2 \left[2a_3(x-1) + a_4\left(3x^2 - 4\right)\right]^2 dx - 2\left(-a_3 - 3a_4\right)$$

on expanding and integrating the terms, we get

$$\Pi = 1.333a_3^{\ 2} + 12.8a_4^{\ 2} + 8a_3a_4 + 2a_3 + 6a_4$$

We differentiate with respect to the variables and equate to zero.

$$\frac{\partial \Pi}{\partial a_3} = 2.667a_3 + 8a_4 + 2 = 0$$

$$\frac{\partial \Pi}{\partial a_4} = 8a_3 + 25.6a_4 + 6 = 0$$

On solving, we get

$$a_3 = -0.74856 \text{ and } a_4 = -0.00045.$$

On substituting in the expression for u, at $x = 1$,

$$u_1 = 0.749$$

This approximation is close to the value obtained in the example problem. ■

1.20

(a) $$\Pi = \frac{1}{2}\int_0^L \sigma^{\mathrm{T}} \varepsilon A dx - \int_0^L T(x) u dx$$

$$\sigma = E\varepsilon \quad \text{and} \quad \varepsilon = \frac{du}{dx}$$

On substitution,

$$\Pi = \frac{1}{2}\int_0^{60} EA\left(\frac{du}{dx}\right)^2 dx - \int_0^{30} Tudx - \int_{30}^{60} Tudx$$

$$\Pi = \frac{1}{2}\left(60 \times 10^6\right)\int_0^{60}\left(\frac{du}{dx}\right)^2 dx - \int_0^{30} 10xudx - \int_{30}^{60} 300udx$$

(b)

Since $u = 0$ at $x = 0$ and $x = 60$, and $u = a_0 + a_1x + a_2x^2$, we have

$$u = a_2 x(x - 60)$$

$$\frac{du}{dx} = a_2(2x - 60)$$

On substituting and integrating,

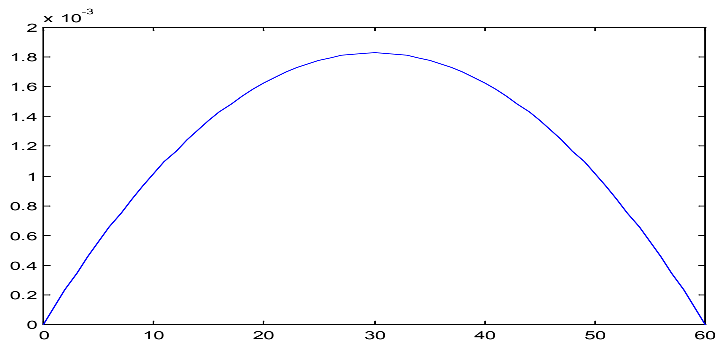
$$\Pi = 216 \times 10^{10} a_2^{\,2} + 8775000a_2$$

Setting $d\Pi/da_2 = 0$ gives

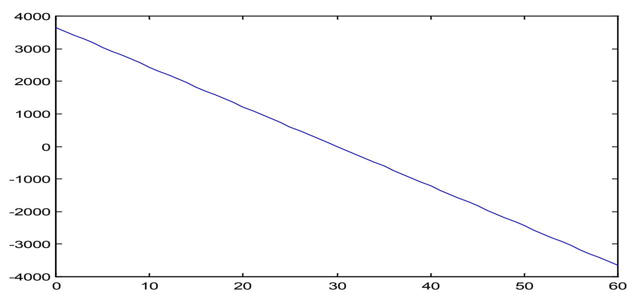
$$a_2 = -2.03125 \times 10^{-6}$$

$$\sigma = E\frac{du}{dx} = -60.935(2x - 60)$$

Plots of displacement and stress are given below:



Displacement u



Stress

. ■

1.21 $y = 20$ at $x = 60$ implies that

$$20 = a_0 + 60a_1 + 3600a_2, \quad \text{which yields}$$
$$a_0 = 20(1 - 3a_1 - 180a_2)$$

Substituting for k, h, L, and a_0 in I, we get

$$I = \int_0^{60} 10(a_1 + 2a_2x)^2\, dx + \frac{1}{2}(25)[20(1 - 3a_1 - 180a_2) - 800]^2$$

$$I = \int_0^{60} 10(a_1{}^2 + 4xa_1a_2 + 4x^2a_2{}^2)dx + 5000(3a_1 + 18a_2 + 39)^2$$

$$I = 45600a_1{}^2 + 612000a_1a_2 + 45 \times 10^5 a_2{}^2 + 117 \times 10^4 a_1 + 702 \times 10^4 a_2 + 7605000$$

$$\frac{dI}{da_1} = 912000a_1 + 612000a_2 + 117\times10^4 = 0$$

$$\frac{dI}{da_2} = 612000a_1 + 90\times10^5 a_2 + 702\times10^4 = 0$$

On solving,

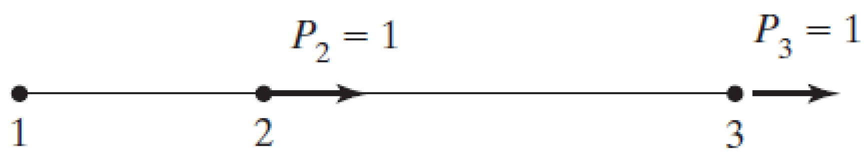
$a_2 = 0.1699$
$a_1 = -13.969$

Substituting into the expression for a_0, we get

$a_0 = 246.538$

. ■

1.22 Since $u = 0$ at $x = 0$, the displacement satisfying the boundary condition is $u = a_1x$. Also the coordinates are $x_2 = 1$, and $x_3 = 3$.



The potential energy for the problem is

$$\pi = \frac{1}{2}\int_0^3 EA\left(\frac{du}{dx}\right)^2 dx - P_2u_2 - P_3u_3$$

We have $u_2 = a_1$, $u_3 = 3a_1$, $E = 1$, $A = 1$, and $\frac{du}{dx} = a_1$. Thus

$$\pi = \frac{1}{2}\int_0^3 (a_1)^2 dx - a_1 - 3a_1 = \frac{3}{2}a_1^2 - 4a_1.$$

For stationary value, setting $\frac{d\pi}{da_1} = 0$, we get

$3a_1 - 4 = 0$, which gives $a_1 = 0.75$.

The approximate solution is $u = 0.75x$. ■

1.23 Use Galerkin approach with approximation $u = a + bx + cx^2$ to solve

$$\frac{du}{dx} + 3u = x \quad 0 \le x \le 1$$
$$u(0) = 1$$

The week form is obtained by multiplying by ϕ satisfying $\phi(0) = 0$.

$$\int_0^1 \phi\left(\frac{du}{dx} + 3u - x\right)dx = 0$$

We now set $u = 1 + bx + cx^2$ satisfying $u(0) = 1$ and $\phi = a_1 x + a_2 x^2$. On introducing these into the above integral,

$$\int_0^1 \left(a_1 x + a_2 x^2\right)\left(b + 2cx + 3 + 3bx + 3cx^2 - x\right)dx = 0$$

$$a_1\int_0^1 \left(bx + 3x - x^2 + 3bx^2 + 2cx^2 + 3cx^3\right)dx + a_2\int_0^1 \left(bx^2 + 3x^2 - x^3 + 3bx^3 + 2cx^3 + 3cx^4\right)dx + = 0$$

On integrating, we get

$$a_1\left(\frac{b}{2} + \frac{3}{2} - \frac{1}{3} + b + \frac{2c}{3} + \frac{3c}{4}\right) + a_2\left(\frac{b}{3} + 1 - \frac{1}{4} + \frac{3b}{4} + \frac{c}{2} + \frac{3c}{5}\right) = 0$$

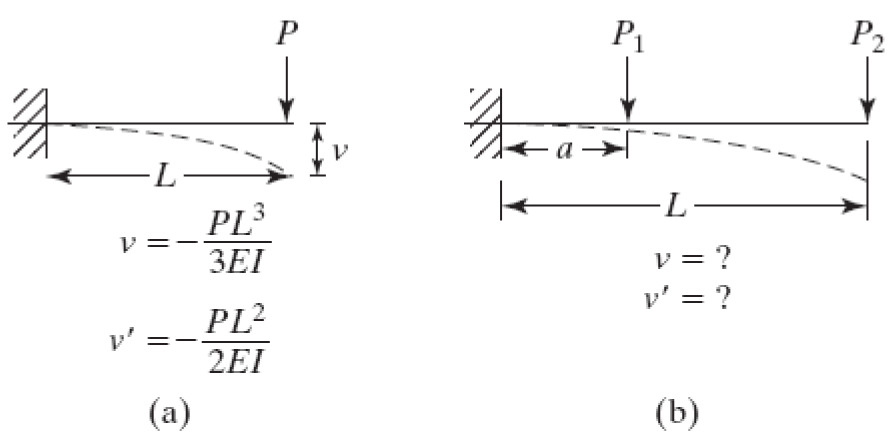
$$a_1\left(\frac{3}{2}b + \frac{17}{12}c + \frac{7}{6}\right) + a_2\left(\frac{13}{12}b + \frac{11}{10}c + \frac{3}{4}\right) = 0$$

This must be satisfied for every a_1 and a_2. Thus the equations to be solved are

$$\frac{3}{2}b + \frac{17}{12}c + \frac{7}{6} = 0$$
$$\frac{13}{12}b + \frac{11}{10}c + \frac{3}{4} = 0$$

The solution is $b = -1.9157$, $c = 1.2048$. Thus $u = 1 - 1.9157x + 1.2048x^2$. ■

1.24



The deflection and slope at a due to P_1 are $-\dfrac{P_1a^3}{3EI}$ and $-\dfrac{P_1a^2}{2EI}$. Using this the deflection and slope at L due to load P_1 are

$$v_1 = -\frac{P_1a^3}{3EI} - \frac{P_1a^2(L-a)}{2EI}$$
$$v_1' = -\frac{P_1a^2}{2EI}$$

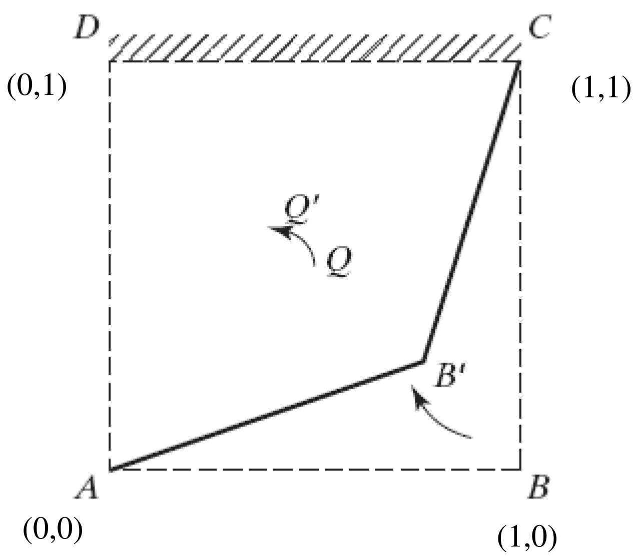
The deflection and slope due to load P_2 are

$$v_2 = -\frac{P_2L^3}{3EI}$$
$$v_2' = -\frac{P_2L^2}{2EI}$$

We then get

$$v = v_1 + v_2$$
$$v' = v_1' + v_2' \quad \blacksquare$$

1.25



(a) The displacement of B is given by (–0.1, 0.1) and A, C, and D remain in their original position. Consider a displacement field of the type

$$u = a_1 + a_2x + a_3y + a_4xy$$
$$v = b_1 + b_2x + b_3y + b_4xy$$

The four constants can be evaluated using the known displacements

At A (0, 0) $\quad a_1 = 0$

$b_1 = 0$

At B (1, 0) $\quad a_1 + a_2 = -0.1$

$b_1 + b_2 = 0.1$

At C (1, 1) $\quad a_1 + a_2 + a_3 + a_4 = 0$

$b_1 + b_2 + b_3 + b_4 = 0$

At D (0, 1) $\quad a_1 + a_3 = 0$

$b_1 + b_3 = 0$

The solution is

$$a_1 = 0,\ a_2 = -0.1,\ a_3 = 0,\ a_4 = 0.1$$

$$b_1 = 0,\ b_2 = 0.1,\ b_3 = 0,\ b_4$$

This gives
$$u = -0.1x + 0.1xy$$
$$v = 0.1x - 0.1xy$$

(b) The shear strain at B is

$$\gamma = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.1x + 0.1 - 0.1y$$
$$\gamma_B = 0.1(1) + 0.1 - 0.1(0) = 0.2$$

■

CHAPTER 2
MATRIX ALGEBRA AND GAUSSIAN ELIMINATION

2.1 $\quad \mathbf{A} = \begin{bmatrix} 8 & -2 & 0 \\ -2 & 4 & -3 \\ 0 & -3 & 3 \end{bmatrix}$, $\mathbf{d} = \begin{Bmatrix} 2 \\ -1 \\ 3 \end{Bmatrix}$

(a) $\mathbf{I} - \mathbf{d}\,\mathbf{d}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 & -6 \\ 2 & 0 & 3 \\ -6 & 3 & -8 \end{bmatrix}$

(b) det **A** = 8 [(4)(3) – (–3) (–3)] – (–2) [(–2)(3) – (0)(–3)] = 12

(c) The characteristic equation is det (**A** - λ **I**) = 0, or

$$\det \begin{bmatrix} 8-\lambda & -2 & 0 \\ -2 & 4-\lambda & -3 \\ 0 & -3 & 3-\lambda \end{bmatrix} = 0$$

which yields

$$\lambda^3 - 15\lambda^2 + 55\lambda - 12 = 0$$

Handbooks (e.g., CRC Mathematical Handbook) give explicit solutions to cubic equations. Here equations given in Chapter 9 are used, which give formulas for finding the eigenvalues of the (3x3) symmetric stress tensor. Referring to Section 9.3 in the text, we have

$$I_1 = A_{11} + A_{22} + A_{33} = 15,\ I_2 = 55,\ I_3 = 12$$

Thus, $a = 20$, $b = 13$, $c = 5.164$, $\theta = 37.4\,°$

whence $\quad \lambda_1 = 0.2325$, $\lambda_2 = 5.665$, $\lambda_3 = 9.103$

Note: Since all $\lambda_i > 0$, **A** is <u>positive definite</u>.

Now, eigenvector $\mathbf{y}^i$ corresponding to eigenvalue λ_i is obtained from

$$(\mathbf{a} - \lambda_i \mathbf{I})\,\mathbf{y}^i = 0,\ i = 1,2,3$$

Thus, $\mathbf{y}^1$ is obtained as

$$\begin{bmatrix} 7.7675 & -2 & 0 \\ -2 & 3.7675 & -3 \\ 0 & -3 & 2.7675 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Thus,

$$7.7675\, y_1 - 2\, y_2 = 0$$
$$-2\, y_1 + 3.7675\, y_2 - 3\, y_3 = 0$$
$$-3\, y_2 + 2.7675\, y_3 = 0$$

Only two of the above three equations are independent. We have

$$y_1 = 0.2575\, y_2$$
$$y_2 = 0.922\, y_3$$

Letting $y_3 = 1$, we get $\mathbf{y}^1 = [0.237, 0.922, 1]^T$

The length of the vector is $\|y^1\| = \sqrt{\mathbf{y}^T\mathbf{y}} = 1.381$. Normalizing $\mathbf{y}^1$ to be a unit vector yields

$$\mathbf{y}^1 = [0.172, 0.668, 0.724]^T.$$

Similarly,

$$\mathbf{y}^2 = [0.495, 0.577, -0.650]^T,\ \mathbf{y}^3 = [0.850, -0.470, 0.232]^T.$$

(d) **Solution to A x = b using Algorithm 1 for general matrix**:

$n = 3$

First Step ($k = 1$)

$\boldsymbol{i} = 2$ (2^{nd} row)

$$\mathbf{A} = \begin{bmatrix} 8 & -2 & 0 \\ -2 & \mathbf{4} & -\mathbf{3} \\ 0 & -\mathbf{3} & \mathbf{3} \end{bmatrix}$$

$c = a_{21} / a_{11} = -2/8 = -1/4$

$a_{22}^{(1)} = 4 - (-1/4)(-2) = 7/2$, $a_{23}^{(1)} = -3$, $d_2^{(1)} = -1-(-1/4)(2) = -1/2$

$i = 3$ (3^{rd} row)

c = 0

$a_{32}^{(1)} = -3$, $a_{33}^{(1)} = 3$, $d_3^{(1)} = 3$

$$\text{Thus } \mathbf{A}^{(1)} = \begin{bmatrix} 8 & -2 & 0 \\ 0 & 7/2 & -3 \\ 0 & -3 & 3 \end{bmatrix},\ \mathbf{d}^{(1)} = \begin{bmatrix} 2 \\ -1/2 \\ 3 \end{bmatrix}$$

Second Step ($k = 2$)

$i = 3$ (3^{rd} row)

c = -6/7, $a_{33}^{(2)} = 3 - (-6/7)(-3) = 3/7$, $d_3^{(2)} = 3 - (-6/7)(-1/2) = 18/7$

$$\text{Thus } \mathbf{A}^{(2)} = \begin{bmatrix} 8 & -2 & 0 \\ 0 & 7/2 & -3 \\ 0 & 0 & 3/7 \end{bmatrix},\ \mathbf{d}^{(1)} = \begin{bmatrix} 2 \\ -1/2 \\ 18/7 \end{bmatrix}$$

Back-Substitution

$$\begin{bmatrix} 8 & -2 & 0 \\ 0 & -7/2 & -3 \\ 0 & 0 & 3/7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1/2 \\ 18/7 \end{bmatrix}$$

Third row gives: $3/7\ x_3 = 18/7$ whence $x_3 = 6$
2nd row then gives $x_2 = 5$,
1st row gives $x_1 = 1.5$
Thus, solution is $\mathbf{x} = [1.5, 5, 6\ ]^T$.

Solution to A x = b using Algorithm 2 for symmetric, banded matrix

A is stored as $\begin{bmatrix} 8 & -2 \\ 4 & -3 \\ 3 & 0 \end{bmatrix}$

$n = 3$, $nbw = 2$

First Step ($k = 1$)
$nbk = \min(3,2) = 2$
2nd row ($i = 2$):
$i1 = 2 \quad c = a_{12}/a_{11} = -1/4$
$j1 = 1 \quad j2 = 2$
$a_{21} = 4 - (-1/4)(-2) = 7/2$

Thus $\mathbf{A}^{(1)} = \begin{bmatrix} 8 & -2 \\ 7/2 & -3 \\ 3 & 0 \end{bmatrix}$

Second Step ($k = 2$)
$nbk = 2$
3rd row ($i = 3$):
$c = -6/7$
$j1 = 1 \quad j2 = 2$
$a_{31} = 3/7$

Thus $\mathbf{A}^{(2)} = \begin{bmatrix} 8 & -2 \\ 7/2 & -3 \\ 3/7 & 0 \end{bmatrix}$

reduction of right-hand-side vector **d** and back-substitution is same as in Algorithm 1 above, resulting in the the same solution $\mathbf{x} = [1.5, 5, 6\ ]^T$. ■

2.2 $\mathbf{N} = \begin{bmatrix} \xi & 1-\xi^2 \end{bmatrix}$

(a) $\int_{-1}^{1} \mathbf{N}\, d\xi = \begin{bmatrix} 0 & \frac{4}{3} \end{bmatrix}$

(b) $\int_{-1}^{1} \mathbf{N}^{\mathrm{T}}\ \mathbf{N}\ d\xi = \begin{bmatrix} \int \xi^2 & \int \xi(1-\xi^2) \\ \int \xi(1-\xi^2) & \int (1-\xi^2)^2 \end{bmatrix} = \begin{bmatrix} 2/3 & 0 \\ 0 & 16/15 \end{bmatrix}$ ■

2.3 $q = x_1 - 6\,x_2 + 3\,x_1^{\,2} + 5\,x_1\,x_2$

$= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{bmatrix} 3 & 2.5 \\ 2.5 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\equiv\ \mathbf{x}^{\mathrm{T}}\,\mathbf{Q}\,\mathbf{x} + \mathbf{c}^{\mathrm{T}}\,\mathbf{x}$

where

$$\mathbf{Q} = \begin{bmatrix} 3 & 2.5 \\ 2.5 & 0 \end{bmatrix} \ \textit{and}\ \mathbf{c} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$$ ■

2.4 The detailed algorithm is given in the text. This is an excercise in computer programming. The solutions are (a) (–2.25, –11.5, –10.5) (b) (1.55, 5.1, 6.1) ■

2.5 $\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$M_{11} = \det \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 3 \qquad \begin{bmatrix} X & X & X \\ X & . & . \\ X & . & . \end{bmatrix}$

$M_{12} = \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 1 \qquad \begin{bmatrix} X & X & X \\ . & X & . \\ . & X & . \end{bmatrix}$

$M_{13} = -1, M_{21} = 1, M_{22} = 3, M_{23} = 1, M_{31} = -1, M_{32} = 1, M_{33} = 3$

$C_{ij} = (-1)^{i+j}\, M_{ij}$

The minors are

Thus, the co – factor matrix is

$$\mathbf{C} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \mathbf{C}^{T} \ \text{yields}$$

$$\mathbf{A}^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

■

2.6 Area = $\frac{1}{2} \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 4 \end{bmatrix} = 4$ ■

2.7 $A_1 = \frac{1}{2} \det \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1.5 \\ 1 & 2.5 & 5 \end{bmatrix} = 1.625$

$A_2 = \frac{1}{2} \det \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2.5 & 5 \\ 1 & 1 & 1 \end{bmatrix} = 1.25$

$A_3 = \frac{1}{2} \det \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & 1.5 \end{bmatrix} = 0.75$

$A = A_1 + A_2 + A_3 = 3.625$

$A_1 / A = 0.448$, $A_2 / A = 0.345$, $A_3 / A = 0.207$ ■

2.8 $A_{i,j} = B_{i,\, j-i+1}$ for $j \geq i$

Thus, $A_{11,14}$ corresponds to $B_{11,4}$

and $B_{6,1}$ 〃 〃 $A_{6,6}$ ■

2.9 Full (10x10) matrix

<u>BANDED</u>

$n = 10$, $nbw = 10$

No. of storage locations = $(n)\ (nbw)$

= 100

<u>SKYLINE</u>

x x x x ...

x x x ...

x x ...

x ...

No. of storage locations = no. of column entries

= 1 + 2 + 3 + ... + 10

= (10) (10+1) / 2

= 55 ■

2.10 $\mathbf{A} = \begin{bmatrix} 4 & 3 & 1 \\ 3 & 6 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ Cholesky decomposition follows the steps as per Eq. (2.22):

<u>k = 1 (1st row)</u>

$$\ell_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$$

<u>k = 2 (2nd row)</u>

$$\ell_{21} = \frac{a_{21}}{\ell_{11}} = \frac{3}{2} \;,\; \ell_{22} = \sqrt{a_{22} - \ell_{21}{}^2} = 6 - (3/2)^2 = 1.936492$$

<u>k = 3 (3rd row)</u>

$$\ell_{31} = \frac{a_{31}}{\ell_{11}} = \frac{1}{2} \quad , \quad \ell_{32} = \frac{a_{32} - \ell_{31}\,\ell_{21}}{\ell_{22}} = \frac{2 - (1/2)(3/2)}{1.936492} = 0.6454971$$

$$\ell_{33} = 2.333333$$

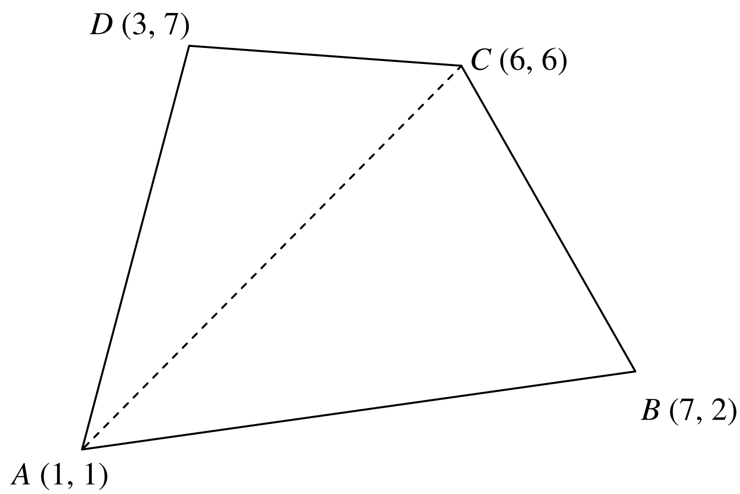
Thus, $\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ 1.5 & 1.936492 & 0 \\ 0.5 & 0.6454971 & 2.333333 \end{bmatrix}$. We may see that $\mathbf{A} = \mathbf{L}\,\mathbf{L}^T$. Also, program implementation is given in Subroutine CHOLESKI within Program GENEIGEN.

■

2.11 By expanding the matrix equations, we obtain a set of simultaneous equations for the ℓ and u coefficients, resulting in the solution

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0.4 & 1 & 0 \\ 0.4 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{U} = \begin{bmatrix} 5 & 3 & 1 \\ 0 & 2.8 & 1.6 \\ 0 & 0 & 5.7143 \end{bmatrix}$$ ■

2.12



We split the quadrilateral into two triangles as shown. Using the expression from Problem 2.6, the area is then given by

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 7 & 2 \\ 1 & 6 & 6 \end{bmatrix} + \frac{1}{2}\det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 6 & 6 \\ 1 & 3 & 7 \end{bmatrix} \\
&= \frac{1}{2}\det\begin{bmatrix} 1 & 1 & 1 \\ 0 & 6 & 1 \\ 0 & 5 & 5 \end{bmatrix} + \frac{1}{2}\det\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 5 \\ 0 & 2 & 6 \end{bmatrix} \\
&= \frac{1}{2}(25) + \frac{1}{2}(20) \\
&= 22.5
\end{aligned}$$

At the second step, we subtract the first row from second row and third row, which does not change the determinant value. ■

2.13 Setting $\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, it is easy to show that $\mathbf{T}\mathbf{T}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. ■

2.14 $\mathbf{A} = \begin{bmatrix} 4 & 3 & 1 \\ 3 & 3 & 2 \\ 3 & 4 & 5 \end{bmatrix}$

(a) Minors

$$M_{11} = \begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 7 \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 3 & 5 \end{vmatrix} = 9 \quad M_{13} = \begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} = 3$$

$$M_{21} = \begin{vmatrix} 3 & 1 \\ 4 & 5 \end{vmatrix} = 11 \quad M_{22} = \begin{vmatrix} 4 & 1 \\ 3 & 5 \end{vmatrix} = 17 \quad M_{23} = \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 7$$

$$M_{31} = \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = 3 \quad M_{32} = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 5 \quad M_{33} = \begin{vmatrix} 4 & 3 \\ 3 & 3 \end{vmatrix} = 3$$

(b) Cofactors

$$C_{11} = M_{11} = 7 \quad C_{12} = -M_{12} = -9 \quad C_{13} = M_{13} = 3$$

$$C_{21} = -M_{21} = -11 \quad C_{22} = M_{22} = 17 \quad C_{23} = -M_{23} = -7$$

$$C_{31} = M_{31} = 3 \quad C_{32} = -M_{32} = -5 \quad C_{33} = M_{33} = 3$$

(c) Adjoint

$$\mathrm{Adj}\mathbf{A} = \mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 7 & -11 & 3 \\ -9 & 17 & -5 \\ 3 & -7 & 3 \end{bmatrix}$$

(d) Determinant

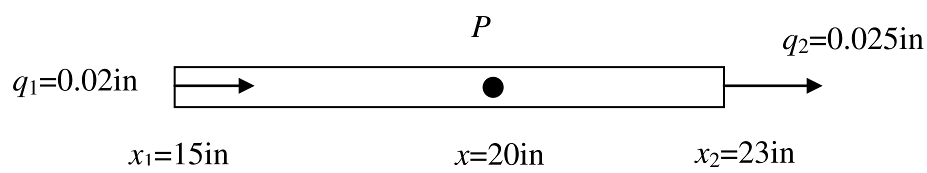
det**A** = 4(7)-3(9)+1(3) = 4

(e) Inverse

$$\mathbf{A}^{-1} = \frac{\mathrm{Adj}\mathbf{A}}{\det \mathbf{A}} = \frac{1}{4}\begin{bmatrix} 7 & -11 & 3 \\ -9 & 17 & -5 \\ 3 & -7 & 3 \end{bmatrix} \quad \blacksquare$$

CHAPTER 3
ONE-DIMENSIONAL PROBLEMS

3.1



$$A = 1.2\ \text{in}^2 \qquad E = 30 \times 10^6\ \text{psi}$$

(a) At point P,

$$N_1 = \frac{x_2 - x}{x_2 - x_1} = \frac{3}{8}$$

$$N_2 = \frac{x - x_1}{x_2 - x_1} = \frac{5}{8}$$

$$u = \mathbf{Nq} = N_1 q_1 + N_2 q_2$$
$$= 0.023125\ \text{in}$$

(b)

$$\varepsilon = \frac{du}{dx} = \frac{q_2 - q_1}{x_2 - x_1}$$

$$\varepsilon = \mathbf{Bq} \quad \Rightarrow \quad \mathbf{B} = \frac{1}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$\mathbf{B} = \frac{1}{8}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$\varepsilon = \frac{1}{8}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} 0.02 \\ 0.025 \end{Bmatrix} = 0.000625$$

$$\sigma = E\varepsilon = 18750\ \text{psi} = 18.75\ \text{ksi}$$

(c)

$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 4.5 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

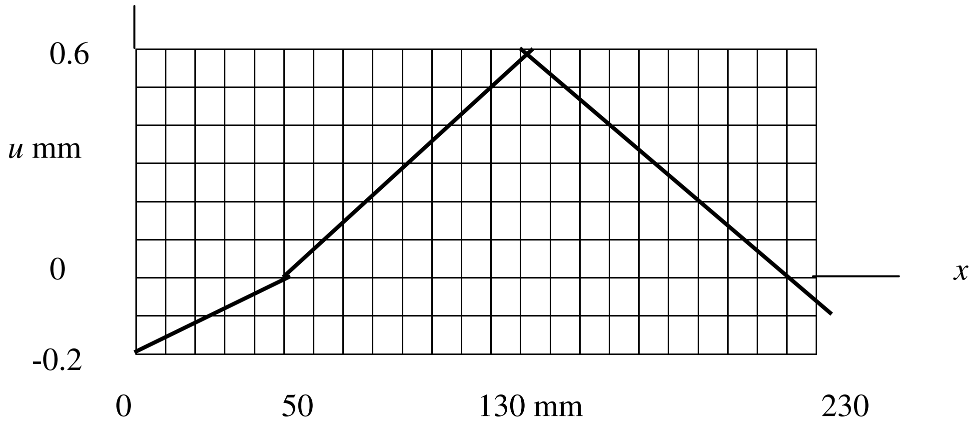
(d) $U_e = \tfrac{1}{2}\, \mathbf{q}^{\mathrm{T}}\mathbf{k}\mathbf{q}$

$$U_e = \frac{1}{2}\begin{bmatrix} 0.02 & 0.025 \end{bmatrix} 4.5 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0.02 \\ 0.025 \end{Bmatrix} = 56.25\ \text{lb-in}$$ ■

3.2

NBW = max [(3-1), (4-3), (5-4), **(5-2)**] + 1 = 4 ■

3.3 **(i)** Plotting the displacement is straight forward



(ii) Strains are constant in each of the three elements

Since the displacements are linear within the elements, the strains are constant over each element.

Strain in Element 1-2: $L_{1\text{-}2} = 50$ mm

$$\varepsilon = \frac{0-(-0.2)}{50} = 0.004$$

Strain in Element 2-3: $L_{2\text{-}3} = 80$ mm

$$\varepsilon = \frac{0.6-0}{80} = 0.0075$$

Strain in Element 3-4: $L_{3\text{-}4} = 100$ mm

$$\varepsilon = \frac{-0.1-(0.6)}{100} = -0.007$$

The plotted curve represents the slope of the curve shown in section (i).

(iii) The **B** matrix for element 2-3 is given by

$$\mathbf{B} = \frac{1}{x_3 - x_2}\begin{bmatrix}-1 & 1\end{bmatrix} = \begin{bmatrix}-0.0125 & 0.0125\end{bmatrix}$$

(iv) The stiffness matrix of element 1-2 is given by

$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{(1)(1)}{50}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\text{N/mm}$$

The element strain energy is given by

$$U = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}\mathbf{q}$$
$$\mathbf{q} = \begin{bmatrix} -0.2 \\ 0 \end{bmatrix}$$

On substituting for **k** and **q**, we get $U = 0.0004$ N-mm. ■

3.4 **(a)** Element stiffness matrices [**k**] are always nonsingular—true or false, make your choice and justify**. FALSE** . Without boundary conditions, and element is a free body.

(b) The strain energy in a structure, $U = ½\, \mathbf{Q}^{\mathbf{T}}\mathbf{K}\, \mathbf{Q}$ is always > 0 for any **Q** , provided <u>**K** is positive definite.</u>

(c) An FE model of a rod is given a displacement **Q** = [1, 1, . . . , 1] $^{\mathrm{T}}$. The associated strainenergy $U = ½\, \mathbf{Q}^{\mathrm{T}}\, \mathbf{K}\, \mathbf{Q}$ equals zero. What can you conclude regarding the stiffnessmatrix **K** ? It implies that $\sum_{j=1}^{N}\sum_{i=1}^{N} K_{ij} = 0$. **K** is neither positive definite nor negative definite and it may be singular.

(d) Consider a rod fixed at its ends, $x = 0$ and $x = 1$, respectively. Is the axial displacementfield $u = (x - 1)^2$ "kinematically admissible" ? NO since $u(0) = 1$. ■

3.5 $h(x) = a_0 + a_1 x + a_2 x^2$. A quadratic expression has three unknown coefficients. ■

3.6 The displacement is defined using shape functions N_1, N_2 as follows:

$$u = N_1 q_1 + N_2 q_2$$
$$\Rightarrow u = \mathbf{N}\mathbf{q}$$
$$\text{where } \mathbf{N} = [N_1 \quad N_2]$$
$$\varepsilon = \frac{du}{dx} = \frac{d\mathbf{N}}{dx}\mathbf{q}$$
$$\text{thus if we denote } \mathbf{B} = \frac{d\mathbf{N}}{dx}$$
$$\varepsilon = \mathbf{B}\mathbf{q}$$

We note here that if we make use of isoparametric representations for x,

$$x = N_1 x_1 + N_2 x_2$$

$$\Rightarrow x = \mathbf{N}\mathbf{x}$$

then $\dfrac{d}{dx} = \left(\dfrac{dx}{d\xi}\right)^{-1} \dfrac{d}{d\xi}$

$$\mathbf{B} = \left(\frac{dx}{d\xi}\right)^{-1} \frac{d\mathbf{N}}{d\xi}$$

■

3.7 S(22,1) = S(22,1) + 150 ■

3.8 **(a)** Let $a = E_1A_1/L_1$ for element 1 and $b = E_2A_2/L_2$ for element 2. The assembled stiffness matrix is given by

$$\mathbf{K} = \begin{bmatrix} a & -a & 0 \\ -a & a+b & -b \\ 0 & -b & b \end{bmatrix}$$

We note that

$$\det(\mathbf{K}) = a \det\begin{bmatrix} a+b & -b \\ -b & b \end{bmatrix} + a \det\begin{bmatrix} -a & 0 \\ -b & b \end{bmatrix}$$

$$= a^2b + ab^2 - ab^2 - a^2b = 0$$

This shows that **K** is singular.

We also observe that the second row is sum of the first row and the last row. Thus one row is a linear combination of the others showing singularity. An easier way of checking singularity is to perform Gaussian elimination to reduce **K** to upper triangular form. If any of the diagonal terms is zero, the matrix is singular.

(b) Let us try to solve **KQ** = **0** using the form of **K** given in **(a)**.

From the first equation

$$aQ_1 - aQ_2 = 0 \Rightarrow Q_1 = Q_2$$

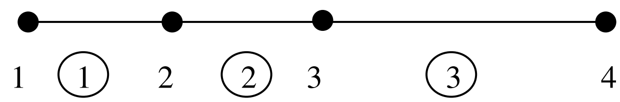
Then from the second equation,

$$-a(Q_1 - Q_2) + b(Q_2 - Q_3) = 0 \Rightarrow Q_2 = Q_3$$

The third equation also yields the same result. Thus $Q_1 = Q_2 = Q_3$. This is rigid body displacement of the entire body. This results in no strain or stress in each element. The strain energy in each element is zero. The strain energy in the structure is zero.

(c) If **K** is nonsingular, it has an inverse. Thus when **K** is nonsingular, **KQ** = **0** implies that **Q** = **0**. A nonzero solution **Q** to **KQ** = **0** implies that **K** is singular. ■

3.9 We introduce a node at the point of load application, and a node at the point where there is a change of cross section. The finite element configuration is shown below.



$E = 200 \times 10^3$ N/mm^2 (MPa)

$A_1 = 250$ mm^2 $A_2 = 250$ mm^2 $A_3 = 400$ mm^2

$L_1 = 150$ mm $L_2 = 150$ mm $L_3 = 300$ mm

Load $P = 300000$ N is applied at node 2. $Q_1 = 0$, $Q_4 = 0$.

$$\begin{bmatrix} \frac{E_1A_1}{L_1} & -\frac{E_1A_1}{L_1} & 0 & 0 \\ -\frac{E_1A_1}{L_1} & \frac{E_1A_1}{L_1}+\frac{E_2A_2}{L_2} & -\frac{E_2A_2}{L_2} & 0 \\ 0 & -\frac{E_2A_2}{L_2} & \frac{E_2A_2}{L_2}+\frac{E_3A_3}{L_3} & -\frac{E_3A_3}{L_3} \\ 0 & 0 & -\frac{E_3A_3}{L_3} & \frac{E_3A_3}{L_3} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ 0 \\ 0 \end{Bmatrix}$$

Since Q_1 and Q_4 are zero, using the elimination approach, as indicated above,

$$10^5 \begin{bmatrix} 6.667 & -3.333 \\ -3.333 & 6 \end{bmatrix} \begin{Bmatrix} Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 30000 \\ 0 \end{Bmatrix}$$

On solvind, we get $Q_2 = 0.623$ mm and $Q_3 = 0.346$ mm.

The data set for using program FEM1D is given below:

```
<< 1D STRESS ANALYSIS USING BAR ELEMENT >>
PROBLEM 3.7
NN NE NM NDIM NEN NDN
 4  3  1   1   2   1
ND NL  NMPC
 2  1   0
Node#   X-Coordinate
 1       0
 2     150
 3     300
 4     600
Elem#  N1  N2  Mat#  Area TempRise
 1      1   2    1    250   0
 2      2   3    1    250   0
```

```
 3      3   4    1     400   0
DOF#  Displacement
 1      0
 4      0
DOF#  Load
 2    300000
MAT#  E        Alpha
 1   200000      0
B1  i  B2 j  B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

The output from the program is given below.

Results from Program FEM1D

PROBLEM 3.7

Node#	Displacement
1	3.11536E-05
2	0.623102751
3	0.346174349
4	1.38464E-05
5	

Element#	Stress
1	830.7621304
2	-369.2378696
3	-230.7736685

Node#	Reaction
1	-207690.5326
4	-92309.4674

The programs gives the reactions at the fixed nodes. ■

3.10 The unmodified system of equations is

$$\frac{10^5}{3}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60\times 10^3 \\ 0 \end{Bmatrix}$$

We impose the boundary conditions $Q_1 = 0$ and $Q_3 = 1.2$ mm.

$Q_1 = \ \ 0 \ \Rightarrow$ first row and first column are eliminated.
$Q_2 = 1.2 \ \Rightarrow$ drop third row and third column on the left hand side (stiffness) and subtract $K_{23}Q_3$ from F_2.

This results in

$$\frac{10^5}{3}2Q_2 = 60\times 10^3 + \frac{10^5}{3}(1.2)$$

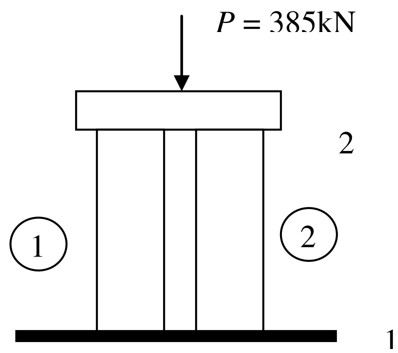
or $Q_2 = 1.5$ mm

Using the unmodified stiffness matrix, the reactions are calculated.

$$\begin{aligned}
R_1 &= K_{11}Q_1 + K_{12}Q_2 + K_{13}Q_3 \\
&= \frac{10^5}{3}(0 - 1.5 + 0) \\
&= -50\,\text{kN} \\
R_3 &= K_{31}Q_1 + K_{32}Q_2 + K_{33}Q_3 \\
&= \frac{10^5}{3}(0 - 1.5 + 1.2) \\
&= -10\,\text{kN}
\end{aligned}$$

■

3.11



$E_1 = 70000$ MPa $\quad A_1 = 30\times 60 = 1800$ mm^2
$E_2 = 105000$ MPa $\quad A_2 = 1800$ mm^2
$L_1 = L_2 = 200$ mm

This problem is easily formulated by defining same node numbers for each element.

Elem#	Node1	Node2	Material#
1	1	2	1
2	1	2	2

The unmodified system is

$$\begin{bmatrix} \dfrac{E_1A_1}{L_1} + \dfrac{E_2A_2}{L_2} & -\dfrac{E_1A_1}{L_1} - \dfrac{E_2A_2}{L_2} \\ -\dfrac{E_1A_1}{L_1} - \dfrac{E_2A_2}{L_2} & \dfrac{E_1A_1}{L_1} + \dfrac{E_2A_2}{L_2} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

On eliminating the first row and the first column,

$$\frac{1800}{200}\left(7\times10^4+10.5\times10^4\right)Q_2=385\times10^3$$

$$Q_2=0.244 \text{ mm}$$

$$\sigma_1=\frac{E_1Q_2}{L_1}=85.4 \text{ MPa}$$

$$\sigma_2=\frac{E_2Q_2}{L_2}=128.1 \text{ MPa}$$

■

3.12 We solve the problem assuming that the gap on the right closes. This means that after, solving, the reaction at the right must be directed along the $-x$ direction. If not, the problem has to be solved again imposing only one boundary condition on the left.

We divide the body into 4 elements, placing nodes at $x = 0$, 150, 300, 500,700. The point loads are at nodes 2 and 4. The input data for FEM1D is given here.

```
PROGRAM FEM1D  << BAR ANALYSIS
PROBLEM 3.10
NN      NE      NM      NDIM    NEN     NDN
5       4       1       1       2       1
ND      NL      NMPC
2       2       0
NODE#   X-COORD
1       0
2       150
3       300
4       500
5       700
EL#     N1      N2      MAT#    AREA    TEMP RISE
1       1       2       1       250     0
2       2       3       1       250     0
3       3       4       1       400     0
4       4       5       1       400     0
DOF#    DISP
1       0
5       3.5
DOF#    LOAD
2       300000
4       600000
MAT#    E       Alpha
1       200000  1.20E-05
B1      I       B2      J       B3
```

The output from program FEM1D follows.

```
Results from Program FEM1D
PROBLEM 3.10
Node#   Displacement                    (mm)
1       8.40903E-05
2       2.018250723
3       3.136417355
4       4.068222883
5       3.50002841
```

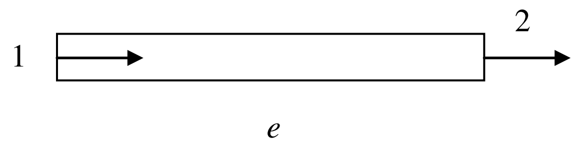
Element#	Stress	(MPa)
1	2690.888843	
2	1490.888843	
3	931.8055271	
4	-568.1944729	

Node#	Reaction	(N)
1	-672722.2109	
5	-227277.7891	

The reaction at node 5 is negative, indicating that the contact at node 5 is established. ■

3.13 This problem requires program modification, in order to try various mesh divisions.

Consider the typical element shown:



The average radius and the element body force are given by

$$\bar{r} = \frac{x_1 + x_2}{2}$$

$$\mathbf{f}^e = A_e L_e \rho \bar{r} \omega^2 \begin{Bmatrix} 0.5 \\ 0.5 \end{Bmatrix}$$

The body force must be computed in the element loop and added to the global locations.

The typical changes in the BASIC program:

(*Just before the element stiffness loop, add the lines*)

```
INPUT "Weight per unit volume=" ; RHO
INPUT "Angular velocity rad/s ="; OMEGA
```

(*In the element stiffness loop, add lines*)

```
RBAR = 0.5 * (X(N2) + X(N1))
FORCE = A(N) * EL * RHO * RBAR * OMEGA^^2
```

(*after temperature load calculation*)

```
F(N1) = F(N1) – TL + 0.5 * FORCE
F(N2) = F(N2) + TL + 0.5 * FORCE
```

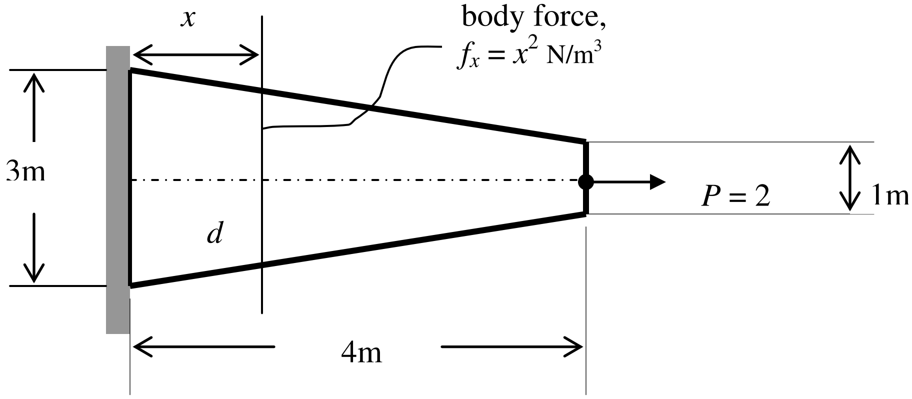
After the modifications, the program can be used for comparisons. ■

3.14 **(a)** Using the geometry of the problem shown in the figure,

$$d = 3 - 0.5x$$
$$A = td = 0.6 - 0.1x$$

$u = 0$ at $x = 0 \Rightarrow a_0 = 0 \;\; \Rightarrow \;\; u = a_1x + a_2x^2,\;\; du/dx = a_1 + 2a_2x,\; u_P = 4a_1 + 16a_2.$

$$\Pi = \frac{1}{2}\int_0^4 E\left(\frac{du}{dx}\right)^2 A dx - \int_0^4 u f_x A dx - u_P P$$



thickness $t = 0.2$m, $E = 50$ N/m^2

On substituting for E, u, A, etc., and performing the integration, we get

$$\Pi = 40a_1^{\;2} + 266.67a_1a_2 + 640a_2^{\;2} - 21.92a_1 - 127.5a_2$$

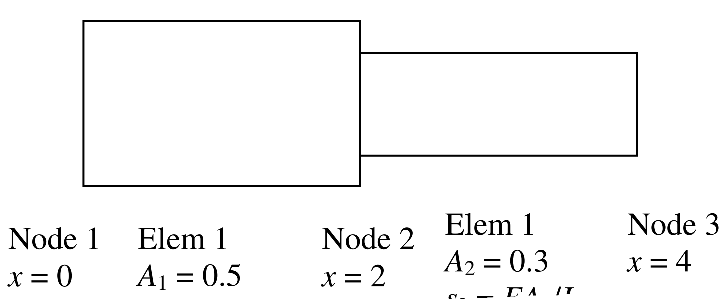
$$\frac{\partial \Pi}{\partial a_1} = 80a_1 + 266.67a_2 - 21.92 = 0$$

$$\frac{\partial \Pi}{\partial a_2} = 266.7a_1 + 1280a_2 - 127.5 = 0$$

On solving, $a_1 = -0.1898$, $\quad a_2 = 0.1391$.

Thus, $u = -0.1898x + 0.1391x^2$, which is a parabola, with $u_P = 1.4664$ m. The stress $\sigma = E(-0.1898 + 0.2782x)$ is linear in x.

(b) We show the two element model of the problem.



From the above figure, we the element stiffness matrices and the global stiffness are easily calculated.

Stiffness matrix of element 1 is $\begin{bmatrix} 12.5 & -12.5 \\ -12.5 & 12.5 \end{bmatrix}$.

Stiffness matrix of element 2 is $\begin{bmatrix} 7.5 & -7.5 \\ -7.5 & 7.5 \end{bmatrix}$.

The assembled global stiffness matrix is $\begin{bmatrix} 12.5 & -12.5 & 0 \\ -12.5 & 20 & -7.5 \\ 0 & -7.5 & 7.5 \end{bmatrix}$.

The body force is calculated using a constant distribution in each element. The body force value at the centroid location is used as representative value.

For element 1, $\int_0^2 f_x uAdx = x_B{}^2 u_B A_e L_e$

Noting $x_B = \dfrac{x_1 + x_2}{2} = 1, u_B = \dfrac{Q_1 + Q_2}{2}$, we have

$$\int_0^2 f_x uAdx = \{Q_1 \quad Q_2\}\begin{bmatrix} \dfrac{1}{2} x_B{}^2 A_1 L_1 \\ \dfrac{1}{2} x_B{}^2 A_1 L_1 \end{bmatrix} = \{Q_1 \quad Q_2\}\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

For element 2, $\int_2^4 f_x uAdx = x_B{}^2 u_B A_e L_e$

Noting $x_B = \dfrac{x_1 + x_2}{2} = 3, u_B = \dfrac{Q_2 + Q_3}{2}$, we have

$$\int_2^4 f_x uAdx = \{Q_2 \quad Q_4\}\begin{bmatrix} \dfrac{1}{2} x_B{}^2 A_2 L_2 \\ \dfrac{1}{2} x_B{}^2 A_2 L_2 \end{bmatrix} = \{Q_2 \quad Q_4\}\begin{bmatrix} 2.7 \\ 2.7 \end{bmatrix}$$

The assembled load vector is now given by $\mathbf{F} = [0.5 \quad 3.2 \quad 4.7]^T$.

For manual solution, we set **KQ** = **F**, and solve it using strike off approach.

Computer input and output are given. If solution is to be obtained for various mesh divisions, the body force calculations may be introduced into the element stiffness loop.

PROGRAM FEM1D << BAR ANALYSIS
PROBLEM 3.12

NN	NE	NM	NDIM	NEN	NDN
3	2	1	1	2	1
ND	NL	NMPC			
1	3	0			
NODE#	X-COORD				
1	0				
2	2				
3	4				
EL#	N1	N2	MAT#	AREA	TEMP RISE
1	1	2	1	0.5	0
2	2	3	1	0.3	0
DOF#	DISP				
1	0				
DOF#	LOAD				
1	0.5				
2	3.2				
3	4.7				
MAT#	E	Alpha			
1	50	0.00E+00			
B1	I	B2	J	B3	

Results from Program FEM1D
PROBLEM 3.12

Node#	Displacement
1	0.000042
2	0.632042
3	1.258708667
Element#	Stress
1	15.8
2	15.66666667
Node#	Reaction
1	-8.4

■

3.15 Following the steps of Eqs 3.87 and 3.88, the penalty term added to the potential energy is

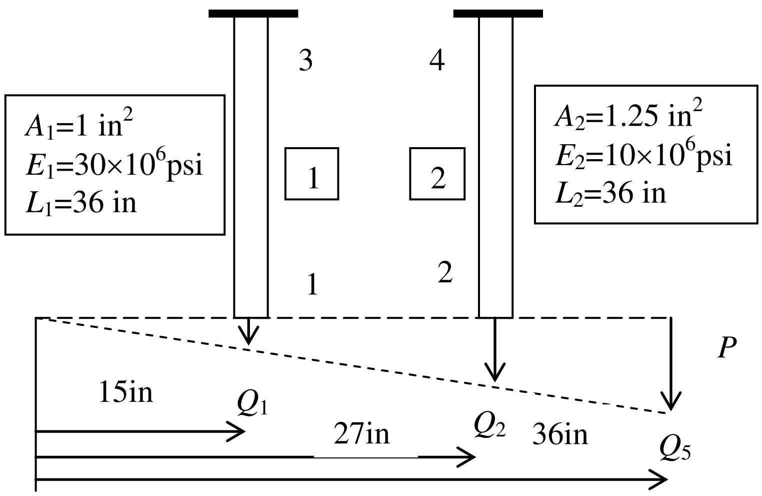
$$\frac{1}{2}C\left(3Q_p - Q_q\right)^2$$

This results in stiffness modification (addition)

$$\mathbf{K} \Leftarrow C \begin{array}{cc} \begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix} & \begin{matrix} p \\ q \end{matrix} \\ \begin{matrix} p \quad & \quad q \end{matrix} & \end{array}$$

New bandwidth = max(n_1, $|p - q| + 1$) ■

3.16 This problem follows the steps used in Example 3.6.



We denote $s_1 = \dfrac{E_1 A_1}{L_1}$ and $s_2 = \dfrac{E_2 A_2}{L_2}$. The unmodified stiffness is

$$\mathbf{K} = \begin{bmatrix} s_1 & 0 & -s_1 & 0 & 0 \\ 0 & s_2 & 0 & -s_2 & 0 \\ -s_1 & 0 & s_1 & 0 & 0 \\ 0 & -s_2 & 0 & s_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The multipoint constraints are

$$Q_1 - \frac{15}{36} Q_5 = 0$$

$$Q_2 - \frac{27}{36} Q_5 = 0$$

The displacement constraints are

$$Q_3 = 0; \quad Q_4 = 0$$

These are now used in preparing the input file for the solution of the problem.

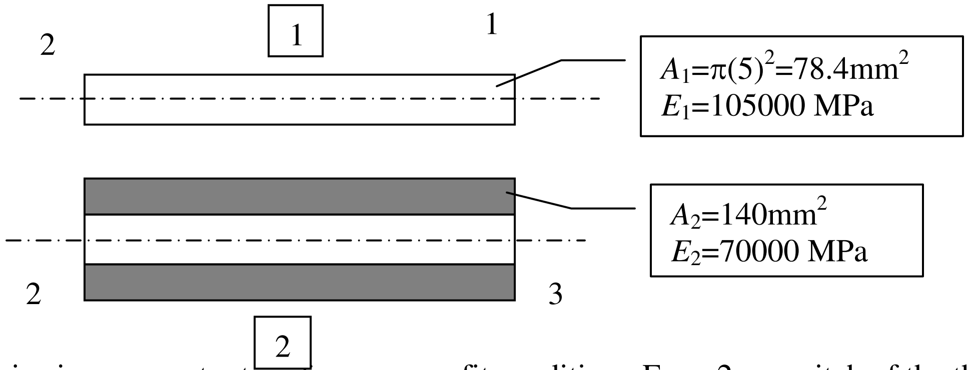
Input Data file

```
PROGRAM FEM1D  << BAR ANALYSIS
EXAMPLE 3.16
NN   NE     NM      NDIM   NEN    NDN
5    2      2       1      2      1
ND   NL     NMPC
2    1      2
NODE#       X-COORD
1    0
2    0
3    -36
4    -36
5    0
EL#  N1     N2      MAT#   AREA   TEMP RISE
1    1      3       1      1      0
2    2      4       2      1.25   0
DOF#        DISP
3    0
4    0
DOF#        LOAD
5    15000
MAT#        E       Alpha
1    3.00E+07       1.20E-05
2    1.00E+07       2.30E-05
B1   I      B2      J      B3
1    1      -0.416665      0
1    2      -0.75   5      0
```

```
Results from Program FEM1D
EXAMPLE 3.14
Node#       Displacement
1    0.018383541
2    0.033092834
3    1.83817E-06
4    1.37881E-06
5    0.044125617
Element#    Stress
1    15318.08586
2    9192.070767
Node#       Reaction
3    -15318.08586
4    -11490.08846
```

■

3.17



The nut is given a quarter turn from a snug fit condition. For a 2 mm pitch of the thread, the undeformed length of the bolt is 0.5mm shorter than the sleeve. If L is the length of the sleeve, the multipoint constraint (MPC) condition is

$$L + Q_3 = L - \delta + Q_1$$
$$\Rightarrow Q_1 - Q_3 = \delta = 0.5$$
$$(\beta_1 Q_1 + \beta_2 Q_3 = \beta_3) \Rightarrow \beta_1 = 1, \beta_2 = -1, \beta_3 = 0.5$$

We also have $Q_2 = 0$.

Denoting
$$a = \frac{E_1 A_1}{L} = 3.675 \times 10^4 \text{ N/mm}$$
$$b = \frac{E_2 A_2}{L} = 1.374 \times 10^4 \text{ N/mm}$$
We choose penalty constant $C = a \times 10^4$

The modified stiffness is given by

$$\mathbf{K} = \begin{bmatrix} a + C\beta_1^{\,2} & -a & C\beta_1\beta_2 \\ -a & a+b+C & -b \\ C\beta_1\beta_2 & -b & b + C\beta_2^{\,2} \end{bmatrix}$$
$$\mathbf{F} = \begin{Bmatrix} C\beta_3\beta_1 \\ 0 \\ C\beta_3\beta_2 \end{Bmatrix}$$

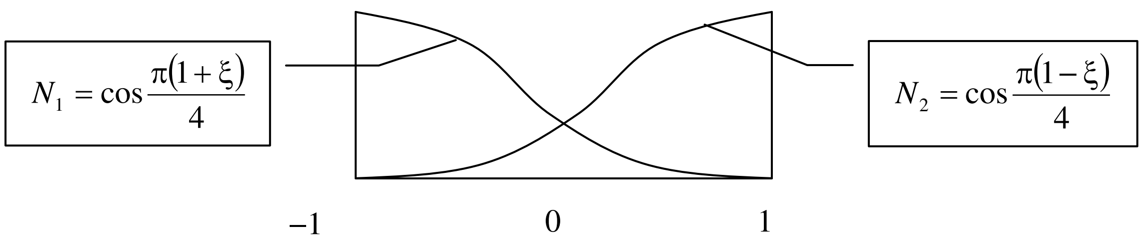
On solving $\mathbf{KQ} = \mathbf{F}$, we get

$$[Q_1 \;\; Q_2 \;\; Q_3]^{\mathrm{T}} = [0.136 \quad 0 \quad -0.364]^{\mathrm{T}} \text{ mm}$$

Stress in bolt $= E_1(Q_1 - Q_2)/L = 35.725$ MPa

Stress in sleeve $= E_2(Q_3 - Q_2)/L = -63.681$ MPa ■

3.18



$$\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1$$

$$u(\xi) = N_1 q_1 + N_2 q_2$$

$$\varepsilon = \frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx}$$

$$= \frac{\pi}{4}\frac{2}{x_2 - x_1}\left[-\sin\frac{\pi(1+\xi)}{4}q_1 + \sin\frac{\pi(1-\xi)}{4}q_2\right]$$

(a) $\varepsilon = \mathbf{Bq} \Rightarrow$

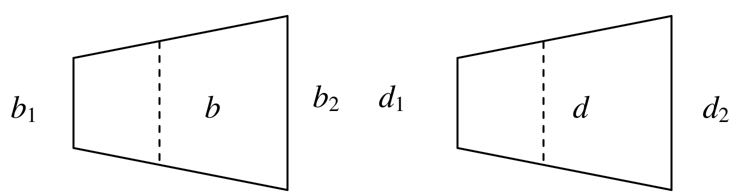
$$\mathbf{B} = \frac{\pi}{2(x_2 - x_1)}\left[-\sin\frac{\pi(1+\xi)}{4} \quad \sin\frac{\pi(1-\xi)}{4}\right]$$

(b) $\mathbf{k} = \int_e \mathbf{B}^{\mathrm{T}}\mathbf{DB}A\,dx \Rightarrow$

$$\mathbf{k} = \frac{\pi^2 EA_e}{2l_e}\int_{-1}^{1}\begin{bmatrix} \sin^2\frac{\pi(1+\xi)}{4} & \sin\frac{\pi(1+\xi)}{4}\sin\frac{\pi(1-\xi)}{4} \\ \mathit{Symmetric} & \sin^2\frac{\pi(1-\xi)}{4} \end{bmatrix} d\xi$$

■

3.19



$$b = N_1 b_1 + N_2 b_2 \qquad d = N_1 d_1 + N_2 d_2$$

$$A = bt \qquad A = \frac{\pi}{4} d^2$$

$$dx = \frac{1}{2} l_e d\xi$$

Noting from Eq 3.20 and Eq 3.21 that

$$\varepsilon = \mathbf{Bq} \qquad \mathbf{B} = \frac{1}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

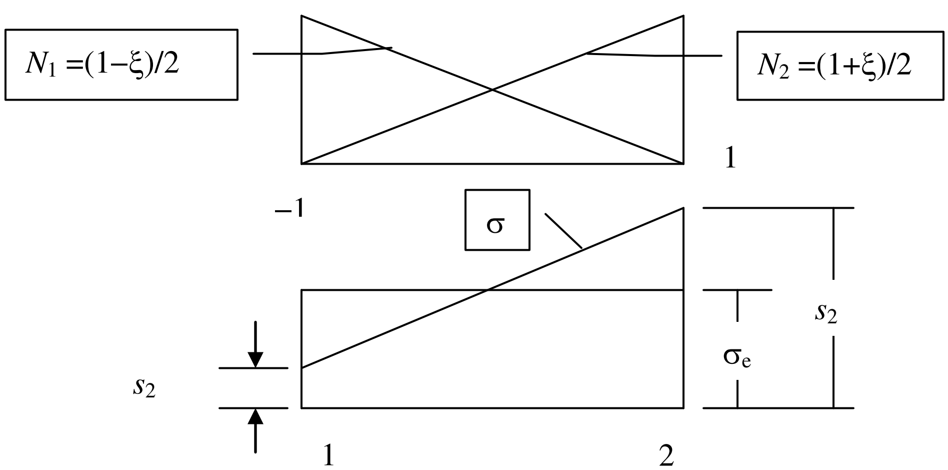
$$U_e = \frac{1}{2}\mathbf{q}^T \mathbf{B}^T \mathbf{B} E \int_e A dx = \frac{1}{2}\mathbf{q}^T \mathbf{k}^e \mathbf{q}$$

Subsituting for A and integrating,

(a) $$\mathbf{k}^e = \frac{Et}{l_e}\left(\frac{b_1 + b_2}{2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(Note: $\int_{-1}^{1} N_1^{\,2} d\xi = \frac{2}{3}$, $\int_{-1}^{1} N_1 N_2 d\xi = \frac{2}{3}$, $\int_{-1}^{1} N_2^{\,2} d\xi = \frac{2}{3}$) ■

3.20



Let σ be the linearly varying stress with continuity at the element connections (nodes).

For least squares error, we need to minimize

$$I = \sum \int_e \frac{1}{2}(\sigma - \sigma_e)^2 dx$$

Using shape functions on an element,

$$\sigma = N_1 s_1 + N_2 s_2 = \mathbf{Ns}$$
$$\mathbf{N} = [N_1 \quad N_2] \qquad \mathbf{s} = [s_1 \quad s_2]^T$$

We consider the error integral on the element.

$$I_e = \int_e \frac{1}{2}(\sigma - \sigma_e)^2 dx$$
$$= \frac{1}{2}\int_e \sigma^2 dx - \int_e \sigma\sigma_e dx - \int_e \sigma_e{}^2 dx$$

The last term above is a constant.

The first term

$$\frac{1}{2}\int_e \sigma^2 dx = \frac{1}{2}\mathbf{s}^T\left(\int_{-1}^{1}\mathbf{N}^T\mathbf{N}\frac{l_e}{2}d\xi\right)\mathbf{s}$$
$$\int_{-1}^{1}\mathbf{N}^T\mathbf{N}d\xi = \int_{-1}^{1}\begin{bmatrix} \frac{(1-\xi)^2}{4} & \frac{1-\xi^2}{4} \\ \frac{1-\xi^2}{4} & \frac{(1+\xi)^2}{4} \end{bmatrix} d\xi$$
$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$
$$\Rightarrow$$
$$\frac{1}{2}\int_e \sigma^2 dx = \frac{1}{2}\mathbf{s}^T \frac{l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\mathbf{s} = \frac{1}{2}\mathbf{s}^T\mathbf{ks}$$

Element stiffness matrix

$$\mathbf{k} = \frac{l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\int_e \sigma\sigma_e dx = \mathbf{s}^T \sigma_e \frac{l_e}{2}\int_{-1}^{1}\begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} d\xi$$
$$= \mathbf{s}^T \sigma_e \frac{l_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \mathbf{s}^T\mathbf{f}^e$$
$$\mathbf{f}^e = \frac{\sigma_e l_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

On assembling,

$$I = \Sigma\, I_e \;=\; (1/2)\, \mathbf{S}^{\mathrm{T}}\mathbf{K}\mathbf{S} \;-\; \mathbf{S}^{\mathrm{T}}\mathbf{F}$$

where $\mathbf{S} = [S_1, S_2, \ldots, S_n]^{\mathrm{T}}$ is the vector of nodal values.

For the least squares error, we get the system of equations

$$\mathbf{KS} = \mathbf{F}$$

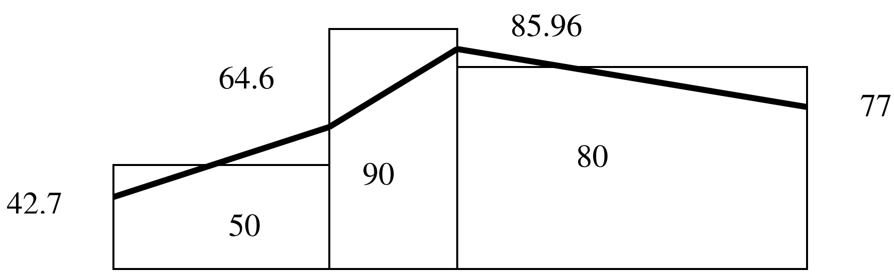
In the present problem, we have

$l_1 = 0.2$ m $\quad l_2 = 0.08$ m $\quad l_3 = 0.3$ m $\quad \sigma_1 = 50$ MPa $\quad \sigma_2 = 90$ MPa $\quad \sigma_3 = 80$ MPa

$$\mathbf{K} = \begin{bmatrix} \frac{l_1}{3} & \frac{l_1}{6} & 0 & 0 \\ \frac{l_1}{6} & \frac{l_1}{3}+\frac{l_2}{3} & \frac{l_2}{6} & 0 \\ 0 & \frac{l_2}{6} & \frac{l_2}{3}+\frac{l_3}{3} & \frac{l_3}{6} \\ 0 & 0 & \frac{l_3}{6} & \frac{l_3}{3} \end{bmatrix} \qquad \mathbf{F} = \begin{Bmatrix} \frac{\sigma_1 l_1}{2} \\ \frac{\sigma_1 l_1}{2}+\frac{\sigma_2 l_2}{2} \\ \frac{\sigma_2 l_2}{2}+\frac{\sigma_3 l_3}{2} \\ \frac{\sigma_3 l_3}{2} \end{Bmatrix}$$

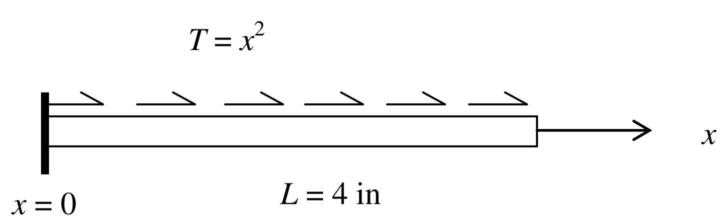
Solving for **S** form **KS** = **F**,

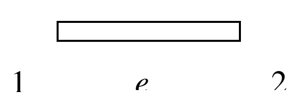
$$\mathbf{S} = [42.691,\; 64.617,\; 85.952,\; 77.024]^{\mathrm{T}} \;\text{MPa}$$



(Note: If a 2-D proble has point loads, the stresses are constant in elements. Smoothing is meaningful in problems with distributed loading) ■

3.21





The element stiffness is $\mathbf{k}^e = \dfrac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

The distributed load is $T = \left(N_1 x_1 + N_2 x_2\right)^2$ over the element.

$$\int_e Tudx = \int \left(N_1 x_1 + N_2 x_2\right)^2 \left(N_1 q_1 + N_2 q_2\right)\left(\frac{l_e}{2} d\xi\right)$$

On expanding and noting that

$$\int_{-1}^{1} N_1^3 d\xi = \frac{1}{2}, \int_{-1}^{1} N_1^2 N_2 d\xi = \frac{1}{6}, \int_{-1}^{1} N_1 N_2^2 d\xi = \frac{1}{6}, \int_{-1}^{1} N_2^3 d\xi = \frac{1}{2}$$

$$\int_e Tudx = \mathbf{q}^T \frac{l_e}{2}\begin{Bmatrix} \dfrac{x_1^2}{2} + \dfrac{1}{6}x_2^2 + \dfrac{1}{3}x_1 x_2 \\ \dfrac{1}{6}x_1^2 + \dfrac{x_2^2}{2} + \dfrac{1}{3}x_1 x_2 \end{Bmatrix} = \mathbf{q}^T \mathbf{T}^e$$

(a) One element model

On substituting $x_1 = 0, x_2 = 4, l_e = 4$,

$$\mathbf{K} = \frac{\left(30\times 10^6\right)(2)}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{F} = 1000 \times \frac{4}{2}\begin{Bmatrix} \dfrac{4^2}{6} \\ \dfrac{4^2}{2} \end{Bmatrix} = \begin{Bmatrix} 5333.3 \\ 16000 \end{Bmatrix}$$

Using the strike off approach with $Q_1 = 0$, we get

$$\frac{\left(30\times 10^6\right)(2)}{4} Q_2 = 16000$$

$$Q_2 = 1.066 \times 10^{-3} \text{ in}$$

Stress $\sigma = (E/l)Q_2 = 8000$ psi

(b) Two element model

We place nodes 1, 2, 3 at $x_1 = 0, x_2 = 2, x_3 = 4$. On substituting the values and assembling,

$$\mathbf{K} = \frac{(30\times10^6)(2)}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{F} = 10^3\begin{Bmatrix} 0.667 \\ 2+7.333 \\ 11.333 \end{Bmatrix} = \begin{Bmatrix} 667 \\ 9333 \\ 11333 \end{Bmatrix}$$

Using the elimination approach with $Q_1 = 0$, we get

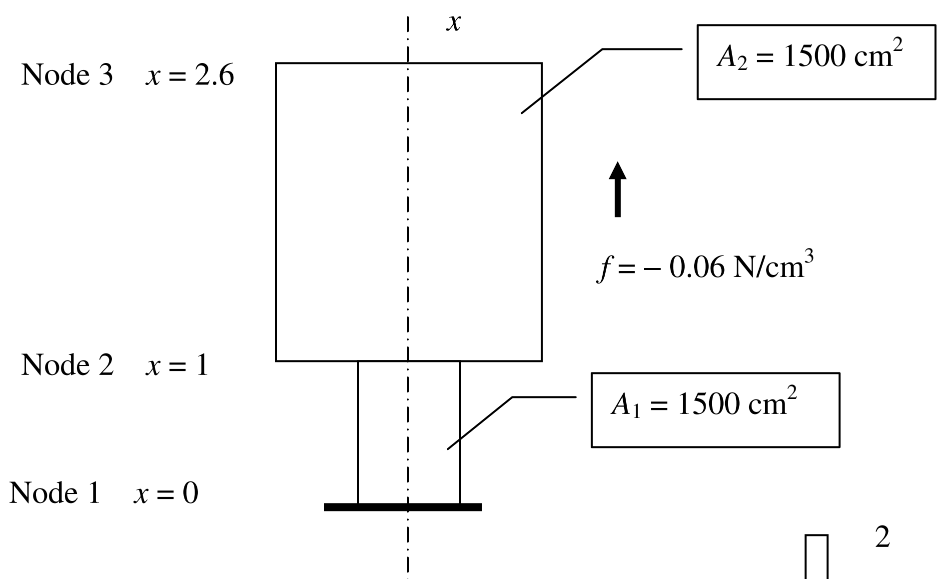
$$30\times10^6\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 9333 \\ 11333 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 6.889\times10^{-4} \\ 1.067\times10^{-3} \end{Bmatrix}$$

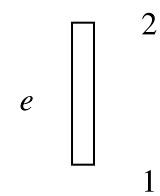
Stresses in elements are $\sigma_1 = (E/l_1)Q_2 = 10333$ psi, $\sigma_2 = (E/l_2)(Q_3 - Q_2) = 5667$ psi.

In this problem, the stress is continuous in the physical problem. The technique presented in the previous problem can be employed to get piecewise linear stress. ■

3.22



For an element e,



$$\mathbf{f}^e = A_e l_e f\begin{Bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{Bmatrix}$$

The weight of the element is distributed equally to the two nodes of the elements when linear shape functions are used.

Consistent units are to be used in calculations. Here the consistent units are f N/cm^3, A cm^2, L cm, E N/cm^2.

Following program modifications are suggested.

Number of material properties NPR is to be increased by 1 in the input data.

We need to read in one more material property weight per unit volume for each element. Add in one more column to the material properties in the input data file. Thus if the current two material properties are E and the coefficient of linear expansion, the third property is N/cm^3. PM(EL#, 3) is this property.

In the element loop where stiffness is calculated, we need to the following modifications (shown in **bold**)

```
FOR N = 1 TO NE
    …
    RHO = PM(N,3)
    …
    WT = AREA(I)*EL*RHO
    …
    F(N1) = F(N1) − TL + 0.5*WT
    F(N2) = F(N1) + TL + 0.5*WT
    …
NEXT N
```

After making these modifications, run the program to obtain results. ■

3.23 In this problem involving uniformly varying cross section, an automatic generation scheme is suggested for area and weight calculation of elements. n elements each having a constant cross section are shown in the figure.

Using the geometry of the body, if the element cross section is taken as that at its mid-point alon x-axis,

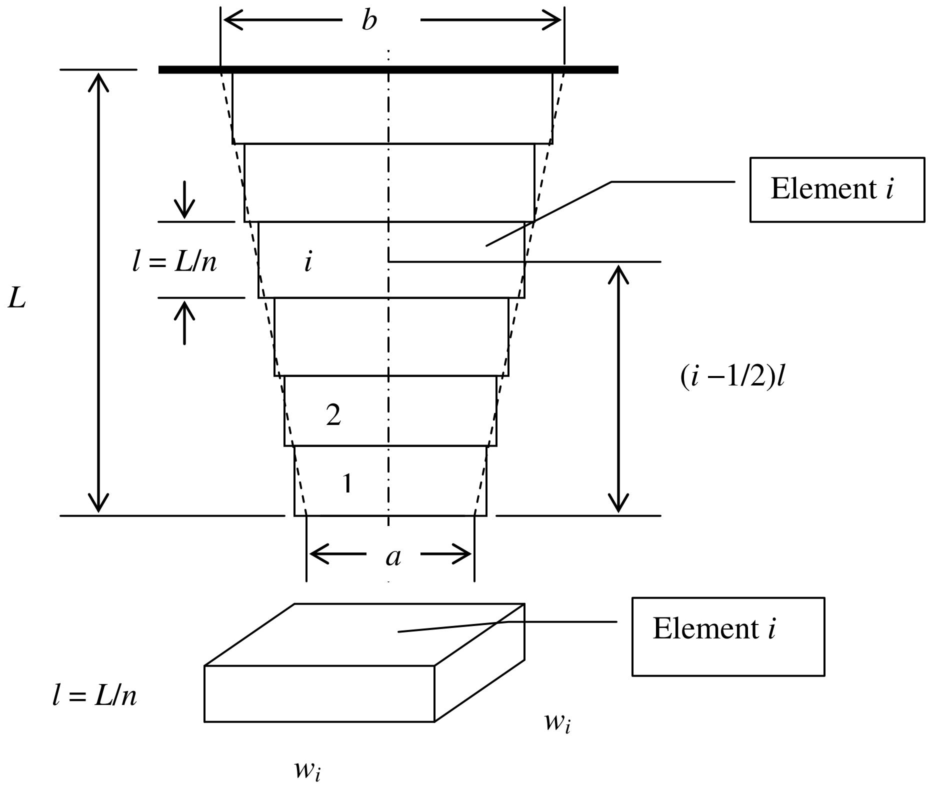
$$w_i = \left[\left(i - \frac{1}{2}\right) l\right] \frac{b-a}{L} + a$$

This relationship needs to be introduced into the program. Once a, b, f, L, and the number of divisions n are read in, the nodal coordinates are generated using

$$x_j = (j-1)\frac{L}{n} \qquad j = 1,2,\ldots,n$$

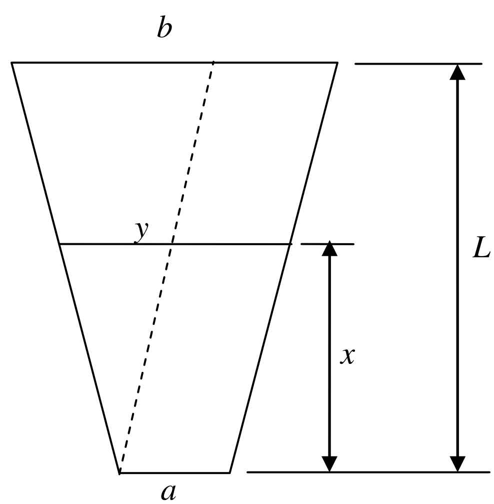
Element area

$$A_i = w_i^{\,2} \qquad i = 1,2,\ldots,n$$



These, together with the changes suggested in the previous problem 3.22 will pave the way for solving problems of this type.

The exact expression for the stress may be obtained as follows.



From the geometry shown,

$$y = a + cx$$

$$c = \frac{b-a}{L}$$

Area $A = y^2$. Weight of the bottom part is given by

$$W = \int_0^x fA dx = f \int_0^x y^2 dx = f \int_0^x (a+cx)^2 \, dx$$

$$W = \frac{f(a+cx)^3}{3c} - \frac{fa^3}{3c}$$

The stress at x is obtained by dividing by the area

$$\sigma_x = \frac{f(a+cx)}{3c} - \frac{fa^3}{3c(a+cx)^2}$$

This expression may be used in comparing the approximate solution with the exact. An expression for u can be obtained by integrating $\frac{du}{dx} = \frac{\sigma_x}{E}$. ■

3.24 We have $u = N_1 q_1 + N_2 q_2$. For element 1-2, $q_1 = 0$, $q_1 = 0.5$. At mid-point, $\xi = 0$.

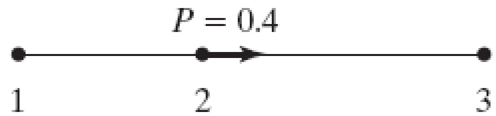
$$u = \frac{(1-\xi)^2}{4} q_1 + \frac{(1+\xi)^2}{4} q_2$$

$$= \frac{(1-0)^2}{4}(0) + \frac{(1+0)^2}{4}(0.5)$$

$$= 0.125$$

■

3.25 In the problem $x_1 = 0$, $x_2 = 1$, $x_3 = 3$, giving $\ell_1 = 1$, $\ell_2 = 2$, and we are given $Q_1 = 0$, $Q_3 = 0.2$.



The element stiffness matrices are

$$\mathbf{k}^{(1)} = \frac{EA}{\ell_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{k}^{(2)} = \frac{EA}{\ell_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

On assembling the matrices, the unmodified set of equations are

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 0 \\ Q_2 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4 \\ 0 \end{bmatrix}$$

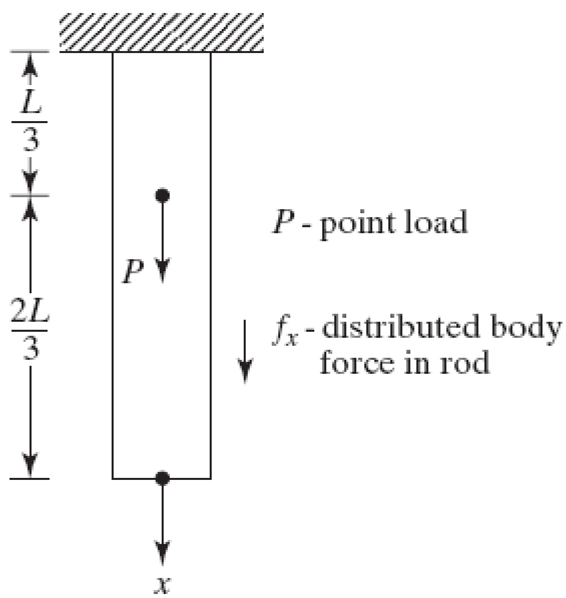
Here the first row and first column are struck off since $Q_1 = 0$. The third row is struck off since Q_3 is specified. But before striking off the third column, the third column times Q_3 (= 0.2) is subtracted from the right hand side as given Eq. 3.74. Thus the equation to solve is

$$1.5Q_2 = 0.4 - (-0.5)(0.2) = 0.5$$

$$Q_2 = 0.333$$

■

3.26 $E = 1$, $A = 1$, $L = 3$, $f_x = 1$, $P = 1$ using consistent N, m units as given.



(a) We first use $u = a_0 + a_1 x$. Setting $u(0) = 0$, $u = a_1 x$. The strain is $\frac{du}{dx} = a_1$ throughout the bar. The displacement at $u(1) = a_1$. The potential energy is

$$\pi = \frac{1}{2}\int_0^3 EA\left(\frac{du}{dx}\right)^2 dx - Pu(1) - \int_0^3 uf_x A dx$$

$$= \frac{1}{2}\int_0^3 a_1^2 dx - a_1 - \int_0^3 a_1 x dx$$

$$= \frac{3}{2}a_1^2 - \frac{11}{2}a_1$$

On differentiating with respect to a_1, we get

$$a_1 = \frac{11}{6} = 1.833$$

Thus $u = 1.833x$, and $u(1) = 1.833$, and $u(3) = 5.5$.

(b) For two element model, the element stiffness matrices and body force loads are

$$\mathbf{k}^{(1)} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad f_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

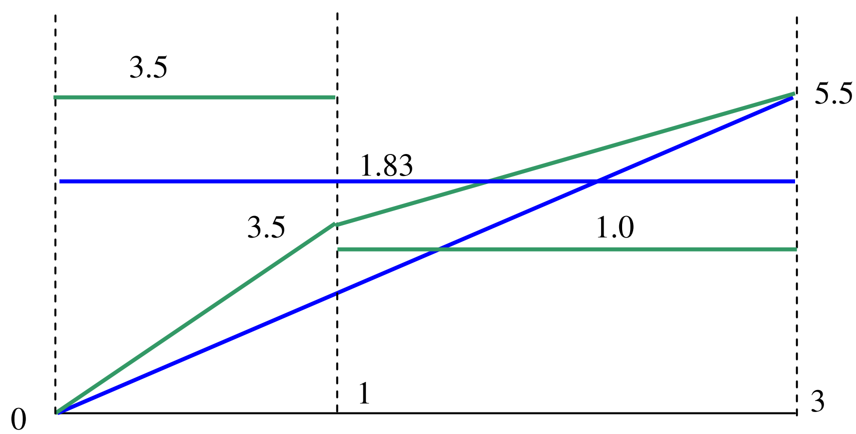
$$\mathbf{k}^{(2)} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \quad f_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We used the weights or elements of 1 and 2. The unmodified assembled equations are

$$\begin{bmatrix} \cancel{1} & \cancel{-1} & \cancel{0} \\ \cancel{-1} & 1.5 & -0.5 \\ \cancel{0} & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} \cancel{Q_1} \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} \cancel{0.5} \\ 1 + 0.5 + 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cancel{0.5} \\ 2.5 \\ 1 \end{bmatrix}$$

On solving, we get $Q_2 = 3.5$, $Q_3 = 5.5$.

(c) The plotted results are shown below



The stress in (a) is a constant value of 1.833 N/m^2. In part (b) the stresses in element 1 and element 2 are 3.5 N/m^2, and 1 N/m^2. ■

3.27 **(a)** Shape functions used $N_1 = \dfrac{(1-\xi)^2}{4} \quad N_2 = \dfrac{(1+\xi)^2}{4}$. They satisfy the requirements $N_1 =$ 1 at $\xi = -1$, and 0 at $\xi = 1$, and $N_2 = 0$ at $\xi = -1$, and 1 at $\xi = 1$. Now we use

$$u = N_1 q_1 + N_2 q_2 = \frac{(1-\xi)^2}{4} q_1 + \frac{(1+\xi)^2}{4} q_2$$

$$x = \frac{1-\xi}{2} x_1 + \frac{1+\xi}{2} x_2$$

If we use same shape functions for x, note that midpoint does not correspond to $\xi = 0$. Using these,

$$\varepsilon_x = \frac{du}{dx} = \frac{\dfrac{du}{dx}}{\dfrac{dx}{d\xi}} = \frac{-\dfrac{(1-\xi)}{2} q_1 + \dfrac{(1+\xi)}{2} q_2}{\dfrac{x_2 - x_1}{2}} = \frac{-(1-\xi) q_1 + (1+\xi) q_2}{x_2 - x_1}$$

This may be put in the form

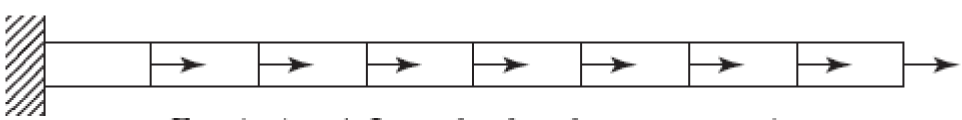
$$\boldsymbol{\varepsilon} = \mathbf{Bq}$$

$$\mathbf{B} = \frac{1}{x_2 - x_2}\left[-(1-\xi) \quad (1+\xi)\right]$$

(b) $\varepsilon_x = \dfrac{-(1-\xi) + (1+\xi)}{x_2 - x_1} = \dfrac{2\xi}{x_2 - x_1}$

This gives zero strain at $\xi = 0$ but equal and opposite non-zero strains at at $\xi = -1$ and $\xi =$ 1. For the displacements specified (rigid body translation) we need to have zero strain throughout. ■

3.28



$E = 1, A = 1$, Length of each segment $= 1$
Total length $= 8$

We first use $u = a_0 + a_1 x$. Setting $u(0) = 0$, $u = a_1 x$. The strain is $\dfrac{du}{dx} = a_1$ throughout the bar. The displacements are at $u(1) = a_1$, $u(2) = 2a_1, \ldots, u(k) = ka_1$, $P = 1$. The potential energy is

$$\pi = \frac{1}{2}\int_0^8 EA\left(\frac{du}{dx}\right)^2 dx - \sum_{k=1}^{8} Pka_1$$
$$= \frac{1}{2}\int_0^8 a_1^2 dx - 36a_1$$
$$= 4a_1^2 - 36a_1$$

On differentiating with respect to a_1, we get

$$a_1 = \frac{9}{2} = 4.5$$

Thus $u = 4.5x$, and the end displacement $u(8) = 36$. Strain is 4.5 throughout.

Fea Solution

PROGRAM FEM1D << BAR ANALYSIS >>

PROBLEM 3.28

NN	**NE**	**NM**	**NDIM**	**NEN**	**NDN**
9	8	1	1	2	1

ND	**NL**	**NMPC**
1	8	0

NODE#	**X-COORD**
1	0
2	1
3	2
4	3
5	4
6	5
7	6
8	7
9	8

EL#	**N1**	**N2**	**MAT#**	**AREA**	**ΔT**
1	1	2	1	1	0
2	2	3	1	1	
3	3	4	1	1	
4	4	5	1	1	
5	5	6	1	1	
6	6	7	1	1	
7	7	8	1	1	
8	8	9	1	1	

DOF#	**DISP**
1	0

DOF#	**LOAD**
2	1
3	1
4	1
5	1
6	1
7	1

8	1			
9	1			
MAT#	**E**	**Alpha**		
1	1	0		
B1	**I**	**B2**	**J**	**B3**

Results

Results from Program FEM1D
PROBLEM 3.28

Node#	Displacement
1	0.0004
2	8.0004
3	15.0004
4	21.0004
5	26.0004
6	30.0004
7	33.0004
8	35.0004
9	36.0004

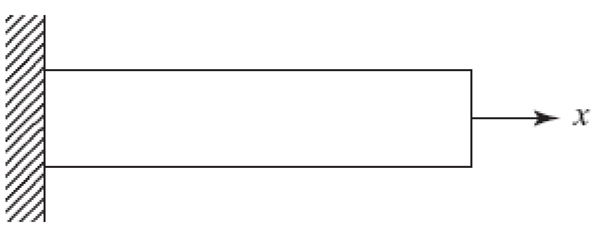
Element #	Stress
1	8
2	7
3	6
4	5
5	4
6	3
7	2
8	1

Node#	Reaction
1	-8

Plot the values and see the difference

■

3.29



$E = 100, A = 1, L = 1$

The body force $f = x$.

$u = a_0 + a_1 x + a_2 x^2$

Since $u(0) = 0$, $a_0 = 0$, $u = a_1 x + a_2 x^2$, $\dfrac{du}{dx} = a_1 + 2a_2 x$.

The potential energy

$$\pi = \frac{1}{2}\int_0^1 EA\left(\frac{du}{dx}\right)^2 dx - \int_0^1 ufAdx$$

$$= \frac{1}{2}\int_0^1 100\left(a_1 + 2a_2 x\right)^2 dx - \int_0^1 \left(a_1 x + a_2 x^2\right) x dx$$

$$= \frac{1}{2}\int_0^1 100\left(a_1^2 + 4a_1 a_2 x + 4a_2^2 x^2\right) dx - \int_0^1 \left(a_1 x^2 + a_2 x^3\right) dx$$

$$= 50\left(a_1^2 + 2a_1 a_2 + \frac{4}{3}a_2^2\right) - \frac{a_1}{3} - \frac{a_2}{4}$$

For stationary value,

$$\begin{bmatrix} 100 & 100 \\ 100 & \dfrac{400}{3} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.3333 \\ 0.25 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.005832 \\ -0.0025 \end{bmatrix}$$

$$u = 0.005832x - 0.0025x^2, \quad u(1) = 0.00333$$

$$\varepsilon_x = \frac{du}{dx} = 0.005832 - 0.005x$$

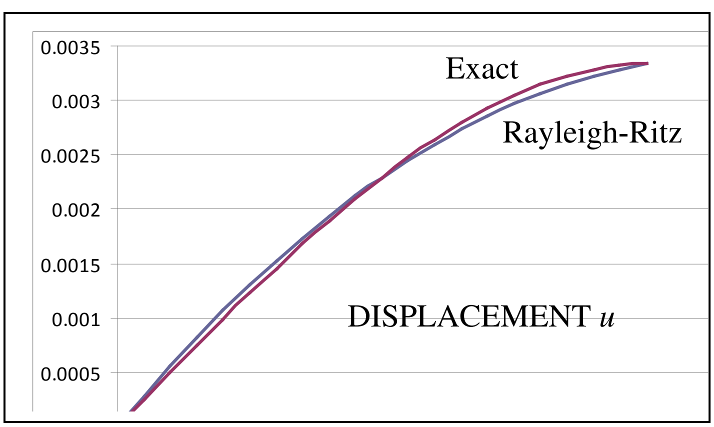
$$\sigma_x = E\varepsilon_x = 0.5832 - 0.5x$$

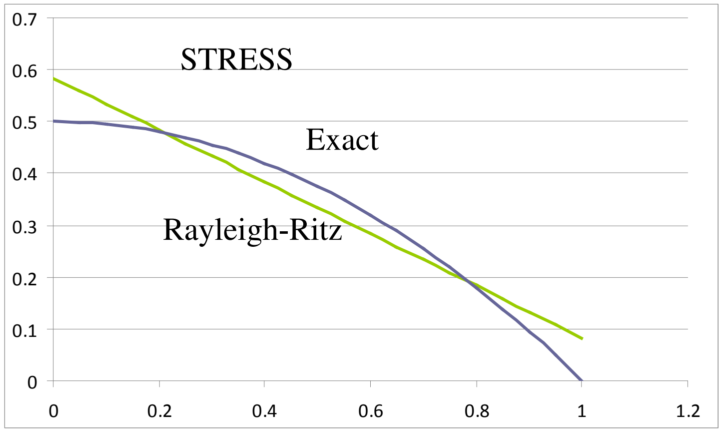
The exact solution from equilibrium consideration is

$$\sigma_x = 0.5\left(1 - x^2\right)$$

$$\varepsilon_x = \frac{\sigma_x}{E} = 0.005\left(1 - x^2\right)$$

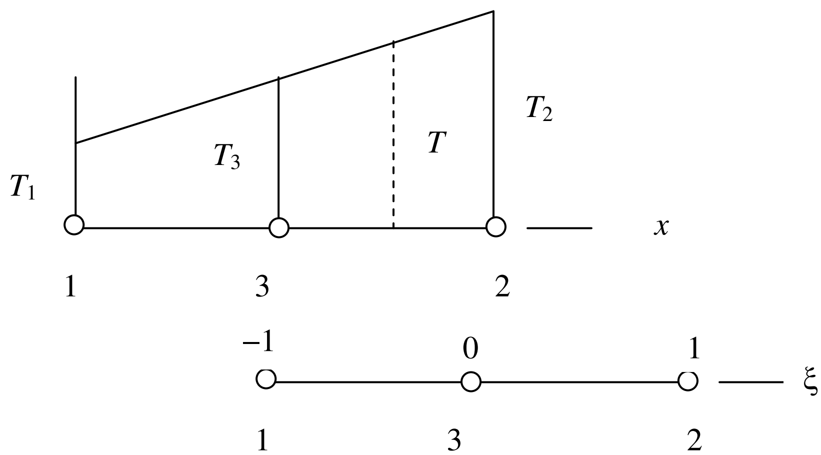
$$u = 0.005\left(x - \frac{x^3}{3}\right)$$





■

3.30



(a)

$$T = N_1T_1 + N_2T_2 + N_3T_3 = \mathbf{NT}$$

where

$$N_1 = -\frac{1}{2}\xi(1-\xi)$$

$$N_2 = -\frac{1}{2}\xi(1+\xi)$$

$$N_3 = (1+\xi)(1-\xi)$$

(b)

$$u = \mathbf{Nq}$$
$$T = \mathbf{NT}$$

$$\int_e uTdx = \mathbf{q}^T \int_{-1}^{1} \mathbf{N}^T \mathbf{N} \frac{l_e}{2} d\xi \mathbf{T}$$

On carryin out the product $\mathbf{N}^T\mathbf{N}$ and integrating each term over the limits, we get

$$\int_e uTdx = \mathbf{q}^T \frac{l_e}{2} \begin{bmatrix} \frac{4}{15} & -\frac{1}{15} & \frac{2}{15} \\ -\frac{1}{15} & \frac{4}{15} & \frac{2}{15} \\ \frac{2}{15} & \frac{2}{15} & \frac{16}{15} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$$

Denoting $\int_e uTdx = \mathbf{q}^T \mathbf{T}^e$,

$$\mathbf{T}^e = \frac{l_e}{30} \begin{Bmatrix} 4T_1 - T_2 + 2T_3 \\ -T_1 + 4T_2 + 2T_3 \\ 2T_1 + 2T_2 + 16T_3 \end{Bmatrix}$$

■

3.31

The data file for running the program FEM1D is as follows:

```
PROGRAM FEM1D  << BAR ANALYSIS
PROBLEM 3.23
NN NE     NM     NDIM   NEN    NDN
4  3      3      1      2      1
ND NL     NMPC
2  2      0
NODE#     X-COORD
1  0
2  800
3  1400
4  1800
EL#N1     N2     MAT#   AREA   TEMP RISE
1  1      2      1      2400   80
2  2      3      2      1200   80
3  3      4      3      600    80
DOF#      DISP
1  0
4  0
DOF#      LOAD
2  -60000
3  -75000
MAT#      E      Alpha
1  83000  1.89E-05
2  70000  2.30E-05
3  200000 1.17E-05
```

B1 I B2 J B3

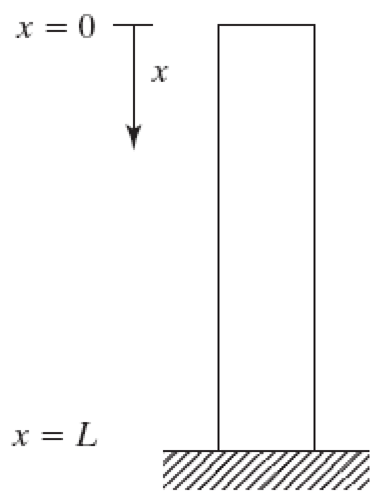
Results from Program FEM1D
PROBLEM 3.23

Node#	Displacement
1	-5.5931E-05
2	0.22120552
3	-0.004053755
4	2.52492E-05

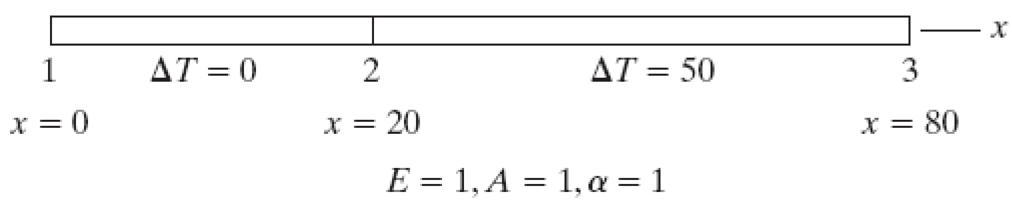
Element#	Stress
1	-102.5401244
2	-155.0802488
3	-185.1604977

Node#	Reaction
1	246096.2986
4	-111096.2986 ■

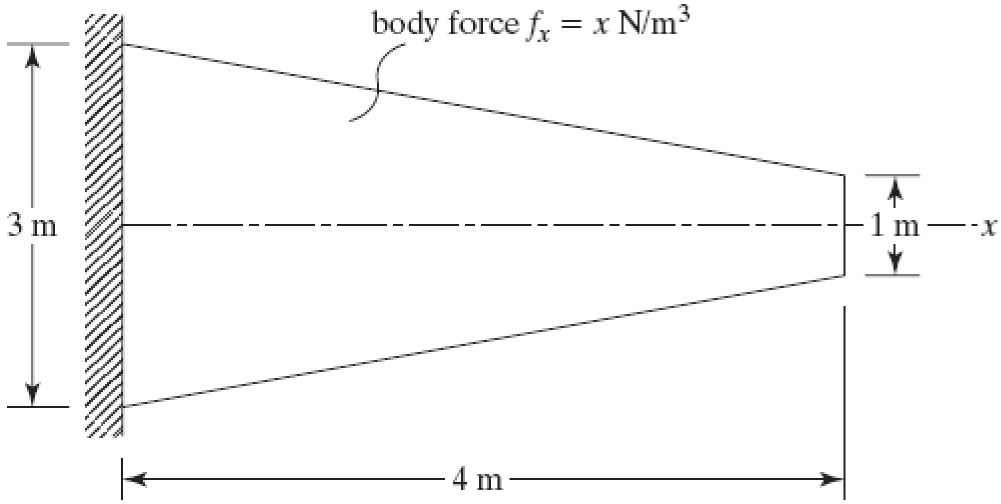
3.32



3.33



3.34



Thickness $t = 1$ m, $E = 100$ Pa

The exact solution for this problem is

$$\sigma_x = \frac{\frac{40}{3} - \frac{3x^2}{2} + \frac{x^3}{6}}{3 - \frac{x}{2}}$$

The denominator is the area at x and the numerator is the weight of the object to the right of x.
Solve the problem as discussed in problem 3.23 and plot the stress values to check the approximation with the exact solution. The exact expression for u can be obtained by integrating $\frac{du}{dx} = \frac{\sigma_x}{E}$. ■

3.35 Add "Dim StrainEnergy" in the opening part of the program. Then in element loop of the stress calculation part of the program shown in the problem, add the line

StrainEnergy = StrainEnergy + EPS * Stress(N) * AREA(N) * Abs(X(N2) – X(N1))

This calculates the strain energy in the structure. This can be printed out in the OUTPUT.

■

3.36 Using $u = \mathbf{N}\mathbf{q}$, we get

$$\int_e u^2 dx = \mathbf{q}^{\mathrm{T}} \left(\int_e \mathbf{N}^{\mathrm{T}} \mathbf{N} dx \right) \mathbf{q}$$

$$= \mathbf{q}^{\mathrm{T}} \left(\int_{-1}^{1} \begin{bmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{bmatrix} \frac{\ell_e}{2} d\xi \right) \mathbf{q}$$

$$= \mathbf{q}^{\mathrm{T}} \frac{\ell_e}{2} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \mathbf{q} = \mathbf{q}^{\mathrm{T}} \mathbf{W} \mathbf{q}$$

$$\mathbf{W} = \frac{\ell_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$ ■

CHAPTER 4

TRUSSES

4.1 (a) $$l = \frac{x_2 - x_1}{l_e} = \frac{50-10}{50} = 0.8$$

$$m = \frac{y_2 - y_1}{l_e} = \frac{40-10}{50} = 0.6$$

$$\mathbf{q}' = \mathbf{L}\ \mathbf{q}$$

$$\begin{Bmatrix} q_1' \\ q_2' \end{Bmatrix} = \begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ 0 & 0 & 0.8 & 0.6 \end{bmatrix} \begin{Bmatrix} 1.5 \\ 1.0 \\ 2.1 \\ 4.3 \end{Bmatrix} \times 10^{-2} = \begin{Bmatrix} 1.80 \\ 4.26 \end{Bmatrix} \times 10^{-2}\ in$$

(b) $$\sigma = \frac{E}{l_e}\begin{bmatrix} -1 & 1 \end{bmatrix}\mathbf{q}'$$

$$= \frac{30\times 10^6}{50}\left[4.26-1.8\right]\times 10^{-2} = 14{,}760\,psi$$

(c) From Eq. (4.13),

$$\mathbf{k} = \frac{30x10^6 \times 2.1}{50}\begin{bmatrix} .64 & .48 & -.64 & -.48 \\ & .36 & -.48 & -.36 \\ & & .64 & .48 \\ Sym. & & & .36 \end{bmatrix}$$

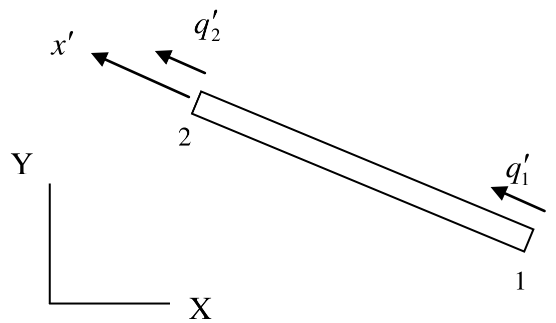
(d)

$$U^e = \frac{1}{2}k\delta^2 = \frac{1}{2}\times\frac{30\times 10^6\times 2.1}{50}(4.26-1.8)^2\times 10^{-4}$$
$$= 381.3\ in-lb$$

■

4.2 $\ell = \dfrac{(2-5)}{\sqrt{(2-5)^2+(14-8)^2}} = \dfrac{-3}{6.708} = -0.447$

$m = (14-8)/6.708 = 0.894$



$$q' = \begin{bmatrix} -0.447 & 0.894 & 0 & 0 \\ 0 & 0 & -0.447 & 0.894 \end{bmatrix} \begin{bmatrix} 0. \\ 0.01 \\ -0.025 \\ -0.05 \end{bmatrix} = \begin{bmatrix} 0.00894 \\ -0.033525 \end{bmatrix} \quad \blacksquare$$

4.3

Element	**Node 1**	**Node 2**
1	3	1
2	2	1

$$\mathbf{k}^{(1)} = \frac{200 \times 10^3 \times 1250}{901.39} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 1 & 2 \end{array} \\ \left[\begin{array}{cc|cc} & & & \\ \hline & & .692 & .462 \\ & & .462 & .308 \end{array}\right] \end{array}$$

(since $\ell = 0.832$, $m = 0.555$)

$$\mathbf{k}^{(2)} = \frac{200 \times 10^3 \times 1000}{750} \begin{bmatrix} & & & \\ & & 1 & 0 \\ & & 0 & 0 \\ & & & \end{bmatrix}$$

Thus

$$\mathbf{k} = 200 \times 10^3 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 2.293 & 0.641 \\ 0.641 & 0.427 \end{bmatrix} \end{array}$$

$$\equiv \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \qquad \blacksquare$$

4.4

$$E = 30 \times 10^6 \; psi$$
$$A = 1.5 \; in^2$$

Element	**Node 1**	**Node 2**
1	1	2
2	2	3
3	4	2
4	4	3

$$\mathbf{k}^{(1)} = \frac{30 \times 10^6 \times 1.5}{30} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\mathbf{k}^{(2)} = \frac{30\times10^6\times1.5}{40}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\mathbf{k}^{(3)} = \frac{30\times10^6\times1.5}{50}\begin{array}{c} \begin{array}{cccc} 7 & 8 & 3 & 4 \end{array} \\ \begin{bmatrix} .36 & -.48 & -.36 & .48 \\ -.48 & .64 & .48 & -.64 \\ -.36 & .48 & .36 & -.48 \\ .48 & -.64 & -.48 & .64 \end{bmatrix} \end{array}$$

$$\mathbf{k}^{(4)} = \frac{30\times10^6\times1.5}{30}\begin{array}{c} \begin{array}{cccc} 7 & 8 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(b)

$$\mathbf{K} = \frac{30\times10^6\times1.5}{600}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 20 & 0 & -20 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 24.32 & -5.76 & 0 & 0 & -4.32 & 5.76 \\ & & & 22.68 & 0 & -15 & 0 & -7.68 \\ & & & & 20 & 0 & -20 & 0 \\ & & & & & 15 & 0 & 0 \\ & \mathit{Sym} & & & & & 24.32 & -5.76 \\ & & & & & & & 7.68 \end{bmatrix} \end{array}$$

(c) Eliminating dof's 1,2,4,7,8,

K Q = F is

$$\begin{array}{cccc} & 3 & 5 & 6 \end{array}$$

$$\frac{30\times10^6\times1.5}{600}\begin{bmatrix}24.32 & 0 & 0\\ 0 & 20 & 0\\ 0 & 0 & 15\end{bmatrix}\begin{Bmatrix}Q_3\\ Q_5\\ Q_6\end{Bmatrix}=\begin{Bmatrix}4000\\ 0\\ 0\end{Bmatrix}$$

The solution is

$$Q_3 = 219.3\times10^{-5} in$$
$$Q_5 = 0$$
$$Q_6 = 0$$

(d)

$$\sigma = \frac{E}{l_e}\begin{bmatrix}-\ell & -m & \ell & m\end{bmatrix}\mathbf{q}$$

$$\sigma_2 = \frac{30\times10^6}{40}\;[0 \quad -1 \quad 0 \quad 1]\begin{Bmatrix}219.3\times10^{-5}\\ 0\\ 0\\ 0\end{Bmatrix} = 0$$

(Check with free body of Node 3)

$$\sigma_3 = \frac{30\times10^6}{50}\;[-.6 \quad .8 \quad +.6 \quad -.8]\begin{Bmatrix}0\\ 0\\ 219.3\times10^{-5}\\ 0\end{Bmatrix} = 789.5\,psi$$

e) Reaction of Node 2 in y- direction is

$$R_4 = \sum_{j=1}^{8} K_{4j}\; Q_j$$

$$= \frac{30\times10^6\times1.5}{600}\;[-5.76\times219.3\times10^{-5}\,] = -\,947.4lb \quad (downward\;\;pull)$$

■

4.5 Due to $Q_4 \equiv a_4 = -0.24"$, the load vector by the elimination approach is

$$\mathbf{F} = [\quad -K_{43}a_4, \quad -K_{45}a_4 \quad -K_{46}a_4 \quad]^{\mathrm{T}} \quad \text{or}$$

$$\mathbf{F} = \frac{30\times10^6\times1.5}{600}\begin{Bmatrix} -1.3824 \\ 0 \\ -3.6 \end{Bmatrix}\begin{array}{c} 3 \\ 4 \\ \underset{\substack{\uparrow \\ \text{dof}}}{5} \end{array}$$

K is as given in Problem 4.4 above

Solution of **K Q = F** yields

$Q_3 = 0.056842", Q_5 = 0,$
$Q_6 = -0.24",$
$Q_4 = -.0.24$

We have $\sigma = \dfrac{E_e}{l_e}[\quad -l \quad -m \quad l \quad m \quad]\mathbf{q}$

$$\sigma_2 = \frac{30\times10^6}{40}\;[0\;\; -1 \quad 0 \quad 1]\begin{Bmatrix} -0.056842 \\ -0.24 \\ 0 \\ -0.24 \end{Bmatrix} = 0$$

Also $\sigma_3 = 94{,}737\,psi$

■

4.6

$E = 70{,}000 \quad Mpa$

$A = 200 mm^2$

$L_1 = L_2 = 500 mm$

Note: 1 N/mm^2 = 1 Mpa

Element connectivity Direction Cosines:

		l	m
1	$2-1$	1	0
2	$1-3$	0.8	-0.6

Element stiffness (Eq. 4.13)

$$\mathbf{k}^1 = C \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \left[\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & 1 & 0 \\ & & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array}$$

$$\mathbf{k}^2 = C \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cc|cc} .64 & -.48 & & \\ -.48 & -.36 & & \\ \hline & & & \\ & & & \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$C = \frac{EA}{L} = \frac{70000 \times 200}{500}$$

Assembly of $\mathbf{K}$:

$$\mathbf{K} = C \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1.64 & -.48 \\ -.48 & .36 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

Elimination Approach :

$$C \begin{bmatrix} 1.64 & -.48 \\ -.48 & .36 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -12000 \end{bmatrix}$$

Solving,

$Q_1 = -0.5714 in$

$Q_2 = -1.9524 in$

Stress in Element 2 :

Eq. (4.16):

$$\sigma_2 = \frac{70000}{500} \begin{bmatrix} -.8 & .6 & .8 & -.6 \end{bmatrix} \begin{bmatrix} -.5714 \\ -.9524 \\ 0 \\ 0 \end{bmatrix} = -100 MPa \quad \text{(Compression)}$$

Verify the above using program TRUSS2D. ■

4.7 Solution Using Truss Program -- Input File

Next line is problem title
Problem 4.7 – Truss

NN	NE	NM	NDIM	NEN	NDN
4	3	1	2	2	2
ND	NL	NCH	NPR	NMPC	
6	1	2	2	0	
Node #	X	Y			
1	0	0			
2	-450	600			
3	800	600			
4	450	600			
Elem#	N1	N2	Mat#	Area	TempRise
1	2	1	1	250	0
2	1	3	1	250	0
3	1	4	1	250	0
DOF#	Displacement				
3	0				
4	0				
5	0				

6	0
7	0
8	0

DOF# Load
2 -18000
MAT# PROP1 PROP2
1 200e3 12E-6
Bl i B2j B3 (Multi-point constr.)

Output

Output for Input Data from File temp. inp

Problem 4.7

Node#	X-Displ	Y-Displ
1	5.6176E-02	-1.8725E-01
2	7.1030E-06	-9.4707E-06
3	-2.6093E-06	-1.9570E-06
4	-4.4937E-06	-5.9917E-06

Elem#	Stress
1	4.893E+01
2	1.348E+01
3	3.096E+01

DOF#	Reaction
3	-7.3398E+03
4	9.7864E+03
5	2.6963E+03
6	2.0222E+03
7	4.6435E+03
8	6.1914E+03

■

4.8

Bandwith = NBW

$2 \max_{e} |j-i+1|$. For the node numbering shown above,

$$NBW = 2 \ |12-1+1| = 24$$

Alternative node numbering:
Idea is to keep differences in node numbers small, best is:

NBW = 8 = minimum. ■

4.9 Program TRUSS is used

Units used are N, mm

$E = 200 \times 10^3$ N/mm^2 = 200×10^3 Mpa

(Connectivity)

Element	Node 1	Node 2
1	1	2
2	2	3
3	3	4
4	4	5
5	5	6
6	6	7
7	2	4
8	4	6
9	1	3
10	3	5
11	5	7

DOF#	Specified Displacement
1	0
2	0
14	0

DOF #	Applied Load
2	-280000
6	-210000
10	-280000
14	-360000

OUTPUT

Node #	X-Displ$^{(mm)}$	Y-displ$^{(mm)}$
1	0	-2.27E –04
2	3.1	-3.5
3	0.75	-6.6
4	1.6	-7.2
5	2.3	-7.0
6	-0.05	-3.7
7	3.13	-2.7E –04

Thus, point R moves, horizontally by 3.13mm.

(The small y-displacements of nodes 1 and 7, which are supposed to be zero, is due to the penalty approach of approximating constraints)

Element #	Stress (Mpa)
1	- 82.9 (Compression)
2	82.0
⋮	⋮
5	91.0
⋮	⋮
11	45.6

DOF#	Reaction Force (N)
1	0.0
2	513,330.0
14	616,667.0

■

4.10 Using symmetry, the model is

Data is prepared for program TRUSS as follows:

Element	*Node* 1	*Node* 2
1	1	3
2	1	2
3	2	3
4	2	4
5	3	4

Area#	*Area*
1	5.0
2	10.0
3	12.5

Element	*Area#*
1	3
2	2
3	3
4	2
5	1

– – Due to symmetry

E = 30 x 10^6 psi

DOF#	**Specified Displacement**
1	0
2	0
5	0 *Due to symmetry*
7	0 *Line*

DOF#	**Applied Load**	
8	$-15,000$	*Half the actual load due to symmetry*

OUTPUT

Node	**X – Displ.** (in)	**Y – Displ.** (in)
1	0.0	0.0
2	0.0050	-0.0057
3	0.0	-0.0229
4	0.0	-0.0409

Element #	**stress (psi)**
1	-1371.0
2	-472.0
3	629.0
4	-629.0
5	-3000.0

DOF#	Re*action force*
1	1371.0
2	15000.0
5	-7420.0
7	-6290.0

■

4.11

a) Data for program TRUSS is as follows:

Co-ordinate data, connectivity areas and material properties as given in the problem and Fig.4.7. Data is prepared as below:

NE = 10
NN = 6
ND = 4
NL = 0 ← No. of component loads
NM = 1
NA = 6 ← No. of area groups
NTEL = 4

Element #	Temperature Change
1	50
3	50
7	50
8	50

Area #	Area (in^2)
1	25
2	12
3	1
4	4
5	17
6	5

Elem#	Node 1	Node 2	Material #	Area #
1	5	3	1	1
2	3	1	1	2
3	6	4	1	1
4	4	2	1	2
5	4	3	1	3
6	2	1	1	4
7	5	4	1	5
8	6	3	1	5
9	3	2	1	5
10	4	1	1	5

MAT	E	ALPHA
1	30E6	6.667E-06

DOF#	SPECIFIED DISPLACEMENT
9	0
10	0
11	0
12	0

OUTPUT

Node	**X-Displ. (in)**	**Y-Displ (in)**
1	0.1178	- 0.0204
2	0.1178	0.0204
3	0.1314	0.0611
4	0.1314	-0.0611
5	0.0	0.0
6	0.0	0.0

Element	**Stress**
1	951.0
2	-1132.6
3	951.1
|	|
6	-3397.8
|	|
10	1130.7

b) Consider elements 9 and 10 which are 1/4" short and which are stretched to fit in place: when fit in place, the elements have an initial strain ε_o, < o, exists. We have

$$\varepsilon_o = -\frac{\Delta L}{L} = -\frac{\frac{1}{4}}{360\sqrt{2}} = -0.000491$$

Now, we may use the temperature option in program TRUSS by requiring

$$\alpha \Delta T = \varepsilon_0$$

For elements 9, 10 we have

$\alpha = 10^{-6}$/deg and $\Delta T = -491^{o}$.

Similarly, element 6 has an initial strain $\varepsilon_o > 0$ with

$$\varepsilon_0 = \tfrac{1/8}{360} = 0.000347$$

We may treat this as a temperature strain as $\alpha = 10^{-6}, \Delta T = 347$.

OUTPUT

Node	X-Displ.(in.)	Y-Displ. (in)
1	-0.0440	-0.0374
2	-0.0440	0.0374
3	0.0226	-0.1167
4	0.0226	0.1167

Element	Stress (psi)
1	1885
2	-5547
⋮	⋮
5	-19450
6	-16640
⋮	⋮
9	5537
10	5537

DOF #	Reaction Force
9	0
10	-47110
11	0
12	+47110

(c) The support movement is handled as below:

DOF#	Specified Displacement
9	0
10	0
11	0
12	- 0.12

OUTPUT

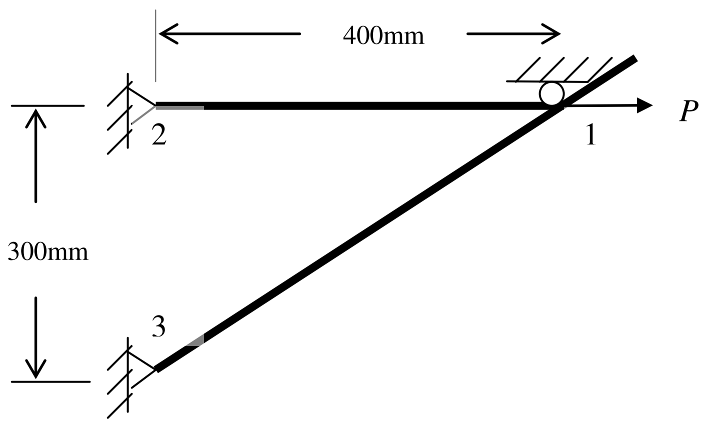
Node	X-Disp.	X-Disp.
1	0.0011	-0.0498
2	0.0011	-0.0702
3	-0.0057	-0.0906
4	-0.0057	-0.0294
5	0.0	0.0
6	0.0	-0.12

Element	Sstress (Psi)
1	-475
2	566
|	|
5	- 5093
6	1698
|	|
10	- 565

DOF #	**Reaction Force (lb)**
9	0
10	11888
11	0
12	–11965

Note that the reaction forces do not exactly sum to zero: there is a 0.6% error, which is acceptably small and is a consequence of using the penalty method for handling boundary conditions. ■

4.12



Member 3-1 has an initial strain of $\varepsilon_0 = +\ 5/500 = 0.01$. Thus, Program TRUSS2D can be used with $\alpha = 0.01$ and $\Delta T = 1.0$, for member 3-1 only. Results are given below.

(a) with only load *P* acting:

Horizontal displacement of node 1 = 0.014 mm
Stress in member 2-1 = +7.05 MPa
Stress in member 3-1 = +4.51 MPa

(b)with load *P* acting AND initial strain in member 3-1:

Horizontal displacement of node 1 = 2.13 mm
Stress in member 2-1 = +1065. MPa (tension)
Stress in member 3-1 = -1318. MPa (compression) ■

4.13 Stress (3-D)

Even in a 3-D truss element, we have σ = E_e ε, or

$$E = \frac{q_2^1 - q_1^1}{l_e}$$

since the element is a two-force member. Thus

$$\sigma = E_E\,[-1\ \ 1]\,\mathbf{q}'$$
$$= E_E\,[-1\ \ 1]\,\mathbf{L}\mathbf{q}'$$

where

$$\mathbf{L} = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix}$$

We have

$$\sigma = \frac{E_e}{l_e}\ \ [-l \quad -m \quad -n \quad l \quad m \quad n\ \ ]\mathbf{q}$$

with **q** a (6x1) vector

Temperature Effect (3-D)

$$\theta' = E_e A_e \alpha\,\Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

since

$\mathbf{q}'^{\mathbf{T}}\ \boldsymbol{\theta}' = \mathbf{q}^{\mathbf{T}}\ \boldsymbol{\theta}$, or $\mathbf{q}^{\mathbf{T}}\ \mathbf{L}^{T}\ \boldsymbol{\theta}' = \mathbf{q}^{\mathbf{T}}\ \boldsymbol{\theta}$

We get

$$\boldsymbol{\theta} = \mathbf{L}^{T}\ \boldsymbol{\theta}'$$

or

$$\theta' = E_e A_e \partial \Delta T \begin{Bmatrix} -\ell \\ -m \\ -n \\ \ell \\ m \\ n \end{Bmatrix}$$

■

4.14

Data preparation follows from the figure in P4.14:

OUTPUT

Bandwidth = 6

Node #	**X - Displ**	**Y – Displ**
1	$0.270in$	$-1.414in$
2	-0.191	-0.606
3	0.135	0.0
4	-0.011	0.101
5	0.0	0.0

Elem#	**Stress(psi)**
1	-18750
2	15020
⋮	⋮
5	-15020
⋮	⋮
7	-15020

DOF#	**Reaction Force (lb)**
6	300,000
9	0.0
10	- 150,000

■

4.15

The load at node 5 in Figure P4.15 is resolved into its X- and Y - components.

OUTPUT

Node#	**(in.)X – Displ**	**(in.)Y – Displ**
1	0	0
2	0.0111	0
3	0.0138	-0.0126
$\vdots$	$\vdots$	$\vdots$
8	-0.0088	-0.1288
$\vdots$	$\vdots$	$\vdots$
10	0.0342	-0.0367

Elem#	**Stress(psi)**
1	695
4	-1870
9	-2012
14	2083
17	-1339

■

4.16

PROJECT PROBLEM

The main modifications to program TRUSS to handle 3-D Trusses are:

(1) Input the z-coordinate of each node.
(2) Half-bandwidth in Eq.(4.37) is modified as

$$\text{NBW} = \max_e \quad m_e$$

$$\text{Where} \quad 3\left[\,\left| i-j \right| + 1\,\right]$$

(3) Direction cosines ℓ, m, n are computed as shown in Fig.4.5 Element stiffness is given in Eq.4.29. Each node I has 3 dof's, with the numbers Q_{3*I-2}, Q_{3*I-1}, Q_{3*I}, respectively.

Assembly in Eqs 4.32 – 4.34 generalise as

k is a (6 x 6) matrix

$\mathbf{K}^{\mathbf{e}}_{\alpha,\beta} \longrightarrow S_{p,,\ q-p+1,\ q\geq p}$

where **S** is the banded stiffness matrix

α, β take values of 1,2, …, 6

p, q take values of 3I-2, 3I-1, 3I, 3J-2, 3J-1, 3J

(4) Stress is calculated as discussed in the solution of Problem 4.13 above. ■

4.17 In the Do-Loop where stresses are printed out, we can insert the following:

For I = 1 to NE

\-------------------------

If Stress <0 Then

$$P_c = \frac{\pi^2 EI}{L^2}$$

$$FS = \frac{P_{cr}}{(Stress * Area)}$$

$\Pr\text{int} \quad "I, FS", I, SF$

Endif

Next I

$$\begin{aligned} Note: E &\equiv "PM(I3,1)" \\ Area &\equiv Area(I) \\ I &\equiv 8.4E5 \\ L &\equiv EL \end{aligned}$$

OUTPUT

Element	Factor of Safety, FS w.r.t. buckling	
4	.475	Not OK
5	.432	Not OK
6	.475	Not OK
8	4.748	OK
11	.432	Not OK

■

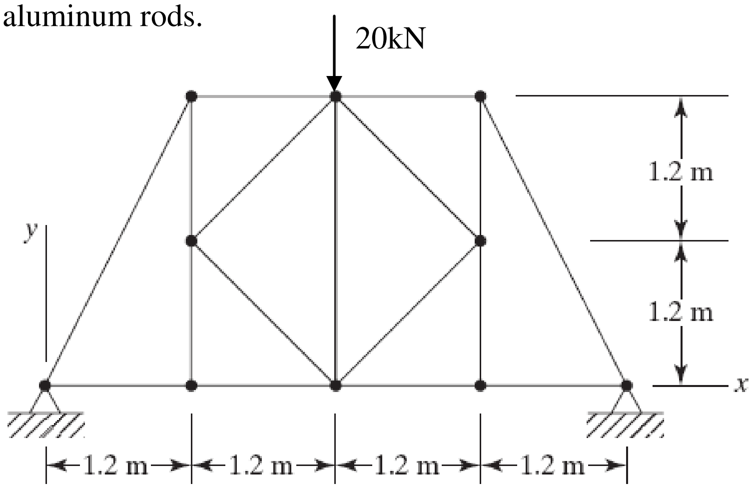
4.18 PROJECT PROBLEM

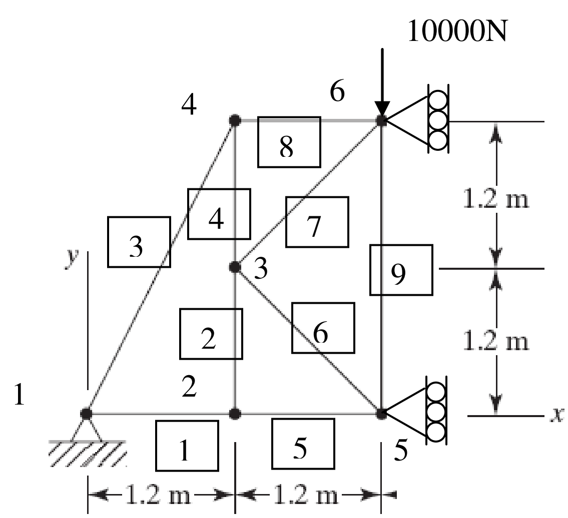
See comments for Problem 4.16 above, regarding the modifications to program TRUSS to handle 3-D Trusses. Here, results (displacements in metres) using ANSYS commercial code is given below:

NODE	UX	UY	UZ
1	.00000	.00000	.00000
2	.00000	.00000	.00000
3	.00000	.00000	.00000
4	-.18875E-03	-.11111E-03	-.55556E-04
5	-.19570E-03	-.42851E-03	-.21258E-19
6	.10206E-03	-.11111E-03	.55556E-04
7	-.44000E-03	-.38889E-03	-.83333E-04
8	-.44695E-03	-.96813E-03	-.13553E-19
9	.11265E-03	-.38889E-03	.83333E-04

Note that *Node* 8 has maximum y-deflection of –0.968mm , as is to be expected. Also, the structure twists as is again to be expected. ■

4.19 The symmetrically applied load of 20kN is missing in the figure. All members are aluminum rods.





All horizontal members 1,4,7 are 20mm dia, area of cross section $\frac{\pi(20)^2}{4} = 314.16\,\text{mm}^2$. All other members 2,3,4,6,7 are 25 mm square section with area of cross section 625 mm^2. The vertical member is on the line of symmetry so we use half the area of 312.5 mm^2. Due to symmetry, the applied load is 10000 N on the left half considered.

Input Data

```
<< 2D TRUSS ANALYSIS  USING BAND SOLVER>>
PROBLEM 4.19
NN NE    NM     NDIM  NEN   NDN
6  9     1      2     2     2
ND NL    NMPC
4  1     0
Node#    X      Y
1  0     0
2  1200  0
3  1200  1200
4  1200  2400
5  2400  0
6  2400  2400
Elem#    N1     N2    Mat#   Area  TempRise
1  1     2      1     314.16 0
2  2     3      1     625    0
3  1     4      1     625    0
4  3     4      1     625    0
5  2     5      1     314.6  0
6  3     5      1     625    0
7  3     6      1     625    0
8  4     6      1     314.6  0
9  5     6      1     312.5  0
DOF#     Displacement
1  0
2  0
9  0
11 0
DOF#     Load
12 -10000
MAT#     E      Alpha
1        70000  2.30E-05
B1 i     B2     j     B3     (Multi-point constr. B1*Qi+B2*Qj=B3)
```

Output

```
Results from Program TRUSS2D
PROBLEM 4.19
Node#    X-Displ
1  -5.066E-06    -1.013E-05
2  -5.066E-06    -1.1772
3  -0.2743       -1.1772
```

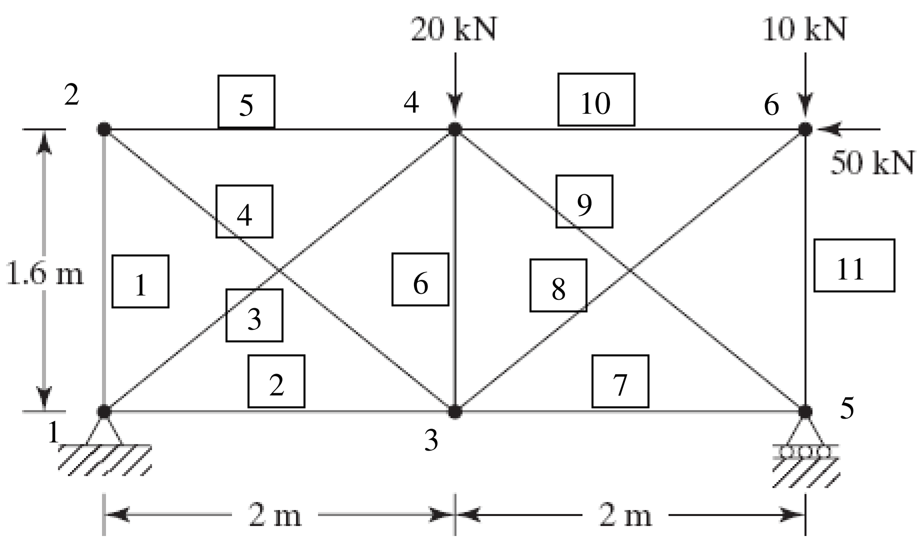
4	0.2725	-0.9029
5	-5.066E-06	-1.2908
6	1.0132E-05	-1.8394

Elem#	Stress
1	3.459E-19
2	0
3	-17.89
4	16
5	2.47E-19
6	11.314
7	-11.314
8	-15.893
9	-16

DOF#	Reaction
1	5000
2	10000
9	5000
11	-10000

■

4.20 Since the boundary condition is not symmetric, and since the number of members is not large, it is easier to solve the full problem. We assume that all members are of circular cross section.



The horizontal and vertical members have a cross sectional area of 200 mm^2, and other members are 90 mm^2. Assuming circular cross section, the diameters are 15.96 mm, and 10.705 mm respectively. The moments of inertia ($\frac{\pi d^4}{64}$) are 3183.1 mm^4 and 644.6 mm^4 respectively. The Euler critical load is $\frac{\pi^2 EI}{l^2}$.

Input Data
<< 2D TRUSS ANALYSIS USING BAND SOLVER>>
PROBLEM 4.20

NN	NE	NM	NDIM	NEN	NDN
6	11	1	2	2	2

ND	NL	NMPC
3	3	0

Node#	X	Y
1	0	0
2	0	1600
3	2000	0
4	2000	1600
5	4000	0
6	4000	1600

Elem#	N1	N2	Mat#	Area	TempRise
1	1	2	1	200	0
2	1	3	1	200	0
3	1	4	1	90	0
4	2	3	1	90	0
5	2	4	1	200	0
6	3	4	1	200	0
7	3	5	1	200	0
8	3	6	1	90	0
9	4	5	1	90	0
10	4	6	1	200	0
11	5	6	1	200	0

DOF#	Displacement
1	0
2	0
10	0

DOF#	Load
8	-20000
11	-50000
12	-10000

MAT#	E	Alpha
1	200000	1.20E-05

B1 i B2 j B3 (Multi-point constr. B1*Qi+B2*Qj=B3)

Output

Results from Program
TRUSS2D
PROBLEM 4.20

Node#	X-Displ	
1	-0.00010	-0.00006
2	-2.99365	-0.43528
3	-1.30512	-2.29184
4	-3.67367	-2.37955
5	-1.38716	0.00000

6	-5.63070	-0.05250

Elem#	Stress MPa	Force N	Mom_In mm^4	Length mm	Critical Load N
1	-54.40	-10880	3183.1	1600	2454
2	-130.50	-26100	3183.1	2000	1571
3	-340.07	-30606	644.6	2561	194
4	193.52				
5	-68.00	-13600	3183.1	2000	1571
6	-10.96	-2193	3183.1	1600	2454
7	-8.20	-1641	3183.1	2000	1571
8	-154.52	-13907	644.6	2561	194
9	23.35				
10	-195.70	-39141	3183.1	2000	1571
11	-6.56	-1313	3183.1	1600	2454

DOF#	Reactn
1	50000
2	30000
10	0

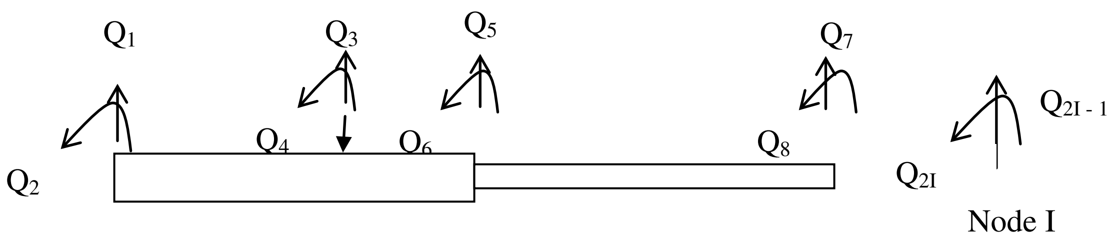
In the above table, columns of Force, Moment of Inertia, Length and Critical Load have been added. Calculations have been carried out in the spreadsheet in the program in VBExcel. The compression loads see to far exceed the critical load values. The cross section areas chosen are not enough. ■

CHAPTER 5

BEAMS AND FRAMES

5.1

$I_1 = 1.25 \times 10^5 \text{ mm}^4$, $I_2 = 4.0 \times 10^4 \text{ mm}^4$



NE = 3, NL = 1 $\rightarrow$ F_3 = -3000.

DOF (degree of freedom)

Boundary Conditions

$Q_1 = Q_7 = 0$ (ND = 2)

i.e.,

DOF #	Specified displacement
1	0.0
7	0.0

Node	X-Coordinate (mm)
1	0.
2	150.
3	225.
4	350.

Solution using program BEAM

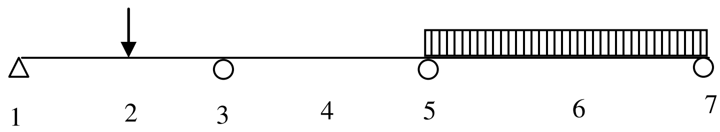
Deflection under load point, Q_3 = - 0.133350 mm (down)

Slope at left end, Q_2 = - 0.001146 rad (i.e., clockwise rotation)

Slope at right end Q_8 = + 0.0015 rad ( )

■

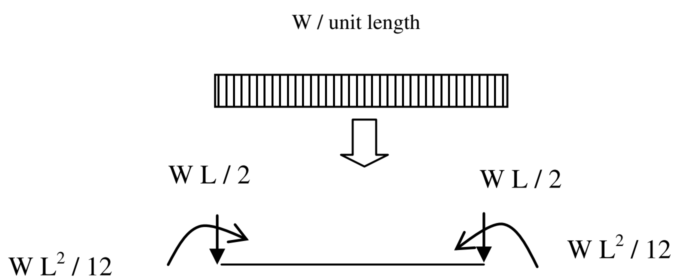
5.2

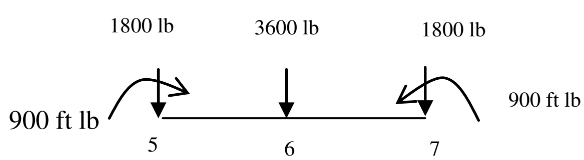


NE = 6

DOF	Specified displacement
1	0
5	0
9	0
13	0

The distributed load is handled as follows:





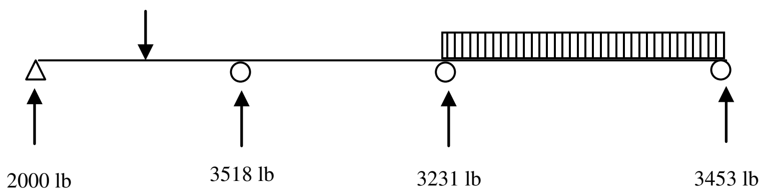
Thus

DOF	Load (inch, lb)
3	- 5000
9	- 1800
10	-10800
11	- 3600
13	- 1800
14	+10800

Solution Using Program BEAM

Node	Displ.(in)	Rotation (rad)
2	-13.76×10^{-3}	3.3×10^{-5}
4	4.45×10^{-3}	-2.2×10^{-5}
6	-3.45×10^{-3}	-0.3×10^{-5}

DOF	Reaction	
1	2,000	(lb)
5	3,518	
9	3,231	
13	3,453	

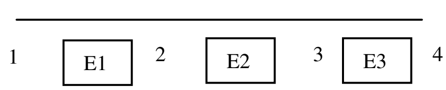


Free Body Diagram

Shear Force and Bending Moment Diagrams may now be drawn from the above results. ■

5.3

1/2 Symmetry Model:



Boundary Conditions: $Q_1 = 0$, $Q_2 = 0$ (fixed support)
& $Q_8 = 0$

$$I_1 = \frac{12.8^3}{12} = 512\ in^4$$

$$I_2 = \frac{12.6^3}{12} = 216\ in^4$$

$$I_3 = \frac{12.4^3}{12} = 64\ in^4$$

$$E = 4.5x10^6\ \ psi$$

$$NE = 3$$

Loading: We treat the self-weight as an uniform load:

$$\left(\omega = 145 \frac{lb}{ft^3} \times \frac{2}{3} ft^2 = 96.67 \, lb/ft \right)$$

Similar to Example 8.2, we obtain

DOF#	**Applied Load (in, lb)**
1	- 241.68
2	- 2416.80
3	- 422.93
4	604.32
5	- 241.65
6	1510.44
7	- 60.41
8	302.04

Note: Load applied at supports do not affect the displacements and stresses but do affect the reactions

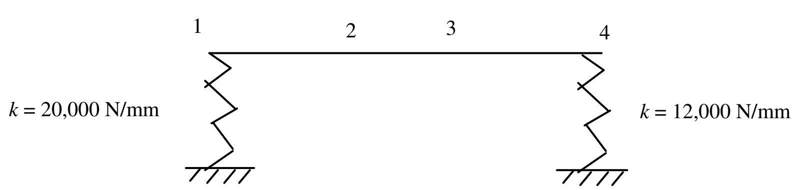
Solution (Using Program BEAM)

Node	**Displ.(in)**	**Rotation (rad)**
2	-2.96×10^{-2}	-8.0×10^{-4}
3	-8.41×10^{-2}	-8.3×10^{-4}
4	-9.70×10^{-2}	0.

DOF #	**Reaction**
1	966.7
2	54,865.
8	8,570.

■

5.4 The bearing stiffnesses shown have to be added to the (1,1) and (7,7) locations of **K**, respectively.



Or add these to the (1,1) and (7,1) locations of the banded stiffness **S** in the BEAM program. Specifically, in Program BEAM, after completion of element assembly and prior to considering boundary conditions, we insert:

S(1,1) = S(1,1) + 20000
S(7,1) = S(7,1) + 12000

Further, we have ND = 0, since there are no specified displacement.

Solution

Q_3= - 0.22825 mm
Q_2= - 0.0012 rad
Q_8= - 0.0014 rad
Q_7= - 0.107 mm ■

5.5

$$k_1 = \frac{EA}{L} = \frac{30\times10^6\times0.08}{12} = 200{,}000\ lb/in$$

$$k_2 = \frac{30\times10^6\times0.08}{20} = 120{,}000\ lb/in$$

$$I_{beam} = 0.64\ in^4$$

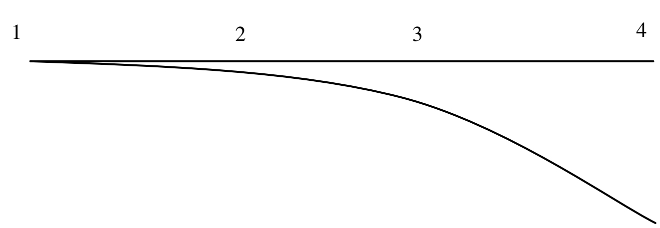
Forces in the rods or springs:

$$f_{s_1} = 200000\times1.186\times10^{-2} = 2{,}372\ lb$$

$$f_{s_2} = 120000\times3.386\times10^{-2} = 4{,}063\ lb$$

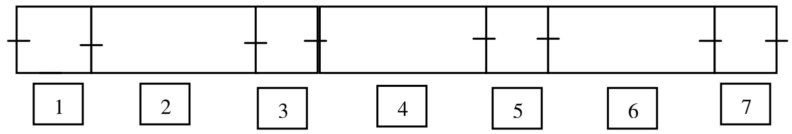
$$\sigma_1 = \frac{2372}{0.08} = 29{,}650.psi$$

$$\sigma_2 = \frac{4063}{0.08} = 50{,}788.psi$$



$Q_1 = 0$ $Q_3 = 1.186$ $Q_5 = 3.386$ $Q_7 = 8.848\ \text{x}\ 10^{-2}$ in. ■

5.6



NE = 7

$I_1 = I_3 = I_5 = I_7 = 4.12^3/12 = 576 \text{ in}^4$

$I_2 = I_4 = I_6 = 4.12^3/12 - 4.6^3/12 = 504 \text{ in}^4$

Without opening:

$\delta = PL^3/3EI = 4.167 \times 10^{-2}$ in

With opening:

Using BEAM program:

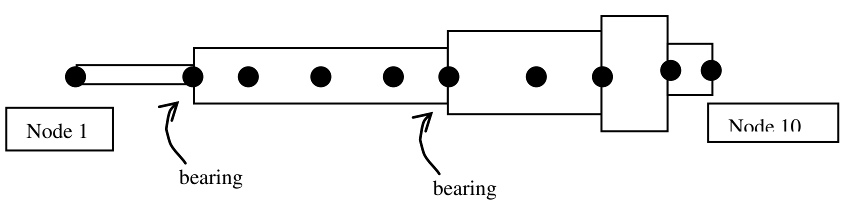
Node	**Displ.(in)**	**Rotation (rad)**
2	-0.060×10^{-2}	-0.2×10^{-3}
3	-0.536×10^{-2}	-0.58×10^{-3}
4	-0.925×10^{-2}	-0.714×10^{-3}
5	-1.935×10^{-2}	-0.952×10^{-3}
6	-2.529×10^{-2}	-1.025×10^{-3}
7	-3.826×10^{-2}	-1.121×10^{-3}
8	-4.503×10^{-2}	-1.131×10^{-3}

DOF #	**Reaction**	
1	10,000	lb
2	60,000	in-lb

■

5.7 Spindle

Some nodes have been introduced to avoid large variations in length between adjacent elements.



Node	**X-Coord (mm)**
1	0.
2	50.
3	125.
4	200.
5	275.
6	350.
7	380.
8	410.
9	425.
10	431.

$$I_1 = \frac{\pi(42)^4}{64} - \frac{\pi(30)^4}{64} = 112{,}984.5\ mm^4$$

$$I_3 = I_3 = I_4 = I_5 = 267{,}036\ mm^4$$

$$I_6 = 596{,}413.1\ mm^4$$

$$I_7 = I_6$$

$$I_8 = 1{,}970{,}863.1\ mm^4$$

$$I_9 = 267{,}036.mm^4$$

The bearing stiffnesses get added to the banded **S** before CALL BANSOL :

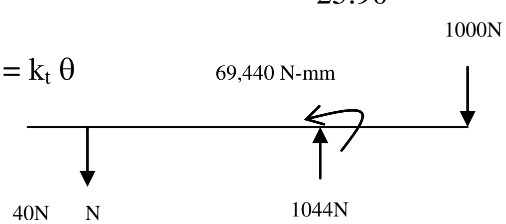
S(3,1) = S(3,1) + 20000
S(12,1)= S(12,1) + 60000
S(11,1) = S(11, 1) + 8 x 10^8

The deformed spindle shape is:

Node	**Displ. x 10^{-3} mm**
1	4.65
2	1.98
3	-2.08
4	-6.46
5	-11.46
6	-17.40
7	-20.30
8	-23.50
9	-25.20
10	-25.90

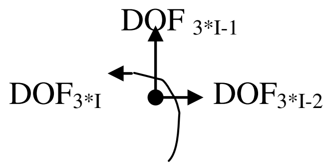
Reactions

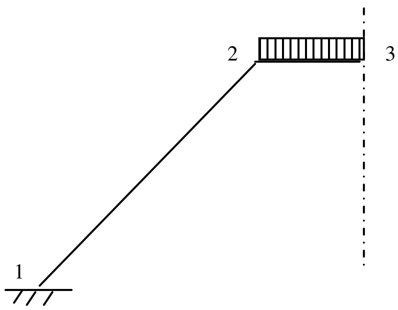
$f_s = k_r \times \delta$, $C_s = k_t\ \theta$



5.8 Frame Program

½ Symmetry Model





Node	**X-Coord (ft.)**	**Y-Coord (ft.)**
1	0	0
2	10	20
3	20	20

Element	**Area (in^2)**	**M_Inertia (in^4)**
1-2	15.0	305.
2-3	7.5	125.

Connectivity:

e	1	2
1	1	2
2	2	3

Boundary Conditions

DOF #	**Specified Displacement**	
1	0.	Fixed support
2	0.	
3	0.	
7	0.	Symmetry
9	0.	

Thus, ND = 5

Loading

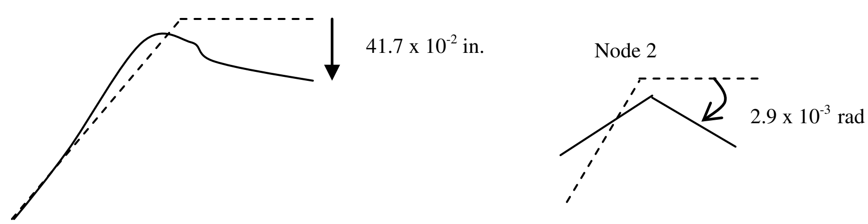
As in previous problems, the distributed load is replaced by equivalent nodal loads, as:

DOF #	Load
5	–6000.
6	–120000.
8	-6000.
9	+120000.

NL = 4

Output:

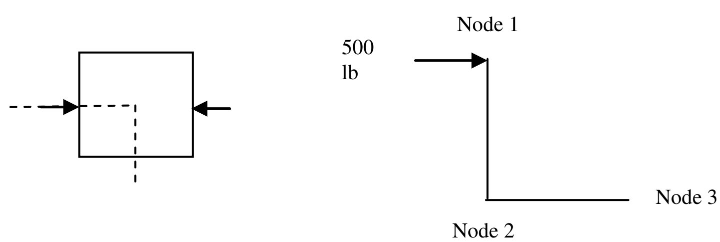
Node	(in.) X-Disp.	(in) Y-Disp.	(rad.) Rot
2	0.45×10^{-2}	-1.19×10^{-2}	-2.9×10^{-3}
3	0.0	-41.7×10^{-2}	0



■

5.9 Railing

(a)



$A = 0.25\ in^2$, $I = 0.001302083\ in^4$

<u>Boundary Conditions:</u> Due to symmetry, $Q_2 = Q_3 = 0$ & $Q_7 = Q_9 = 0$ (ND =4)

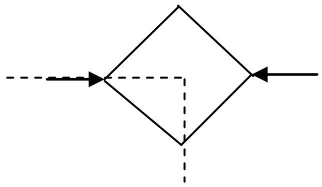
<u>Loading</u>: DOF # 1, Load = 500 lb

Node **X-Coord.** **Y-Coord**

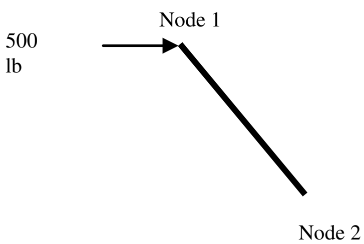
1	0.	6.
2	0.	0.
3	6.	0.

Solution: $Q_1 = 0.576$"

(b)



¼ Symmetry Model: only 1 element



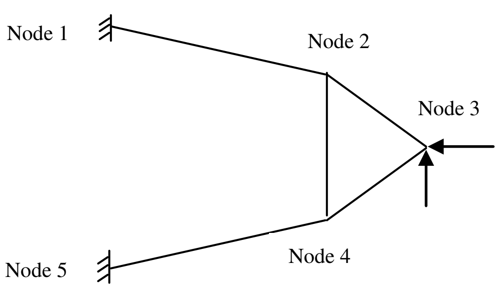
Node	**X-Coord**	**Y-Coord**
1	$\frac{-12}{\sqrt{2}}$	$\frac{12}{\sqrt{2}}$
2	0	0

Symmetry Boundary Conditions
$Q_2 = Q_3 = 0$ (ND = 4)
$Q_4 = Q_6 = 0$

Area, Inertia as in part (a)

Solution: $Q_1 = 0.9221$ ■

5.10



INPUT FILE for FRAME2D:

```
<<2-D  Frame Analysis using FRAME2D >>
Problem 5.10
NN NE NM NDIM NEN NDN
5   5  1  2    2   3
ND NL  NMPC
6  2    0
Node#  X   Y
 1     0  300
 2   800  175
 3   1250   0
 4   800   -175
  5  0     -300
ELEM#  N1  N2  MAT#  Area  Inertia  Distr_load
 1      1   2   1    3770    678.6e4      0.
 2      2   3   1    3770    678.6e4        0.
 3      5   4   1    3770    678.6e4        0.
 4      4   3   1    3770    678.6e4        0.
 5      2   4   1    3770    678.6e4        0.
DOF#  Displacement
 1     0
 2     0
 3     0
13     0
14     0
15     0
DOF#  Load
 7    -10000
 8    6000
MAT#   E
 1    200e3
B1  i  B2 j  B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

OUTPUT:

```
Output for Input Data from file test.inp
Problem 8.10

NODE#     X-Displ.        Y-Displ.        Z-Rot.
    1    -4.6264E-011     1.1335E-011     1.8130E-009
    2    -7.8918E-003     5.9955E-002     3.4005E-005
    3    -9.2043E-003     8.0195E-002     4.3788E-005
    4    -3.0406E-003     5.9491E-002     3.8869E-005
    5     1.6377E-011     6.5969E-012     1.8099E-009
  DOF#     Reaction
    1     1.5480E+004
    2    -3.7927E+003
    3    -6.0661E+005
   13    -5.4797E+003
   14    -2.2073E+003
   15    -6.0557E+005
 Member End-Forces
 Member # 1
    1.5880E+004    -1.3576E+003    -6.0661E+005
   -1.5880E+004     1.3576E+003    -4.9262E+005
```

```
Member # 2
   1.3366E+004     5.6761E+001    -1.3797E+004
  -1.3366E+004    -5.6761E+001     4.1203E+004
Member # 3
  -5.7548E+003    -1.3349E+003    -6.0557E+005
   5.7548E+003     1.3349E+003    -4.7528E+005
Member # 4
  -2.7476E+003    -2.2795E+002    -6.8858E+004
   2.7476E+003     2.2795E+002    -4.1203E+004
Member # 5
  -9.9897E+002     3.0016E+003     5.0641E+005
   9.9897E+002    -3.0016E+003     5.4414E+005
```

Discussion & Stresses

Area $A \approx 2\pi Rt = 37.7x10^2 \ mm^2$ Inertia I $\approx \pi R^3 = 678.6x10^4 \ mm^4$

Stresses may be calculated from the member-end-forces that are output. It is important o note that the end-forces are in the local (primed) system:

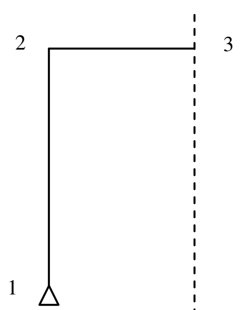
Axial Stress: $\sigma_a = \dfrac{R_4^{'}}{A} (> 0 \, tension)$

Banding Stress: $\sigma_b = \dfrac{R_3^{'}}{S}, \dfrac{R_6^{'}}{S}$

Shear Stress: $\gamma = \dfrac{R_2^{'}Q}{I \ t} \ or \ \dfrac{R_5^{'}Q}{I \ t}$ ■

5.11 ½ Symmetry Model

(a)



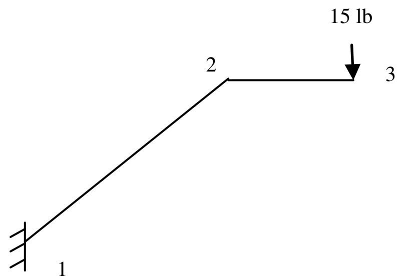
Boundary Conditions : $Q_1 = Q_2 = 0$ (pin), $Q_7 = Q_9 = 0$ (Symmetry)

Note:1) Using more nodes will allow detailed plot of deformed shape:
Note 2) Accurate model of the fillet will required use of ‘curved beam’ elements.

The modeling and solution of other cases proceed in a similar manner. ■

5.12

(a) Without tie rod:

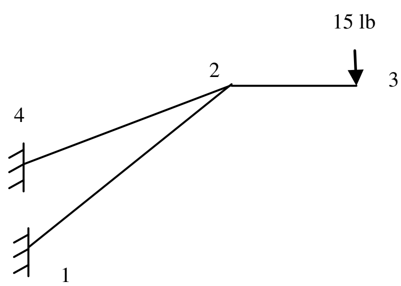


$A \approx 2\pi Rt \qquad R = 1/2 in$

$I \approx \pi t R^3 \qquad t = 1/8 in$

$B.C.'s : Q_1 = Q_2 = Q_3 = 0$

(b) With tie rod:



$Q_1 = Q_2 = Q_3 = 0$

$Q_{10} = Q_{11} = Q_{12} = 0$

$A_{tie} = \pi(1/4)^2/4, I_{tie} = \pi(0.25)^4/64$

Data : A = .3436 in^2
I = .0336 in^4
$A_{\text{tie rod}}$ = .0491 in^2
$I_{\text{tie rod}}$ = .00019175 in^4
E = 30 E6 psi

Δ	w/o tie	$with$ tie
Δ_{2x}	1.3"	.02"
Δ_{2y}	−7.0"	−.11"
Δ_{3x}	1.3"	.02"
Δ_{3y}	−11.6"	−.87"

Thus the tie rod has a dramatic effect.

Note: If the tie rod is lightly attached at node 2 as opposed to being welded or rigidly riveted, then it should be considered as a truss element. Solution of this is a good exercise for the student. ■

5.13

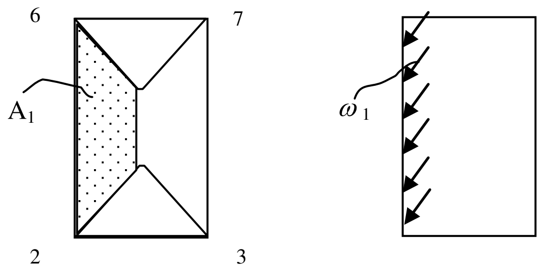
Bandwidth must be kept relatively small in this problem. Thus, nodes should run along y- and then along X- , for example, nodes 1, 11, 11', 1' can be numbered as 1, 2, 3, 4; nodes 6, 10, 10', 6' can be numbered as 5,6,7,8, etc. ■

5.14

Input data file using frame 3d and corresponding output is given below:

Load Estimation

We will assume that the total roof load is transferred as uniformly distributed loads (UDL's) to the beams. Roughly, these are proportional to the areas shown below:



$$\omega_1 = \frac{(A_1)(100/144\ lb/in^2)}{L_1} = 27.778\ lb/in$$

$$\omega_2 = \frac{A_2(100/144)}{L_2} = 20.833\ lb/in$$

Note: These are UDL's along +y' –direction owing to the choice of the "reference nodes" – see input file.

INPUT DATA FILE

```
<<3-D  Frame Analysis using FRAME3D >>
Problem 5.14
NN NE NM NDIM NEN NDN  NNREF
8   8  1  3    2   6    0
ND NL  NMPC
24  2    0
Node#  X   Y    Z
 1     0  0     0
 2     0  0     180
 3    120  0    180
 4    120  0      0
 5     0   180   0
 6     0   180   180
 7    120  180   180
 8    120  180   0
ELEM#  N1  N2  REF_PT  MAT#  Area  Iy    Iz   J   UDLy'  UDLz'
```

```
 1      1   2   4       1     6.  3.75   51. 0.24   0.    0.
 2      5   6   8       1     6.  3.75   51. 0.24   0.    0.
 3      8   7   5       1     6.  3.75   51. 0.24   0.    0.
 4      4   3   1       1     6.  3.75   51. 0.24   0.    0.
 5      2   6   1       1     3.  1.26   17. 0.08   27.778    0.
 6      6   7   5       1     3.  1.26   17. 0.08   20.833    0.
 7      7   3   4       1     3.  1.26   17. 0.08   27.778    0.
 8      3   2   1       1     3.  1.26   17. 0.08   20.833    0.
DOF#  Displacement
 1     0
 2     0
 3     0
 4  0
 5  0
 6  0
 19  0
 20  0
 21  0
 22  0
 23  0
 24  0
 25  0
 26  0
 27  0
 28  0
 29  0
 30  0
 43  0
 44  0
 45  0
 46  0
 47  0
 48  0
DOF#  Load
 7    3000
 31   3000
MAT#   E      G
 1    30E6   11.54E6
B1  i  B2 j  B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

OUTPUT

```
Output for Input Data from file TEST.INP
Problem 8.14

NODE#     X-Displ.        Y-Displ.        Z-Displ.
    1     2.6169E-009    -3.7505E-010    -4.0495E-009
    2     2.2500E-008     3.1196E-007     2.3348E-024
    3     8.3687E-001     1.9129E-004    -2.0655E-003
    4    -9.1842E-003     4.5874E-003     7.7397E-017
    5     8.3465E-001     1.9129E-004    -5.4346E-003
    6    -9.1842E-003     3.3961E-003     7.8129E-017
    7     3.2649E-009    -3.7505E-010    -1.0655E-008
    8     2.2500E-008     3.5043E-007     2.3334E-024
NODE#     X-Displ.        Y-Displ.        Z-Rot.
    1     2.2500E-008     3.1196E-007     2.3348E-024
    2     8.3687E-001     1.9129E-004    -2.0655E-003
```

```
    3    -9.1842E-003     4.5874E-003     7.7397E-017
    4     8.3465E-001     1.9129E-004    -5.4346E-003
    5    -9.1842E-003     3.3961E-003     7.8129E-017
    6     3.2649E-009    -3.7505E-010    -1.0655E-008
    7     2.2500E-008     3.5043E-007     2.3334E-024
    8     2.6169E-009     3.7505E-010    -4.0495E-009
  DOF#     Reaction
    1    -1.3347E+003
    2     1.9129E+002
...

 Member End-Forces
 Member # 1
    2.0654E+003    -1.3347E+003     1.9129E+002    -1.1909E-012    -
1.1476E+004    -1.5912E+005
   -2.0654E+003     1.3347E+003    -1.9129E+002     1.1909E-012    -
2.2957E+004    -8.1136E+004
 Member # 2
    2.0654E+003    -1.3347E+003    -1.9129E+002    -1.1820E-012
1.1476E+004    -1.5912E+005
   -2.0654E+003     1.3347E+003     1.9129E+002     1.1820E-012
2.2957E+004    -8.1136E+004
 Member # 3
    5.4346E+003     1.6653E+003     1.9129E+002    -1.2014E-012    -
1.1476E+004     1.7874E+005
   -5.4346E+003    -1.6653E+003    -1.9129E+002     1.2014E-012    -
2.2957E+004     1.2101E+005
 Member # 4
    5.4346E+003     1.6653E+003    -1.9129E+002    -1.2021E-012
1.1476E+004     1.7874E+005
   -5.4346E+003    -1.6653E+003     1.9129E+002     1.2021E-012
2.2957E+004     1.2101E+005
 Member # 5
    1.9129E+002     2.5000E+003    -4.9963E-013     4.9738E-013     4.5042E-
011     1.2704E+005
   -1.9129E+002     2.5000E+003     4.9963E-013    -4.9738E-013     4.4608E-
011    -1.2704E+005
 Member # 6
    1.6653E+003     2.9345E+003     7.0792E-013     2.8422E-014    -4.2080E-
011     1.3114E+005
   -1.6653E+003    -4.3457E+002    -7.0792E-013    -2.8422E-014    -4.2871E-
011     7.1011E+004
 Member # 7
    1.9129E+002     2.5000E+003    -4.9049E-013     4.9738E-013     4.4493E-
011     1.2704E+005
   -1.9129E+002     2.5000E+003     4.9049E-013    -4.9738E-013     4.4420E-
011    -1.2704E+005
 Member # 8
    1.6653E+003    -4.3457E+002     7.1775E-013     4.2633E-014    -4.3296E-
011    -7.1011E+004
   -1.6653E+003     2.9345E+003    -7.1775E-013    -4.2633E-014    -4.2835E-
011    -1.3114E+005
```

Maximum Bending Moments

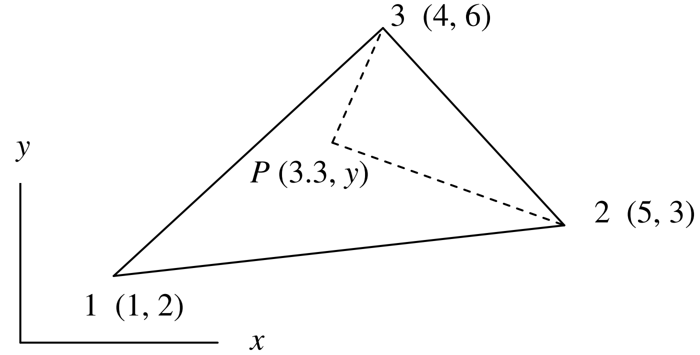
Max. $M_{y'}$ = 22,960 in-lb in the columns

Max. $M_{z'}$ = 178,700 in-lb in the columns. ■

5.15 Derive the expressions by using the shape function ideas in the distributed load for the element in $\mathbf{q}^{\mathrm{T}}\mathbf{f} = \int_e vp\,dx = \int_e (\mathbf{Hq})(\mathbf{Np})\,dx$ etc. Here **N** is the linear shape function vector. ■

CHAPTER 6
TWO DIMENSIONAL PROBLEMS USING CONSTANT STRAIN TRIANGLES

6.1



$$N_1 = \xi = 0.3 \qquad N_2 = \eta \qquad N_3 = 1 - \xi - \eta$$

$$x = \xi(x_1 - x_3) + \eta(x_2 - x_3) + x_3 \Rightarrow$$

at P,

$$3.3 = 0.3(1-4) + \eta(5-4) + 4 \Rightarrow$$

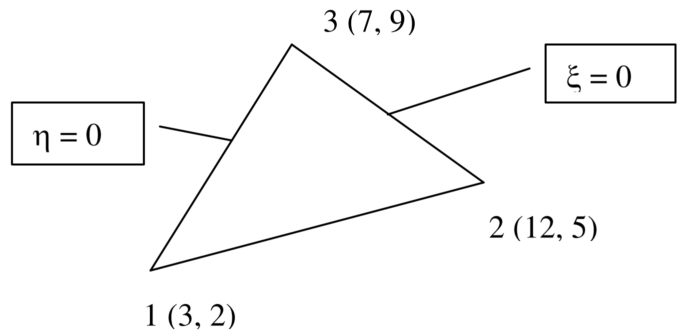
$$\eta = 0.2$$

$$N_2 = 0.2, \quad N_3 = 0.5$$

$$\begin{aligned} y &= N_1 y_1 + N_2 y_2 + N_3 y_3 \\ &= (0.3)(2) + (0.2)(3) + (0.5)(6) \\ &= 4.2 \end{aligned}$$

■

6.2



Jacobian **J** from Eq. 6.18 is

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

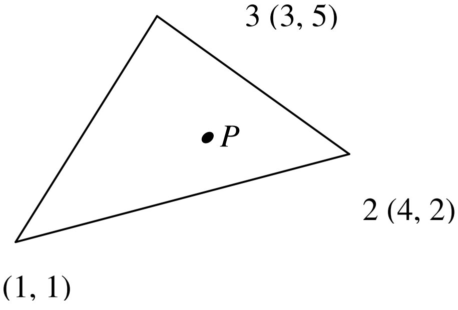
where $x_{ij} = x_i - x_j$.

$$\mathbf{J} = \begin{bmatrix} (3-7) & (2-9) \\ (12-7) & (5-9) \end{bmatrix} = \begin{bmatrix} -4 & -7 \\ 5 & -4 \end{bmatrix}$$

$$\text{Area} = \frac{1}{2}\det\mathbf{J} = \frac{1}{2}(16+35) = 25.5$$

■

6.3

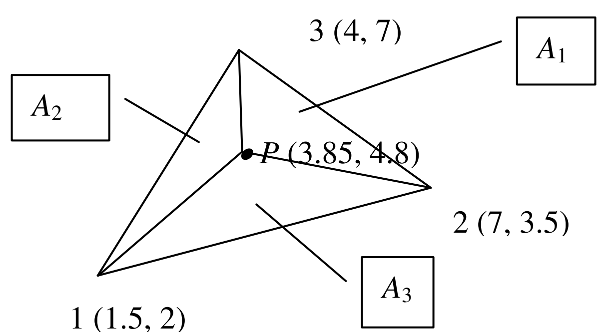


At P, $N_1 = 0.15$, $N_2 = 0.25$ $\Rightarrow$ $N_3 = 1 - N_1 - N_2 = 0.6$.

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 = 2.95$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 = 3.65$$

■

6.4 .



$x_p = 3.85$, $y_p = 4.8$, $x_1 = 1.5$, $y_1 = 2$, $x_2 = 7$, $y_2 = 3.5$, $x_3 = 4$, $y_3 = 7$.

$$A = 0.5\,(x_{13}y_{23} - x_{23}y_{13}) = 11.875$$
$$A_1 = 0.5\,(x_{p3}y_{23} - x_{23}y_{p3}) = 3.5625$$
$$A_2 = 0.5\,(x_{p1}y_{31} - x_{31}y_{p1}) = 2.375$$
$$A_3 = 0.5\,(x_{p2}y_{12} - x_{12}y_{p2}) = 5.935$$

$$N_1 = \frac{A_1}{A} = 0.3 \quad N_2 = \frac{A_2}{A} = 0.2 \quad N_3 = \frac{A_3}{A} = 0.5$$ ■

6.5

$$\boldsymbol{\varepsilon} = \mathbf{Bq}$$

$$\mathbf{B} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$\det \mathbf{J} = x_{13} y_{23} - x_{32} y_{31}$$

From the given coordinates,

$x_{13} = -1,\ x_{21} = 7,\ x_{32} = -6$

$y_{23} = -3,\ y_{31} = 5,\ y_{12} = -3$

$\det \mathbf{J} = 39$

$$\mathbf{B} = \frac{1}{39} \begin{bmatrix} -3 & 0 & 6 & 0 & -3 & 0 \\ 0 & -6 & 0 & -1 & 0 & 7 \\ -6 & -3 & -1 & 6 & 7 & -3 \end{bmatrix}$$

$$\mathbf{q}^T = [0.001 \quad -0.004 \quad 0.003 \quad 0.002 \quad -0.002 \quad 0.005]^T$$

$$\boldsymbol{\varepsilon} = \mathbf{Bq} = [5.897\times10^{-4} \quad 0.001 \quad -3.59\times10^{-4}]^T$$ ■

6.6 **(a)**

Elem#	Node1	Node2	Node3	Max Difference
1	1	2	6	5
2	2	6	7	5
3	2	3	7	5
4	3	7	8	5
5	3	4	8	5
6	4	8	9	5
7	4	5	9	5
8	5	9	10	5
8	6	7	11	5
10	7	11	12	5
11	7	8	12	5
12	8	12	13	5
Maximum of the Maximum difference				5

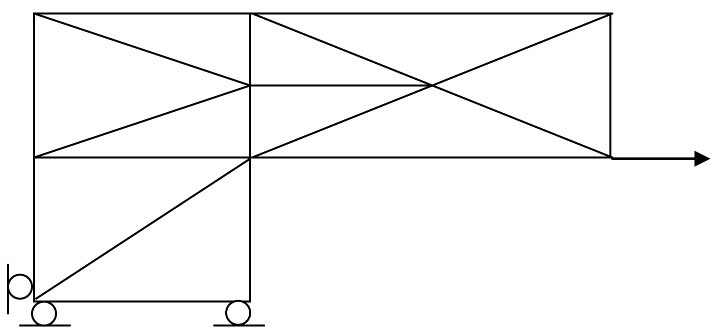
The bandwidth NBW = dof per node*(max diff + 1) = 2*(5 + 1) = 12

(b) When a multipoint constraint $Q_1 = Q_{18}$ is introduced and we use penalty approach, this results in adding ½*C*($Q_1 - Q_{18})^2$ to the potential energy. Stiffness is now added to locations (1,1),(1,18), and (18,1) in the banded matrix. Thus the bandwidth changes to 18. ■

6.7 **(a)** Only two degrees of freedom of the rigid body are constrained. The body can rotate freely about the pinned pivot point. At least one other node needs to have a roller support to constrain another degree of freedom.

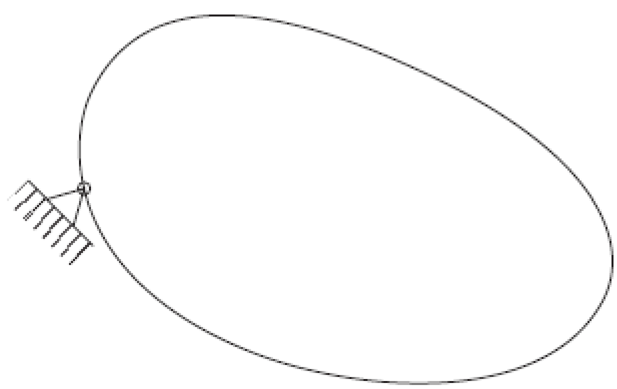
(b) There are two mistakes in the model shown.

1) The roller supports only suppress two degrees of freedom in the plane. The body is free to move in the horizontal direction. There needs to be a roller support to prevent motion in the horizontal direction.
2) The CST elements must be connected at the corners. The large CST element at the top has a node at the midpoint of the side. This is not allowed. Introduce additional elements to make it a viable mesh. Figure below shows a corrected model. The mesh shown is not an optimal division.



■

6.8



(a) Is the model valid? **No**. The pinned boundary condition implies that only two degrees of freedom are constrained. A two dimensional problem needs to have at least three degrees of freedom constrained.

(b) Based on the comment above, it is easy to see that the answer is element independent.

■

6.9 We assume plane stress condition.

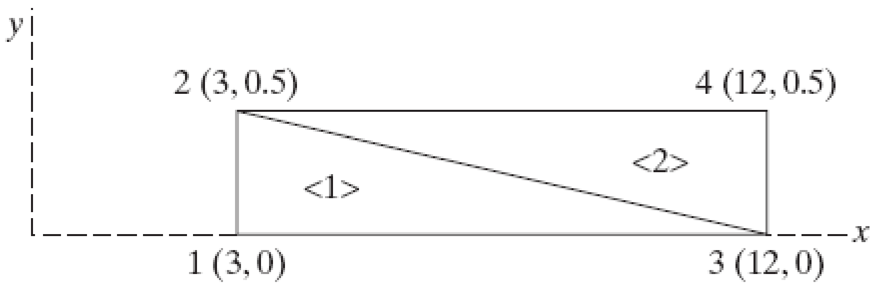
$$\varepsilon_0 = [\alpha\Delta T \quad \alpha\Delta T \quad 0]^T$$
$$= [0.001 \quad 0.001 \quad 0]^T$$

The temperature load $\boldsymbol{\theta}^e = t_e A_e \mathbf{B}^T \mathbf{D} \boldsymbol{\varepsilon}_0$

Noting that $t_e A_e$ = Volume = 25 mm^3 and $\mathbf{B}^T\mathbf{D} = (\mathbf{DB})^T$, we get

$$\boldsymbol{\theta}^e = [200 \quad 155 \quad 65 \quad 95 \quad -147.5 \quad 55]^T \qquad \blacksquare$$

6.10



$$\mathbf{f} = \begin{bmatrix} f_x & f_y \end{bmatrix}^T = \begin{bmatrix} x & 0 \end{bmatrix}^T$$

We have

$$\int_e \mathbf{u}^T \mathbf{f} t dA = \int_e u f_x t dA = \int_e uxt dA$$
$$= \int_e (N_1 q_1 + N_2 q_3 + N_3 q_5)(N_1 x_1 + N_2 x_2 + N_3 x_3) t \det \mathbf{J} d\xi d\eta$$
$$= 2A_e t \int_0^1 \int_0^{1-\xi} \left(\xi^2 x_1 + \xi\eta x_2 + \xi(1-\xi-\eta) x_3\right) q_1 d\xi d\eta$$
$$+2A_e t \int_0^1 \int_0^{1-\xi} \left(\eta\xi x_1 + \eta^2 x_2 + \eta(1-\xi-\eta) x_3\right) q_3 d\xi d\eta$$
$$+2A_e t \int_0^1 \int_0^{1-\xi} \left((1-\xi-\eta)\xi x_1 + (1-\xi-\eta)\eta x_2 + (1-\xi-\eta)^2 x_3\right) q_3 d\xi d\eta$$

Using Eqn 6.46 $\int_0^1 \int_0^{1-\xi} \xi^a \eta^b (1-\xi-\eta)^c \, d\xi d\eta = \dfrac{a!b!c!}{(a+b+c+2)}$, the squared terms integrate to 1/6, and each of the other terms integrate to 1/12. Thus

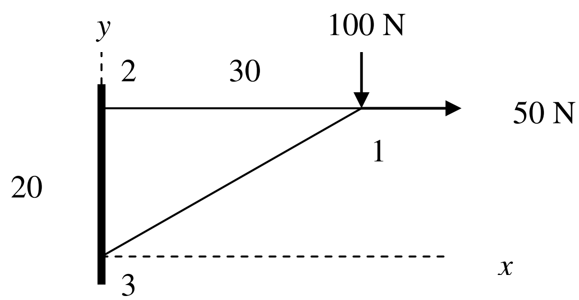
$$\int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}t dA = 2A_e t\int_0^1\int_0^{1-\xi}\left(\xi^2 x_1 + \xi\eta x_2 + \xi(1-\xi-\eta)x_3\right)q_1 d\xi d\eta$$
$$+2A_e t\int_0^1\int_0^{1-\xi}\left(\eta\xi x_1 + \eta^2 x_2 + \eta(1-\xi-\eta)x_3\right)q_3 d\xi d\eta$$
$$+2A_e t\int_0^1\int_0^{1-\xi}\left((1-\xi-\eta)\xi x_1 + (1-\xi-\eta)\eta x_2 + (1-\xi-\eta)^2 x_3\right)q_3 d\xi d\eta$$
$$= A_e t\begin{bmatrix}(2x_1+x_2+x_3)/6\\0\\(x_1+2x_2+x_3)/6\\0\\(x_1+x_2+2x_3)/6\\0\end{bmatrix}$$

With connectivity defined by (1,3,2) for element 1, (x_1, x_2, x_3) = (3,12,3) ,
$A_e = \dfrac{(9)(0.5)}{2} = 2.25$, t = 1, $\mathbf{f}$ = 2.25[3.5, 0, 5, 0, 3.5, 0]$^{\mathrm{T}}$, with corresponding dof (1 2 5 6 3 4).

With connectivity defined by (2,3,4) for element 2, (x_1, x_2, x_3) = (3,12,12) ,
$A_e = \dfrac{(9)(0.5)}{2} = 2.25$, t = 1, $\mathbf{f}$ = 2.25[5, 0, 6.5, 0, 6.5, 0]$^{\mathrm{T}}$, with corresponding dof (3 4 5 6 7 8).

Now **F** = 2.25[3.5, 0, 3.5+5, 0, 5+6.5, 0, 6.5, 0]$^{\mathrm{T}}$ = 2.25[3.5, 0, 8.5, 0, 11.5, 0, 6.5, 0]$^{\mathrm{T}}$.
■

6.11 Loading at the pointed end is not practical. However, the problem has a finite element solution.



det**J** = 600

$$\mathbf{B} = \frac{1}{600}\begin{bmatrix}20 & 0 & -20 & 0 & 0 & 0\\0 & 0 & 0 & 30 & 0 & -30\\0 & 20 & 30 & -20 & -30 & 0\end{bmatrix}$$

For plane stress condition,

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 76920 & 23080 & 0 \\ 23080 & 76920 & 0 \\ 0 & 0 & 26920 \end{bmatrix}$$

$$\mathbf{K} = t_e A_e \mathbf{B}^T \mathbf{D}\mathbf{B} = 10^5 \begin{bmatrix} 2.564 & 0 & -2.564 & 1.154 & 0 & -1.154 \\ 0 & 0.897 & 1.364 & -0.897 & -1.346 & 0 \\ -2.564 & 1.364 & 4.583 & -2.5 & -2.019 & 1.154 \\ 1.154 & -0.897 & -2.5 & 6.667 & 1.346 & -5.77 \\ 0 & -1.346 & -2.019 & 1.346 & 2.019 & 0 \\ -1.154 & 0 & 1.154 & -5.77 & 0 & 5.77 \end{bmatrix}$$

By using the elimination approach, we have

$$10^5 \begin{bmatrix} 2.564 & 0 \\ 0 & 0.897 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} 50 \\ -100 \end{Bmatrix}$$

$Q_1 = 0.000195$ mm $\qquad Q_2 = -0.00111$ mm ■

6.12

Elem#	Node1	Node2	Node3	Max Diff.
1	1	2	7	6
2	1	7	6	6
3	2	3	7	5
4	3	7	8	5
5	3	4	8	5
6	4	8	9	5
7	4	5	9	5
8	5	9	10	5
8	6	7	11	5
10	7	11	12	5
11	7	8	13	6
12	7	12	13	6
13	8	9	14	6
14	8	14	13	6
15	9	10	15	6
16	9	15	14	6
17	11	12	17	6
18	11	17	16	6
19	12	13	17	5
20	13	18	17	5
Maximum of the Maximum difference				6

The bandwidth NBW = dof per node*(max diff + 1) = 2*(6 + 1) = 14

Numbering the nodes in the vertical direction first gives a lower bandwidth in this problem. ■

6.13 The element force vector for each element needs to be determined. Let the connectivity be established as

Element 1	1, 2, 4
Element 2	2, 5, 4
Element 3	2, 3, 5
Element 4	3, 6, 5

The simplest model is to assume the body force f to be represented for an element by its value at the centroid. We calculate y_{bar}, f_{bar} , and the element force for each element. Area of each element is 0.75 m^2. The thickness is 1m. The element volume is 0. 75 m^3.

Element#	y_{bar}	$f_{bar} = (y_{bar})^2$	Elem Force $F = f_{bar} * V_e$	$F/3$
1	(2+1+2)/3 = 1.667	2.778	2.0833	0.6944
2	(1+1+2)/3 = 1.333	1.778	1.3333	0.4444
3	(1+0+1)/3 = 0.667	0.444	0.3333	0.1111
4	(0+0+1)/3 = 0.333	0.111	0.0833	0.0278

The last column is the distributed load to the dof in the y direction for the element.

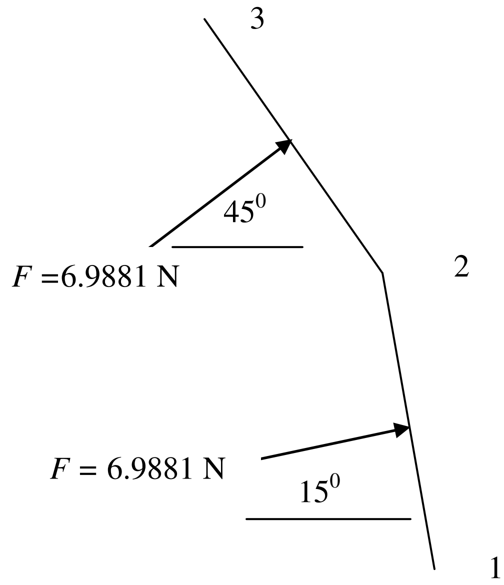
The body force vectors for elements are now given

$$\mathbf{f}^{(1)} = \begin{Bmatrix} 0 \\ 0.6944 \\ 0 \\ 0.6944 \\ 0 \\ 0.6944 \end{Bmatrix} \quad \mathbf{f}^{(2)} = \begin{Bmatrix} 0 \\ 0.4444 \\ 0 \\ 0.4444 \\ 0 \\ 0.4444 \end{Bmatrix} \quad \mathbf{f}^{(3)} = \begin{Bmatrix} 0 \\ 0.1111 \\ 0 \\ 0.1111 \\ 0 \\ 0.1111 \end{Bmatrix} \quad \mathbf{f}^{(4)} = \begin{Bmatrix} 0 \\ 0.0278 \\ 0 \\ 0.0278 \\ 0 \\ 0.0278 \end{Bmatrix}$$

The force F_4 at node 2 is (0.6944 + 0.4444 + 0.1111) = 1.2499. Similarly assembling the remaining forces, we get the the global load vector.

$$\mathbf{F} = [0 \quad 0.6944 \quad 0 \quad 1.2499 \quad 0 \quad 0.1389 \quad 0 \quad 1.1388 \quad 0 \quad 0.5833 \quad 0 \quad 0.0278]^T$$ ■

6.14 We assume a thickness of 1mm.
The length of the edge $l = 2(15\sin 15^0) = 7.7646$ mm.
The normal force is $F = pl(1) = (0.9)(7.7646)\,(1) = 6.9881$ N.



$$\mathbf{f}^{(1)} = \begin{Bmatrix} 0.5*\cos 15*F \\ 0.5*\sin 15*F \\ 0.5*\cos 15*F \\ 0.5*\sin 15*F \end{Bmatrix} = \begin{Bmatrix} 3.375 \\ 0.9043 \\ 3.375 \\ 0.9043 \end{Bmatrix} \quad \mathbf{f}^{(2)} = \begin{Bmatrix} 0.5*\cos 45*F \\ 0.5*\sin 45*F \\ 0.5*\cos 45*F \\ 0.5*\sin 45*F \end{Bmatrix} = \begin{Bmatrix} 2.4707 \\ 2.4707 \\ 2.4707 \\ 2.4707 \end{Bmatrix}$$

On assembling, we get the force vector

$$\mathbf{F} = [3.375 \quad 0.9043 \quad 5.8457 \quad 3.375 \quad 2.4707 \quad 2.4707]^T \quad \blacksquare$$

6.15 $I = \int_e y^2 dA$

On the triangle, we have $y = N_1 y_1 + N_2 y_2 + N_3 y_3 = \mathbf{Ny}$, and $dA = \det\mathbf{J}\, d\xi d\eta$, $\det\mathbf{J} = 2A_e$.

$$\begin{aligned} I &= \int_e \mathbf{y}_e{}^T \mathbf{N}^T \mathbf{N} \mathbf{y}_e dA \\ &= \mathbf{y}_e{}^T \left[\int_e \mathbf{N}^T \mathbf{N} dA \right] \mathbf{y}_e \\ &= \mathbf{y}_e{}^T \mathbf{R} \mathbf{y}_e \end{aligned}$$

We determine **R**.

$$\mathbf{R} = \int_e \begin{bmatrix} N_1^{\;2} & N_1N_2 & N_1N_3 \\ N_1N_2 & N_2^{\;2} & N_2N_3 \\ N_1N_2 & N_2N_3 & N_3^{\;2} \end{bmatrix} dA$$

Note that all the diagonal terms will have the same values on integration. Similarly all off diagonal terms will be same.

$$\int_e N_1^{\;2} dA = \int_0^1 \xi^2 \int_0^{1-\xi} \det \mathbf{J} d\eta d\xi = 2A_e \int_0^1 \xi^2 (1-\xi) d\xi = \frac{A_e}{6}$$

$$\int_e N_1 N_2 dA = \int_0^1 \xi \int_0^{1-\xi} \eta \det \mathbf{J} d\eta d\xi = 2A_e \int_0^1 \xi \frac{(1-\xi)^2}{2} d\xi = \frac{A_e}{12}$$

Thus

$$\mathbf{R} = \frac{A_e}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Note that in the above integral evaluations, the polynomial formula (such as the one given on page 278 for tetrahedral) is

$$\int_0^1 \int_0^{1-\xi} \xi^m \eta^n d\xi d\eta = \frac{m!\,n!}{(m+n+2)!}.$$

This formula may be used in the integral evaluations. ■

6.16

$$I = \int_e N_1 N_2 N_3 dA$$

$$= \int_0^1 \int_0^{1-\xi} \xi\eta(1-\xi-\eta) \det \mathbf{J} d\xi d\eta$$

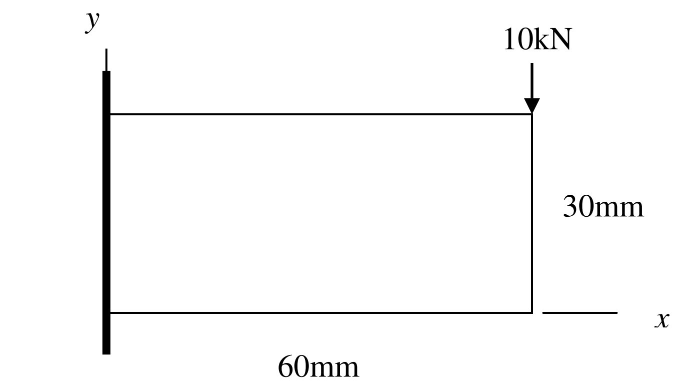
$$= 2A_e \int_0^1 \int_0^{1-\xi} (\xi\eta - \xi^2\eta - \xi\eta^2) d\xi d\eta$$

Noting that $\int_0^1 \int_0^{1-\xi} \xi\eta d\xi d\eta = \frac{1}{24}$, $\int_0^1 \int_0^{1-\xi} \xi^2\eta d\xi d\eta = \int_0^1 \int_0^{1-\xi} \xi\eta^2 d\xi d\eta = \frac{1}{60}$,

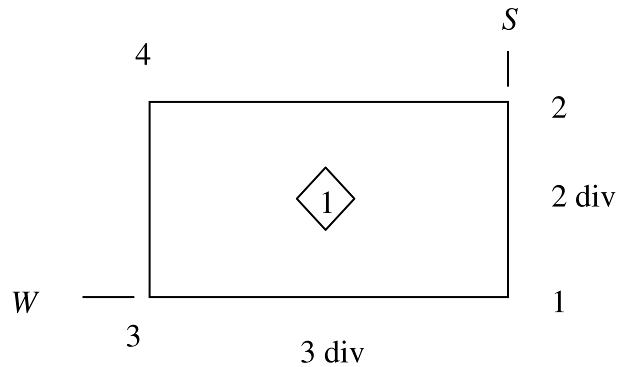
$$I = \frac{A_e}{60}$$

The polynomial formula given in the solution of problem 5.13 may be used. ■

6.17 We provide the complete data preparation aspects for one mesh model using MESHGEN, PLOT2D, and CST in solving this plane stress problem.



E = 70000 MPa ν = 0.33 t = 10 mm



Block diagram for MESHGEN

<u>Input Data File for MESHGEN (you may name this file P515.MSH)</u>

```
Mesh Generation
Problem 5.15
Number of Nodes per Element <3 or 4>
  3
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ)
  1             1             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
  1        2
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        3
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
  1      60        0
  2      60        30
  3      0         0
  4      0         30
  0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
```

```
  0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
```

After running the program MESHGEN, the data output may be stored in file P515.INP. The output from MESHGEN is as given here.

```
Program MESHGEN - CHANDRUPATLA & BELEGUNDU
Problem 5.15
NN  NE  NM  NDIM  NEN  NDN
 12  12  1  2  3  2
ND   NL   NMPC
 0  0  0                    ← Edit here
Node#   X    Y
 1  60  0
 2  60  15
 3  60  30
 4  40  0
 5  40  15
 6  40  30
 7  20  0
 8  20  15
 9  20  30
 10  0  0
 11  0  15
 12  0  30
Elem#  Node1  Node2  Node3  Material#         ← Edit here
 1  1  2  5  1                   ← Edit here
 2  5  4  1  1                   ← Edit here
 3  2  3  6  1                   ← Edit here
 4  6  5  2  1                   ← Edit here
 5  4  5  8  1                   ← Edit here
 6  8  7  4  1                   ← Edit here
 7  5  6  9  1                   ← Edit here
 8  9  8  5  1                   ← Edit here
 9  7  8  11  1                  ← Edit here
 10  11  10  7  1                ← Edit here
 11  8  9  12  1                 ← Edit here
 12  12  11  8  1                ← Edit here
```

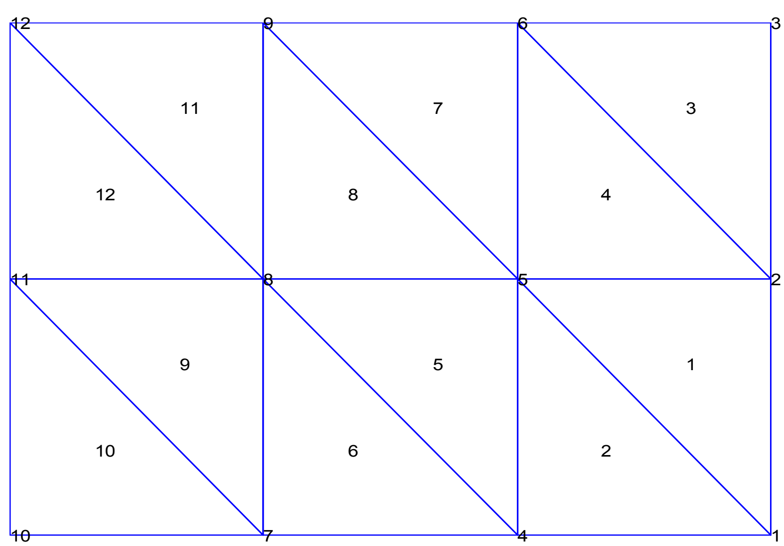
<Add lines here using the data file format given at the end of the chapter>
<See the file below for the fully edited version>

The first step before editing is to use PLOT2D to see how the mesh looks like. After that, the changes suggested above must be made before running CST program.
Program MESHGEN - CHANDRUPATLA & BELEGUNDU

```
Problem 5.15
NN  NE  NM  NDIM  NEN  NDN
 12  12  1  2  3  2
ND   NL   NMPC
 4  1  0
Node#   X    Y
 1  60  0
 2  60  15
```

```
 3  60  30
 4  40  0
 5  40  15
 6  40  30
 7  20  0
 8  20  15
 9  20  30
 10  0  0
 11  0  15
 12  0  30
Elem#  Node1  Node2  Node3  Material#  Thickness  TempRise
 1  1  2  5  1  10   0
 2  5  4  1  1  10   0
 3  2  3  6  1  10   0
 4  6  5  2  1  10   0
 5  4  5  8  1  10   0
 6  8  7  4  1  10   0
 7  5  6  9  1  10   0
 8  9  8  5  1  10   0
 9  7  8  11  1  10   0
 10  11  10  7  1  10   0
 11  8  9  12  1  10   0
 12  12  11  8  1  10   0
DOF#  Displacement
 19     0
 21     0
 22     0
 23     0
DOF#  Load
 6    -10000
MAT#   E      Nu    Alpha
 1    70000   .33    23E-6
B1  i  B2 j  B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

Running PLOT2D shows the mesh given below. PLOT2D is available in QBASIC, VisualBasic, and MATLAB. The graphics from MATLAB are easy to transport to a word processor.



Note that we fix node 11 in both *x* and *y* directions, and nodes 10 and 12 in *x* direction. Other boundary conditions can be tried.

The output from the program is given below. This file may be named P515.OUT

```
Output for Input Data in File --- p515.inp
Problem 5.15
Plane Stress Analysis
NODE#   X-Displ     Y-Displ
   1  -8.753E-02  -3.050E-01
   2   2.775E-04  -3.108E-01
   3   9.641E-02  -3.302E-01
   4  -7.897E-02  -1.736E-01
   5  -1.507E-03  -1.741E-01
   6   7.776E-02  -1.795E-01
   7  -5.195E-02  -6.509E-02
   8  -2.604E-03  -6.154E-02
   9   4.897E-02  -6.825E-02
  10  -7.370E-07  -2.114E-02
  11  -1.328E-07  -4.017E-07
  12   8.698E-07   6.244E-04
DOF#        Reaction
   19      1.8347E+04
   21      3.3066E+03
   22      1.0000E+04
   23     -2.1653E+04
ELEM#    SX          SY          TXY         S1          S2      ANGLE SX->S1
   1 -2.913E+00 -2.775E+01 -2.581E+01  1.331E+01 -4.397E+01 -3.215E+01
   2 -3.441E+01 -1.348E+01 -3.700E+01  1.451E+01 -6.240E+01 -5.290E+01
   3  3.963E+01 -7.771E+01 -2.972E+01  4.673E+01 -8.481E+01 -1.343E+01
   4 -2.305E+00 -2.591E+01 -4.080E+01  2.837E+01 -5.658E+01 -3.693E+01
   5  3.524E+00 -9.568E-01 -1.219E+01  1.367E+01 -1.111E+01 -3.979E+01
   6 -9.999E+01 -1.642E+01 -5.624E+01  1.186E+01 -1.283E+02 -6.331E+01
   7  1.038E+02  9.090E+00 -7.294E+00  1.043E+02  8.531E+00 -4.380E+00
   8 -7.285E+00 -3.371E+01 -5.761E+01  3.861E+01 -7.960E+01 -3.854E+01
   9 -4.087E+00  1.523E+01  5.604E+00  1.674E+01 -5.595E+00  7.494E+01
  10 -1.675E+02  4.337E+01 -5.783E+01  5.819E+01 -1.823E+02 -7.563E+01
  11  1.808E+02  2.834E+01 -1.372E-01  1.808E+02  2.834E+01 -5.158E-02
  12 -9.147E+00 -1.029E-01 -8.097E+01  7.647E+01 -8.572E+01 -4.660E+01
```

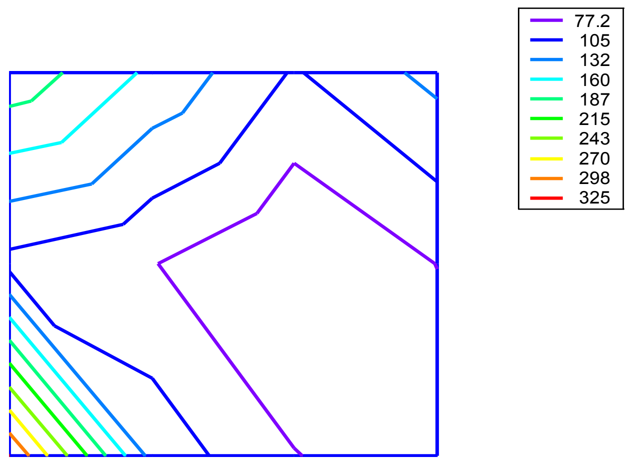
The plot data file created is given below. This file may be named P515.ELE

```
Von Mises Stress (Element) for Data in File p515.inp
 51.92271
 70.7776
 115.493
 74.90855
 21.49833
 134.5997
 100.3209
 104.4041
 20.12767
 217.3501
 168.3779
 140.5386
```

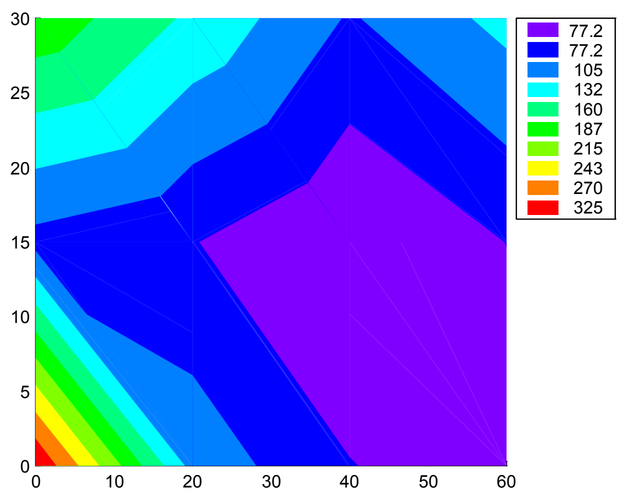
We run BESTFIT to get nodal data file P515.NOD

```
Nodal Values for Data in Files p515.inp & p515.ele
 58.90577
 77.63367
 141.1053
 78.3159
 49.61431
 102.1277
 122.5266
 78.47304
 154.5225
 325.2864
 96.30095
 206.9741
```

Program CONTOURA needs files P515.INP and P515.NOD. The CONTOURA output is given below.



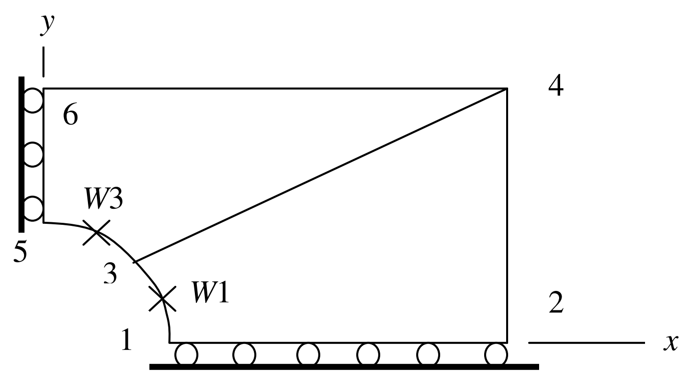
Program CONTOURB needs files P515.INP and P515.NOD. The CONTOURB output is given below.



■

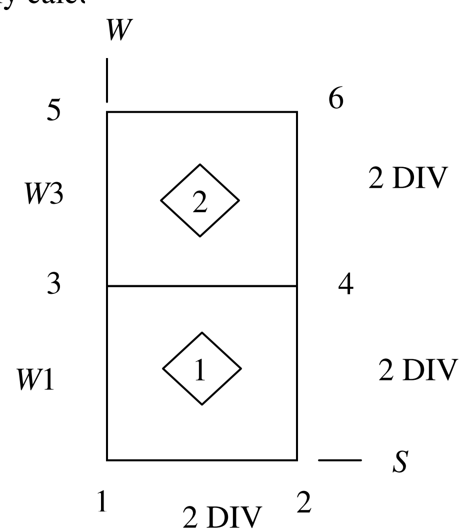
6.18 The quarter symmetry consideration for a rectangular plate discussed in section 5.4 applies to this problem. This is a plane stress problem.

The quarter model for the problem is shown in the figure. Use of MESHGEN helps in the generation of coordinate and connectivity data with utmost ease.



1-W1-3-W3-5 is a circular arc and the three inside points are at 22.5^0, 45^0, 67.5^0, The coordinates for the block division are easily calculated.

Corner/Side	X-Coord	Y-Coord
1	0.75	0
2	3	0
3	0.5303	0.5303
4	3	2
5	0	0.75
6	0	2
W1	0.693	0.287
W3	0.287	0.693



Input Data File for MESHGEN (you may name this file P516.MSH)

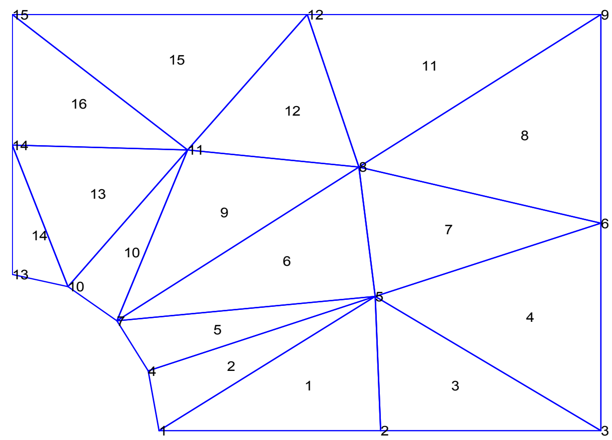
```
Mesh Generation
Problem 5.16
Number of Nodes per Element <3 or 4>
  3
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ)
  1             2             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
  1        2
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        2
  2        2
BLOCK MATERIAL DATA (for Material Number other than 1)
```

```
Block#   Material  (Void => 0   Block# = 0 completes this data)
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
  1      0.75      0
  2      3         0
  3      0.5303    0.5303
  4      3         2
  5      0         0.75
  6      0         2
  0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  1      0.693     0.287
  3      0.287     0.693
  0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
```

Edited MESHGEN output. File P516.INP

```
Program MESHGEN - CHANDRUPATLA & BELEGUNDU
Problem 5.16
NN  NE  NM  NDIM  NEN  NDN
 15  16  1  2  3  2
ND   NL   NMPC
 6  3  0
Node#   X    Y
 1  .75  0
 2  1.875  0
 3  3  0
 4  .693  .287
 5  1.8465  .6435
 6  3  1
 7  .5303  .5303
 8  1.76515  1.26515
 9  3  2
 10  .287  .693
 11  .8935  1.3465
 12  1.5  2
 13  0  .75
 14  0  1.375
 15  0  2
Elem#  Node1  Node2  Node3  Material#   Thickness  TempRise
 1  1  2  5  1   1   0
 2  5  4  1  1   1   0
 3  2  3  5  1   1   0
 4  6  5  3  1   1   0
 5  4  5  7  1   1   0
 6  8  7  5  1   1   0
 7  5  6  8  1   1   0
 8  9  8  6  1   1   0
 9  7  8  11  1   1   0
 10  11  10  7  1   1   0
 11  8  9  12  1   1   0
```

```
 12  12  11  8  1   1   0
 13  10  11  14  1   1   0
 14  14  13  10  1   1   0
 15  11  12  15  1   1   0
 16  15  14  11  1   1   0
DOF#  Displacement
 2     0
 4     0
 6     0
 25     0
 27     0
 29     0
DOF#  Load
 5    1000
 11   2000
 17   1000
MAT#   E      Nu    Alpha
 1    30E6   .3    12E-6
B1  i  B2 j  B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

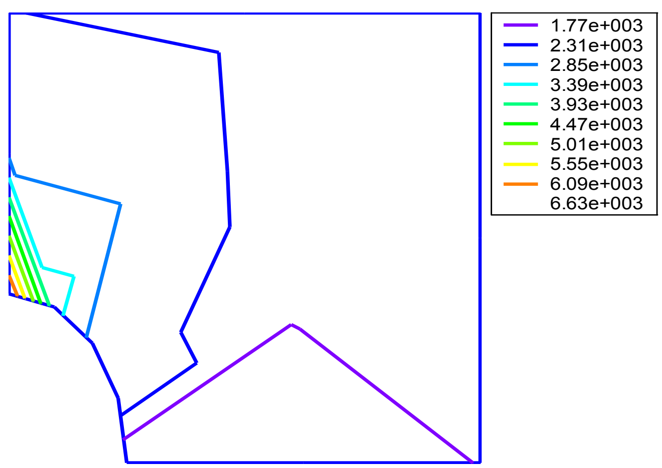


Mesh Configuration

The output from the program is given below. This file may be named P516.OUT

```
Output for Input Data in File --- p516.inp
Problem 5.16
Plane Stress Analysis
NODE#   X-Displ     Y-Displ
   1   1.355E-04  -2.727E-10
   2   1.839E-04   7.731E-11
   3   2.514E-04   1.954E-10
   4   1.216E-04  -1.169E-05
   5   1.735E-04  -8.051E-06
   6   2.388E-04  -6.467E-06
   7   9.036E-05  -2.295E-05
   8   1.482E-04  -2.240E-05
   9   2.215E-04  -2.420E-05
  10   4.742E-05  -4.109E-05
  11   7.749E-05  -4.179E-05
  12   1.123E-04  -4.408E-05
```

```
  13   1.155E-09  -3.923E-05
  14   1.179E-09  -6.293E-05
  15   3.732E-10  -7.760E-05
DOF#        Reaction
    2      4.0292E+02
    4     -1.1422E+02
    6     -2.8870E+02
   25     -1.7064E+03
   27     -1.7422E+03
   29     -5.5133E+02
ELEM#     SX          SY          TXY         S1          S2      ANGLE SX->S1
   1  1.294E+03  1.295E+01 -1.646E+02  1.315E+03 -7.842E+00 -7.202E+00
   2  1.488E+03 -6.869E+02 -2.581E+02  1.518E+03 -7.171E+02 -6.675E+00
   3  1.854E+03  1.809E+02 -1.559E+02  1.869E+03  1.665E+02 -5.277E+00
   4  1.930E+03  3.851E+02 -1.067E+02  1.938E+03  3.777E+02 -3.933E+00
   5  1.952E+03 -5.127E+02 -7.747E+02  2.176E+03 -7.359E+02 -1.608E+01
   6  1.963E+03 -5.196E+01 -2.189E+02  1.986E+03 -7.548E+01 -6.130E+00
   7  1.975E+03 -6.799E+01 -2.755E+02  2.011E+03 -1.045E+02 -7.550E+00
   8  2.121E+03  1.044E+02 -9.486E+01  2.126E+03  9.999E+01 -2.687E+00
   9  2.207E+03 -2.879E+02 -3.520E+02  2.256E+03 -3.366E+02 -7.878E+00
  10  3.787E+03 -1.646E+02 -3.130E+02  3.812E+03 -1.893E+02 -4.500E+00
  11  2.154E+03 -9.519E+01 -1.069E+02  2.159E+03 -1.003E+02 -2.716E+00
  12  2.392E+03  5.108E+01 -1.571E-01  2.392E+03  5.108E+01 -3.844E-03
  13  2.603E+03  1.098E+02 -1.217E+02  2.608E+03  1.038E+02 -2.789E+00
  14  5.072E+03  3.842E+02 -1.617E+02  5.078E+03  3.786E+02 -1.973E+00
  15  2.229E+03 -5.838E+01  7.134E+01  2.232E+03 -6.060E+01  1.784E+00
  16  2.627E+03  8.338E+01  2.642E+02  2.654E+03  5.621E+01  5.869E+00
```



Contours for vonMises Stress

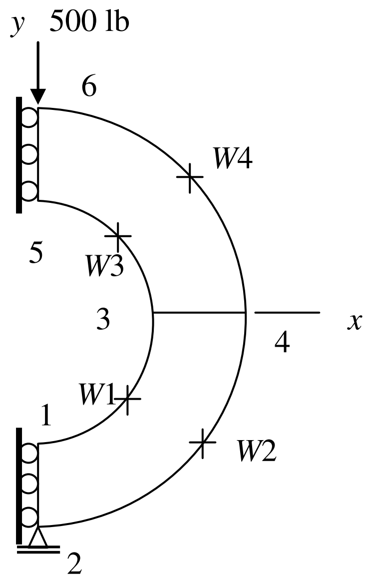
The stress concentration factor K is defined as

$$K = \frac{\sigma_{max}}{\sigma_{average}}$$

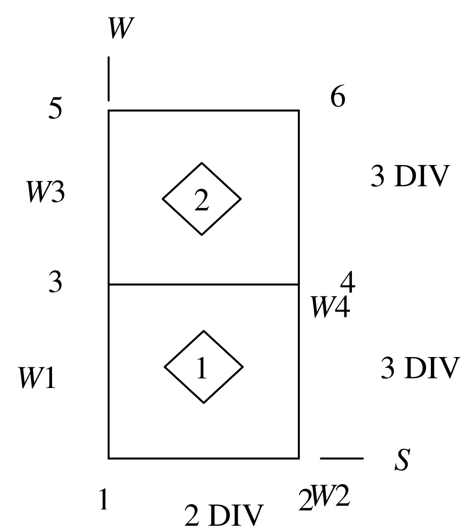
σ_{max} is calculated as $(2000\times4\times1)/((4-1.5)\times1) = 3200$ psi. Maximum vonMises stress obtained from the contour plot is 6630 psi, giving a value of $K = 1.97$.

Theory from mechanics of solids shows that K approaches a maximum of 3 as the hole becomes small relative to the plate. ■

6.19 We use the diametral symmetry in modeling this problem.



Corner/Side	X-Coord	Y-Coord
1	0	–1
2	0	–2
3	1	0
4	2	0
5	0	1
6	0	2
$W1$	0.707	–0.707
$W2$	1.414	–1.414
$W3$	0.707	0.707
$W4$	1.414	1.414

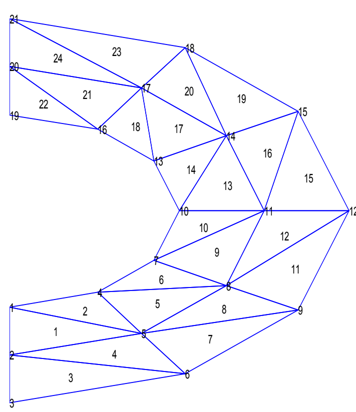


The file for program MESHGEN is given here.

```
Mesh Generation
Problem 5.17
Number of Nodes per Element <3 or 4>
```

```
  3
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ)
  1             2             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
  1        2
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        3
  2        3
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
  1      0         -1
  2      0         -2
  3      1         0
  4      2         0
  5      0         1
  6      0         2
  0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  1      0.707     -0.707
  2      1.414     -1.414
  3      0.707      0.707
  4      1.414      1.414
  0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
```

The mesh created is shown in the figure. The horizontal and vertical scales are not proportional



in the figure. The input

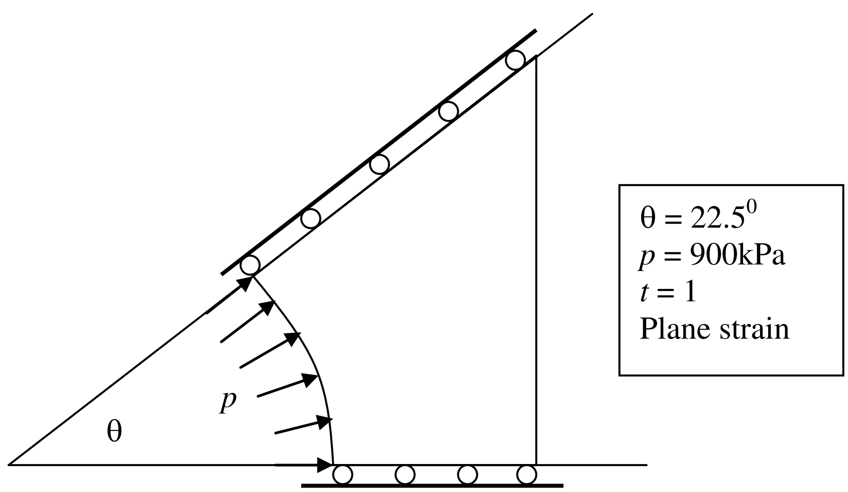
We need to edit the file created by MESHGEN as discussed in the previous problem. Note that ND = 7, and NL = 1. The dof with zero displacement are 1,3,5,6,37,39,41. The load of –500 lb is along dof 42. In addition, the material property data must be edited in.

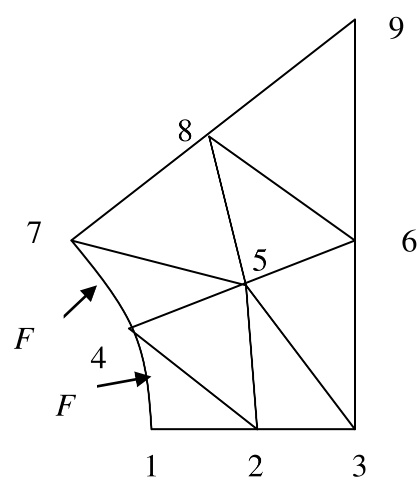
Comment: If the maximum shear stress is plotted, the constant shear stress contours are precisely the fringe patterns obtained in photoelasticity. ■

6.20 Using the multipoint constraint steps developed in Chapter 3, the filling steps are straightforward. The MPC is $\beta_1 Q_5 + \beta_2 Q_9 = \beta_0$ with $\beta_1 = 3$, $\beta_2 = -2$, $\beta_0 = 0.1$.

$S(5,1) = S(5,1) + 9C$	$< \beta_1^2 C = 9C >$
$S(9,1) = S(9,1) + 4C$	$< \beta_2^2 C = 4C >$
$S(5,5) = S(5,5) - 6C$	<note second 5 = $9 - 5 + 1$, and $\beta_1\beta_2 C = -6C$ >
$F(5) = F(5) + 0.3C$	$<\beta_0\beta_1 C = 0.3C>$
$F(9) = F(9) - 0.2C$	$<\beta_0\beta_2 C = -0.2C>$ ■

6.21 Using the octagonal symmetry, we model the 22.5^0 segment.





The mesh model is created by defining a single block in the MESHGEN. A model with 2 divisions in the *S* direction and two divisions in the *W* direction is shown above.

Loads are calculated using the steps given in Problem 5.12. Let $l = L_{14} = L_{47} = 15\sin(\theta/2)$, and $F = plt$. Then the forces are

$F_1 = F\cos(\theta/4)$ $\quad$ $F_2 = F\sin(\theta/4)$
$F_7 = F\cos(\theta/4) + F\cos(3\theta/4)$ $\quad$ $F_8 = F\sin(\theta/4) + F\sin(3\theta/4)$
$F_{13} = F\cos(3\theta/4)$ $\quad$ $F_{14} = F\sin(3\theta/4)$

These loads need to be calculated and edited into the data file by setting NL = 6.

The boundary conditions $Q_2 = 0$, $Q_3 = 0$, $Q_5 = 0$ are easily taken care of by setting ND = 3 and editing in dof# and displacement. At nodes 7, 8, and 9 we need to apply the multipoint constraint condition (MPC). We set NMPC = 3. The three MPC conditions are

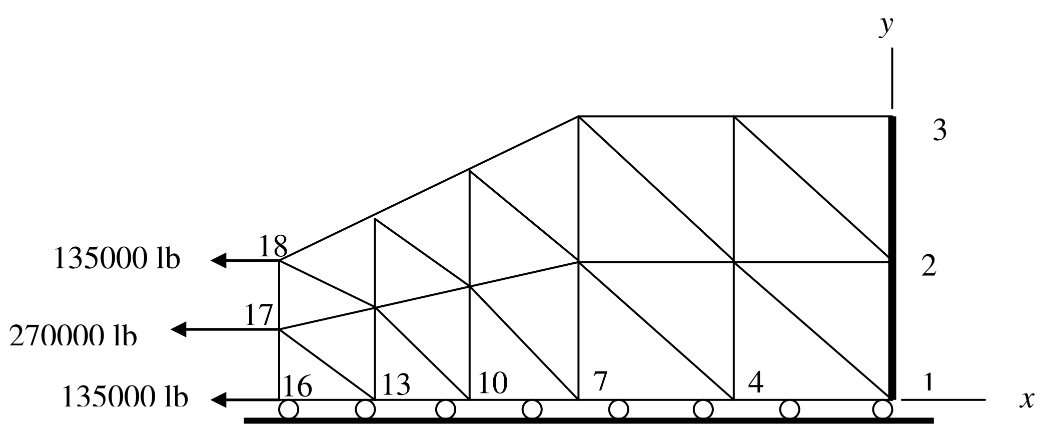
$$Q_{13}\sin\theta - Q_{14}\cos\theta = 0 \qquad (\sin\theta = 0.3827 \qquad \cos\theta = 0.9239)$$
$$Q_{15}\sin\theta - Q_{16}\cos\theta = 0$$
$$Q_{17}\sin\theta - Q_{17}\cos\theta = 0$$

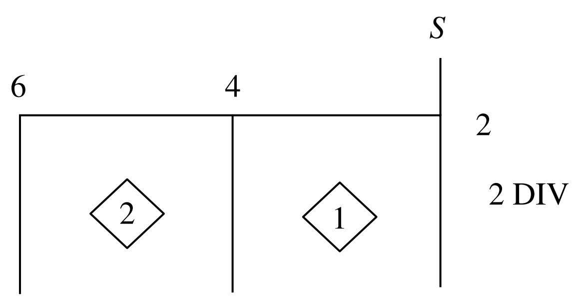
The MPC data follow the last line in the input file as follows:

```
B1   i   B2  j   B3   (Multi-point constr. B1*Qi+B2*Qj=B3)
0.3827  13  0.9239  14  0
0.3827  15  0.9239  16  0
0.3827  17  0.9239  17  0
```

■

6.22





W

5 3 DIV 3 2 DIV 1

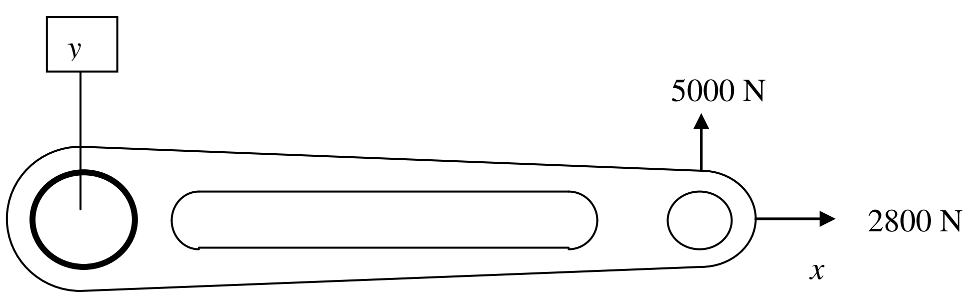
Block diagram and the mesh division for the problem are shown above. MESHGEN needs two blocks with the spans and divisions indicated on the diagram.

The data file generated by MESHGEN needs following modifications.

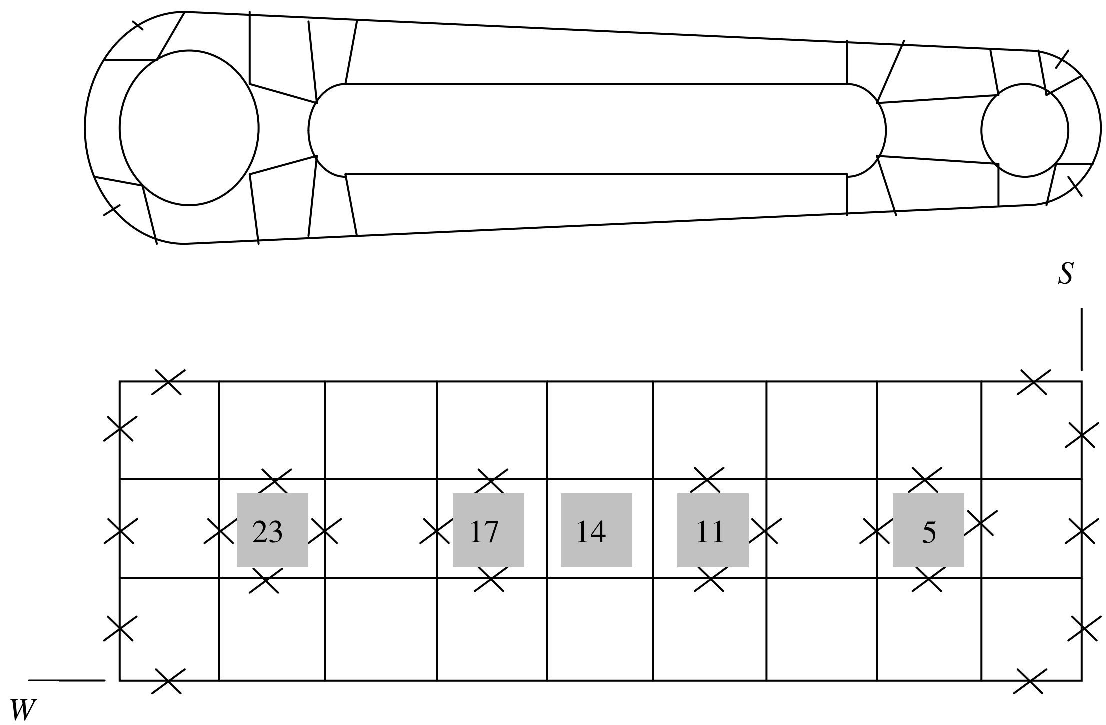
```
ND   NL   NMPC
 11  3   0
Node#   X    Y
 …
Elem#  Node1  Node2  Node3  Material#   Thickness  TempRise
 …  1   0
 …
DOF#  Displacement
 1     0
 2     0
 3     0
 4     0
 5     0
 6     0
 8     0
 14     0
 20     0
 26     0
 32     0
DOF#  Load
 31    -135000
 33    -270000
 35    -135000
MAT#   E      Nu    Alpha
 1    30E6   .3    12E-6
B1  i  B2 j  B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

The data file generated is edited as above and input to the CST program choosing the plane stress option. ■

6.23



In this problem, the displacement constraint (displacement = 0) may be placed on the boundary of the larger hole. The main work in this problem is to get the mesh data using MESHGEN or some other CAD program. The block diagram steps are indicated here.



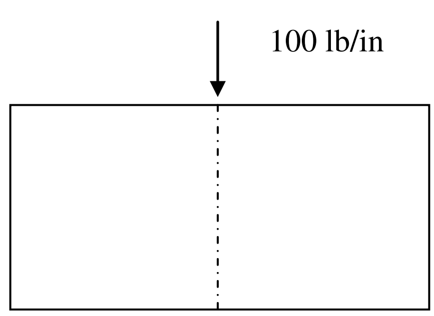
9 SPANS

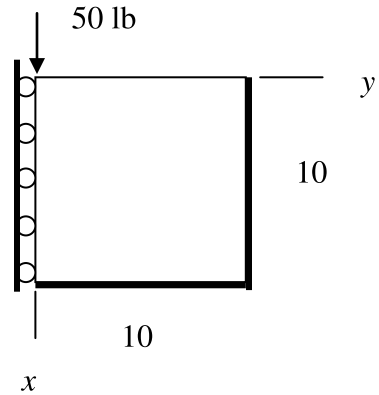
Material number for the shaded blocks 5, 11, 14, 17, and 23 needs to be defined as zero.

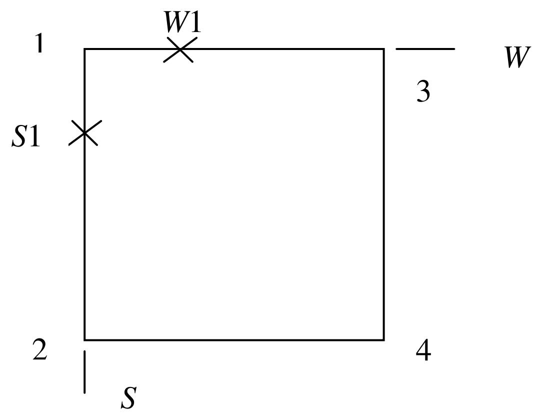
All sides with cross mark S_1, S_2, S_3, S_5, S_8, S_{11}, S_{20}, S_{23}, S_{26}, S_{28}, S_{29}, S_{30}, W_1, W_4, W_6, W_7, W_{14}, W_{15}, W_{22}, W_{23}, W_{30}, W_{31}, W_{33}, W_{36} are curved and the coordinates of these points (midpoints) must be given.

Further division of the spans will result in large number of elements and nodes. Crude meshes (for example 1 division for each span) may be tried out in the initial runs to check if the results are physically reasonable. The boundary conditions and load values have to be edited into the data file generated from MESHGEN. ■

6.24



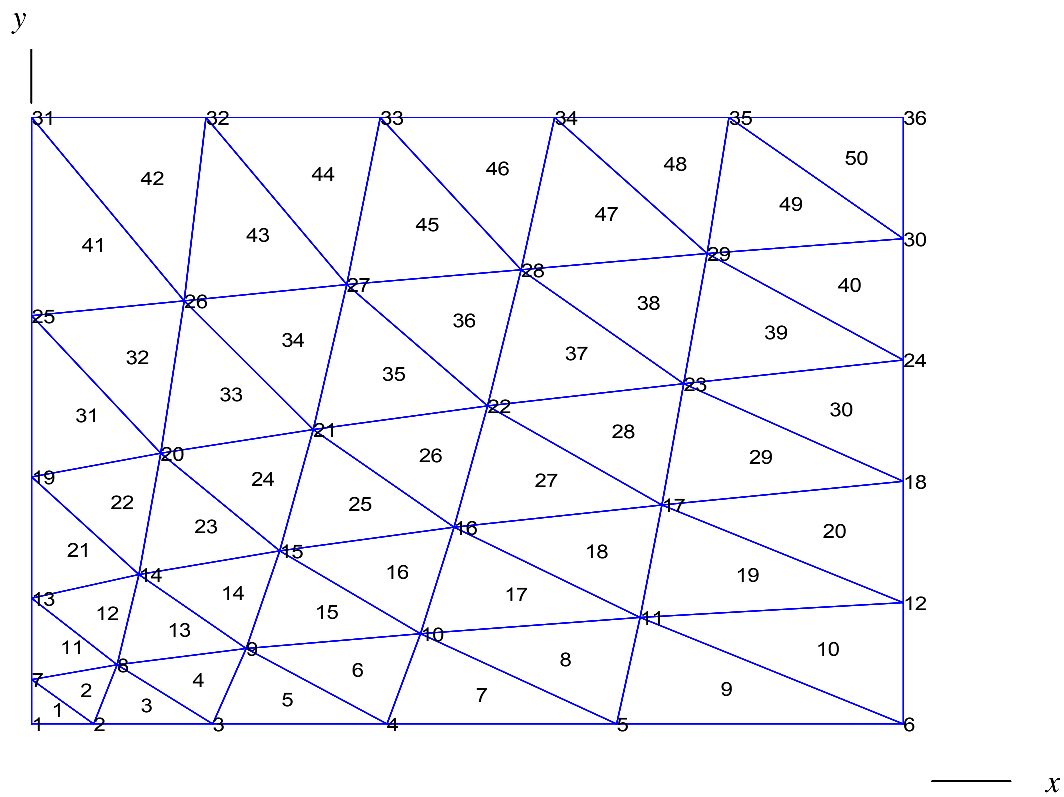




Note that in the *S*1 is defined at (3, 0) and *W*1 is defined at (0, 3) even for straight sides. This results in a graded mesh with smaller elements close to corner 1, which is at (0, 0). The created mesh is shown in the figure following the mesh data set prepared for MESHGEN.

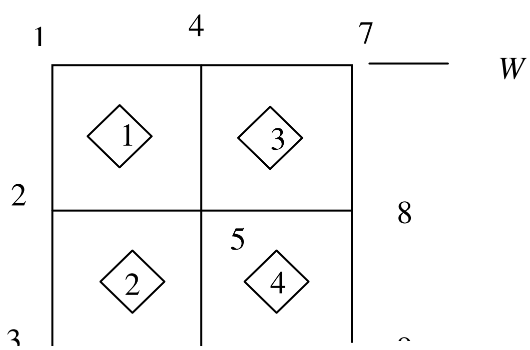
```
Mesh Generation
Problem 5.22
Number of Nodes per Element <3 or 4>
  3
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ)
  1             1             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
  1        5
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        5
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
  1      0         0
  2      10        0
  3      0         10
```

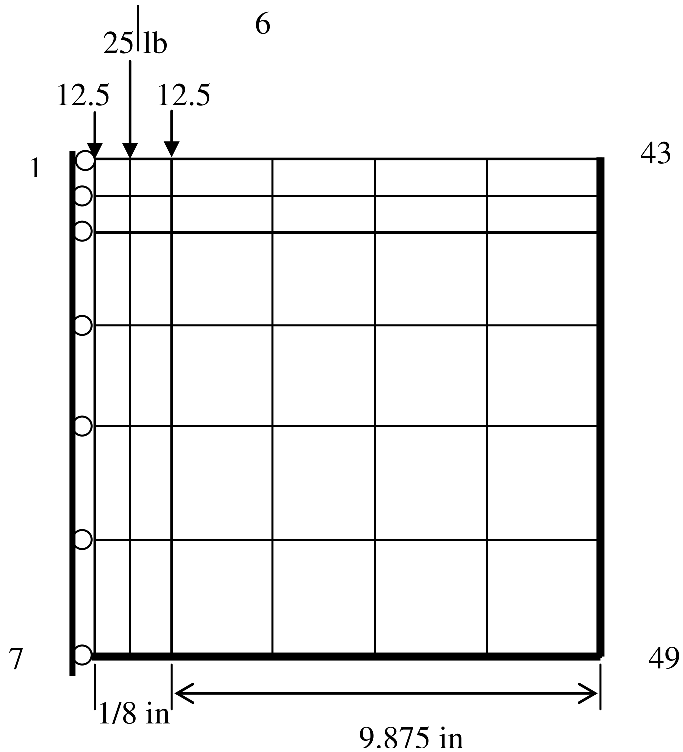
```
  4      10        10
  0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#   X-Coord   Y-Coord (Side# = 0 completes this data)
  1      3         0
  0
W-Side#   X-Coord   Y-Coord (Side# = 0 completes this data)
  1      0         3
  0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
```



The input data generated from MESHGEN needs to be edited in manner similar to problems 5.15 to 5.17. Note that nodes 1,7,…,31 and 31,32,..36 are completely fixed and nodes 1,2,…6 are fixed in the *y*-direction. The 50 lb load is applied along dof 1. Plane strain option has to be chosen interactively from CST program. On solving the problem, the deformation under the load is about 7.113×10^{-6}in. ■

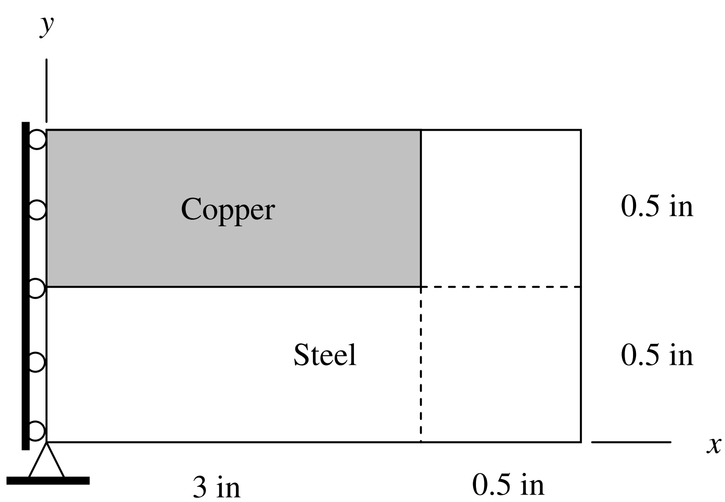
6.25

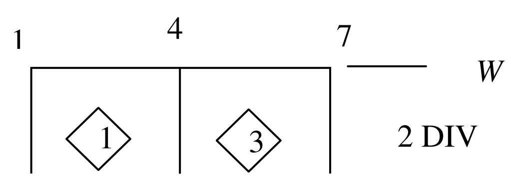




The block division and mesh division are shown above. The problem is solved using plane strain condition in CST. Discussion provided for problem 5.22 also holds for this problem. After trying out the coarse mesh suggested, finer meshes can be tried out. ■

6.26 We model one half of the configuration using the symmetry of the problem.





2 DIV

3 DIV 2 DIV

Blocks 2,3,4 Material 1 (Steel) Block 1 Material 2 (Copper)

File for the MESHGEN, the edited data file, and the output from CST are now given.

Input File for MESHGEN

```
Mesh Generation
Problem 5.24
Number of Nodes per Element <3 or 4>
  3
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ)
  2             2             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
  1        2
  2        2
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        3
  2        2
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  1        2
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
  1      0         1
  2      0         0.5
  3      0         0
  4      2.5       1
  5      2.5       0.5
  6      2.5       0
  7      3         1
  8      3         0.5
  9      3         0
  0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
```

Input File for CST with edited lines

```
Program MESHGEN - CHANDRUPATLA & BELEGUNDU
Problem 5.24
NN  NE  NM  NDIM  NEN  NDN
 30  40  2  2  3  2
ND   NL   NMPC
 6  0  0
Node#   X    Y
 1  0  1
 2  0  .75
 3  0  .5
 4  0  .25
 5  0  0
 6  .8333333  1
 7  .8333333  .75
 8  .8333333  .5
 9  .8333333  .25
 10  .8333334  0
 11  1.666667  1
 12  1.666667  .75
 13  1.666667  .5
 14  1.666667  .25
 15  1.666667  0
 16  2.5  1
 17  2.5  .75
 18  2.5  .5
 19  2.5  .25
 20  2.5  0
 21  2.75  1
 22  2.75  .75
 23  2.75  .5
 24  2.75  .25
 25  2.75  0
 26  3  1
 27  3  .75
 28  3  .5
 29  3  .25
 30  3  0
Elem#  Node1  Node2  Node3  Material#  Thickness  TempRise
 1  1  2  7  2    1   80
 2  7  6  1  2    1   80
 3  2  3  8  2    1   80
 4  8  7  2  2    1   80
 5  6  7  12  2    1   80
 6  12  11  6  2    1   80
 7  7  8  13  2    1   80
 8  13  12  7  2    1   80
 9  11  12  17  2    1   80
 10  17  16  11  2    1   80
 11  12  13  18  2    1   80
 12  18  17  12  2    1   80
 13  3  4  9  1    1   80
 14  9  8  3  1    1   80
 15  4  5  10  1    1   80
 16  10  9  4  1    1   80
```

```
 17  8  9  14  1    1   80
 18  14  13  8  1    1   80
 19  9  10  15  1    1   80
 20  15  14  9  1    1   80
 21  13  14  19  1    1   80
 22  19  18  13  1    1   80
 23  14  15  20  1    1   80
 24  20  19  14  1    1   80
 25  16  17  22  1    1   80
 26  22  21  16  1    1   80
 27  17  18  23  1    1   80
 28  23  22  17  1    1   80
 29  21  22  27  1    1   80
 30  27  26  21  1    1   80
 31  22  23  28  1    1   80
 32  28  27  22  1    1   80
 33  18  19  24  1    1   80
 34  24  23  18  1    1   80
 35  19  20  25  1    1   80
 36  25  24  19  1    1   80
 37  23  24  29  1    1   80
 38  29  28  23  1    1   80
 39  24  25  30  1    1   80
 40  30  29  24  1    1   80
DOF#  Displacement
 1     0
 3     0
 5     0
 7     0
 9     0
 10    0
DOF#  Load
MAT#   E      Nu    Alpha
 1    30e6   .3     6.5e-6
 2    18e6   .25    10e-6
B1  i  B2 j  B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

Note that there are no applied loads in this problem. Temperature loads cause deformation.

Output from program CST

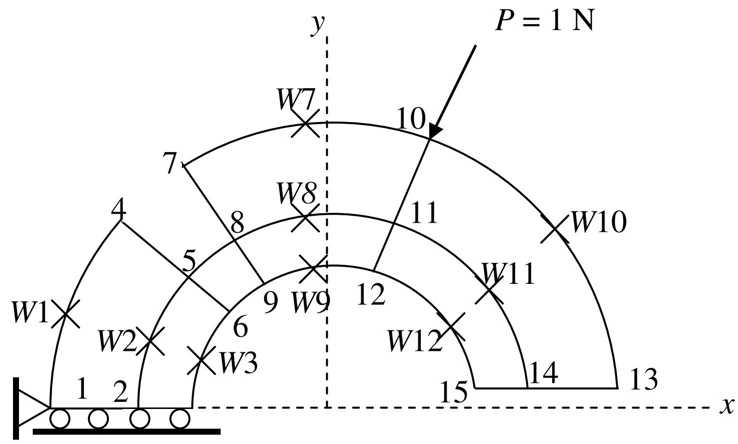
```
Output for Input Data in File --- p524.inp
Problem 5.24
Plane Stress Analysis
NODE#   X-Displ     Y-Displ
   1   6.780E-11   6.802E-04
   2  -2.225E-10   4.709E-04
   3   9.535E-11   2.586E-04
   4   2.056E-10   1.313E-04
   5  -1.462E-10   5.438E-15
   6   6.185E-04   5.990E-04
   7   5.735E-04   3.934E-04
   8   5.294E-04   1.843E-04
   9   4.856E-04   6.083E-05
  10   4.424E-04  -6.634E-05
  11   1.239E-03   3.789E-04
  12   1.147E-03   1.748E-04
```

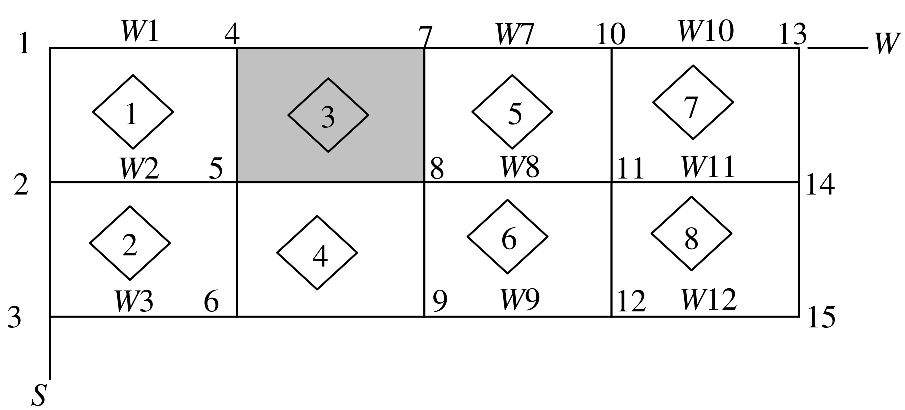
```
  13   1.060E-03  -3.366E-05
  14   9.738E-04  -1.573E-04
  15   8.889E-04  -2.839E-04
  16   1.885E-03  -4.196E-05
  17   1.740E-03  -2.155E-04
  18   1.591E-03  -3.897E-04
  19   1.458E-03  -5.108E-04
  20   1.341E-03  -6.368E-04
  21   2.019E-03  -2.296E-04
  22   1.859E-03  -3.645E-04
  23   1.715E-03  -5.045E-04
  24   1.591E-03  -6.365E-04
  25   1.468E-03  -7.651E-04
  26   2.148E-03  -3.836E-04
  27   1.990E-03  -5.127E-04
  28   1.845E-03  -6.385E-04
  29   1.720E-03  -7.641E-04
  30   1.597E-03  -8.927E-04
DOF#        Reaction
    1     -1.5370E+02
    3      5.0452E+02
    5     -2.1616E+02
    7     -4.6612E+02
    9      3.3151E+02
   10     -1.2328E-02
ELEM#    SX         SY          TXY         S1          S2     ANGLE SX->S1
   1 -1.969E+03  1.786E+02 -6.697E+02  3.704E+02 -2.161E+03 -7.402E+01
   2 -1.003E+03  1.500E+02  5.955E+02  4.024E+02 -1.255E+03  6.703E+01
   3 -2.926E+03  1.575E+02 -6.415E+02  2.857E+02 -3.054E+03 -7.870E+01
   4 -1.973E+03  1.605E+02  5.993E+02  3.173E+02 -2.130E+03  7.534E+01
   5 -2.032E+03 -1.073E+02 -5.908E+02  5.957E+01 -2.199E+03 -7.423E+01
   6 -9.869E+02  4.911E+01  7.377E+02  4.325E+02 -1.370E+03  6.254E+01
   7 -2.960E+03 -8.626E+01 -6.142E+02  3.950E+01 -3.086E+03 -7.843E+01
   8 -1.976E+03  1.168E+02  6.255E+02  2.895E+02 -2.149E+03  7.456E+01
   9 -1.618E+03 -1.085E+02 -7.339E+02  1.895E+02 -1.916E+03 -6.790E+01
  10 -9.741E+02 -2.148E+03  5.397E+02 -7.637E+02 -2.358E+03  2.130E+01
  11 -2.960E+03 -1.294E+02 -5.623E+02 -2.177E+01 -3.068E+03 -7.917E+01
  12 -2.192E+03 -2.407E+03  9.249E+02 -1.369E+03 -3.231E+03  4.168E+01
  13  1.961E+03  2.625E+02 -9.754E+02  2.405E+03 -1.814E+02 -2.447E+01
  14  3.542E+03  2.805E+02  9.915E+02  3.820E+03  2.760E+00  1.565E+01
  15  4.098E+02  2.756E+02 -9.186E+02  1.264E+03 -5.783E+02 -4.291E+01
  16  1.957E+03  2.474E+02  1.019E+03  2.432E+03 -2.278E+02  2.500E+01
  17  1.913E+03 -2.081E+02 -1.000E+03  2.311E+03 -6.054E+02 -2.166E+01
  18  3.597E+03  3.146E+02  9.585E+02  3.856E+03  5.513E+01  1.514E+01
  19  4.071E+02 -2.176E+02 -1.018E+03  1.160E+03 -9.702E+02 -3.647E+01
  20  2.038E+03  2.055E+02  9.019E+02  2.407E+03 -1.640E+02  2.228E+01
  21  1.761E+03 -2.360E+02 -9.190E+02  2.120E+03 -5.944E+02 -2.131E+01
  22  3.518E+03 -6.434E+00  1.210E+03  3.893E+03 -3.820E+02  1.724E+01
  23  6.131E+02 -2.218E+02 -9.637E+02  1.246E+03 -8.546E+02 -3.329E+01
  24  1.852E+03  6.420E+01  5.042E+02  1.984E+03 -6.822E+01  1.472E+01
  25  2.855E+02  5.312E+03 -1.816E+02  5.319E+03  2.790E+02 -8.793E+01
  26  6.431E+02  7.771E+02 -1.306E+03  2.018E+03 -5.980E+02 -4.647E+01
  27  9.629E+02  5.590E+03  1.589E+03  6.083E+03  4.700E+02  7.276E+01
  28 -1.041E+03  8.918E+02 -2.157E+02  9.155E+02 -1.064E+03 -8.371E+01
  29  3.298E+02  6.831E+02  5.094E+02  1.046E+03 -3.275E+01  5.456E+01
  30 -1.538E+02 -1.539E+02  1.539E+02  4.730E-03 -3.077E+02  4.500E+01
  31  4.195E+02  1.330E+03  4.732E+02  1.531E+03  2.181E+02  6.694E+01
```

```
  32 -3.052E+01 -5.178E+02 -1.454E+02  9.577E+00 -5.579E+02 -1.541E+01
  33  5.254E+01 -1.046E+03  3.389E+02  1.487E+02 -1.142E+03  1.584E+01
  34 -7.039E+02  3.420E+01  4.261E+02  2.289E+02 -8.985E+02  6.545E+01
  35 -5.445E+02 -6.546E+02 -5.231E+02 -7.361E+01 -1.126E+03 -4.200E+01
  36  3.448E+02 -7.167E+01 -1.267E+02  3.804E+02 -1.072E+02 -1.566E+01
  37 -3.724E+01  2.342E+02 -1.614E+02  3.093E+02 -1.124E+02 -6.503E+01
  38 -1.533E+02 -5.793E+02 -4.116E+02  9.720E+01 -8.298E+02 -3.132E+01
  39 -2.022E+02 -2.358E+02 -2.157E+02 -2.671E+00 -4.353E+02 -4.278E+01
  40 -1.722E+02 -2.157E+02 -2.022E+02  9.420E+00 -3.973E+02 -4.193E+01
```

Plotting of deformed configuration needs some modifications to the PLOT2D program. This may be taken up as a project. ■

6.27

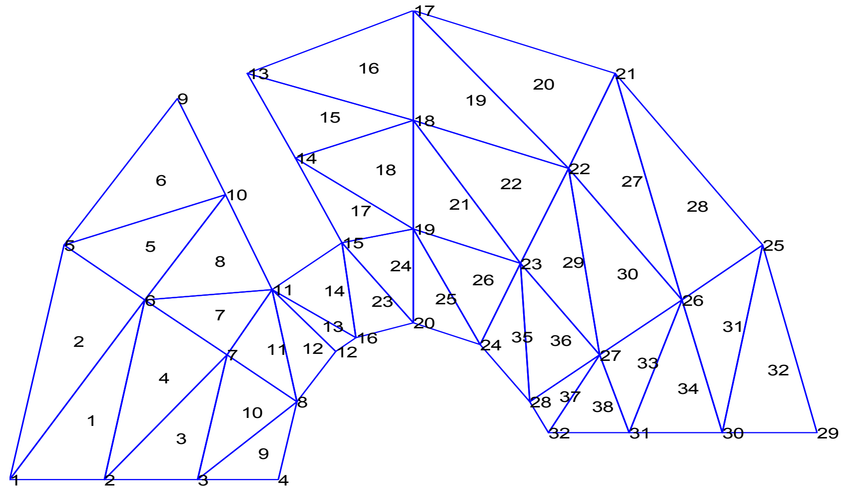




The region map and block diagram for the problem are shown above. The geometry is used in creating the data file for MESHGEN.

```
Mesh Generation
Problem 5.25
Number of Nodes per Element <3 or 4>
  3
BLOCK DATA
```

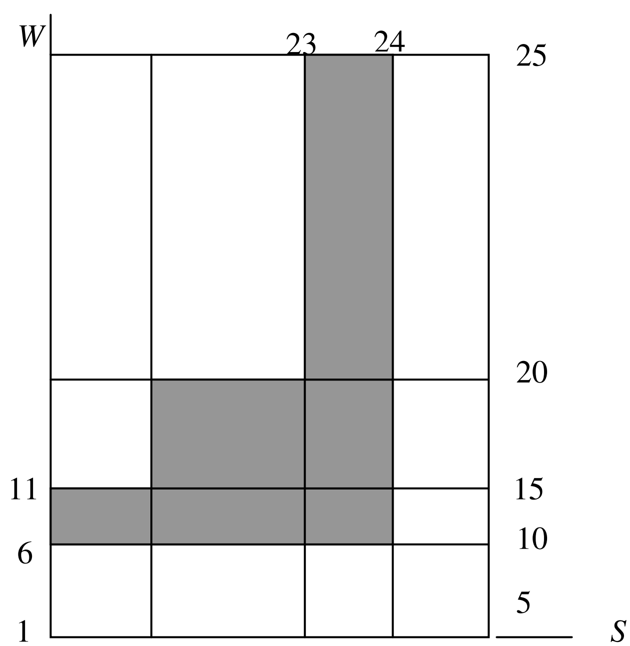
```
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ)
  2             4             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
  1        2
  2        1
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        2
  2        1
  3        2
  4        2
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  3        0
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
  1      -15       0
  2      -8        0
  3      -5        0
  4      -8.761    12.175
  5      -5.228    6.055
  6      -2.868    4.096
  7      -6.163    12.99
  8      -2.637    7.555
  9      -2.113    4.532
  10     7.5       12.99
  11     4         6.928
  12     2.5       4.33
  13     15        1.5
  14     8         1.5
  15     5         1.5
  0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  1      -12.99    7.5
  2      -6.928    4
  3      -4.33     2.5
  7      0         15
  8      0         8
  9      0         5
  10     12.99     7.5
  11     6.928     4
  12     4.33      2.5
  0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
```
The input for generated by MESHGEN using the above data file is edited as discussed in problems 5.15 and 5.16. The mesh plot is shown below.



The deflection is calculated for unit load. Once Q_{58} is determined for the unit load, the load P required to close the gap will be

$$P = \frac{1.5}{Q_{58}} \qquad \blacksquare$$

6.28 The ideas of mesh generation and imposition of multipoint constraints are discussed here. Two dimensional mesh generation details presented in Chapter 12 are to be studied at this stage. We make use of the symmetry about the vertical center line. Schematic block division is now shown.



The six shaded blocks 5,6,7,9,10,11,15 are defined to be of material 0. This will make the two bodies independent of each other. The number of divisions of each S and W spans are to be set at appropriate values. We need to make S–span 4 into larger number of divisions (say 5 initially), so that contact can be established by imposing multipoint constraints on the coincident degrees of freedom.

The coordinates of corner points are defined in the positions occupied by the corresponding points on the bodies. As an example, we give coordinates of few points here.

Point	x-coord	y-coord
1	0	0
6	0	15
…		
10	65	15
11	0	15
12	20	15
…		
14	50	25
15	65	25
…		
18	51.5	35
19	50	35
…		
23	51.5	75
24	50	75
25	65	75

The MPC condition is obtained as follows. If the dof on a node T-piece is Q_i and the corresponding dof on the U-piece is Q_j then the MPC condition is

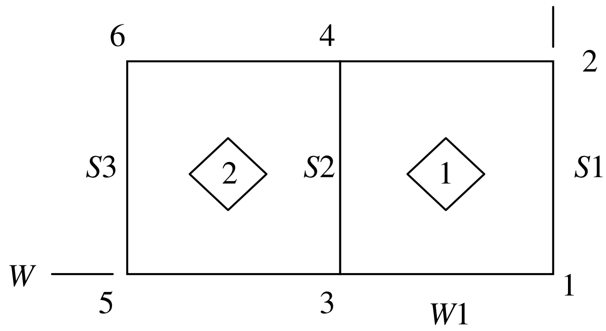
$$Q_i + x_{(i)} = Q_j + x_{(j)} \quad \text{which is } \; Q_j - Q_j = -1.5.$$

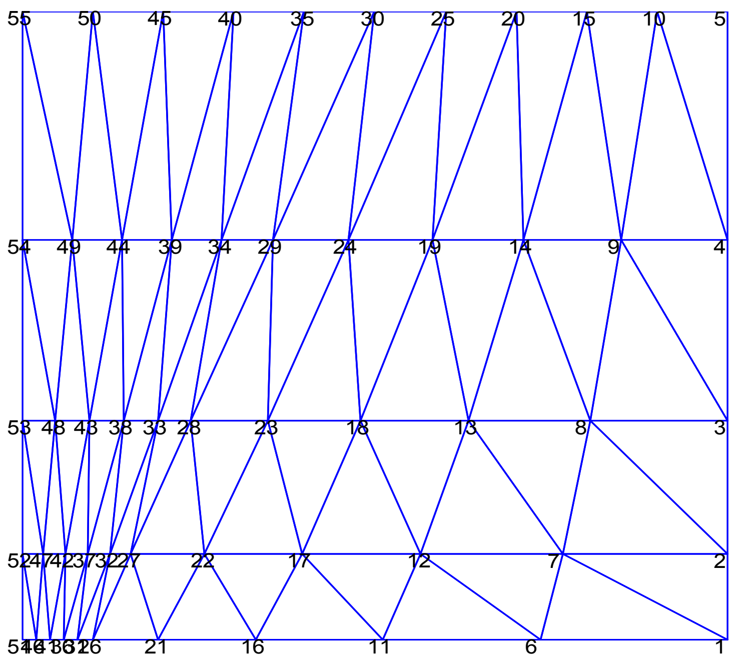
Establishment of the contact region $L_{\text{interface}}$ needs trial and error. This needs the evaluation of reaction components corresponding to the dof i and j. If these reaction components are pushing on the surface, the contact is established. If the reaction components are pulling on the surface, there is no contact at the pair of dof. This pair of dof must be removed from the MPC set in the next trial.

At least one trial may be given as an assignment and the next steps may be discussed. ■

6.29 The problem is modeled using symmetry about the central horizontal axis. A two block configuration is used. Midside nodes are defined for S_1, S_2, S_3, and W_1 so that smaller elements are near to the crack region. The two block configuration and the corresponding mesh generated are shown in the figures below. The data file for MESHGEN is also given. The load calculations and the boundary conditions follow.

S





Note that in the mesh above, 1-5-26-30 is Block 1 and 26-30-51-55 is Block 2. The Mesh data file used in creating the above mesh is given below.

```
Mesh Generation
Problem 5.27
Number of Nodes per Element <3 or 4>
   3
BLOCK DATA
#S-Spans(NS)   #W-Spans(NW)   #PairsOfEdgesMergedNSJ)
  1              2              0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
  1         4
```

```
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        5
  2        5
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
  1       85.5     0
  2       85.5     200
  3       0        0
  4       38       200
  5      -9.5      0
  6      -9.5      200
  0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  1      85.5      70
  2      13.3      70
  3     -9.5       70
  0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  1      30        0
  0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
```

In the mesh shown, the edge 5-10-…-55 has 10 element boundaries each of length 9.5 mm with a distributed load of 450MPa. Using thickness of 1 mm, the load on each element edge is

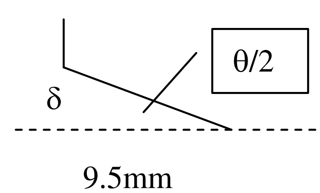
$9.5 \times 450 \times 1 = 4275$ N. The outermost nodes 5 and 55 each get a vertical load (along *y*) of half this value equal to 2137.5N. The inner nodes 10, 15,…, 50 each get a load of 4275N along *y* direction.

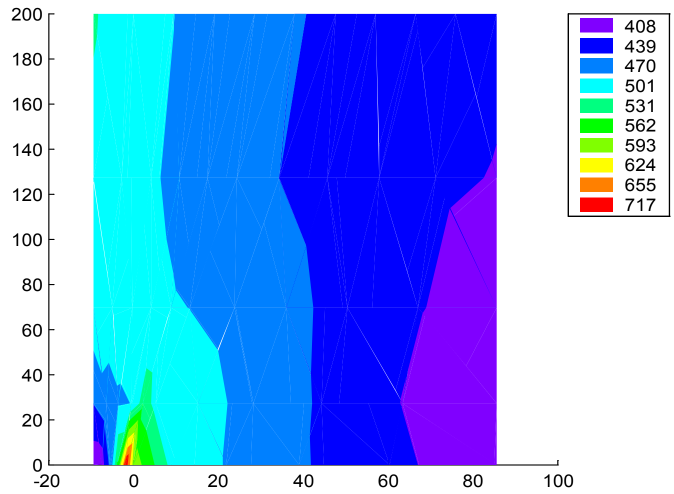
The degrees of freedom with 0 displacement are 1, 2, 12, 22, 32, 42, 52.

The input data file is edited to introduce these. The resulting stress distribution is shown below. The crack tip deflection is obtained as

$$\delta = 0.06093 \text{ mm}$$

$$\theta = 2\tan^{-1}(\delta/9.5) = 0.735^0$$





■

6.30
6.31 These problems require computer program (CST) modification. The fiber orientation angle θ of each element is to be read in. This can be done by dimensioning angle variable. Add the THETA(NE) in the dimension statement. We refer to the QuickBasic program CST.BAS here. See the corresponding lines in other programs.

DIM THETA(NE)

In the line where we read the connectivity, we need to read angle theta. If the angle theta for element 1 is 30^0 (for data from Prob 5.15), we add an additional column as follows.

```
Elem#   Node1   Node2   Node3   Material#   Thickness   TempRise   ThetaDeg
1       1       2       5       1           10          0          30
```

In the data read section add THETA(N) in the following line.

INPUT #1, MAT(N), TH(N), DT(N), THETA(N)

Having read the angle in degrees, convert to radians for all calculations.

The **D** matrix section of the program needs to be modified. The **D** matrix for orthotropic materials is defined in Eq.6.81. Angle THETA(N) needs to be available in this subroutine.

This **D** is again used later to calculate the stresses. The equation is

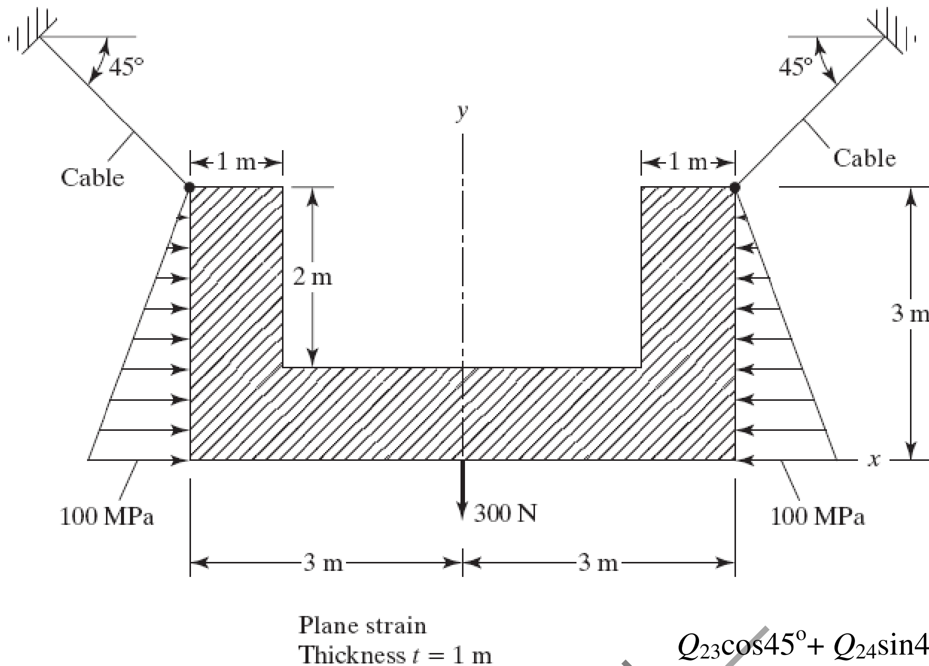
$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

This evaluation results in stresses in the x, y system. The transformation matrix **T** is now used to get the stresses with respect to the fiber orientations.

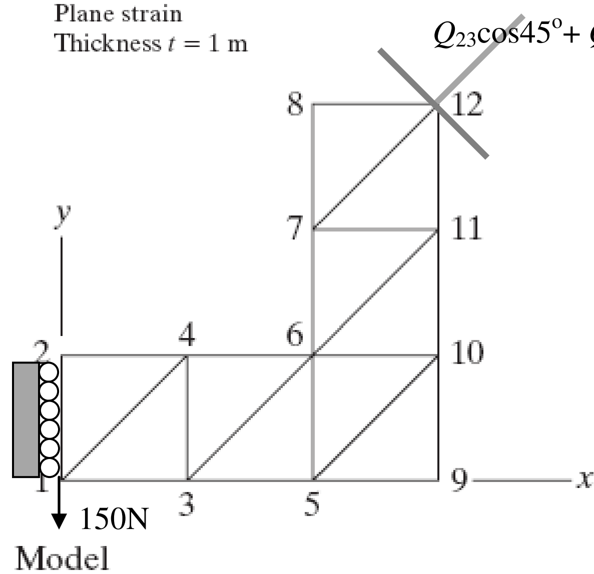
$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \mathbf{T} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

These calculations need to be introduced in the section of the program where the stresses are evaluated. ■

6.32



$$Q_{23}\cos 45^{\circ} + Q_{24}\sin 45^{\circ} = 0$$
$$Q_{23} + Q_{24} = 0$$



We assume the cable to be inextensible, which requires that there is no deflection component along its axis. Thus $Q_{23} + Q_{24} = 0$, which is a multipoint constraint. For the half symmetric model, nodes 1 and 2 are constrained to move along the y axis, thus

$Q_1 = 0 \quad Q_3 = 0$. Also symmetry implies that we apply a load of 150N (300/2) downward at node 1.

Input Data

2D STRESS ANALYSIS USING CST
PROBLEM 6.32

NN	NE	NM	NDIM	NEN	NDN	
12	10	1	2	3	2	
ND	**NL**	**NMPC**				
2	1	1				
Node#	**X**	**Y**				
1	0	0				
2	0	1				
3	1	0				
4	1	1				
5	2	0				
6	2	1				
7	2	2				
8	2	3				
9	3	0				
10	3	1				
11	3	2				
12	3	3				
Elem#	**N1**	**N2**	**N3**	**Mat#**	**Thick**	**deltaT**
1	1	3	4	1	1	0
2	1	4	2	1	1	0
3	3	5	6	1	1	0
4	3	6	4	1	1	0
5	5	9	10	1	1	0
6	5	10	6	1	1	0
7	6	10	11	1	1	0
8	6	11	7	1	1	0
9	7	11	12	1	1	0
10	7	12	8	1	1	0
DOF#	**Displacement**					
1	0					
3	0					
DOF#	**Load**					
4	-1000					
MAT#	**E**	**Nu**	**Alpha**			
1	3.00E+07	0.25	1.20E-05			
B1	**i**	**B2**	**j**	**B3**	**(Multi-point**	**constr.**
1	23	1	24	0		

Output

Program CST - Plane Stress Analysis

PROBLEM 6.32

Node#	X-Displ	Y-Displ
1	0.00000	-0.00084
2	0.00000	-0.00088
3	0.00003	-0.00077
4	0.00004	-0.00079
5	0.00001	-0.00077
6	0.00013	-0.00074
7	0.00054	-0.00066
8	0.00096	-0.00064
9	0.00000	-0.00098
10	0.00016	-0.00099
11	0.00052	-0.00102
12	0.00098	-0.00098

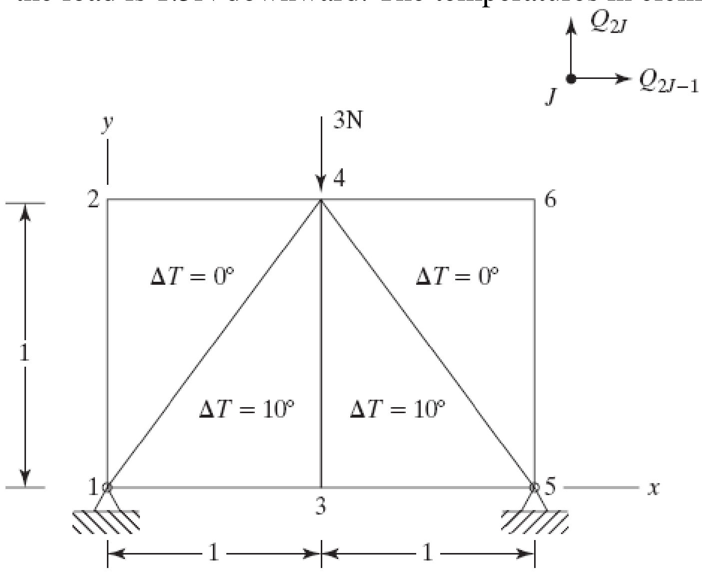
Elem#	SX	SY	Txy	S1	S2	S3
1	940.90	-428.68	940.90	1419.82	-907.60	26.98
2	1059.10	-940.90	1059.10	1515.71	-1397.50	23.32
3	-622.03	715.90	1377.97	1578.70	-1484.83	57.95
4	2622.03	-8.39	622.03	2761.71	-148.07	12.66
5	-500.63	-500.63	-500.63	0.00	-1001.27	-45.00
6	1165.14	1162.70	-1499.37	2663.28	-335.45	-44.98
7	720.79	-614.71	1385.29	1590.87	-1484.79	32.13
8	60.00	2614.71	614.71	2754.92	-80.21	77.15
9	-309.80	1135.50	1135.50	1758.80	-933.10	61.24
10	864.50	864.50	864.50	1729.01	0.00	45.00

DOF#	Reaction
1	-1000
3	0

■

6.33 Use MATLAB program CST.m to solve the problem. Observe the matrices in the code. You may also code the problem in MATLAB as suggested. ■

6.34 We model the left half of the problem. The BC are $Q_1 = 0 \quad Q_2 = 0 \quad Q_5 = 0 \quad Q_7 = 0$, and the load is 1.5N downward. The temperatures in elements are to be entered in the data.



Input data

2D STRESS ANALYSIS USING CST
PROBLEM 6.32

NN	NE	NM	NDIM	NEN	NDN	
4	2	1	2	3	2	
ND	**NL**	**NMPC**				
4	1	0				
Node#	**X**	**Y**				
1	0	0				
2	0	1				
3	1	0				
4	1	1				
Elem#	**N1**	**N2**	**N3**	**Mat#**	**Thickness**	**TempRise**
1	1	3	4	1	1	10
2	1	4	2	1	1	0
DOF#	**Displacement**					
1	0					
2	0					
5	0					
7	0					
DOF#	**Load**					
4	-1.5					
MAT#	**E**	**Nu**	**Alpha**			
1	1	0.3	0.1			
B1	**i**	**B2**	**j**	**B3**	**(Multi-point**	**constr.**

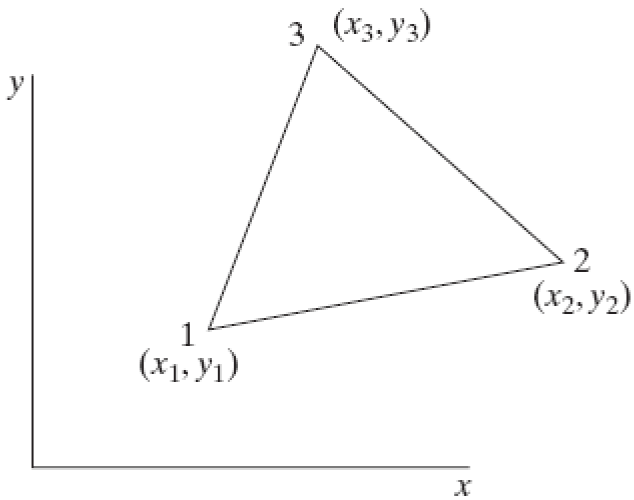
Output

Program CST2 - Plane Stress Analysis
PROBLEM 6.32

Node#	X-Displ	Y-Displ				
1	-0.00005	-0.00020				
2	-1.21692	-2.68329				
3	0.00005	-1.17728				
4	0.00000	-0.28930				
Elem#	**SX**	**SY**	**Txy**	**S1**	**S2**	**S3**
1	-1.13573	-0.45274	-0.45274	-0.22714	-1.36132	-63.5134
2	0.45274	-2.54726	0.45273	0.51957	-2.6141	8.39757
			8	3		7
DOF#	**Reaction**					
1	0.34150					
2	1.50000					
5	-0.34150					
7	0.00000					

■

6.35



S in the equation $\varepsilon_x = \mathbf{Sq}$ is the first row or the **B** matrix (6.26) in the relation $\boldsymbol{\varepsilon} = \mathbf{Bq}$ (6.25) derived in the text.

$$\mathbf{S} = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \end{bmatrix}^{\mathrm{T}}$$

$$\det \mathbf{J} = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

■

6.36 We need to get expression **W** in $\int_e u^2 dA = \mathbf{q}^{\mathrm{T}}\mathbf{W}\mathbf{q}$. We use $u = \mathbf{Nq}$. Then

$$\int_e u^2 dA = \mathbf{q}^{\mathrm{T}} \int_0^1 \int_0^{1-\xi} \mathbf{N}^{\mathrm{T}} \mathbf{N} \det \mathbf{J} d\xi d\eta \mathbf{q}$$

$$= \mathbf{q}^{\mathrm{T}} 2A_e \int_0^1 \int_0^{1-\xi} \begin{bmatrix} \xi^2 & \xi\eta & \xi(1-\xi-\eta) \\ \xi\eta & \eta^2 & \eta(1-\xi-\eta) \\ \xi(1-\xi-\eta) & \eta(1-\xi-\eta) & (1-\xi-\eta)^2 \end{bmatrix} d\xi d\eta \mathbf{q}$$

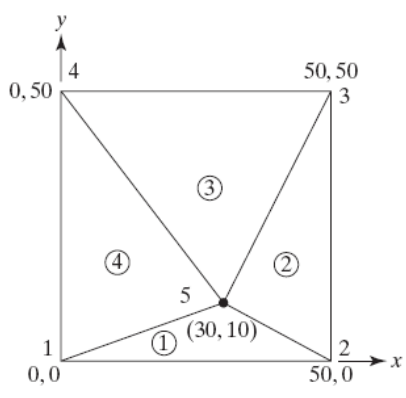
Now using the result (6.46), $\int_0^1 \int_0^{1-\xi} \xi^a \eta^b (1-\xi-\eta)^c \, d\xi d\eta = \dfrac{a!b!c!}{(a+b+c+2)}$ we get

$$\int_e u^2 dA =$$

$$= \mathbf{q}^{\mathrm{T}} 2A_e \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{12} \end{bmatrix} \mathbf{q}$$

$$\mathbf{W} = A_e \begin{bmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & \frac{1}{6} \end{bmatrix} \qquad \blacksquare$$

6.37



BC	Node	u	v
	1	0	0
	2	0.005	0.0025
	3	0.0075	0.0075
	4	0.0025	0.0050

$E = 10^6$ $\qquad$ $\nu = 0.25$

The displacements of the four corner nodes are specified using the values calculated from

$$u = 10^{-4}\left(x + \frac{y}{2}\right)$$

$$v = 10^{-4}\left(\frac{x}{2} + y\right)$$

This should result in a constant state of strain $\varepsilon_x = 0.0001, \quad \varepsilon_y = 0.0001, \quad \gamma_{xy} = 0.0001$. The expected plane stress solution for the patch test is
$u = 0.0035 \quad v = 0.0025 \quad \sigma_x = 133.3 \quad \sigma_y = 133.3 \quad \tau_{xy} = 40$.

Input Data

2D STRESS ANALYSIS USING CST
PROBLEM 6.37

NN	NE	NM	NDIM	NEN	NDN
5	4	1	2	3	2

ND	NL	NMPC
8	0	0

Node#	X	Y
1	0	0
2	50	0
3	50	50
4	0	50
5	30	10

Elem#	N1	N2	N3	Mat#	Thick	DeltaT
1	1	2	5	1	1	0
2	2	3	5	1	1	0
3	3	4	5	1	1	0
4	4	1	5	1	1	0

DOF#	Displacement
1	0
2	0
3	0.005
4	0.0025
5	0.0075
6	0.0075
7	0.0025
8	0.005

DOF#	Load

MAT#	E	Nu	Alpha
1	1000000	0.25	0

B1	i	B2	j	B3	<=MPC

Plane Stress Solution

Program CST2 - Plane Stress Analysis
PROBLEM 6.37

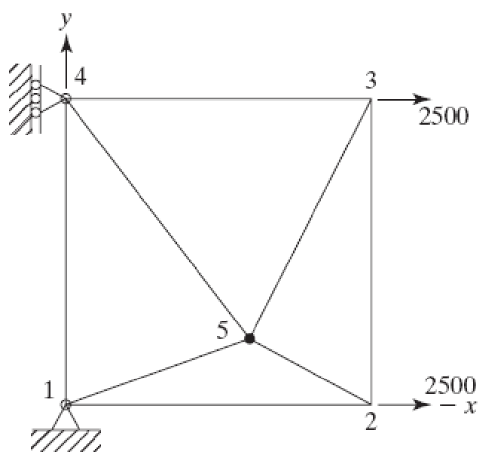
Node#	X-Displ	Y-Displ
1	1.04E-7	1.04E-7
2	0.005	0.0025
3	0.0075	0.0075
4	0.0025	0.005
5	0.0035	0.0025

Elem#	SX	SY	Txy	S1	S2	S3

1	133.329	133.329	39.999	173.328	93.33	45
2	133.329	133.329	39.999	173.328	93.33	45
3	133.329	133.329	39.999	173.328	93.33	45
4	133.329	133.329	39.999	173.328	93.33	45

The displacement of node 5 is obtained as predicted.
Use the same input data to verify the **plane strain** solution. ■

6.38



The expected plane stress solution for the patch test is $\sigma_x = 100 \quad \sigma_y = 100 \quad \tau_{xy} = 0$.

Plane Stress Input Data

2D STRESS ANALYSIS USING CST
PROBLEM 6.38

NN	NE	NM	NDIM	NEN	NDN
5	4	1	2	3	2

ND	NL	NMPC
3	2	0

Node#	X	Y
1	0	0
2	50	0
3	50	50
4	0	50
5	30	10

Elem#	N1	N2	N3	Mat#	Thickness	TempRise
1	1	2	5	1	1	0
2	2	3	5	1	1	0
3	3	4	5	1	1	0
4	4	1	5	1	1	0

DOF#	Displacement
1	0
2	0
7	0.005

DOF#	Load
3	2500

5	2500

MAT#	E	Nu	Alpha
1	1000000	0.25	1.20E-05

B1	i	B2	j	B3	MPC

Plane Stress Solution

Program CST2 - Plane Stress Analysis

PROBLEM 6.38

Node#	X-Displ	Y-Displ
1	0.0000	0.0000
2	0.0050	-0.0050
3	0.0100	-0.0063
4	0.0050	-0.0013
5	0.0040	-0.0033

Elem#	SX	SY	Txy	S1	S2	S3
1	100.000	0.000	0.000	100.000	0.000	0.000
2	100.000	0.000	0.000	100.000	0.000	0.000
3	100.000	0.000	0.000	100.000	0.000	0.000
4	100.000	0.000	0.000	100.000	0.000	0.000

■

CHAPTER 7

AXISYMMETRIC SOLIDS

7.1

Point	**(R, Z –coord)**	**(R, Z – disp.)**
1	(1,1)	(0, 0)
2	(10,4)	(-0.2, -0.1)
3	(6,7)	(0.6, 0.8)

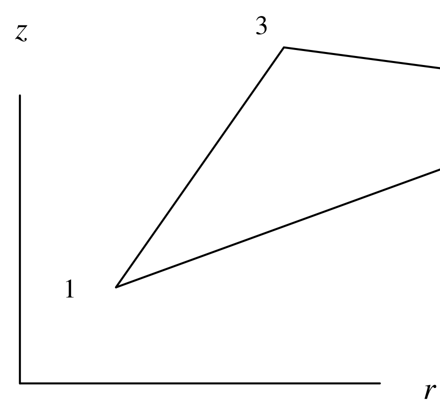


Figure P6.1

Material is steel : E = 30e6 psi, $\nu = 0.3$

a) Tangential (or hoop) stress at the centroid is given by

$$\sigma = D\,\overline{B}\,q$$

Noting that $r_{centroid} = 17/3 = 5.667$ in
we have (Matlab is convenient for "hand-calculations")

$$[D] = 1.0e{+}007 * \begin{matrix} 4.0385 & 1.7308 & 0 & 1.7308 \\ 1.7308 & 4.0385 & 0 & 1.7308 \\ 0 & 0 & 2.3077 & 0 \\ 1.7308 & 1.7308 & 0 & 4.0385 \end{matrix}$$

det [J] = 39

$$[B] = \begin{matrix} -0.0769 & 0 & 0.1538 & 0 & -0.0769 & 0 \\ 0 & -0.1026 & 0 & -0.1282 & 0 & 0.2308 \\ -0.1026 & -0.0769 & -0.1282 & 0.1538 & 0.2308 & -0.0769 \\ 0.0588 & 0 & 0.0588 & 0 & 0.0588 & 0 \end{matrix}$$

q = $[0, 0, -.2, -.1, .6, .8]^T$

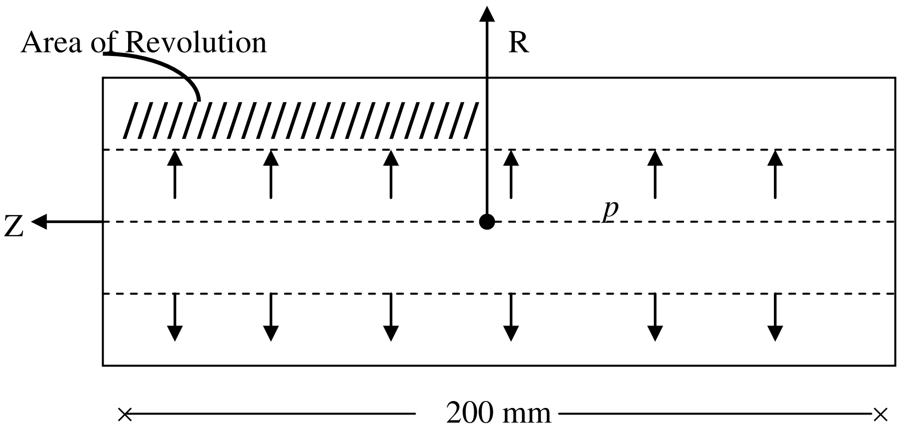
$\boldsymbol{\sigma}$ = $[0.7179 \quad 7.0493 \quad 2.0118 \quad 3.0360]^T \times 10^6$ psi, whence hoop stress = 3.036 e6 psi.

b) Principal stresses in the r-z plane:
σ_1 = 7.6344e+006 psi, σ_2 = 1.3270e+005 psi, ANGLE = 16.2183degrees
and σ_3 = hoop stress = 3.036e+006 psi.

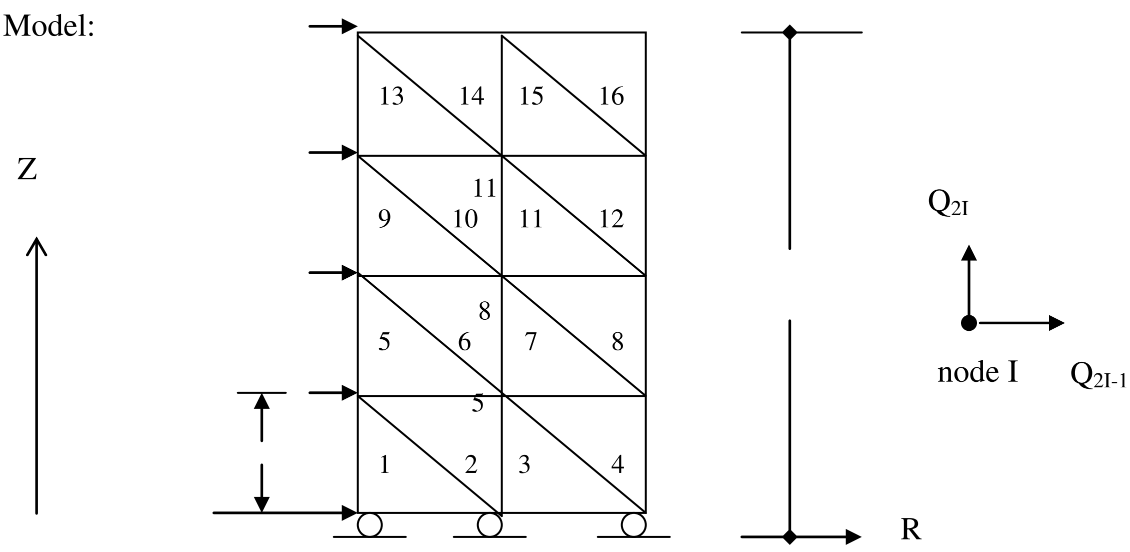
c) From above, (and formula in Chapter 1, text), we get vonMises stress = 6.5517e+006 psi.

■

7.2 Open Cylinder



Model:



E = 200 x 10^3 MPa, ν = 0.3

Load calculations

$$F_1 = F_{25} = \frac{1 \times 2\pi \times 34 \times 25}{2} = 2670.36N$$

$$F_7 = F_{13} = F_{19} = 5340.72N$$

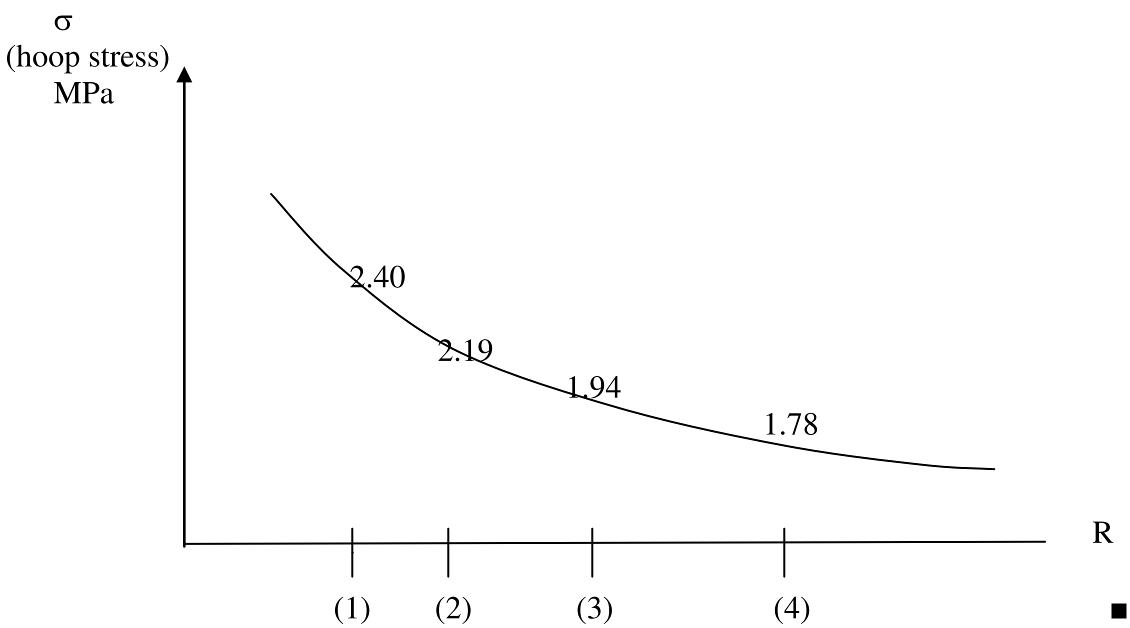
Boundary calculations

$$Q_2 = Q_4 = Q_6 = 0\,(Due\ to\ symmetry)$$

Solution using Program AXISYM

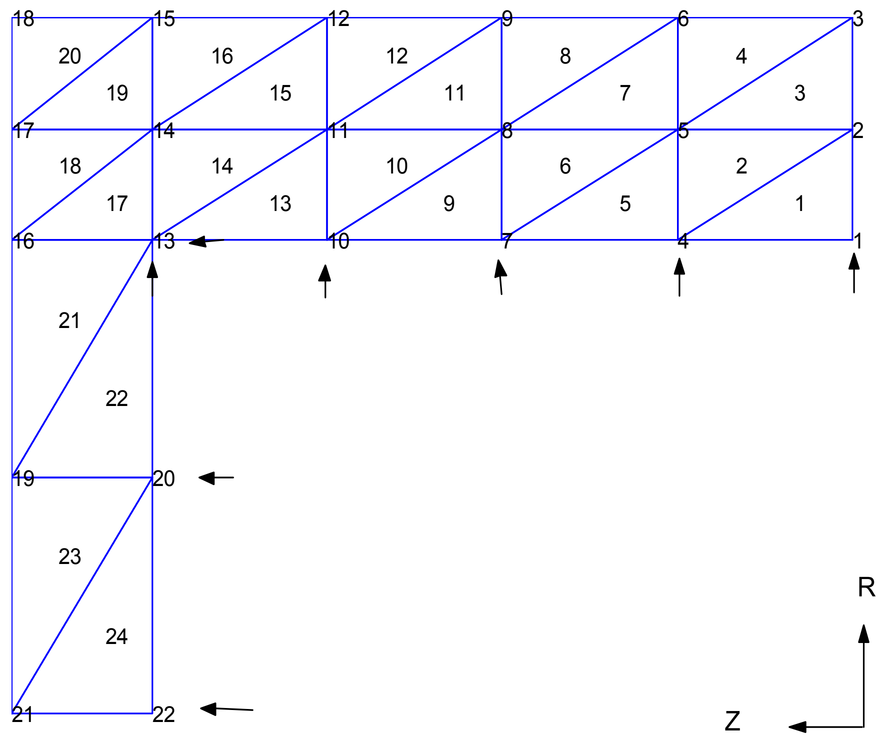
Node	(mm) R-Displacement	(mm) Z-Displacement
1	5.15×10^{-4}	0.0
4	5.12×10^{-4}	-0.60×10^{-4}
7	5.14×10^{-4}	-1.25×10^{-4}
10	5.13×10^{-4}	-1.91×10^{-4}
13	5.07×10^{-4}	-2.59×10^{-4}

Element	σ_1 , MPa	σ_2 , MPa
1	4.47 E-02	-7.44 E-02
2	-4.24 E-02	-7.09 E-02
...	...	...
8	-3.2 E-02	-2.03 E-02
...	...	...
16	-2.67 E-02	-1.07 E-02



■

7.3 Closed Cylinder



Boundary conditions

$Q_2 = Q_4 = Q_6 = 0$

Load calculations : R – Loads are as in P6.1 above:

DOF #	Load (N)
1	2670.36
7	5340.72
13	5340.72
19	5340.72
25	2670.36

Consider now the Z-loads on the closed end. We use Eqs. (7.38) – (7.40) in the text.

22	20
(2)	(1)

$l = 17mm$

$r_1 = 17, r_2 = 0$

$$a = \frac{2 \times 17}{6} = \frac{17}{3}$$

$$b = \frac{17}{6}$$

$$F_{40} = \frac{17}{3} \times 1 \times 2\pi \times 17 = 605.3N$$

$$F_{44} = \frac{17}{6} \times 1 \times 2\pi \times 17 = 302.6N$$

20	13
(2)	(1)

$l = 17mm$

$r_1 = 34, r_2 = 17$

$a = 13.5$

$b = 11.33$

$$F_{26} = 2\pi \times 17 \times 13.5 \times 1 = 1442.0N$$

$$F_{40} = 1210.6N$$

Thus, we also have:

DOF #	**Load (N)**
44	302.6
40	1815.9
26	1442.0

Solution Using Program AXISYM:

Z-Stress:

Element	**σ_z (Mpa)**
1	0.7625
2	0.7439
3	0.7888
4	0.7854

Average σ_z in (1) – (4) = **0.77 Mpa**

Displacements

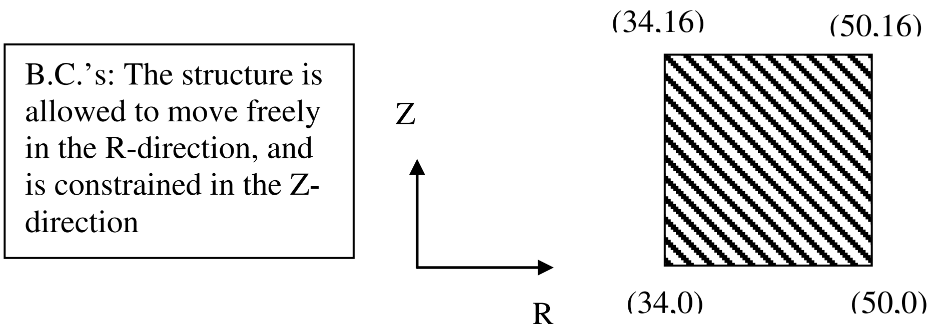
Node	R-disp.(mm)	Z-disp.(mm)
1	4.94×10^{-4}	0.0
7	4.70×10^{-4}	0.36×10^{-4}
13	0.89×10^{-4}	1.48×10^{-4}
22	-0.07×10^{-4}	3.38×10^{-4}

Element	σ_t (hoop), Mpa
1	2.57
2	2.31
3	2.02
4	1.88

■

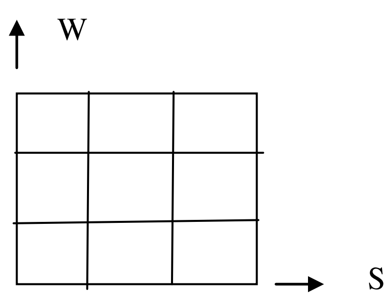
7.4 Infinite Cylinder

The area of revolution to be modeled is shown below, after considering a 16 mm long piece of the shaft:



The boundary conditions approximate a plane strain ($\varepsilon_Z = 0$) condition since the pipe is long.

USE OF MESHGEN PROGRAM: The use of MESHGEN is discussed in the text in Chps. 5,12, and in Chp.5 of the Solutions Manual. A one block diagram with 3x3 divisions is used here.



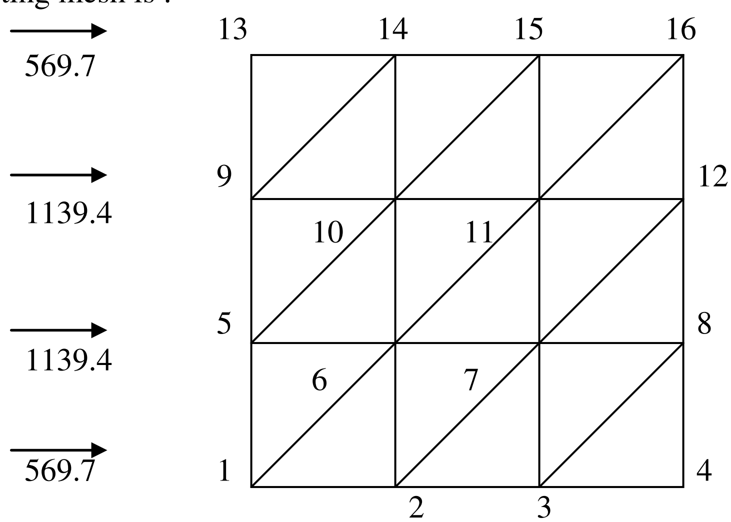
Block Diagram with Corner Nodes

Corner Node **Coordinates**

1	34, 0
2	50, 0
3	34, 16
4	50, 16

No. of S-spans = 1 , No. of W-spans = 1
No. of S-span divisions = 3 , No. of W-span divisions = 3

The resulting mesh is :



Loads:

$F_9 = F_{17} = 1 \text{ x } 2\pi\text{x}34\text{x}16/3 = 1139.35\text{N}$

$F_1 = F_{25} = 1139.35/2 = 569.7\text{N}$

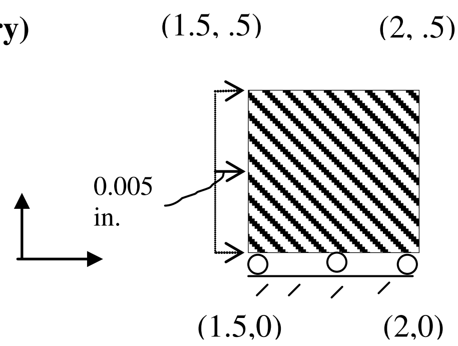
Solution

Node	**R-disp. (mm)**
4	$3.93\text{x}10^{-4}$
8	$3.91\text{x}10^{-4}$
12	$3.91\text{x}10^{-4}$
16	$3.89\text{x}10^{-4}$

Thus, the outer *diameter* is 100mm before deformation and 107.8mm after deformation. ■

7.5 Press Fit on Rigid Shaft

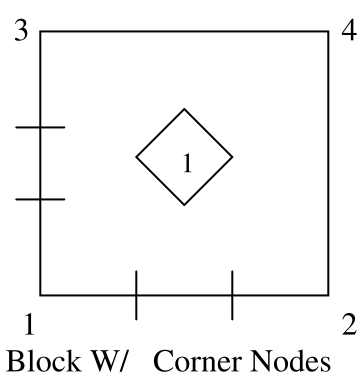
Model (1/2 symmetry)



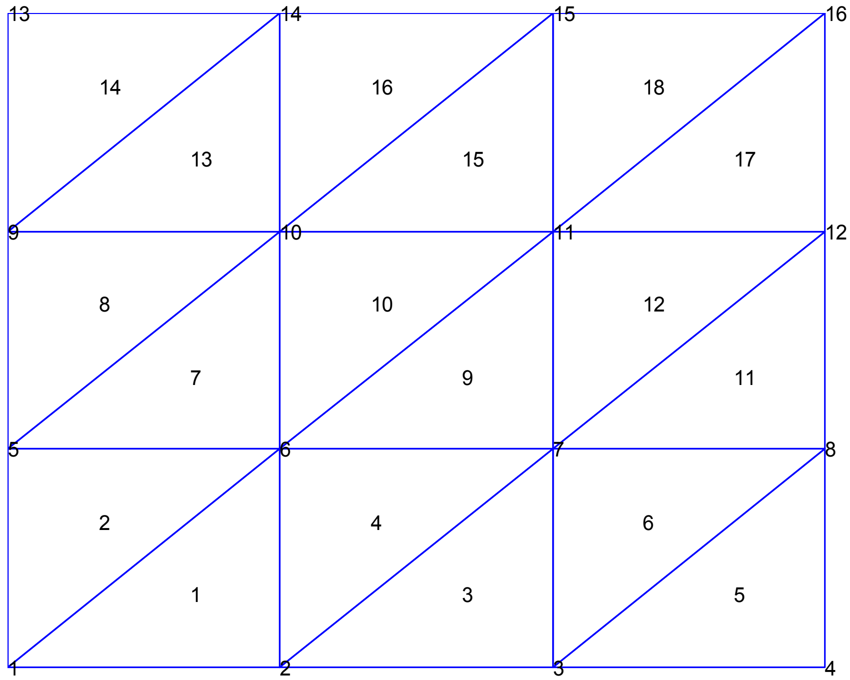
Radial Interference = 0.01/2 = 0.005"

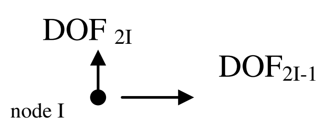
MESHGEN PROGRAM

As in P6.3 above, we use a one-block, 3 x 3 divisions model:



The resulting mesh is shown below (use PLOT 2D.m program to see the mesh)





Boundary Conditions

Rollers: $Q_2 = Q_4 = Q_6 = Q_8 = 0$

Interference : $Q_1 = Q_9 = Q_{17} = Q_{25} = 0.005$

Thus ND = 8

Loading: NL = 0
(the interference causes the stress)

Material : $E = 30 \times 10^6$ psi
$\nu = 0.3$

The above information is supplied to the DATAFEM Program. Thus, we execute

MESHGEN	→	PLOT 2D	→	DATAFEM
To Generate the Mesh file		To view Node numbers		B.C's Loads, Material

The data file, say P6_4.DAT is supplied to AXISYM.

Solution

Element	**σ_γ (psi)**
1	-19,604.
2	-21,455.
7	-19,000.
8	-20,725
13	-18,431.
14	-20,000.

These stresses given an approximate value for the contact stress. More accuracy may be obtained by interpolation, as discussed in E 6.2 in text. Alternatively, we may modify program AXISYM to output only the σ_γ values into a file, say SR.DAT. Then, we supply this and the mesh file to program BESTFIT:

$$\left.\begin{array}{l}\text{SR, DAT}\\ \\ \text{P6\_4.DAT}\end{array}\right\} \Longrightarrow \text{BESTFIT} \Longrightarrow \text{SRG.DAT}$$

The output file SRG.DAT contains the grid point or nodal values of σ_γ . Result:

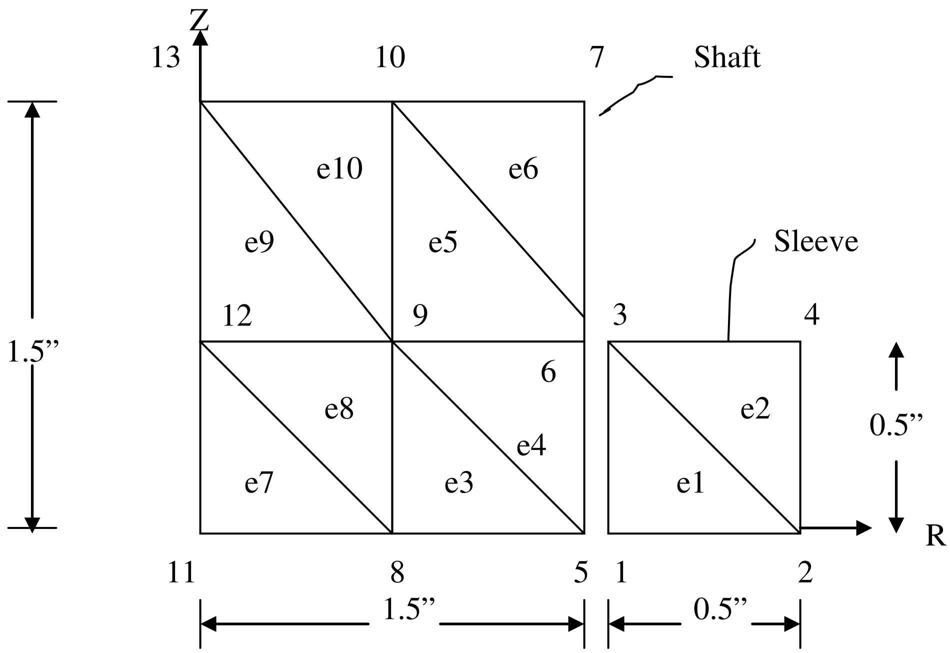
Node	**σ_γ (psi)**
1	-19,581.
5	-22,214.
9	-20,882.
13	-21,801.

Thus, the contact pressure is averaged to be

$$\boldsymbol{P_c = 21{,}120 \text{ psi}} \qquad \blacksquare$$

7.6 Press fit on flexible shaft

Model



Nodes 1,3 are on the sleeve, while 5,6 are on the shaft. A copy of AXISYM.INP (see \EXAMPLES directory) is made and this file is then directly edited to input data corresponding to above mesh.

Assume: Length of shaft = 3”, and sleeve is fitted at the center.

Boundary conditions
ND = 5 : $Q_{22} = Q_{16} = Q_{10} = Q_2 = Q_4 = 0$

Multi-point constraints (MPC’s)

$$Q_5 - Q_{11} = 0.005$$
$$Q_1 - Q_9 = 0.005$$

The MPC’s are input into Program AXISYM (towards end of file) as:
1.0, 5, -1.0, 1, 0.005
1.0, 1, -1.0, 9, 0.005

Solution

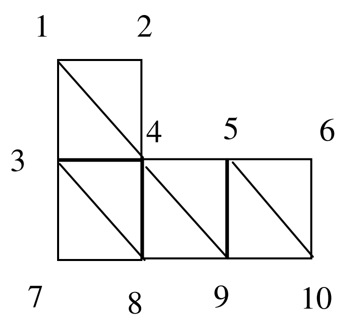
Element	**σ_γ (psi)**
1	-16,660.
2	-3,324.
3	-26,610.
4	-22,550

It is expected that the contact pressure is less than in P6.4 with shaft assumed to be rigid. A finer mesh may be needed in this problem for accuracy. ■

7.7 (Flywheel)

Speed of rotation, $\omega = 300$ rpm

Mesh:



Load Calculations

$\bar{r} = (r_I + r_J + r_K)/3$ = r - Coordinate of centroid of element

$A_e = \frac{1}{2} \left| \det \mathbf{J} \right|$ = area of element

Total inertial force on element is

f_I = mass x acceleration

$$= (\rho\ 2\pi\ \bar{r}\, A_e) \text{ x } (\bar{r}\omega^2) = 2\pi\rho\omega^2 \left(\bar{r}\right)^2 A_e$$

We may divide this as $F_{2*I-1} = F_{2*J-1} = F_{2*K-1} = f_I/3$

Solution:

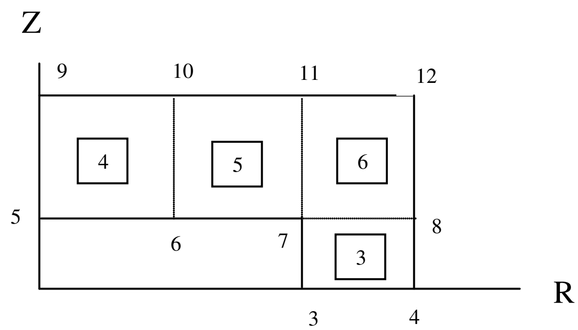
Node	**R-disp.(x10^{-3} in)**	**Z – disp.(x10^{-3})**
1	0.93	-1.77
2	0.71	-0.63
3	2.59	-1.06
⋮	⋮	⋮
5	2.84	0.
8	3.39	0
⋮	⋮	⋮
10	3.66	0.

Element	σ_γ (psi)	σ_t (psi) (hoop)
1	10,140	54,020
2	15,350	51,070
3	9,624	41,660
4	389.8	25,670
⋮	⋮	⋮
7	10,220	24,350
8	8,011	22,380

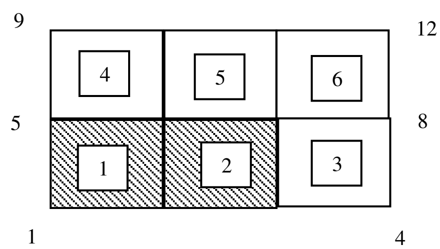
■

7.8 Hydrostatic Bearing

Model/MESHGEN



Region



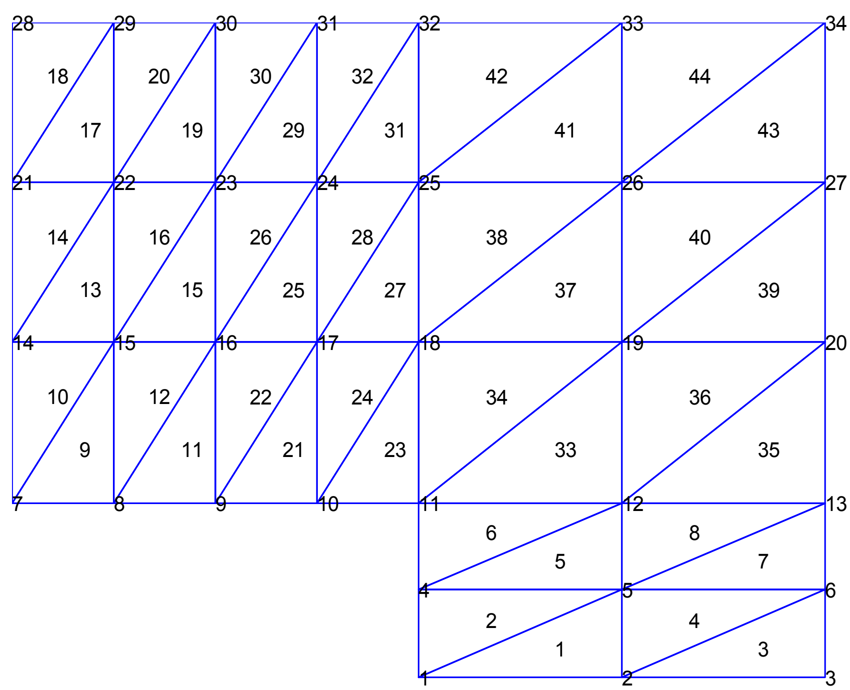
Block Diagram w/ corner nodes (Void Blocks: 1, 2)

The Meshgen input file is:

```
Mesh Generation - hydrostatic bearing
P 7.7
Number of Nodes per Element <3 or 4>
  3
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ)
  3             2             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
 1       2
 2       2
 3       2
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
```

```
  1      2
  1      3
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  1      0
  2      0
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
  3         25        0
  4         50        0
  5          0        8
  6         12.5      8
  7         25        8
  8         50        8
  9          0        30
  10        12.5      30
  11        25        30
  12        50        30
  0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  0
```

Plot2d.m then yields the mesh plot:



Boundary Conditions

ND = 6: $Q_{55} = Q_{56} = 0$
$Q_{57} = Q_{58} = 0$
$Q_{59} = Q_{60} = 0$

Loading

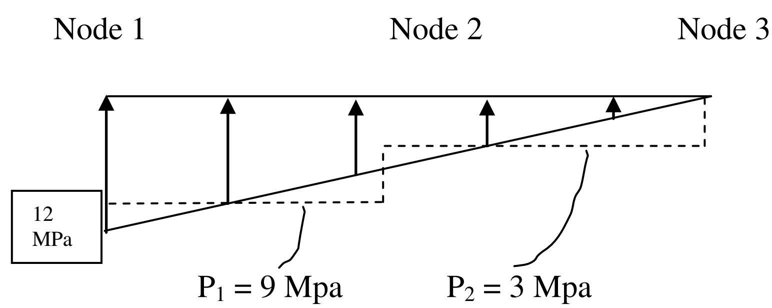
NL = 11 $\left\{\begin{array}{l} \text{Nodes 7, 8, 9, 10, 11,} \\ \quad \text{1, 2, 3 along Z} \\ \\ \text{Nodes 1, 4, 11 along R} \end{array}\right.$

Z-Loads

These follow the steps shown in the solution of P6.2. For a typical edge, the calculations are:

$$a = (2r_1 + r_2)/6$$
$$b = (r_1 + 2r_2)/6$$
$$F_1 = 2\pi\ l_{1-2}\ a\ p$$
$$F_2 = 2\pi\ l_{1-2}\ b\ p$$

On 1 – 2 – 3, the triangular distribution of pressure may be handled as follows:



R-Loads

$$F_1 = F_{21} = \frac{2\pi \times 25 \times 4 \times 12}{2} = 3769.9N$$

$$F_7 = 7540.0 \text{ N}$$

Material: Take it as Steel, E = 200 GPa = 200,000 MPa

Program AXISYM may now be used to solve the problem.

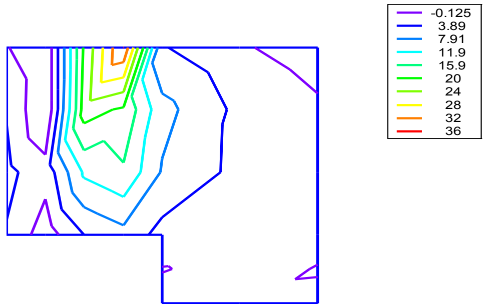
To summarize, The loads and B.C. input are input as:

```
DOF#  Displacement
 55  0
 56   0
 57   0
 58   0
 59   0
 60   0
DOF#  Load
 1   3769.9
 7   7539.8
21   3769.9
 2   10308.3
 4   15217
 6   5399.6
14   490.9
16   2945.3
18   5890.5
20   8835.8
22   5399.6
```

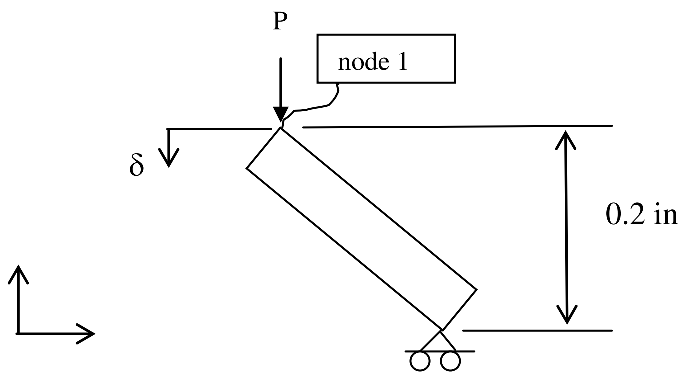
RESULTS

Node#	R-Displ	Z-Displ
1	4.3654E-003	7.7437E-003
2	3.7917E-003	1.0105E-002
3	3.5389E-003	1.2108E-002
28	-1.8555E-016	2.4637E-010
29	-6.7005E-011	1.6729E-009
30	-7.0316E-009	1.1433E-008
31	-8.1856E-004	4.1033E-003
32	-1.2168E-003	6.3971E-003
33	-1.3038E-003	9.4238E-003
34	-1.2267E-003	1.1684E-002

Stress results consist of several components : SR, SZ, TRZ, ST and the derived principal stresses S1 & S2. Below, only contours for in-plane shear, TRZ is given. Also, from output file, element #30 has max. TRZ = 45 MPa.



7.9 Belleville Spring -- Consider the model:



As discussed in the text, the basic steps are:

Step 0 : Choose an incremental load ΔP, say, such that the deflection $\Delta\delta$ is 10% of 0.2", based on current geometry (this may require a preliminary analysis). Set P = 0

Step 1 : Update $P = P + \Delta P$
In AXISYM, zero the load vector **F** and define F(2) = - P

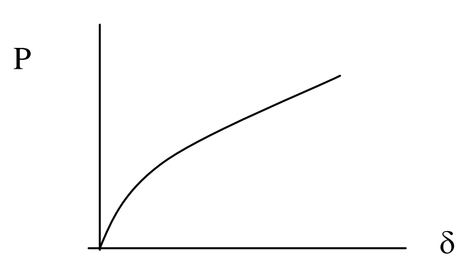
Step 2 : Solve for all nodal displacements. Note that these will be over-written in **F** .

Step 3 : Update the geometry by adding the displacements to the coordinates:

X(I, 1) = X (I, 1) + F(2*I-1)
X(I, 2) = X (I, 2) + F(2*I)   I = 1 to NN.

Go to Step 1

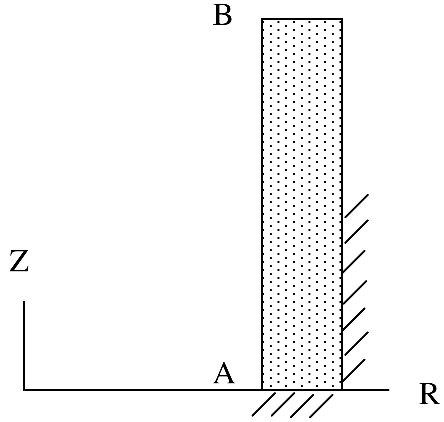
The above procedure is repeated until the spring is flattened. A graph of P and δ at each iteration will appear as



■

7.10 (Thermal Stress)

Model:

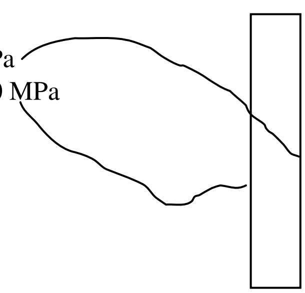


Meshgen is used to create the mesh, followed by Program Axisym.

RESULTS: $\delta_B = 24.4$ mm

τ_{rz} max = 27 MPa

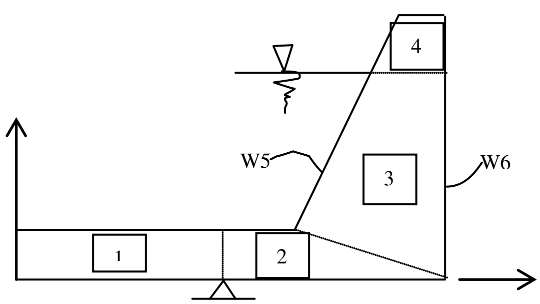
σ_{hoop} max = -110 MPa



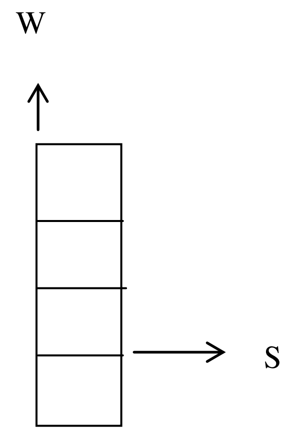
■

7.11

Model:



Region with Block Numbers Shown



Block Diagram

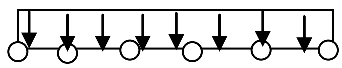
No. of S-spans = 1

No. of W-spans = 4

Note that the MESHGEN program allows 'grading' the mesh by specification of the mid-side node. For example, lowering the mid-sides of W-5 and W-6 would imply a finer mesh nearer edge S-6 of the block.

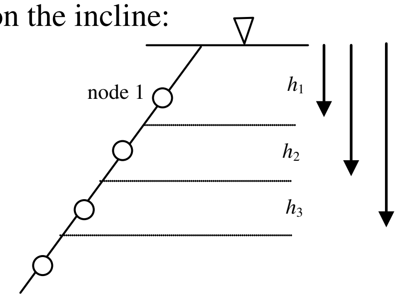
Load calculations

On the bottom surface (vertical) water pressure is uniform $p = \rho g \times 3000$ MPa, with $\rho = 10^{-6}$ kg/mm^3 and g = 9810 mm/s^2, or p = 20.4 MPa.

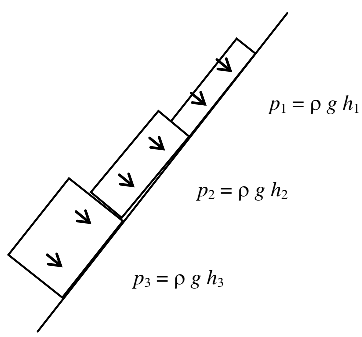


The distributed load is readily converted to equivalent point loads at the nodes.

Consider the pressure on the incline:

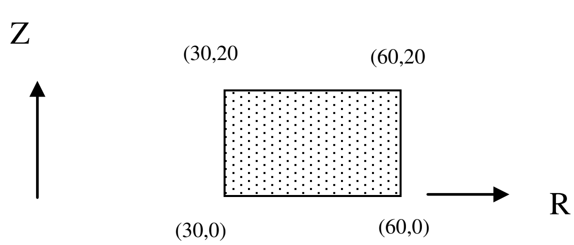


We may treat the pressure as piecewise constant:



Consideration of this has been discussed in text and in solution of 6.2 and previous problems. ■

7.12



Edge 1 –2

$$r_1 = 60mm = r_2$$
$$a = b = 30mm \quad See \quad eqn..6.39$$
$$\mathbf{T}^{1-2} = [F_1 \ , F_3] = 2\pi(20)\left[(1-.5)(30)(-.5)(30)\right]$$
$$= \left[-1885.0,-1885.0\right] N$$

Edge 4-2

$$r_1 = 30mm \ , \quad r_2 = 60mm$$
$$a = \frac{2(30)+60}{6} = 20mm$$
$$b = \frac{30+2(60)}{6} = 25mm$$
$$\mathbf{T}^{4\text{-}2} = [F_8, F_4] = 2\pi(30)\left[(-.5)(20),(-.5)(25)\right]$$
$$\left[-1885.0,-2356.0\right] N$$
$$F_2 = F_7 = 0$$

■

7.13

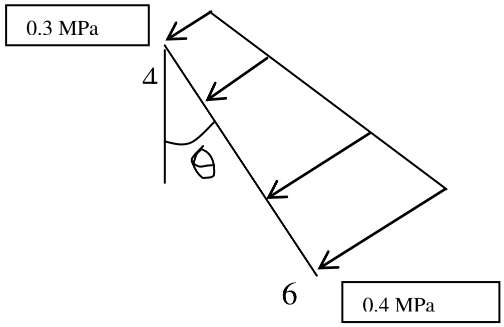
(a) Eq, (7.36): Consider r-traction:

$$\int \mathbf{U}^T \mathbf{T} \, \mathrm{dS}$$
$$= \int (\mathrm{N}_1 q_1 + \mathrm{N}_2 q_3)(\mathrm{N}_1 T_{r1} + \mathrm{N}_2 T_{r2}) dS$$

$$Along \ 1-2,$$

$$N_1 = \xi, N_2 = 1-\xi, N_3 = 0$$
$$dS = 2\pi r l_{1-2} d\xi$$
$$r = N_1 r_1 + N_2 r_2 = \xi r_1 + (1-\xi) r_2$$
Thus we have
$$\int \mathbf{U}^T \mathbf{T} \, \mathrm{dS} =$$
$$\mathrm{q}_1 \left\{ 2\pi \, l \int_0^1 [\xi T_{r_1} + (1-\xi) T_{r_2}][\xi r_1 + (1-\xi) r_2] d\xi \right\} + q_3 \{...\}$$
$$= q_1 \left\{ \frac{2\pi \, l_{1-2}}{12} (3r_1 + r_2) T_{r_1} + \frac{2\pi \, l_{1-2}}{12} (r_1 + r_2) T_{r_2} \right\} + q_3 \{...\}$$
$$\equiv q_1 [a T_{r1} + b T_{r_2}] + q_3 [...]$$

(b) Edge 6-4 (see Example 6.1 Text)



Cos θ = .6
Sin θ = .8

$$T_{r_1} = .4 \ \cos \ \theta = .24 \ \ MPa$$
$$T_{r_2} = .3 \ \cos \ \theta = .18 \ \ MPa$$
$$T_{z_1} = .4 \ \sin \ \theta = .32 \ \ MPa$$
$$T_{r_2} = .4 \ \sin \ \theta = .24 \ \ MPa$$

$$a = \frac{2\pi(25)}{12}[3(60)+40] = 2879.8$$
$$b = 1309.0$$
$$c = 2356.2$$

$$-\left[F_{6x}, F_{4x}\right] = -\left[F_{6z}, F_{4z}\right] = \left[691.15+235.62, 314.16+424.12\right] = [926., 738.3] \ N$$
$$-\left[F_{6z}, F_{4z}\right] = [1235.7, 984.4] \ N$$
Similarly, for edge $4-2$,
$$a = 1832.6, \ b = 785.4, \ c = 1309.$$
$$T_{r1} = .18 \ \ MPa, \ T_{r2} = .12 \ \ MPa, \ T_{z1} = .18 \ \ MPa, \ T_{z2} = .16 \ \ MPa,$$
$$-\left[F_{4x}, F_{2x}\right] = [424.1, \ 298.5] \ N, \ -\left[F_{4z}, F_{2z}\right] = [455.5, \ 350.8] \ N$$
Upon Assembly

$$[F_3, \ F_4, \ F_7, \ F_8, \ F_{11}, \ F_{12} \ ] = -\left[298.4, \ 350.8, \ 1162.4, \ 1439.9, \ 926.8, \ 1235.7\right]N$$

■

7.14 Die Block

(a) **Model without (w/o) shrink ring**

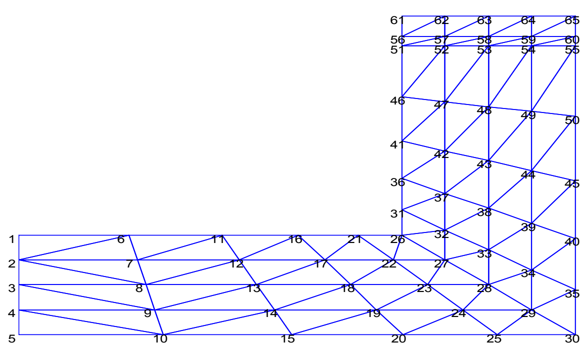
dimensions in mm

Block Diagram for Meshgen program

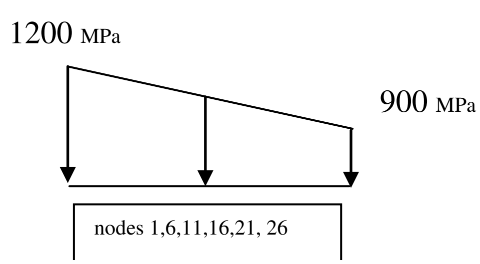
To obtain smaller sized elements near the corner, "mesh grading" was implemented using displaced mid-side points in meshgen2.bas, as given below:

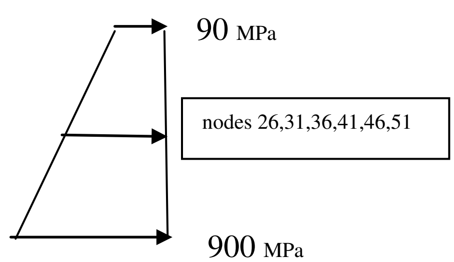
W-side #	Mid-point coordinates
1	(70, 100)
2	(95, 0)
3	(110, 175)
4	(160, 125)

Mesh:



Traction





Automatic Load calculations

The above tractions were converted into nodal loads (see eq 7.38) by inserting the following 'problem-specific' code into program AXISYM.BAS:

'Loads due to traction for P.6.13

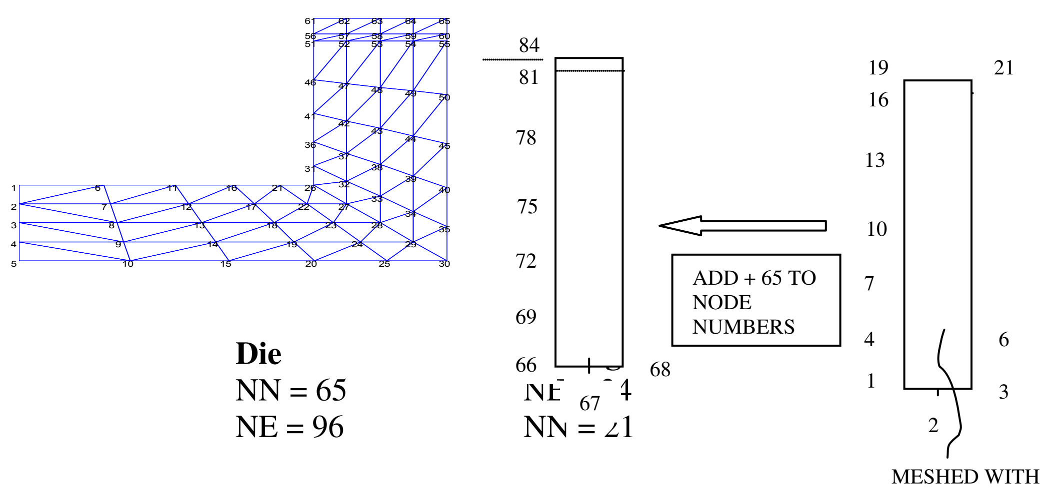
```
PI = 1200: P2 = 900: P3 = 90

FOR II = I TO 5
I1 = 5*(II – 1)+1
12 = I1+5
XI = X(I1, 1): X2 = X(I2,1)
TZ = P1+(P2-P1)/110*(X1+X2)/2
A = (2*X1+X2)/6: B = (X1 + 2*X2)/6
EL = X2 – X1: PI = 3.1415927
F(2*I1)=F(2*I1) – 2 *PI * EL *A*TZ
F(2*I2)=F(2*I2) – 2 *PI * EL *B*TZ
NEXT II

FOR II = 1 TO 5
I1 = 5*(II – I)+26: I2 = I1+5
X1 = X(I1,2) – 100: X2 =X(I2,2) –100
TR = P2+(P3-P2)/190*(X1 + X2)/2
A= 110/2: B =A
EL =X2-X1:
F(2*I1-1) = F(2*I1-1) +2*PI*EL*A*TR
F(2*I2-1) = F(2*I2-1) +2*PI*EL*B*TR
NEXT II
```

Results will be discussed below after modeling the shrink ring:

Die with Shrink Ring -- <u>With</u> slip



To model the shrink ring, a separate mesh was created as shown above - - - this was done to obtain duplicate nodes at the die-ring interface. Then, +65 was added to the node numbers and +96 to the element numbers. (This was achieved by writing a simple program not give here) Lastly, multipoint constraints or MPC's were enforced for interface nodal pairs (input data):

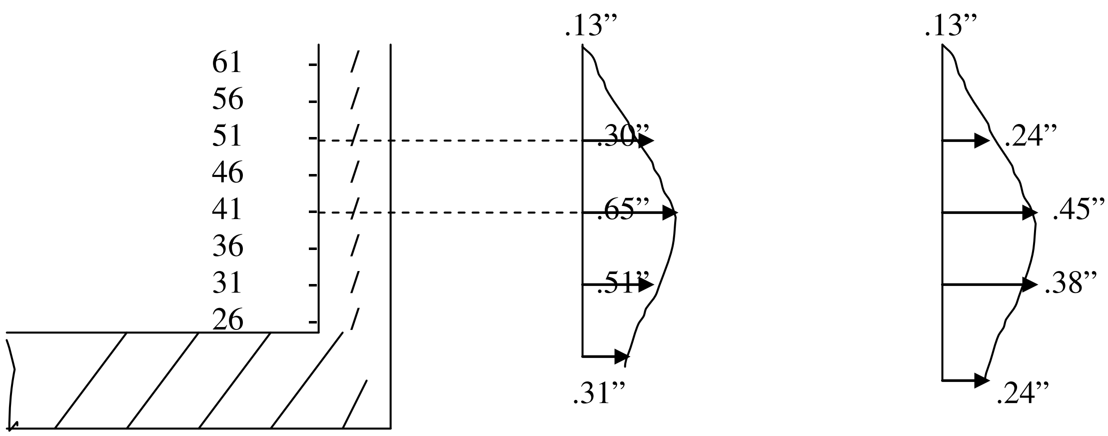
β_1	dof_1	β_2	dof_2	β_3
1.	69	-1.	137	0.
1.	79	-1.	143	0.
1.	89	-1.	149	0.
1.	99	-1.	155	0.
1.	109	-1.	161	0.
1.	119	-1.	167	0.

Note: Slip along *Z*- is allowed

The **base** was fixed along r- and Z on both die & ring.

Results

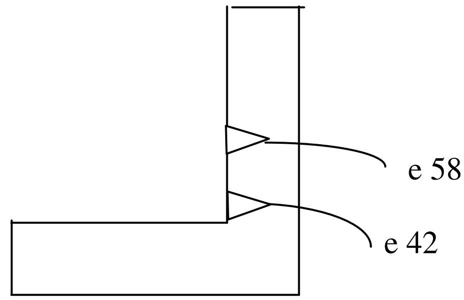
r-displacement of inner surface of die:



w/o shrink ring **With ring (lesser in magnitude)**

Maximum Hoop Stresses

Critical locations (elements)

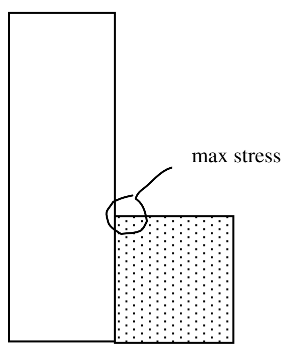


Element	w/o ring	With ring (axial slip allowed)
e 58	989 MPa	647 MPa
e 42	942 MPa	708 MPa

Thus, max stress is less with ring.
(NOTE: max. σ_t in ring = 342 MPa). ■

7.15

Model Creation



Assuming no slip at the sleeve – shaft interface, the modeling using MESHGEN is straight forward: No. of S-spans = 2, No. of W spans = 2, 4 Divisions were used in each span, Block # 4 = void (Material “0”). All elements in Block # 2 were given a ΔT = - 200 ^{o}C (manual editing of file) -- To make this input easy, Block #2 to identify the elements.

von Mises Stress (Minor modification inAXISYM)

$$\sigma_{VM} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_t)^2+(\sigma_t-\sigma_1)^2}$$

■

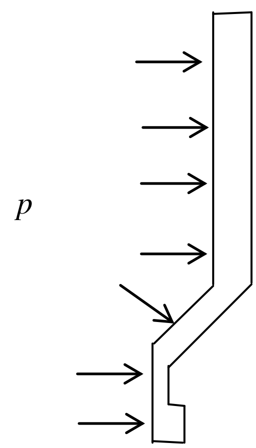
$\text{Maximum } \sigma_{VM} = 607 MPa$ --- See location in figure above

7.16 Syringe

Key step is to model the problem. Pressure applied is determined from

$$p = 50 / (\pi \ 10.5^2) \text{ N/mm}^2 \text{ or MPa}$$

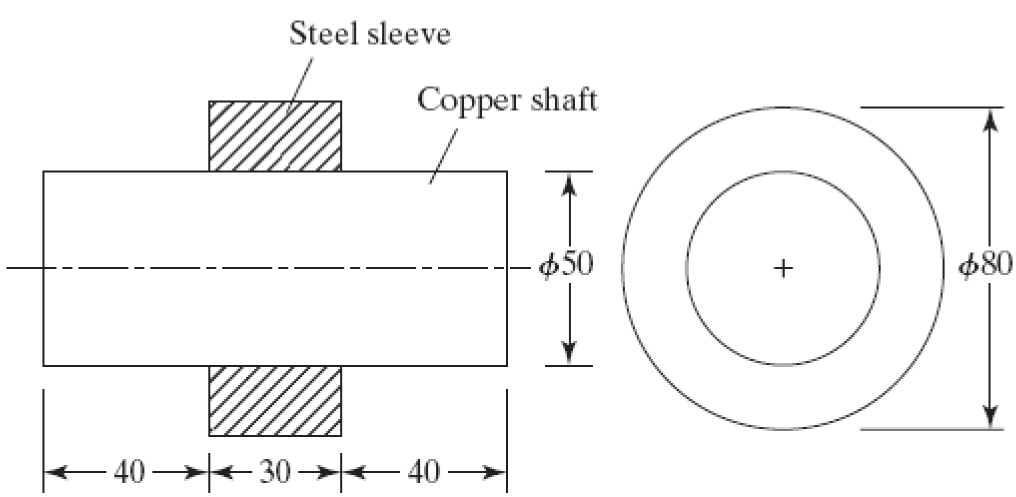
Since fluid pressure is uniformly distributed, the model resembles:



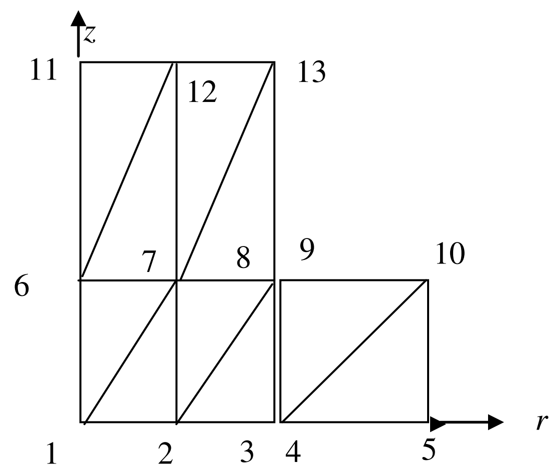
above model must be constrained in the z-direction, before solving

Automatic generation of nodal forces may be introduced in the program where there is pressure distribution on edges. This will reduce hand calculations. ■

7.17



We use the symmetry and try a simple mesh as shown in the following diagram. Nodes 1, 2, 3, 4, and 5 are restrained in the z – direction. We also set the two multipoint constraints $Q_5 = Q_7$, and $Q_{15} = Q_{17}$.



The problem is solved using the following data.

<< AXISYMMETRIC STRESS ANALYSIS USING TRIANGULAR ELEMENT >>

PROBLEM 7.17

NN	NE	NM	NDIM	NEN	NDN
13	10	2	2	3	2
ND	**NL**	**NMPC**			
5	0	2			
Node#	**R**	**Z**			
1	0	0			
2	12.5	0			
3	25	0			
4	25	0			
5	40	0			
6	0	15			
7	12.5	15			
8	25	15			
9	25	15			
10	40	15			
11	0	55			
12	12.5	55			
13	25	55			
Elem#	**N1**	**N2**	**N3**	**Mat#**	**ΔT**
1	1	2	7	1	30
2	1	7	6	1	30
3	2	3	8	1	30
4	2	8	7	1	30
5	4	5	10	2	0
6	4	10	9	2	0

7	6	7	12	1	30
8	6	12	11	1	30
9	7	8	13	1	30
10	7	13	12	1	30

DOF#	**Displ.**
2	0
4	0
6	0
8	0
10	0

DOF#	**Load**

MAT#	**E**	**Nu**	**Alpha**
1	120000	0.33	0.000017
2	200000	0.3	0.000012

B1	**i**	**B2**	**j**	**B3**	**MPC**
1	5	-1	7	0	
1	15	-1	17	0	

Results

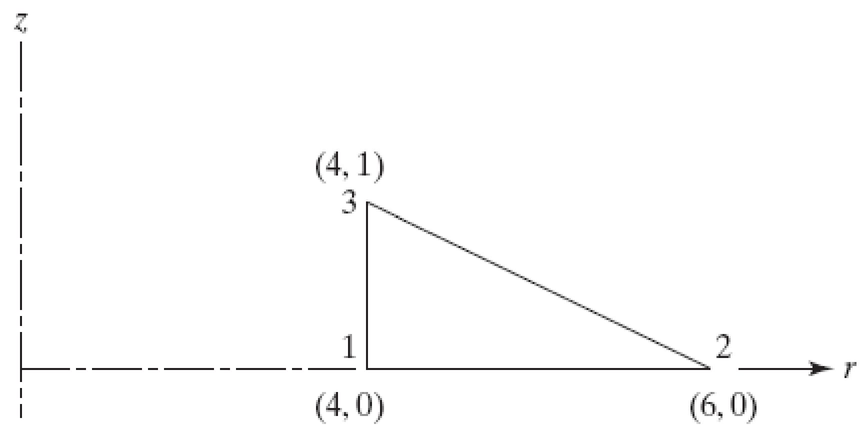
Program AXISYM - Triangular Element
PROBLEM 7.17

Node#	**R-Displ**	**Z-Displ**
1	-0.00025	0.00000
2	0.00448	0.00000
3	0.00947	0.00000
4	0.00947	0.00000
5	0.00802	0.00000
6	-0.00116	0.00991
7	0.00498	0.00943
8	0.01071	0.00884
9	0.01071	-0.00064
10	0.00801	-0.00112
11	0.00054	0.03081
12	0.00721	0.03078
13	0.01352	0.03041

Elem#	**SR**	**SZ**	**Trz**	**ST**	**S1**	**S2**	**Angle°**
1	-25.466	-2.850	1.516	-26.335	-2.749	-25.567	86.182
2	-9.739	5.510	-4.480	-28.330	6.728	-10.958	-74.782
3	-22.821	-5.728	3.711	-23.282	-4.957	-23.592	78.263
4	-8.195	7.208	-0.631	-13.130	7.233	-8.220	-87.659
5	-6.609	-3.300	-0.094	45.605	-3.297	-6.611	-88.379
6	-17.260	3.850	3.850	58.610	4.530	-17.940	79.980
7	-7.214	-3.413	0.791	-11.755	-3.255	-7.372	78.700
8	6.835	5.822	1.819	6.267	8.217	4.440	37.219
9	-10.410	-3.062	1.022	-9.575	-2.923	-10.550	82.226
10	1.414	4.079	1.196	2.320	4.537	0.956	69.041

DOF#	**Reaction**
2	-1496.966
4	-6833.000
6	8329.966
8	-5288.271
10	5288.271

7.18 The body shown is rotating about z axis with an angular velocity of ω rad/s. The material density is ρ.



The body force term in the work potential is now considered.

$$2\pi\int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}r\,dA = 2\pi\int_e \mathbf{u}^{\mathrm{T}}\begin{bmatrix}\rho r\omega^2\\ 0\end{bmatrix} r\,dA$$

$$= 2\pi\int_e u\rho r^2\omega^2 dA$$

$$= 2\pi\int_e \rho\omega^2\left(N_1q_1 + N_2q_3 + N_3q_5\right)\left(N_1r_1 + N_2r_2 + N_3r_3\right)^2 dA$$

$$= 2\pi\rho\omega^2\int_e \left(N_1q_1 + N_2q_3 + N_3q_5\right)\left(N_1^2r_1^2 + N_2^2r_2^2 + N_3^2r_3^2 + 2N_1N_2r_1r_2 + 2N_2N_3r_2r_3 + 2N_1N_3r_1r_3\right)dA$$

$$= 2\pi\rho\omega^2\begin{pmatrix} q_1\int_e\left(N_1^3r_1^2 + N_1N_2^2r_2^2 + N_1N_3^2r_3^2 + 2N_1^2N_2r_1r_2 + 2N_1N_2N_3r_2r_3 + 2N_1^2N_3r_1r_3\right)dA \\ +q_3\int_e\left(N_1^2N_2r_1^2 + N_2^3r_2^2 + N_3^2N_2r_3^2 + 2N_1N_2^2r_1r_2 + 2N_2^2N_3r_2r_3 + 2N_1N_2N_3r_1r_3\right)dA \\ +q_5\int_e\left(N_1^2N_3r_1^2 + N_2^2N_3r_2^2 + N_3^3r_3^2 + 2N_1N_2N_3r_1r_2 + 2N_2N_3^2r_2r_3 + 2N_1N_3^2r_1r_3\right)dA \end{pmatrix}$$

Now using $dA = \det\mathbf{J}d\xi d\eta$, $\det\mathbf{J} = 2A$, $N_1 = \xi$, $N_2 = \eta$, $N_3 = 1 - \xi - \eta$, and the integral

$$\int_0^1\int_0^{1-\xi} \xi^a\eta^b\left(1-\xi-\eta\right)^c d\xi d\eta = \frac{a!b!c!}{\left(a+b+c+2\right)} \qquad (6.46)$$

$$\int_0^1\int_0^{1-\xi} N_j^3 d\xi d\eta = \frac{3!}{5!} = \frac{1}{20}$$

and noting that $\displaystyle\int_0^1\int_0^{1-\xi} N_j^2N_k d\xi d\eta = \frac{2!1!}{5!} = \frac{1}{60} \quad j \neq k$

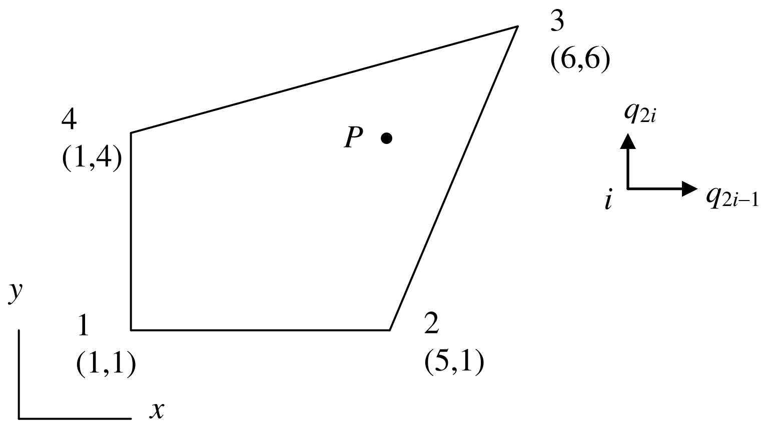
$$\int_0^1\int_0^{1-\xi} N_1N_2N_3 d\xi d\eta = \frac{1!1!1!}{5!} = \frac{1}{120}$$

we get

$$\mathbf{f} = \frac{\pi\rho\omega^2 A}{15}\begin{bmatrix} 3r_1^2 + r_2^2 + r_3^2 + 2r_1r_2 + 2r_1r_3 + r_2r_3 \\ 0 \\ r_1^2 + 3r_2^2 + r_3^2 + 2r_1r_2 + r_1r_3 + 2r_2r_3 \\ 0 \\ r_1^2 + r_2^2 + 3r_3^2 + r_1r_2 + 2r_1r_3 + 2r_2r_3 \\ 0 \end{bmatrix}$$ ■

CHAPTER 8
TWO-DIMENSIONAL ISOPARAMETRIC ELEMENTS AND NUMERICAL INTEGRATION

8.1



(a)

$$x = \sum_{i=1}^{4} N_i x_i$$

$$y = \sum_{i=1}^{4} N_i y_i$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

at $\xi = 0.5, \quad \eta = 0.5$

$$N_1 = \frac{1}{16}, \quad N_2 = \frac{3}{16}, \quad N_3 = \frac{9}{16}, \quad N_4 = \frac{3}{16}$$

$$\Rightarrow x = \frac{1}{16}[(1)(1) + (3)(5) + (9)(6) + (3)(4)]$$

$$x = 4.5625$$

$$x = \frac{1}{16}[(1)(1) + (3)(1) + (9)(6) + (3)(4)]$$
$$x = 4.3725$$

(b) $\mathbf{q} = [0 \quad 0 \quad 0.20 \quad 0 \quad 0.15 \quad 0.10 \quad 0 \quad 0.05]^T$

$\mathbf{u} = [u \quad v]^T$
$\mathbf{u} = \mathbf{Nq}$

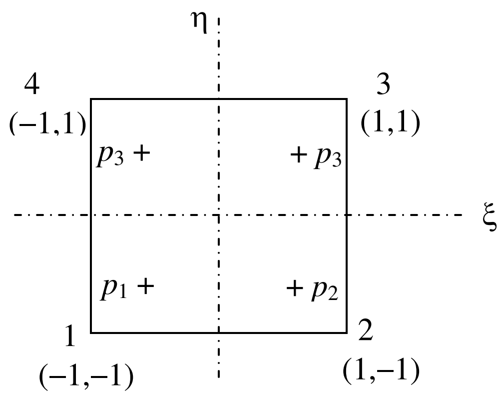
where

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

On carrying out the product **Nq**, we have

$$\mathbf{u} = \begin{Bmatrix} 0.121875 \\ 0.065625 \end{Bmatrix} \quad \blacksquare$$

8.2



$$I = \int_{-1}^{1}\int_{-1}^{1} \left(x^2 + xy^2\right) \det \mathbf{J} d\xi d\eta$$
$$x = \sum_{i=1}^{4} N_i x_i$$
$$y = \sum_{i=1}^{4} N_i y_i$$

Jacobian

$$\mathbf{J} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

where

$$J_{11} = -x_1 + x_2 + x_3 - x_4 + \eta(x_1 - x_2 + x_3 - x_4)$$
$$J_{12} = -y_1 + y_2 + y_3 - y_4 + \eta(y_1 - y_2 + y_3 - y_4)$$

$$J_{21} = -x_1 - x_2 + x_3 + x_4 + \xi(x_1 - x_2 + x_3 - x_4)$$
$$J_{22} = -y_1 - y_2 + y_3 + y_4 + \xi(y_1 - y_2 + y_3 - y_4)$$

Denoting, $t = 0.5773502692$, the coordinates of the integration points are p_1 $(-t, -t)$, p_2 $(t, -t)$, p_3 (t, t), p_4 $(-t, t)$, and the corresponding weight factors are $w_1 = w_2 = w_3 = w_4 = 1$.

Also denoting

$f = (x^2 + xy^2)\det\mathbf{J}$, the integral may be evaluated by the summation of the values at the integration points 1,2,3,4.

$$I = f_1 w_1 + f_2 w_2 + f_3 w_3 + f_4 w_4$$

On substituting the values and evaluating,

$$I = 3712 \qquad \blacksquare$$

8.3 (a) Two shape functions are linear along an edge. Consider an edge say with $\eta = -1$. This represents edge 1-2. Along this edge,
$N_1 = 0.5(1 - \xi)$, $N_2 = 0.5(1 + \xi)$, $N_3 = 0$, $N_4 = 0$. N_1 and N_2 are linear.

(b) No. Point corresponding to $\xi = 0$ and $\eta = 0$ is the intersection of lines joining the mid-points of opposite sides.

(c) No. At the Gauss points, the calculated stresses are close to the stresses in the material.

(d) Yes. There are two points and two weight factors, allowing for a cubic polynomial to be handled. In two dimensions, we make use of the product rule. ■

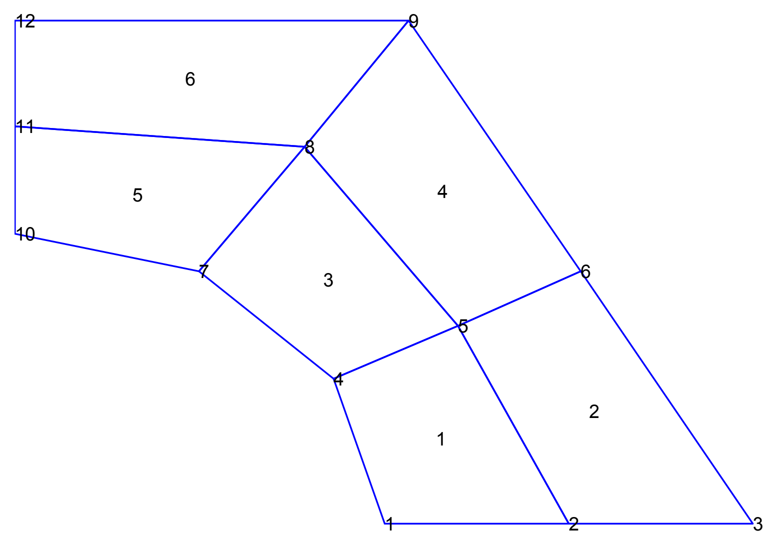
8.4 The discussion for this problem is same as that for Problem 6.17. In the input data set for MESHGEN the number of nodes

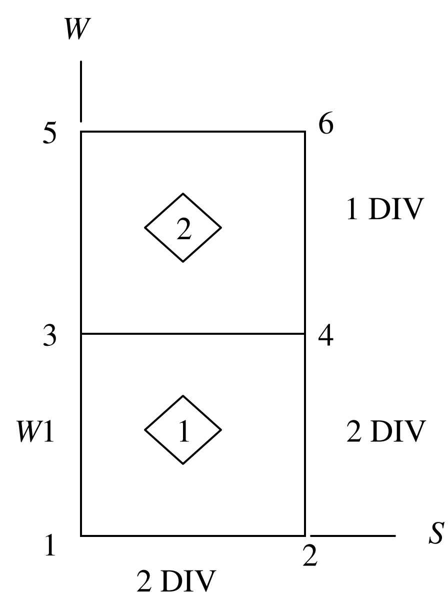
```
Mesh Generation
Problem 8.4
Number of Nodes per Element <3 or 4>
  4                      ←====  4 NODES PER ELEMENT MAKES IT QUADRILATERAL
BLOCK DATA
…
```

The output of this program can be checked using PLOT2D to visually observe the mesh. This file needs editing to add the material thickness and temperature rise to the element data lines. The boundary condition data, load data, and material property data are exactly same. Run program QUAD using this input file. The output from the QUAD program gives displacements of the nodes and stresses evaluated at the integration points. The program asks for interactive input on the type of the problem - plane stress/plane strain, and if contour data file is to be generated.

The input data file and the element output file are then used with BESTFITQ to generate nodal data file. The input file generated above and the nodal data file are then used as input to CONTOURA or CONTOURB to generate contours. ■

8.5





The block configuration and the generated mesh are shown above. The data file for program MESHGEN follows.
Input Data File for MESHGEN (you may name this file P75.MSH)

```
Mesh Generation
Problem 7.5
Number of Nodes per Element <3 or 4>
  4
```

```
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ)
  1             2             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
  1        2
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        2
  2        1
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
  1      3         0
  2      6         0
  3      1.5       2.598
  4      3.2       5.196
  5      0         3
  6      0         5.196
  0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  1      2.598     1.5
  0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
```

In editing the input file generated by the MESHGEN program, we use the following information. DOF with zero displacement are 1, 2, 3, 4, 5, 6, 19, 21, 23, with ND = 9.

There is only one edge with load. Total edge load is $5000 \times 3.2 \times 1 = 16000$N. We use 1m length of the culvert for the *plane strain* model. The loads −8000 N each at DOF 18 and 24, with NL = 2.

After running the program, compare the results with those obtained using triangular elements. ■

8.6
8.7
8.8
These three problems correspond to problems **6.18**, **6.19**, and **6.22**. Follow the steps suggested in problems 7.4 and 7.5. Compare the results from the QUAD elements with those obtained from the triangular elements. ■

8.9 The theory for the axisymmetric quadrilateral element is presented in the text. Program AXIQUAD has been provided. Example 6.2 is a simple model. The input data file for AXIQUAD is given below.

```
<< AXISYMMETRIC STRESS ANALYSIS USING AXIQUAD>>
PROBLEM 7.9
NN NE NM NDIM NEN NDN
 4  1  1  2    4   2
```

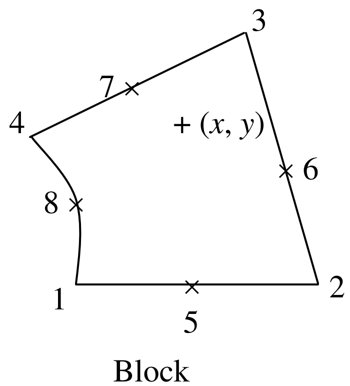
```
ND NL  NMPC
 6  2   0
Node#   X    Y   (r  z  coordinates)
 1      40   10
 2      40   0
 3      60   0
 4      60   10
Elem#  N1  N2  N3  N4  Mat#  TempRise
 1      1   2   3   4   1     0
DOF#  Displacement
 2     0
 4     0
 5     0
 6     0
 7     0
 8     0
DOF#  Load
 1    2514
 3    2514
MAT#   E      Nu    Alpha
 1    200000  .3    12E-6
B1  i  B2 j  B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

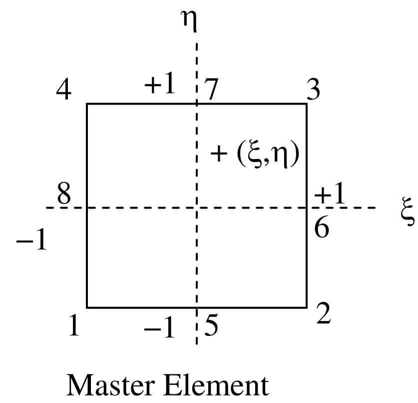
The output from program AXIQUAD follows.

```
Output for Input Data in File --- p79.inp
PROBLEM 7.9
NODE#   R-Displ     Z-Displ
   1   1.339E-04   2.448E-09
   2   1.339E-04  -2.448E-09
   3   4.170E-09  -3.428E-09
   4   4.170E-09   3.428E-09
DOF#        Reaction
    2     -1.6173E+03
    4      1.6173E+03
    5     -2.7544E+03
    6      2.2643E+03
    7     -2.7544E+03
    8     -2.2643E+03
von Mises Stress in each element:
ELEM#  1
  1.25E+00  1.25E+00  1.07E+00  1.07E+00
```

The stresses above are vonMises stress values at the four integration points. The displacements from triangular elements are Q_1 = 1.4E–6, and Q_3 = 1.33E–6. ■

8.10 The basic ideas from this problem are implemented in Chapter 12 for mesh generation. The master element as discussed in this chapter is given below, together with the block. The shape functions are given by Eq 8.56, 8.58.



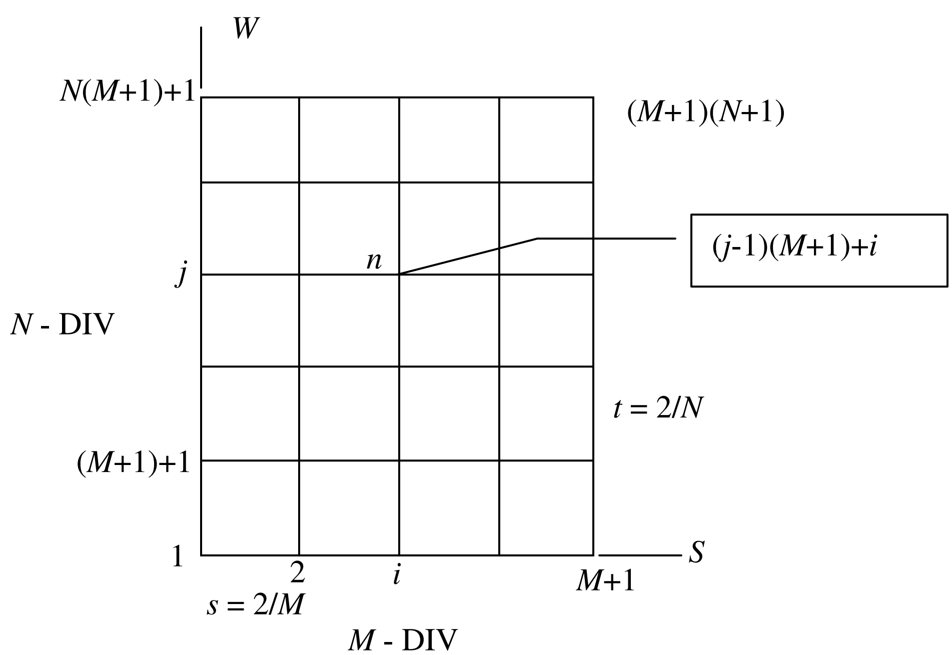


Using the eight shape functions,

$$x = \sum_{i=1}^{8} N_i x_i$$

$$y = \sum_{i=1}^{8} N_i y_i$$

We now consider the division of the master element, with M divisions in the ξ direction and N divisions in the η direction. $M = 3$ and $N = 3$ will give the desired division for the problem.



We scan along S with i going from 1 to $M + 1$, and continue with next row along W. j goes from 1 to $N + 1$ along W.

$$s = 2/M \qquad t = 2/N$$

Limits $\quad i = 1$ to $M + 1$

$j = 1$ to $N + 1$

$$n = (j-1)(M+1) + i \qquad \text{(Node)}$$

$$\xi_n = -1 + (i-1)s$$

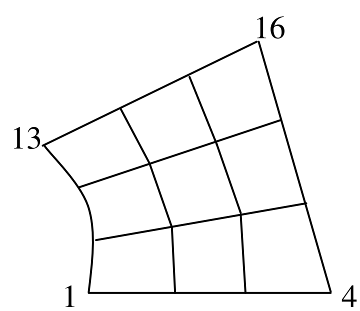
$$\eta_n = -1 + (j-1)t$$

We then define $N_1, N_2, \ldots, N_8$ using ξ_n and η_n, and write

$$x = \sum_{i=1}^{8} N_i x_i$$

$$y = \sum_{i=1}^{8} N_i y_i$$

(x,y) represents coordinates corresponding to point n. The generated mesh is shown below.



This algorithm has been implemented in program MESHGEN. These ideas may be used for other applications. ■

8.11 The program development should follow the steps used in program QUAD. In fact, it is best accomplished by making a copy of this program as QUAD8 and modifying it. The eight shape functions defined in Eqs 8.56 and 7.58 are to be introduced. The element stiffness matrix will be 16×16. The limits on the DO loops must be based on the sizes of **G**, **A**, and **B** matrices.

G, **J**, and **A** matrices need to be derived using the new shape functions.

2×2 integration should work well for the 8 node quadrilateral. The data file may be prepared by editing the current QUAD.INP file. The changes are number of nodes per element NEN, and the connectivity. ■

8.12 The steps of dividing the region into blocks and generating the mesh are same as before. The key step is to calculate the loads at the nodes as suggested in problem 7.16. Automatic calculation in the program will enable trying out various pressures an easy chore. ■

8.13

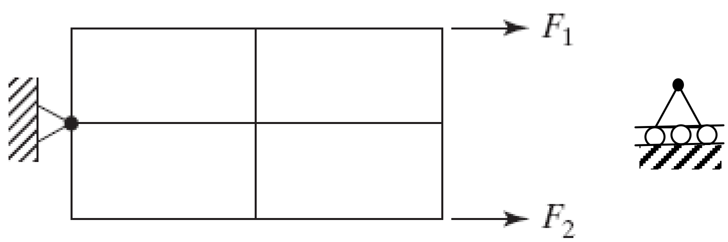
(a) What is meant by the term "higher order elements?" ("Higher" than what?)
Ans: Higher than the element with linear displacements (CST or QUAD in 2D).

(b) How many independent material properties exist for an isotropic material?
Ans: Two. Modulus of elasticity E and Poisson's ratio v.

(c) If a 2-dimensional element for plane stress/plane strain is shaped as a six-node hexagon, what are the dimensions of **k** and **B**?
Ans: **B** relates 3 strains to 12 nodal displacements. **B** is a 3x12 matrix. Element stiffness matrix $\mathbf{k} = \mathbf{B}^T\mathbf{D}\mathbf{B}$ is a 12x12 matrix.

(d) Comment on what is meant by convergence in finite element analysis (FEA). Search internet sources to learn h -convergence and p -convergence.
Ans: The solution approaching the expected exact solution as the finite element solution is refined is convergence. Convergence as the number of elements is increased (size of element h is decreased) is h-convergence. In another approach, the mesh may be fixed but the degree of the interpolating polynomial (p) is increased. This results in p-convergence. ■

8.14 Indicate what is wrong with the model?
In a two dimensional model, there should be at least three degrees of freedom specified. The model shown has a pinned support which suppresses two dof. We need one more dof specified. Any of the other nodes may be placed on a roller support shown on the right in some direction.



■

8.15

(a) Are stresses constant within a four-node quadrilateral element?
Ans: No. Shape functions are of the type $(1+\xi_i\xi)(1+\eta_i\eta)/4$ having a product term of $\xi\eta$. So the strains are not constant. The stresses are therefore not constant.

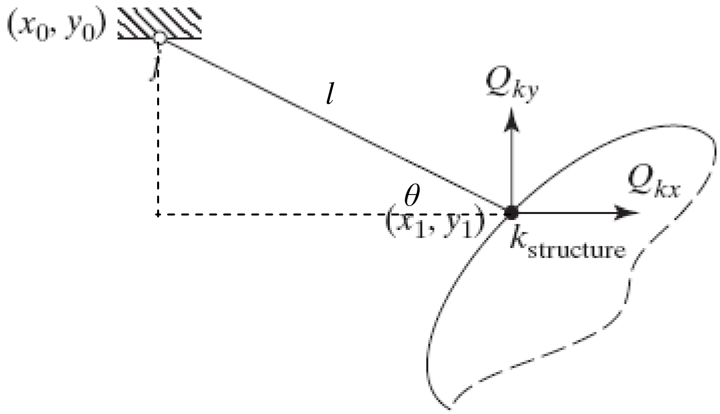
(b) Are shape functions linear on the edge of a four-node quadrilateral element?
Ans: Yes. On each edge $\xi = \pm 1$ or $\eta = \pm 1$, thus it is linear in the other variable.

(c) Why do you need numerical integration?
Ans: Exact integration of the expressions may not exist. Computers only handle numbers and in general do not handle symbolic expressions. The idea of numerical integration is to evaluate a definite integral by summing weighted values at specific points called integration points.

(d) How many integration points are commonly used in computing matrices for the four node quadrilateral element?

Ans: Four integration points. This is called 2x2 scheme based on two integration points in one dimension.

(e) A node k on a structure is connected by a rigid link to a fixed node j , as shown. Assuming small deformations, write the corresponding boundary conditions (constraint equation) in the form of $\beta_1 Q_{kx} + \beta_2 Q_{ky} = \beta_0$. (Hint : Refer to discussion in Chapter 3 .)



From the figure, we have

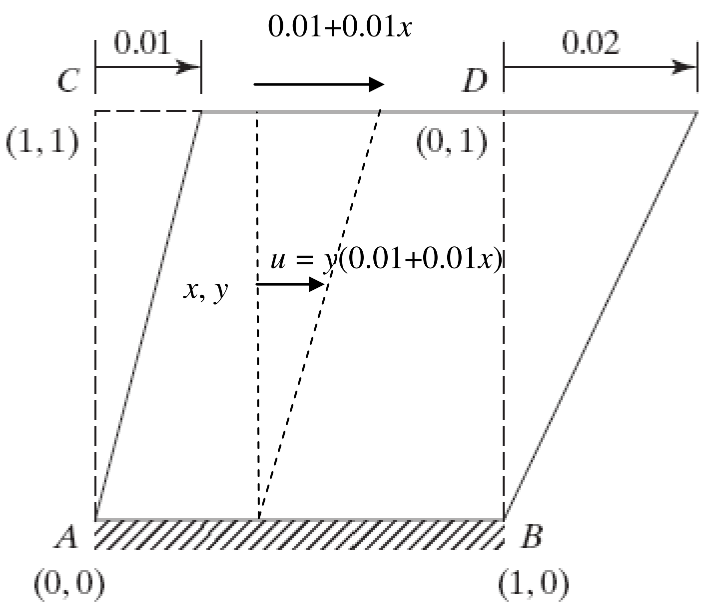
$$l = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)}$$

$$\sin\theta = \frac{y_0 - y_1}{l} \quad \cos\theta = \frac{x_1 - x_0}{l}$$

$$Q_{kx}\cos\theta - Q_{ky}\sin\theta = 0$$

■

8.16



Using the geometry, we observe that the displacement is

$$u = y(0.01 + 0.01x)$$

$$v = 0$$

Thus

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0.01y$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 0$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.01 + 0.01x$$

■

8.17

$$\Pi_e = 3q_1^2 - 6q_1q_2 + 9q_2^2 + 9q_1$$

$$= \frac{1}{2}\begin{bmatrix} q_1 & q_2 \end{bmatrix}\begin{bmatrix} 6 & -6 \\ -6 & 18 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} - \begin{bmatrix} q_1 & q_2 \end{bmatrix}\begin{bmatrix} -9 \\ 0 \end{bmatrix}$$

$$\mathbf{k} = \begin{bmatrix} 6 & -6 \\ -6 & 18 \end{bmatrix}$$

$$\mathbf{f} = \begin{bmatrix} -9 \\ 0 \end{bmatrix}$$

■

8.18

(a) What are the independent material constants for an orthotropic material in two dimensions?
Ans: E_1, E_2, and v_{12}.

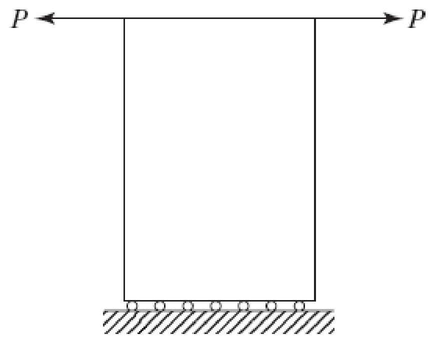
(b) Comment on "modeling error" and "mesh-dependent error."
Ans: *Modeling error* occurs when the boundary conditions are not properly accounted for. A fixed condition in a model may be between fixed and pinned in a real situation.
Mesh-dependent error occurs due to improper matching at the boundary and orientation, size, and distribution of the mesh division.

(c) The main advantage of an eight-node quadrilateral element over a four-node quadrilateral for plane elasticity problem is that the sides of the element can be curved. Comment.
Ans: Yes. A curved boundary can be matched better with a eight node quadrilateral. In addition the stresses are more accurately represented.

(d) An element is subjected to hydrostatic stress, $\sigma_x = \sigma_y = \sigma_z = 0$, and all shear stresses are zero. What is the von Mises stress in the element?
Ans: von Mises stress is zero. von Mises stress depends on
$(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2$ when shear stresses are absent.

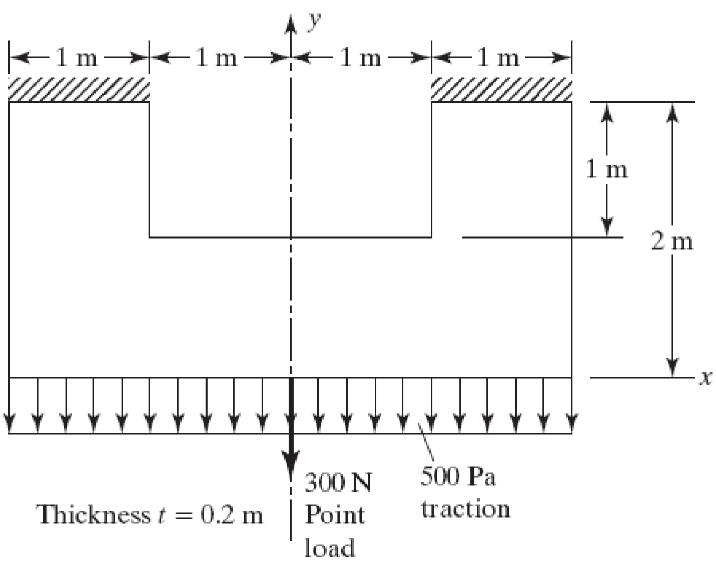
(e) Does the structure shown exhibit rigid body motion?

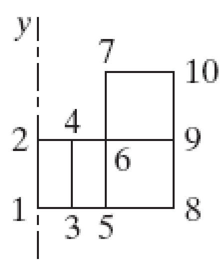
Ans: Yes. It is not properly restrained. Since the forces are equal and opposite theoretically it is in equilibrium. We can model it by using symmetry and restraining it on the line of symmetry.



■

8.19





$\ell_{1-3} = 0.5$ m

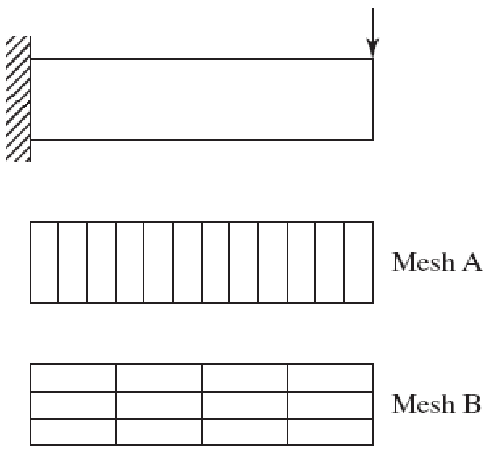
Coordinates and connectivity are easily defined. Boundary conditions and loading must take the symmetry into account.

Boundary conditions: $Q_{13} = 0$, $Q_{14} = 0$, $Q_{19} = 0$, $Q_{20} = 0$, $Q_1 = 0$, $Q_3 = 0$.

Loads: The load on 1-3 is (0.5)(0.2)(500) = 50N distributed equally along the negative direction of dof 2 and 6. Edge 3-5 is similar. Edge 5-8 has a load of 100N distributed along negative directions of dof 10 and 16. We only consider a load of 150N at node 1 due to symmetry. The loads are $F_2 = -150 - 25 = -175$N, $F_6 = -25 - 25 = -50$N, $F_{10} = -25 - 50 = -75$N, $F_{16} = -50$N.

```
DOF#  Displacement
  1      0
  3      0
 13      0
 14      0
 18      0
 10      0
DOF#  Load
 2     -175
 6     -50
10     -75
16     -50
```
■

8.20 Which mesh will give a better solution to the beam problem?

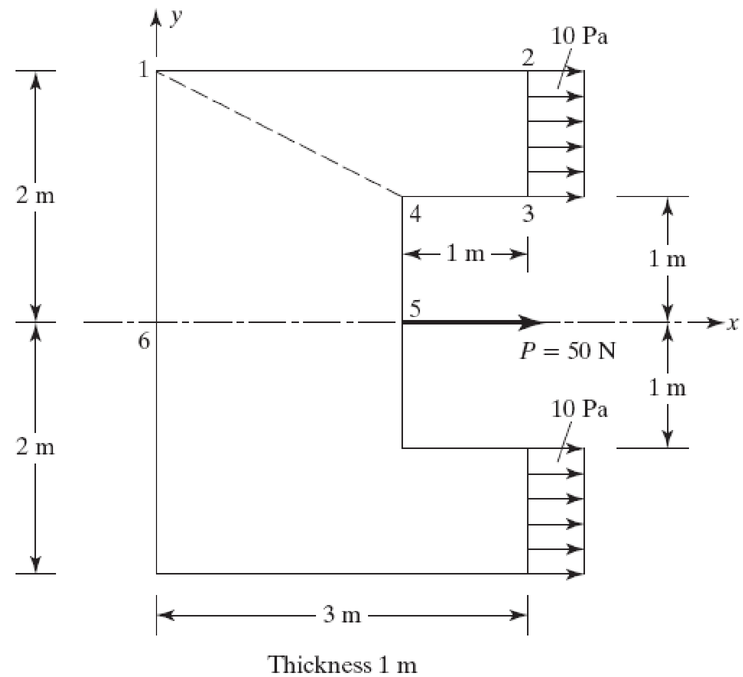


Mesh B gives a better solution. Both meshes have the same number of elements but mesh B has three elements along the depth of the beam. This represents the stress distribution better. ■

8.21

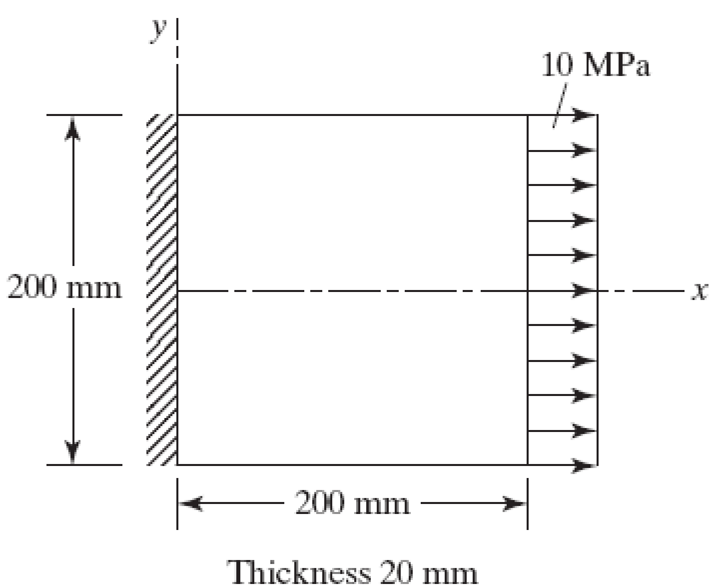
Due to symmetry Nodes 5 and 6 are restrained in the y – direction. Node 6 is also restrained in the x – direction to prevent rigid body motion. 10Pa distributed load gives two 5N loads along dof 3 and 5.

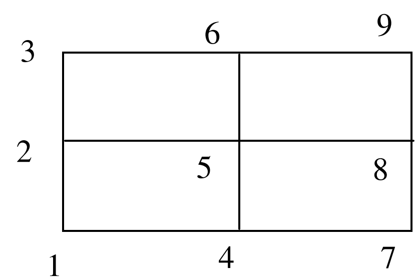
```
DOF#  Displacement
 10      0
 11      0
 12      0
DOF#  Load
 3      5
 5      5
```



■

8.22






```
DOF#  Displacement
 1      0
 2      0
```

```
   3      0
   4      0
   5      0
   6      0
 DOF#   Load
  13     5000
  15    10000
 17     5000
```

■

8.23

$$N_1 = \frac{(1-\xi)(1-\eta)}{4}$$

$\eta = -1$ along 1-2 and $\xi = 1$ along 2-3.

$$I = \int_{node1}^{node2} N_1 dS + \int_{node2}^{node3} N_1 dS$$

The second part of the integral is thus zero. $x = \dfrac{1+\xi}{2} x_2$ along 1-2. Thus $dx = d\xi$

$$I = \int_{node1}^{node2} N_1 dS = \int_{-1}^{1} \frac{1-\xi}{2} d\xi$$

For one point integration, $\xi = 0$, and weight is 2. Thus

$$I = 2\left(\frac{1-0}{2}\right) = 1$$

■

8.24

The key in finding the equivalent nodal loads for Example 8.2 with body force $\mathbf{f} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$ is to develop the load vector for this case. For the rectangular element, we have $x_1 = 0$, $y_1 = 0$, $x_2 = 2$, $y_2 = 0$, $x_3 = 2$, $y_3 = 1$, $x_4 = 0$, $y_4 = 1$.

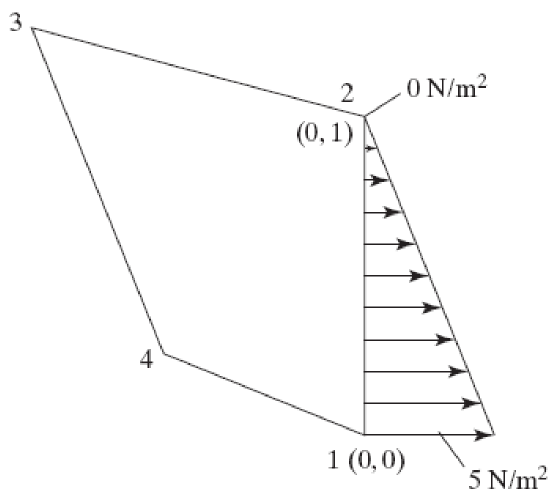
$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 = 2N_2 + 2N_3$$
$$= 2\frac{(1+\xi)(1-\eta)}{4} + 2\frac{(1+\xi)(1+\eta)}{4}$$
$$= 1+\xi$$
$$dx = d\xi$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 = N_3 + N_4$$
$$= \frac{(1+\xi)(1+\eta)}{4} + \frac{(1-\xi)(1+\eta)}{4}$$
$$= \frac{1+\eta}{2}$$
$$dy = \frac{d\eta}{2}$$
$$dxdy = \frac{1}{2}d\xi d\eta$$
$$\det \mathbf{J} = \frac{1}{2}$$
$$t = 1.$$

$$\int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}t dA = \mathbf{q}^{\mathrm{T}} t \int_{-1}^{1}\int_{-1}^{1} \mathbf{N}^{\mathrm{T}}\mathbf{f} \det \mathbf{J} d\xi d\eta$$
$$= \mathbf{q}^{\mathrm{T}} tA \sum_{i=1}^{4} w_i \mathbf{N}^{(i)\mathrm{T}}\mathbf{f}$$
$$= \mathbf{q}^{\mathrm{T}}\left(\frac{1}{2}\right)\left(\sum_{i=1}^{4}(1)\begin{bmatrix} N_1^{(i)} & 0 \\ 0 & N_1^{(i)} \\ N_2^{(i)} & 0 \\ 0 & N_2^{(i)} \\ N_3^{(i)} & 0 \\ 0 & N_3^{(i)} \\ N_4^{(i)} & 0 \\ 0 & N_4^{(i)} \end{bmatrix}\right)\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \mathbf{q}^{\mathrm{T}}\left(\frac{1}{2}\right)\begin{bmatrix} \left(N_1^{(1)}+N_1^{(2)}+N_1^{(3)}+N_1^{(4)}\right)f_x \\ \left(N_1^{(1)}+N_1^{(2)}+N_1^{(3)}+N_1^{(4)}\right)f_y \\ \left(N_2^{(1)}+N_2^{(2)}+N_2^{(3)}+N_2^{(4)}\right)f_x \\ \left(N_2^{(1)}+N_2^{(2)}+N_2^{(3)}+N_2^{(4)}\right)f_y \\ \left(N_3^{(1)}+N_3^{(2)}+N_3^{(3)}+N_3^{(4)}\right)f_x \\ \left(N_3^{(1)}+N_3^{(2)}+N_3^{(3)}+N_3^{(4)}\right)f_y \\ \left(N_4^{(1)}+N_4^{(2)}+N_4^{(3)}+N_4^{(4)}\right)f_x \\ \left(N_4^{(1)}+N_4^{(2)}+N_4^{(3)}+N_4^{(4)}\right)f_y \end{bmatrix}$$
$$= \mathbf{q}^{\mathrm{T}}\mathbf{f}^e$$

We note that for the 2x2 integration $N_j^{(1)} + N_j^{(2)} + N_j^{(3)} + N_j^{(4)} = 1$ for each of the shape functions (you may verify that this property is true at any point in the master element). Thus

$$\mathbf{f}^e = \begin{bmatrix} 0.5f_x \\ 0.5f_y \\ 0.5f_x \\ 0.5f_y \\ 0.5f_x \\ 0.5f_y \\ 0.5f_x \\ 0.5f_y \end{bmatrix} \quad \blacksquare$$

8.25



We use the following

$$N_1 = \frac{1-\xi}{2} \quad N_2 = \frac{1+\xi}{2}$$

$$\mathbf{u} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \mathbf{Nq}$$

$$y = N_1 y_1 + N_2 y_2 = N_2$$

$$dy = \frac{1}{2} d\xi$$

$$\mathbf{p} = \begin{bmatrix} 5N_1 \\ 0 \end{bmatrix}$$

Now the distributed force term in the potential energy is considered.

$$\int_e \mathbf{u}^{\mathrm{T}}\mathbf{p}t\,d\ell = \mathbf{q}^{\mathrm{T}} t\int_{-1}^{1} \mathbf{N}^{\mathrm{T}} \begin{bmatrix} 5N_1 \\ 0 \end{bmatrix} \left(\frac{1}{2}\right) d\xi$$

$$= \mathbf{q}^{\mathrm{T}} t\int_{-1}^{1} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} 5N_1 \\ 0 \end{bmatrix} \left(\frac{1}{2}\right) d\xi$$

$$= \mathbf{q}^{\mathrm{T}} \frac{t}{2}\int_{-1}^{1} \begin{bmatrix} 5N_1^2 \\ 0 \\ 5N_1N_2 \\ 0 \end{bmatrix} d\xi$$

(a) For one point integration $w = 2$, $\xi = 0$, $N_1 = 0.5$, $N_2 = 0.5$,

$$\int_e \mathbf{u}^{\mathrm{T}}\mathbf{p}t\,d\ell = \mathbf{q}^{\mathrm{T}} \frac{t}{2}(2) \begin{bmatrix} 1.25 \\ 0 \\ 1.25 \\ 0 \end{bmatrix} = \mathbf{q}^{\mathrm{T}} \begin{bmatrix} 1.25t \\ 0 \\ 1.25t \\ 0 \end{bmatrix} = \mathbf{q}^{\mathrm{T}}\mathbf{f}^{e}$$

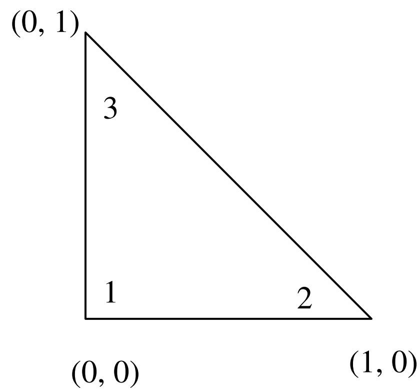
(b) For two point integration $w_1 = 1, \xi_1 = -\frac{1}{\sqrt{3}}, w_2 = 1, \xi_2 = \frac{1}{\sqrt{3}}$

$$\int_e \mathbf{u}^{\mathrm{T}}\mathbf{p}t\,d\ell = \mathbf{q}^{\mathrm{T}} \frac{t}{2} \left((1) \begin{bmatrix} 5\left(\frac{1-\xi_1}{2}\right)^2 \\ 0 \\ 5\left(\frac{1-\xi_1}{2}\right)\left(\frac{1+\xi_1}{2}\right) \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 5\left(\frac{1-\xi_2}{2}\right)^2 \\ 0 \\ 5\left(\frac{1-\xi_2}{2}\right)\left(\frac{1+\xi_2}{2}\right) \\ 0 \end{bmatrix} \right)$$

$$= \mathbf{q}^{\mathrm{T}} \begin{bmatrix} 1.6667t \\ 0 \\ 0.8333t \\ 0 \end{bmatrix} = \mathbf{q}^{\mathrm{T}}\mathbf{f}^{e}$$

Note that the one point integration gives same result as for a uniform distribution with $p/2$ and the two point integration gives the precise load obtained by exact integration as given by (6.42). ■

8.26



We make use of the exact integration formula from Chapter 6.

$$\int_0^1\int_0^{1-\xi} \xi^a \eta^b \left(1-\xi-\eta\right)^c \, d\xi d\eta = \frac{a!b!c!}{\left(a+b+c+2\right)} \qquad (6.46)$$

The exact integral is

$$I = \int_e \left(\xi^2 + \xi\eta\right) dA = 2A\left(\frac{2!}{4!} + \frac{1!1!}{4!}\right) = 2\left(0.5\right)\left(\frac{1}{12} + \frac{1}{24}\right)$$

$$= \frac{1}{8}$$

Using one point integration w = 1/2, $\xi = 1/3, \eta = 1/3$,

$$I = \int_e \left(\xi^2 + \xi\eta\right) dA = 2A\left(\frac{1}{2}\right)\left(\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\right) = \frac{1}{9}$$

Using three point integration, A = 0.5, w = 1/6, $\xi = 2/3, \eta = 1/6, \zeta = 1/6$, where ξ, η, ζ are taken cyclically

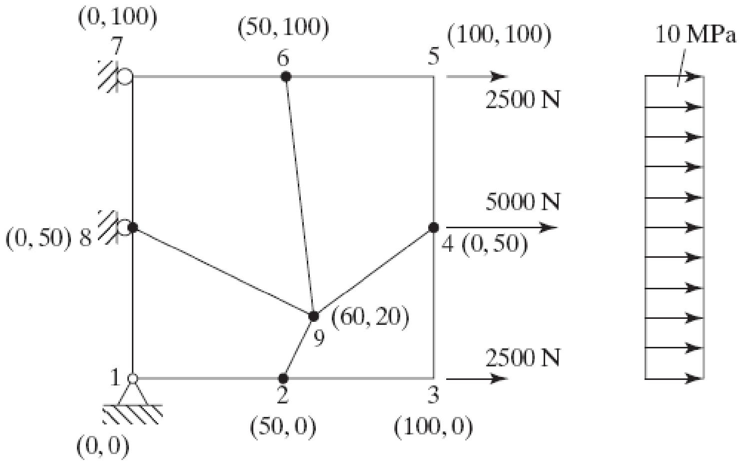
$$I = \int_e \left(\xi^2 + \xi\eta\right) dA$$

$$= 2A\left(\frac{1}{6}\right)\left(\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)\left(\frac{2}{3}\right)\right)$$

$$= 0.125$$

The three point integration gives the exact value for this function. ■

8.27 E and ν may be chosen for any real material. In the data here we use aluminum.



Input data

PROBLEM 8.27

NN	NE	NM	NDIM	NEN	NDN		
9	4	1	2	4	2		
ND	**NL**	**NMPC**					
4	3	0					
Node#	**X**	**Y**					
1	0	0					
2	50	0					
3	100	0					
4	0	50					
5	100	100					
6	50	100					
7	0	100					
8	0	50					
9	60	20					
Elem#	**N1**	**N2**	**N3**	**N4**	**Mat#**	**Thick**	**ΔT**
1	8	1	2	9	1	10	0
2	2	3	4	9	1	10	0
3	4	5	6	9	1	10	0
4	6	7	8	9	1	10	0
DOF#	**Displ.**						
1	0						
2	0						
13	0						
15	0						
DOF#	**Load**						
5	2500						
7	5000						
9	2500						
MAT#	**E**	**Nu**	**Alpha**				
1	70000	0.33	1.2E-05				

B1	i	B2	j	B3	MPC

Output

Program Quad - Plane Stress Analysis
PROBLEM 8.27

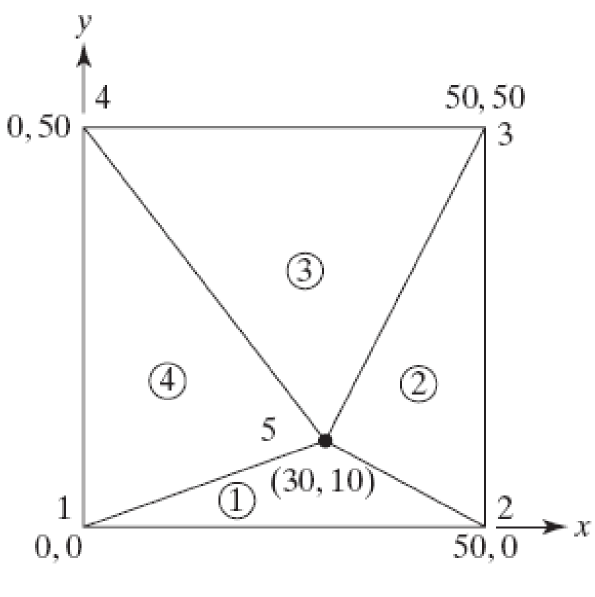
Node#	X-Displ	Y-Displ
1	0.00000	0.00000
2	0.00714	0.00000
3	0.01429	0.00000
4	0.00000	-0.00236
5	0.01429	-0.00471
6	0.00714	-0.00471
7	0.00000	-0.00471
8	0.00000	-0.00236
9	0.00857	-0.00094

Elem#	Iteg1	Iteg2	Iteg3	Iteg4	<== vonMises Stresses
1	10.0000	10.0000	10.0000	10.0000	
2	10.0000	10.0000	10.0000	10.0000	
3	10.0000	10.0000	10.0000	10.0000	
4	10.0000	10.0000	10.0000	10.0000	

DOF#	Reactn
1	-2500.01
2	0.00
13	-2500.01
15	-4999.98

The vonMises stress is constant 10MPa at each of the integration points. This is the constant state of stress for the load patch test. ■

8.28



BC	Node	u	v
	1	0	0
	2	0.005	0.0025
	3	0.0075	0.0075
	4	0.0025	0.0050

$E = 10^6$ $\quad\quad$ $\nu = 0.25$

The expected displacement of 5 is (0.0035, 0.0025).

Input Data

<< 2D STRESS ANALYSIS USING QUAD >>

PROBLEM 8.28

NN	**NE**	**NM**	**NDIM**	**NEN**	**NDN**
5	4	1	2	4	2

ND	**NL**	**NMPC**
8	0	0

Node#	**X**	**Y**
1	0	0
2	50	0
3	50	50
4	0	50
5	30	10

Elem#	**N1**	**N2**	**N3**	**N4**	**Mat#**	**Thick**	**ΔT**
1	5	1	2	5	1	1	0
2	5	2	3	5	1	1	0
3	5	3	4	5	1	1	0
4	5	4	1	5	1	1	0

DOF#	**Displacement**
1	0
2	0
3	0.005
4	0.0025
5	0.0075
6	0.0075
7	0.0025
8	0.005

DOF#	**Load**

MAT#	**E**	**Nu**	**Alpha**
1	1000000	0.25	1.2E-05

B1	**i**	**B2**	**j**	**B3**	**MPC**

Output (Plane Stress)

Program Quad - Plane Stress Analysis

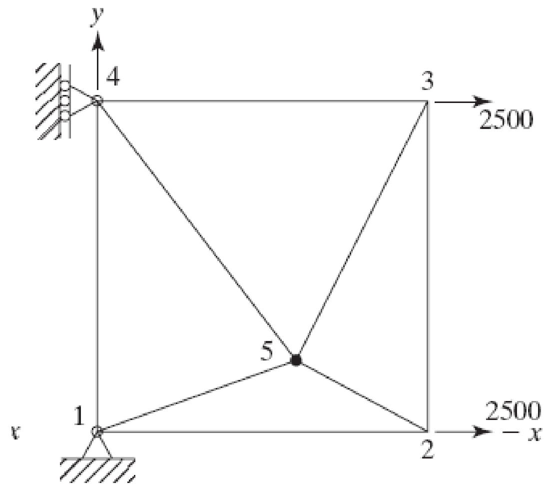
PROBLEM 8.28

Node#	**X-Displ**	**Y-Displ**
1	1.04E-7	1.04E-7
2	0.005	0.0025
3	0.0075	0.0075
4	0.0025	0.005
5	0.0035	0.0025

Elem#	**Iteg1**	**Iteg2**	**Iteg3**	**Iteg4**
1	150.255	150.255	150.255	150.255
2	150.255	150.255	150.255	150.255
3	150.255	150.255	150.255	150.255
4	150.255	150.255	150.255	150.255

Displacement of node 5 is now verified. The vonMises stress calculated corresponds to (133.3,133.3,40) for plane stress. ■

8.29



Distributed load 10
Thickness 10
Point loads $= \dfrac{50 \times 10 \times 10}{2} = 2500$

Exact solution

$$\sigma_x = 10$$
$$\sigma_y = 10$$
$$\tau_{xy} = 0$$

Input Data

<< 2D STRESS ANALYSIS USING QUAD >>
PROBLEM 8.29

NN	**NE**	**NM**	**NDIM**	**NEN**	**NDN**			
5	4	1	2	4	2			
ND	**NL**	**NMPC**						
3	2	0						
Node#	**X**	**Y**						
1	0	0						
2	50	0						
3	50	50						
4	0	50						
5	30	10						
Elem#	**N1**	**N2**	**N3**	**N4**	**Mat#**	**Thickn**	**ΔT**	
1	5	1	2	5	1	10		0
2	5	2	3	5	1	10		0
3	5	3	4	5	1	10		0
4	5	4	1	5	1	10		0
DOF#	**Displacement**							
1	0							
2	0							
7	0							
DOF#	**Load**							
3	2500							
5	2500							

MAT#	E	Nu	Alpha
1	1000000	0.25	1.20E-5

B1	i	B2	j	B3	MPC

Output (Plane Stress)

Program Quad - Plane Stress Analysis

PROBLEM 8.29

Node#	X-Displ	Y-Displ
1	0.00000	0.00000
2	0.00050	0.00000
3	0.00050	-0.00013
4	0.00000	-0.00013
5	0.00030	-0.00002

Elem#	Iteg1	Iteg2	Iteg3	Iteg4
1	10	10	10	10
2	10	10	10	10
3	10	10	10	10
4	10	10	10	10

The stress calculated is vonMises stress. ■

8.30

The element in Fig. E8.2 is subjected to a body force $\mathbf{f} = [f_x, f_y]^T = [\, x, 0]^T$ lb/in^3. Determine the equivalent point loads at the four nodes in the x−direction. Take thickness = 1 in, and use 2x2 Gauss quadrature for numerical integration.

We have

$$\int_e u\, f_x\, dV = q_1\, f_1 + q_3\, f_3 + q_5\, f_5 + q_7\, f_7$$

where degrees of freedom 1,3,5,7 correspond to x directions at nodes 1,2,3,4, respectively. We use expression for u in Eq. (8.7a), x as given in Eq. (8.9), dV = det **J** dη dξ with det **J** = 0.5 from Example 8.2 above, and the 2x2 rule in Fig. 8.4. This gives

$$f_1 = \phi(-c, -c) + \phi(c, -c) + \phi(c, c) + \phi(-c, -c)$$

where $c = 1/\sqrt{3}$, and ϕ = (thickness) (N_1) ($\sum_{i=1}^{4} N_i\ x_i$) (0.5). For f_3, replace N_1 by N_2 into the expression, etc. The reader is encouraged to see the implementation of this in a simple Matlab code below. Results are:

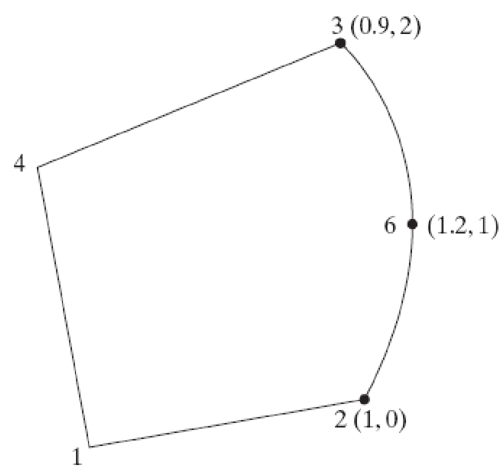
$$[f_1, f_3, f_5, f_7] = [\ 1/3, 2/3, 2/3, 1/3]\ \text{N}.$$

```
%Example 8.2b
clear all; close all;
x1=0; y1=0; x2=2; y2=0; x3=2; y3=1; x4=0; y4=1;
```

```
c=1/sqrt(3); NIP = 4;
XNI(1,1)= -c; XNI(2,1)= -c;
XNI(1,2)= c; XNI(2,2)= -c;
XNI(1,3)= c; XNI(2,3)= c;
XNI(1,4)= -c; XNI(2,4)= c;
f(1:4) = 0;
detJ = 0.5;
for j=1:4
  for i=1:NIP
    xi= XNI(1,i); eta = XNI(2,i);
    N(1) = .25*(1-xi)*(1-eta);   N(2) = .25*(1+xi)*(1-eta);
    N(3) = .25*(1+xi)*(1+eta);   N(4) = .25*(1-xi)*(1+eta);
    x = N(1)*x1 + N(2)*x2 + N(3)*x3 + N(4)*x4;
    f(j) = f(j) + N(j)*x*detJ;
  end
end
f
```

■

8.31



A traction load $T_x = 1$ N/m^2, $T_y = 0$, is applied on an edge of an eight–node quadrilateral as shown in Fig. E8.2c. Determine the equivalent nodal forces.

The starting point is the potential term $\int_{edge\,1-6-3} u\, T_x\, t_e\, dS$. Substituting for $u = N_2\, q_3 + N_3\, q_5 + N_6\, q_{11}$

where $q_{(2*I-1)}$ = x- degree of freedom of node I as per the convention, t_e = thickness of the element. On the edge 2–6–3, we have ξ = 1, dξ = 0 (Fig. 8.7). From calculus, we have

$dS = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{d\eta}\right)^2 + \left(\frac{dy}{d\eta}\right)^2}\, d\eta$. From Eqs. (8.12), (8.56)–(8.58),

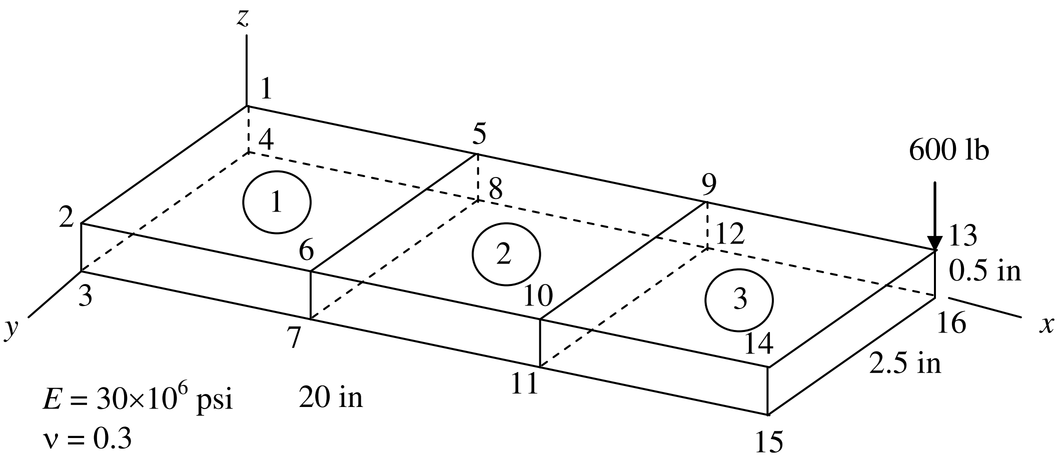
$x = \sum_{i=1}^{8} N_i\, x_i$, $y = \sum_{i=1}^{8} N_i\, y_i$, two–point Gauss quadrature along the edge gives the result [T_3, T_{11}, T_5] = [0.3393, 1.3893, 0.3553] N. The Matlab code used to generate this result is given below.

```
%Example 8.2c - traction on 8-node quad
clear all; close all;
x2=1; y2=0; x6=1.2; y6=1; x3=0.9; y3=2;
c=1/sqrt(3);
te=1; Tx=1; NIP=2;
eta1(1) = -c; eta1(2) = c; %weight = 1
T2x=0; T6x=0; T3x=0;
  for i=1:NIP
    eta = eta1(i);
    N2 = -.5*(1-eta)*eta;
    N2prime = -0.5*(1-2*eta);
    N3 = .5*(1+eta)*eta;
    N3prime = .5*(1+2*eta);
    N6 = 1-eta^2;
    N6prime = -2*eta;
    J21 = x2*N2prime + x6*N6prime + x3*N3prime;
    J22 = y2*N2prime + y6*N6prime + y3*N3prime;
    T2x = T2x + N2*Tx*te*sqrt(J21^2+J22^2);
    T6x = T6x + N6*Tx*te*sqrt(J21^2+J22^2);
    T3x = T3x + N3*Tx*te*sqrt(J21^2+J22^2);
  end
T2x, T6x, T3x
```

■

CHAPTER 9
THREE-DIMENSIONAL PROBLEMS IN STRESS ANALYSIS

9.1



The plate is divided into three hexahedral elements as shown. Nodes 1,2,3, and 4 are fixed. The −600 lb load is along the degree of freedom direction 39. Restraining 12 dof as given in the model here may be too severe. The student may try restraining 6 dof – for example, fully restraining node 4, restraining node 3 in *z* directions, and nodes 1 and 2 in the *x* direction. Note that restraining along less than 6 dof will not prevent rigid body motion. Rigid body motion must be restrained by making a proper choice of at least 6 dof.

<u>Input data file for program HEXAFRON</u>

```
PROBLEM 9.1
3-D ANALYSIS USING HEXAHEDRAL ELEMENT
NN  NE  NM  NDIM  NEN  NDN
16  3   1   3     8    3
ND   NL   NMPC
12   1     0
Node#   X       Y       Z
 1      0       0       0.5
 2      2.5     0       0.5
 3      2.5     0       0
 4      0       0       0
 5      0       6.667   0.5
 6      2.5     6.667   0.5
 7      2.5     6.667   0
 8      0       6.667   0
 9      0       13.333  0.5
10      2.5     13.333  0.5
11      2.5     13.333  0
12      0       13.333  0
13      0       20      0.5
14      2.5     20      0.5
15      2.5     20      0
16      0       20      0
```

```
Elem# N1  N2  N3  N4  N5  N6  N7  N8  MAT#   TempRise
 1     1   2   3   4   5   6   7   8   1      0
 2     5   6   7   8   9  10  11  12   1      0
 3     9  10  11  12  13  14  15  16   1      0
DOF#  Displacement
 1     0
 2     0
 3     0
 4     0
 5     0
 6     0
 7     0
 8     0
 9     0
 10     0
 11     0
 12     0
DOF#  Load
 39   -600
MAT#     E         Nu        Alpha
 1       30e6      0.3       6.67e-6
B1  i  B2 j  B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

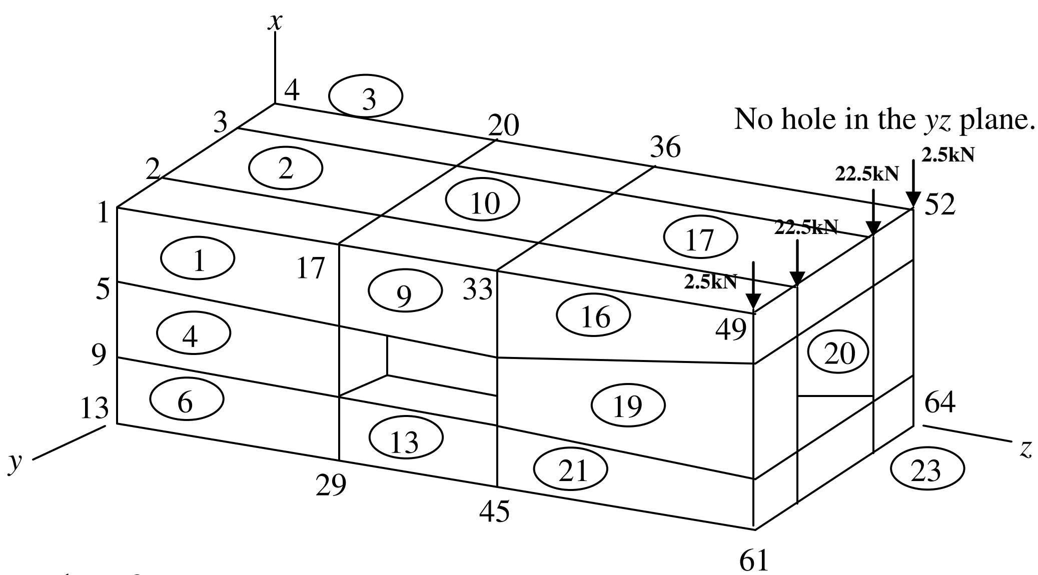
<u>Output from program HEXAFRON</u>

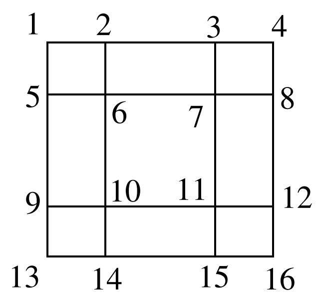
```
Node#   X-Displ.    Y-Displ.    Z-Displ.
   1 -1.3319D-21  7.5254D-17 -1.3401D-18
   2 -1.4060D-18  4.6807D-17 -1.4532D-19
   3  1.4031D-18 -4.6807D-17 -1.3615D-19
   4  4.3034D-21 -7.5254D-17 -1.4299D-18
   5 -5.2302D-04  4.5503D-04 -6.9895D-03
   6 -5.2969D-04  1.5718D-04 -1.7266D-03
   7  5.3055D-04 -1.5730D-04 -1.7267D-03
   8  5.2358D-04 -4.5462D-04 -6.9893D-03
   9 -1.4933D-03  6.6840D-04 -2.2711D-02
  10 -1.4955D-03  3.1062D-04 -7.7772D-03
  11  1.4948D-03 -3.1048D-04 -7.7769D-03
  12  1.4937D-03 -6.6943D-04 -2.2712D-02
  13 -2.5427D-03  7.3782D-04 -4.2142D-02
  14 -2.5419D-03  3.6653D-04 -1.6656D-02
  15  2.5493D-03 -3.6682D-04 -1.6657D-02
  16  2.5465D-03 -7.3265D-04 -4.2138D-02
DOF#         Reaction
    1      2.6189D-01
    2     -1.4797D+04
    3      2.6350D+02
    4      2.7646D+02
    5     -9.2033D+03
    6      2.8572D+01
    7     -2.7588D+02
    8      9.2033D+03
    9      2.6771D+01
   10     -8.4614D-01
   11      1.4797D+04
   12      2.8115D+02
Von Mises Stress at 8 Integation Pts. in Elem#   1
    1.1090D+04    4.9600D+03    4.9605D+03    1.1089D+04
```

```
    7.4113D+03    5.7488D+03    5.7486D+03    7.4107D+03
Von Mises Stress at 8 Integation Pts. in Elem#   2
    7.0189D+03    4.9735D+03    4.9701D+03    7.0191D+03
    4.4628D+03    5.0276D+03    5.0289D+03    4.4675D+03
Von Mises Stress at 8 Integation Pts. in Elem#   3
    4.6313D+03    3.7595D+03    3.7776D+03    4.6359D+03
    3.5332D+03    3.7379D+03    3.7269D+03    3.5099D+03
```

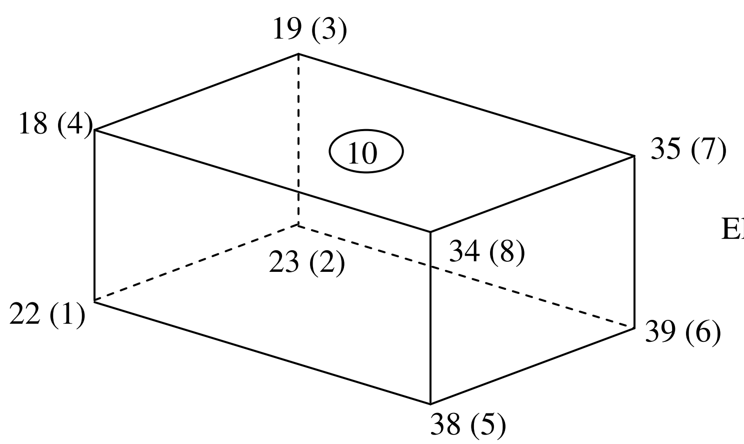
Note that the vertical deflection of node 13 is 0.042 in in the direction of the load. Tetrahedral division may be tried out and the results compared. ■

9.2





Repeating pattern for node numbers.
For next section add 16.



Element 10

The figures above represent a coarse mesh division, which may be used in modeling the problem. The nodal coordinates are easily generated by the repeating pattern approach shown in the second figure.

In the initial model, there are 23 elements and 64 nodes. Nodes 1 through 16 may all be fixed. The distributed load on the edge is equivalent to the point loads shown as applied at nodes 49, 50, 51, and 52.

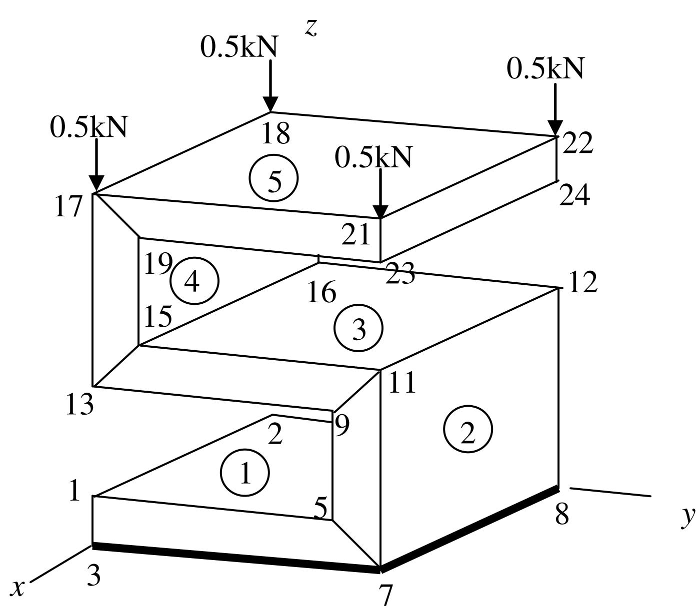
Each hexahedral element is as shown in the third figure. A typical element 10 is shown. The eight nodes of the element are shown in the connectivity as

```
Elem# N1   N2   N3   N4   N5   N6   N7   N8   MAT#    TempRise
 10   22   23   19   18   38   39   35   34   1       0
```

Tetrahedral elements may be introduced by subdividing the hexahedral element shown into five or six elements and the pattern repeated.

Finer meshes increase the degrees of freedom rapidly. The frontal approach used in the program HEXFRON enables one to solve fairly large problems. ■

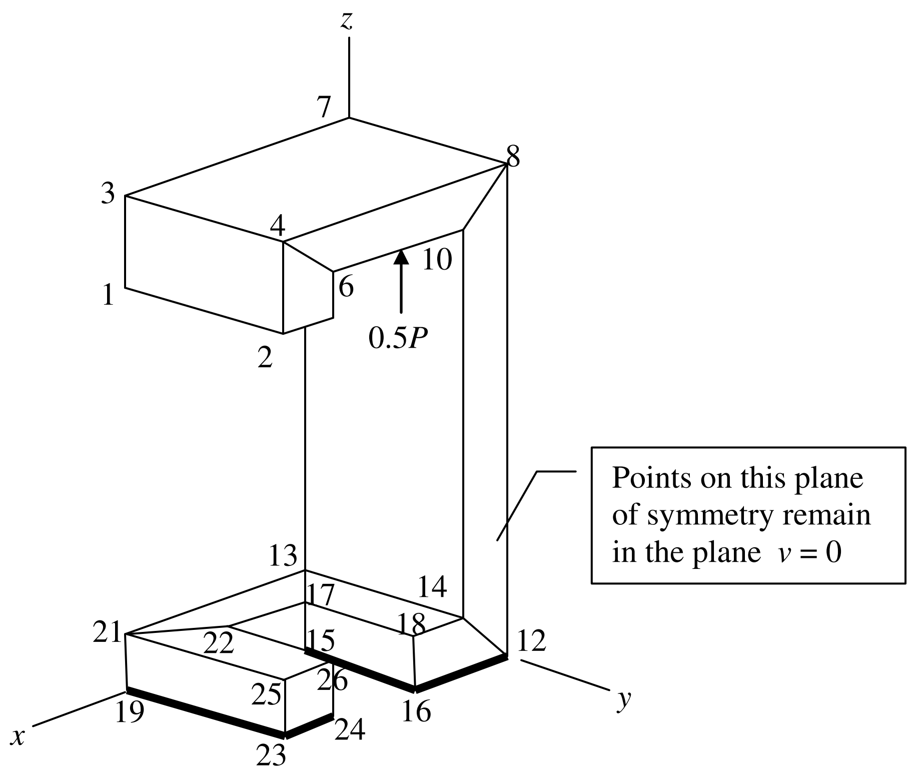
9.3



The division of the S-shaped piece into 5 hexahedral elements is shown in the figure. In this model, nodes 3, 4, 7, 8 may be fixed in all directions. The load of 2kN (20 × 10 × 10 N) may be divided equally as –0.5kN along dof directions 51, 54, 63, and 66 as shown.

The data file HEXFRON.INP may be edited to prepare the input file P93.INP. ■

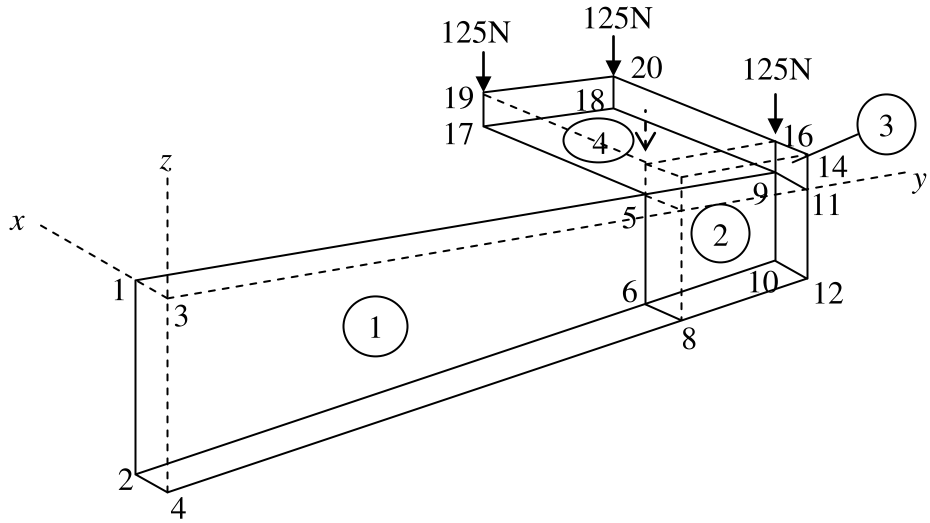
9.4



The problem is modeled by cutting the piece symmetrically about a vertical plane as shown. The nodal division is as shown. We have 26 nodes and 6 elements. The points on the bottom plane (y-z) are constrained along x, y, and z. The points on the plane of symmetry are fixed in the y direction ($v = 0$). Half the load is applied as shown.

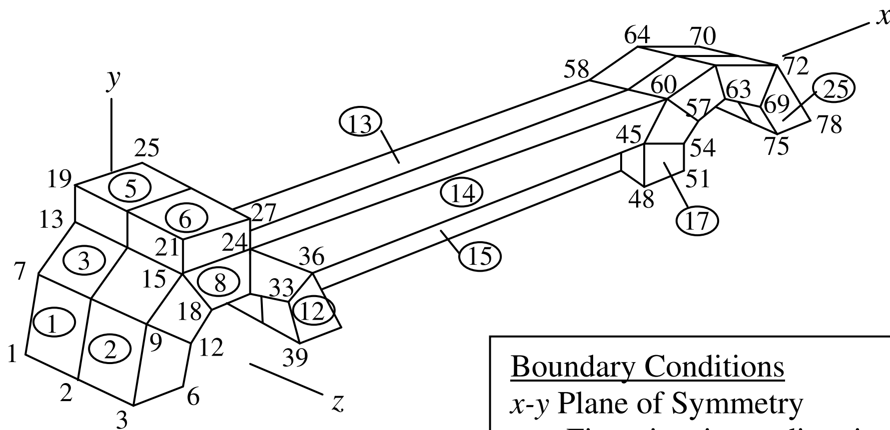
We may a try a load of 1000N. Once the coordinates, connectivity and the boundary conditions, loads and material properties are put in, the problem is ready to run using program HEXAFRON. The problem asks for finding the magnitude and location of maximum principal stresses. Tetrahedral element are easily considered by subdividing the hexahedral elements. ■

9.5 We choose a convenient coordinate system as shown. A simple division of 4 hexahedral elements is chosen. The nodes and connections are established. The nodes 1,2,3, and 4 are fixed in all three directions. The load of 500 N is equally divided among the four nodes 15, 16, 19, and 20 in the negative z direction. The input data file is prepared as given in problem 9.1. Once the data file is prepared running the program is an easy step.



■

9.6

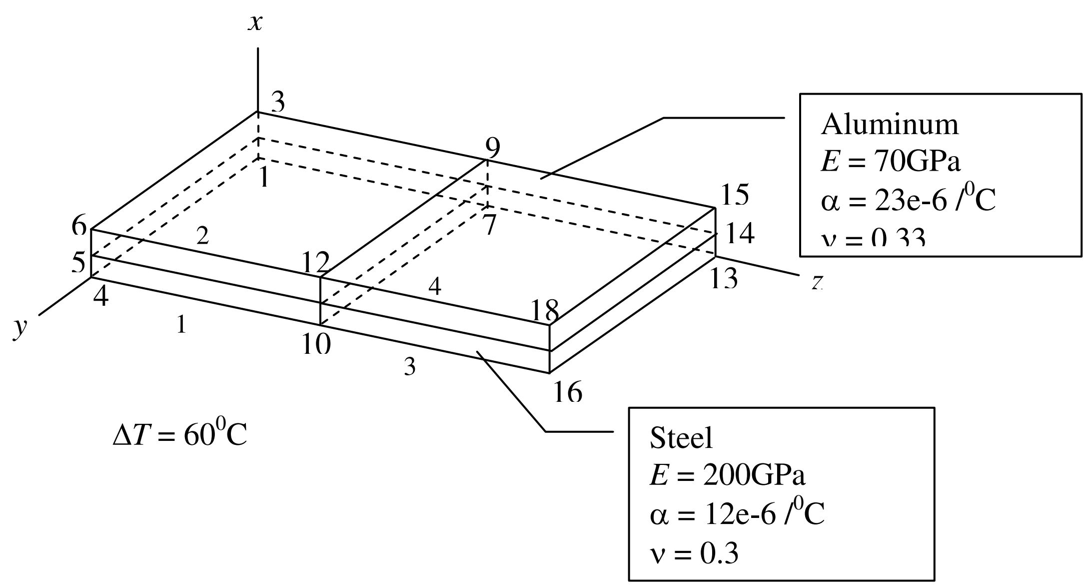


Boundary Conditions
x-y Plane of Symmetry
Fix points in z – direction
x-z Plane of Symmetry
Fix points in y – direction
Fix Nodes 37,38,39 in x – direction

Loading
Quarter model load $F = 0.25P$
$-0.25F$ along dof 145 at node 49
$-0.5F$ along dof 148 at node 50
$-0.25F$ along dof 151 at node 51

The connecting rod problem is included here to show that modeling of complex problems involve the same steps as those in other simple problems. The data preparation involves tedious work. CAD programs or other commercial mesh generators may be used for creating finer meshes. The connecting rod is modeled using a quarter portion using the symmetry. The part is divided into 25 hexahedral elements and 78 nodes. Considerations of boundary conditions and loading are given in the blocks below the figure. Use the aid of a calculator or a spreadsheet in generating the nodal coordinates. The input data file is then prepared by editing the provided data file HEXAFRON.INP. ■

9.7



$\Delta T = 60^0$C

We consider the 18 node 4 element configuration shown. There is no external load in this problem. The temperature rise generates deformation and stresses. The input data for program HEXAFRON and the output from the program are given.

Input data for program HEXAFRON

```
PROGRAM HEXAFRON
Problem 9.7 << 3-D ANALYSIS USING HEXAHEDRAL ELEMENT >>
NN  NE  NM  NDIM  NEN  NDN
18  4   2   3     8    3
ND   NL   NMPC
18   0    0
Node#    X       Y       Z
 1       0       0       0
 2       10      0       0
 3       20      0       0
 4       0       20      0
 5       10      20      0
 6       20      20      0
 7       0.5     0       45
 8       10      0       45
 9       20      0       45
 10      0       20      45
 11      10      20      45
 12      20      20      45
 13      1       0       90
```

```
 14     10      0       90
 15     20      0       90
 16     0       20      90
 17     10      20      90
 18     20      20      90
Elem# N1  N2  N3  N4  N5  N6  N7  N8  MAT#   TempRise
 1     1   2   5   4   7   8   11  10  1       60
 2     2   3   6   5   8   9   12  11  2       60
 3     7   8  11  10  13  14  17  16   1       60
 4     8   9  12  11  14  15  18  17   2       60
DOF#  Displacement
 1     0
 2     0
 3     0
 4     0
 5     0
 6     0
 7     0
 8     0
 9     0
 10     0
 11     0
 12     0
 13     0
 14     0
 15     0
 16     0
 17     0
 18     0
DOF#  Load
MAT#     E          Nu        Alpha
 1     200000      0.3       12e-6
 2      70000      0.33      23e-6
B1  i  B2 j  B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

Output fro program HEXAFRON

```
Node#   X-Displ.    Y-Displ.    Z-Displ.
   1 -1.0572D-15 -5.4345D-16 -8.7348D-17
   2  3.7951D-17 -1.0239D-15  1.7323D-16
   3  1.0167D-15 -5.0495D-16 -8.5884D-17
   4 -1.0535D-15  5.3702D-16 -8.4577D-17
   5  3.8652D-17  1.0315D-15  1.7062D-16
   6  1.0175D-15  5.0377D-16 -8.6042D-17
   7 -5.1722D-03 -1.2479D-03  7.9659D-03
   8 -3.9947D-03 -1.7531D-03  9.7255D-03
   9 -6.4962D-04 -3.1168D-03  1.1292D-02
  10 -5.0707D-03  1.4218D-03  7.8730D-03
  11 -3.8432D-03  1.7670D-03  9.6949D-03
  12 -4.9979D-04  2.9867D-03  1.1273D-02
  13 -1.4878D-02 -9.3486D-04  1.3837D-02
  14 -1.3981D-02 -1.3390D-03  1.6121D-02
  15 -1.1495D-02 -2.3087D-03  1.9252D-02
  16 -1.4843D-02  1.2099D-03  1.3549D-02
  17 -1.3848D-02  1.4905D-03  1.6091D-02
  18 -1.1359D-02  2.3211D-03  1.9219D-02
DOF#        Reaction
```

```
    1      4.6823D+03
    2      2.4068D+03
    3      3.8685D+02
    4     -1.6808D+02
    5      4.5348D+03
    6     -7.6721D+02
    7     -4.5025D+03
    8      2.2363D+03
    9      3.8036D+02
   10      4.6657D+03
   11     -2.3784D+03
   12      3.7458D+02
   13     -1.7118D+02
   14     -4.5685D+03
   15     -7.5564D+02
   16     -4.5062D+03
   17     -2.2311D+03
   18      3.8106D+02
Von Mises Stress at 8 Integation Pts. in Elem#   1
    2.6068D+01    2.8455D+01    2.8263D+01    2.5831D+01
    1.3725D+01    1.6989D+01    1.6954D+01    1.3712D+01
Von Mises Stress at 8 Integation Pts. in Elem#   2
    9.1423D+00    9.6026D+00    9.5711D+00    9.1011D+00
    6.0619D+00    5.7172D+00    5.7616D+00    6.0908D+00
Von Mises Stress at 8 Integation Pts. in Elem#   3
    4.8353D+00    7.0998D+00    7.1725D+00    4.9263D+00
    5.3320D+00    6.6389D+00    6.6364D+00    5.2746D+00
Von Mises Stress at 8 Integation Pts. in Elem#   4
    8.6506D+00    7.8027D+00    7.8092D+00    8.6545D+00
    6.7512D+00    5.5281D+00    5.5294D+00    6.7476D+00
```

Note that the node 13 moves

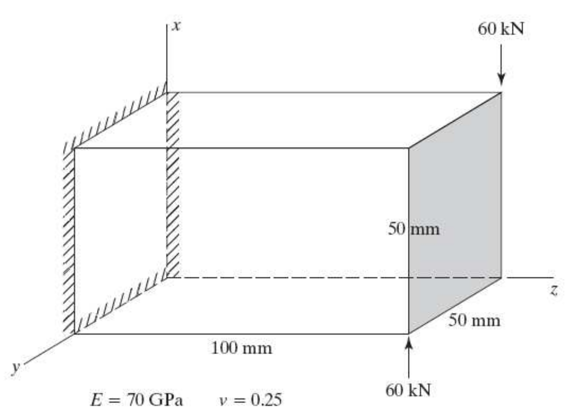
0.0149 mm in the x - direction

0.000935 mm in the y - direction

0.0138 mm in the z – direction

The bonding is making the unit expand in the z – direction and bending down.
The maximum vonMises stress of 28.45MPa is at integration point 2 of element 1. ■

9.8



We solve the problem using two hexahedral elements.

Input Data

3-D ANALYSIS USING HEXAHEDRAL ELEMENT
PROBLEM 9.8

NN	NE	NM	NDIM	NEN	NDN
12	2	1	3	8	3

ND	NL	NMPC
12	2	0

Node#	X	Y	Z
1	0	0	0
2	50	0	0
3	50	50	0
4	0	50	0
5	0	0	50
6	50	0	50
7	50	50	50
8	0	50	50
9	0	0	100
10	50	0	100
11	50	50	100
12	0	50	100

Elem#	N1	N2	N3	N4	N5	N6	N7	N8	MAT#	Temp_Ch
1	1	2	3	4	5	6	7	8	1	0
2	5	6	7	8	9	10	11	12	1	0

DOF#	Displ.
1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	0
9	0
10	0
11	0
12	0

DOF#	Load
28	-60000
34	60000

MAT#	E	Nu	Alpha
1	70000	0.25	0

B1 i B2 j B3 (Multi-point constr. B1*Qi+B2*Qj=B3)

Output

Program HexaFront - 3D Stress Analysis
PROBLEM 9.8

Node#	**X-Displ**	**Y-Displ**	**Z-Displ**

1	0.00000	0.00000	0.00000
2	0.00000	0.00000	0.00000
3	0.00000	0.00000	0.00000
4	0.00000	0.00000	0.00000
5	-0.12915	0.13303	-0.00518
6	-0.12109	-0.13102	0.00316
7	0.12915	-0.13303	-0.00518
8	0.12109	0.13102	0.00316
9	-0.25004	0.23399	-0.00680
10	-0.29845	-0.24609	0.01487
11	0.25004	-0.23399	-0.00680
12	0.29845	0.24609	0.01487

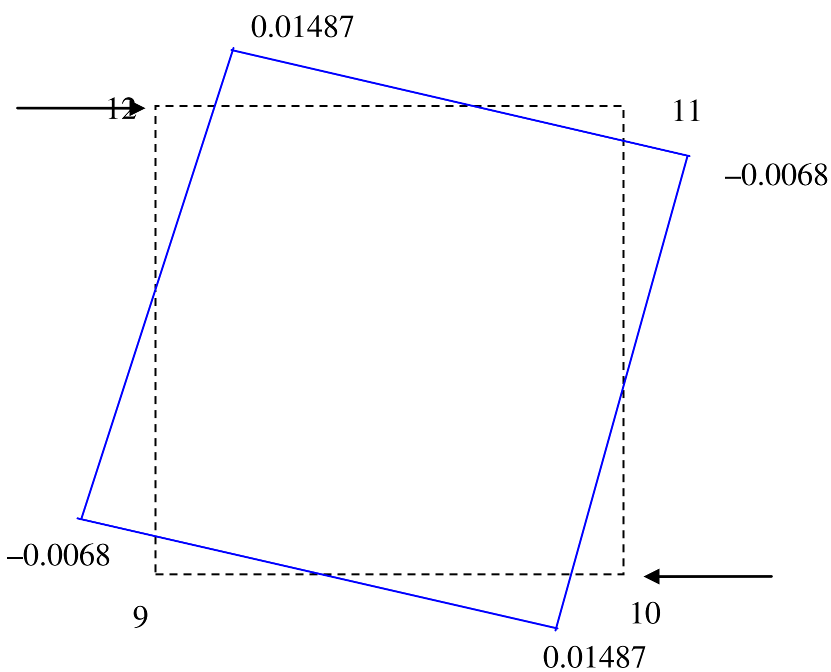
vonMises Stresses in Elements

Elem# 1 vonMises Stresses at 8 Integration Points

103.98	100.03	103.98	100.03	104.99	101.12	104.99	101.12

Elem# 2.00 vonMises Stresses at 8 Integration Points

88.18	117.43	88.18	117.43	111.87	135.87	111.87	135.87



x, *y* displaced position is shown relative to the initial position. Displacements along *z* direction are given at the corners. ■

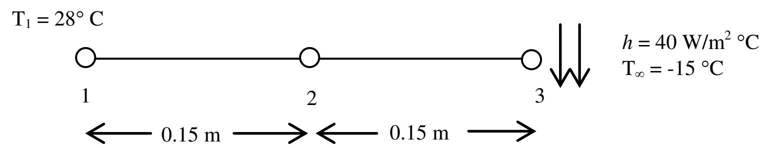
9.9 This is a project problem. Some experimentation is needed in determining the forces needed. Boundary conditions must be carefully determined. ■

CHAPTER 10

SCALAR FIELD PROBLEMS

10.1

Model:



2-elements:

$$\mathbf{k}_T^{(1)} = \frac{0.7}{0.15}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

$$\mathbf{k}_T^{(2)} = \frac{0.7}{0.15}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \end{array}$$

$$\mathbf{K} = \frac{0.7}{0.15}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Convection at node 3:

h is added to K(3,3)

$h\,T_\infty$ is added to R(3)

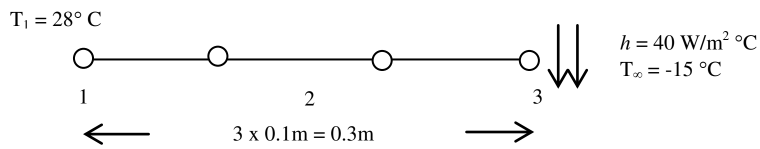
Specified temperature at Node l: A large constant C is added to K(1,1) and C x 28 is added to R(1). Choosing $C = \max\left|K_{ij}\right| \times 10{,}000 = 93{,}333.33$ yields **K T = R** as

$$\frac{0.7}{0.15}\begin{bmatrix} 20001 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 9.5714 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 2{,}613{,}333.3 \\ 0 \\ -600 \end{Bmatrix}$$

This may be solved using GAUSS_R Program:

$$T_1 = 28.0^o, T_2 = 12.6^o, T_3 = 2.89^o \quad \text{in centigrade units } (^oC)$$

3- Element model of above problem using HEAT1D program:



Number of Elements = 3, Number of B.C.'s = 2 (B.C.'s = Boundary Conditions)

Element	**Thermal Conductivity**
1	0.7
2	0.7
3	0.7
Node	**Co-ordinate**
1	0.
2	0.1
3	0.2
4	0.3

B C Data:
Node 1
TEMP
28^o

Node 4
CONV
40, -15

Solution: $\mathbf{T} = (28, 14.46, 0.913, -12.63)^T\ ^oC$

Input Data file for 10.1:

(The easiest way to create the data file is to make a copy and then edit HEATID.INP in the \EXAMPLES Sub-Directory)

```
HEAT1D DATA FILE
Problem 10.1
NE   #BOUNDARY CONDITIONS (B.C.'S)   #NODAL HEAT SOURCES
 3       2                              0
ELEM#   THERMAL CONDUCTIVITY
 1       0.7
 2       0.7
 3       0.7
```

```
NODE COORDINATE
 1       0
 2      .1
 3      .2
 4      .3
NODE  BC-TYPE followed by T0(if TEMP) or q0(if FLUX) or H and Tinf(if CONV)
1    TEMP
 28.
4    CONV
 40.  -15.
NODE    HEAT SOURCE
```

Output Data file for 10.1:

```
Problem 10.1

NODE#   TEMPERATURE
1          27.999
2          14.456
3         0.91314
4          -12.63
```

■

10.2

```
HEAT1D DATA FILE
Problem 10.2
NE   #BOUNDARY CONDITIONS (B.C.'S)   #NODAL HEAT SOURCES
 2       2                              0
ELEM#   THERMAL CONDUCTIVITY
 1       1.
 2       1.
NODE COORDINATE
 1       0
 2      .0125
 3      .025
NODE  BC-TYPE followed by T0(if TEMP) or q0(if FLUX) or H and Tinf(if CONV)
3    TEMP
 10.
1    FLUX
 -300.
NODE    HEAT SOURCE
```

Note: Not "HFLUX"

-ve sign because heat is coming into body

NODE#	TEMPERATURE
1	17.5
2	13.75
3	10

■

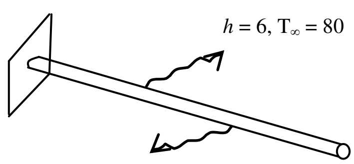
10.3

```
HEAT1D DATA FILE
Problem 10.3
NE   #BOUNDARY CONDITIONS (B.C.'S)   #NODAL HEAT SOURCES
 2       2                              0
ELEM#   THERMAL CONDUCTIVITY
 1       16.6
 2       16.6
NODE COORDINATE
 1       0
 2       .01
 3      .02
NODE  BC-TYPE followed by T0(if TEMP) or q0(if FLUX) or H and Tinf(if CONV)
1    FLUX
 -500.
3    CONV
 5.    20.
NODE    HEAT SOURCE
```

NODE# TEMPERATURE
1 120.6 -------------> temperature at heated face, °C
2 120.3
3 120

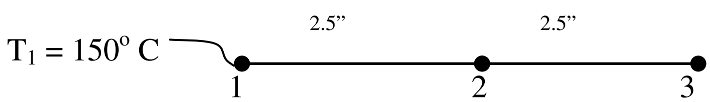
■

10.4



L = 5in, d = 5/16 in, k = 24.8 BTU/(hr.ft.^{o}F)

Model:



Element Matrices:

$$\mathbf{h}_T = \frac{Ph}{A_c}\frac{le}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{k} = \frac{k}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

P = perimeter = πd

A_c = area of cross section = $\pi d^2 / 4$

$$\mathbf{r}_\infty = \frac{PhT_\infty}{A_c}\frac{le}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$$

Upon assembly of the above matrices, we obtain

$$\mathbf{H}_T = 32\begin{bmatrix}2 & 1 & 0\\1 & 4 & 1\\0 & 1 & 2\end{bmatrix}$$

$$\mathbf{K}_\mathbf{T} = \frac{24.8}{2.5}\begin{bmatrix}1 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}$$

$$\mathbf{R}_\infty = 7680\begin{Bmatrix}1\\2\\1\end{Bmatrix}$$

The B.C $T_1 = 150^oF$ is handled by the elimination approach, where the 1st row and column are deleted:

$$\left[32\begin{bmatrix}4 & 1\\1 & 2\end{bmatrix}+\frac{24.8}{2.5}\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\right]\begin{Bmatrix}T_2\\T_3\end{Bmatrix} = 7680\begin{Bmatrix}2\\1\end{Bmatrix}$$

The solution is

$[T_2, T_3] = [92.5, 76.3]\ ^oF$

Heat Loss

$$H = \sum_e H_e$$

$$H_e = hA_s(T_{av} - T_\infty),$$

T_{av} = average temperature, A_s = surface area

= 2.5π d

= 2.45 in^2

H = H$_1$ + H$_2$

$$= 6\times\frac{2.454}{12^2}(121.25-80)+6\times\frac{2.454}{12^2}(84.4-80) = 4.67BTU/hr$$ ■

10.5 Tip not insulated

In Eq. (10.35), we have the boundary term

$$\phi k\frac{dT}{dx}\int_o^L = \phi(L)k\frac{dT(L)}{dx} - \phi 0(o)k\frac{dT(o)}{dx} = -\psi_L h(T_L - T_\infty)$$

Thus, h is added to $\mathbf{K}_T$ at (L,L), h T_∞ is added to $\mathbf{R}_\infty$ at (L, I). The solution of 10.3 will now be,

$$\begin{bmatrix}147.84 & 22.08\\ 22.08 & 73.92+h\end{bmatrix}\begin{Bmatrix}T_2\\ T_3\end{Bmatrix} = \begin{Bmatrix}15360\\ 7680+hT_\infty\end{Bmatrix}$$

Substituting for h = 6, T_∞ = 80, We get [T_2, T_3] = [92.5, 76.6]^o F.

The solution is essentially the same as in 10.3. Hence, we may neglect convection at the tip of the pin fin as was done in 10.3. ■

10.6 Using the isoparametric relations

$x_P = 7 = \xi\,(1) + \eta\,(10) + (1-\xi-\eta)\,(6)$

$y_P = 4 = \xi\,(1) + \eta\,(4) + (1-\xi-\eta)\,(7)$

yields $\xi = 0.2308$, $\eta = 0.5385$. Now,

$T_P = .2308\ T_1 + .5385\ T_2 + (1-.2308-.5385)\ T_3$. Upon substituting for $T_1 = 120$, $T_2 = 140$, $T_3 = 80$,

we get $T_P = 121.4$ deg. ■

10.7 NBW = max (node numbers for element) + 1

= nax (4, 4, 3, 4) + 1 = 5 ■

10.8

(a) If $T_1 = 30°$, then $\psi_1 = 0$ as per Galerkin's method. Thus,

ψ_2 { -2 (30) + 4 T_2 – 20 } = 0 for arbitrary ψ_2 which gives $T_2 = 20°$.

(b) Taking the variation of $T_1 - T_2 - 20° = 0$ yields $\psi_1 - \psi_2 = 0$. Thus, we may let $\psi_1 = 1$ and $\psi_2 = 1$. This gives 4 T_1 + 2 T_2 – 30 = 0 which along with $T_1 - T_2 - 20° = 0$ yields $T_1 = 11.67°$, $T_2 = -8.33°$. ■

10.9

(a) $\int \phi^{\mathrm{T}}\, Q\, dA = \boldsymbol{\psi}^{\mathrm{T}} \left[\int_0^1 \int_0^{1-\xi} \mathbf{N}^{\mathrm{T}}\mathbf{N} \det \mathbf{J}\, d\eta\, d\xi \right] \mathbf{Q}^e \equiv \boldsymbol{\psi}^{\mathrm{T}}\, \mathbf{r}_Q$,

where $\mathbf{N} = [\xi, \eta, 1-\xi-\eta]^{\mathrm{T}}$, $\mathbf{Q}^{e} = [Q_1, Q_2, Q_3]^{\mathrm{T}}$. Upon substituting, we obtain

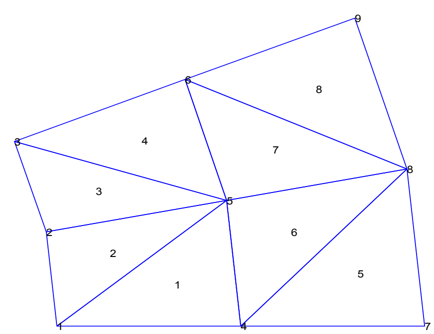
$\mathbf{r}_Q = \dfrac{A_e}{12} \begin{Bmatrix} (2Q_1 + Q_2 + Q_3) \\ (Q_1 + 2Q_2 + Q_3) \\ (Q_1 + Q_2 + 2Q_3) \end{Bmatrix}$. In the case of a constant Q, we obtain $\mathbf{r}_Q = \dfrac{Q\, A_e}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$.

(b) $\mathbf{r}_Q = Q_0 \begin{Bmatrix} \xi_0 \\ \eta_0 \\ 1-\xi_0-\eta_0 \end{Bmatrix}$ ■

10.10 In *cm* units,

h= 0.04 w/cm^2 ^{o}C
k = 0.15 w/cm^2 ^{o}C
q_o= -10 w/cm^2

Now, HEAT2D.INP in \EXAMPLES DIRECTORY was **edited** to create the data file shown below:



input file

```
TWO DIMENSIONAL HEAT ANALYSIS USING HEAT2D
PROBLEM  10.10
NN  NE  NM  NDIM  NEN  NDN
 9   8   1   2     3    1
ND   NL    NMPC
 0   0      0
Node#   X        Y
 1      3         0
 2      2.9424   .5853
 3      2.7716  1.1481
 4      4       0.
 5      3.9231   .7804
 6      3.6955  1.5307
 7      5.      0.
```

```
 8      4.9039  .9755
 9      4.6194  1.9134
Elem# N1  N2  N3  MAT#  Elem_Heat_Source
 1     1   4   5   1      0
 2     1   5   2   1      0
 3     2   5   3   1      0
 4     3   5   6   1      0
 5     4   7   8   1      0
 6     4   8   5   1      0
 7     5   8   6   1      0
 8     6   8   9   1      0
DOF#  Displacement (SPECIFIED TEMPERATURE)
DOF#  Load (NODAL HEAT SOURCE)
MAT#    ThermalConductivity
 1      0.15
No. of edges with Specified Heat flux FOLLOWED BY two edges & q0 (positive if
out)
2
1  2  -10.
2  3  -10.
No.of Edges with Convection FOLLOWED BY edge(2 nodes) &  h & Tinf
2
7 8 .04  120.
8 9 .04  120.
```

Results

```
Node#    Temperature
    1        371.1125
    2        371.1105
    3        371.1164
    4        314.2428
    5        314.2449
    6        314.2457
    7        270.0097
    8        270.0106
    9        270.0103
```

$T_1 = T_2 = T_3 = 371^o = T_i$

$T_7 = T_8 = T_9 = 270^o = T_o$ ■

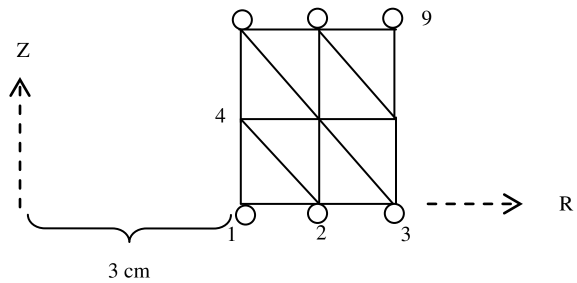
10.11 Re-do Example 10.4 with a finer mesh. Also, total heat flow into/out of the body is computed along each element edge on the boundary using

$$\sum(-k\frac{\partial T}{\partial n}) \times area_o \equiv q_o \times area_i$$

■

10.12 Thermal Stresses

Referring to Problem 10.10, an axisymmetric model (chapter 6, program AXISYM) can be used to determine the thermal stresses in the tube. A 2 cm long section of the pipe has been chosen.



We now need to get ΔT_e = average change in temperature in each element. This is straight forward. For eg., in element 1,

$$T^{average}{}_{e1} = \frac{T_1 + T_2 + T_4}{3} = \frac{271 + 314 + 271}{3} = 352 \text{ , from which } \Delta T = T^{average} - T^{reference} = 322 \text{ deg.}$$

Similarly, ΔT_e in all of the 8 elements are calculated. These are inputs to Program AXISYM. ■

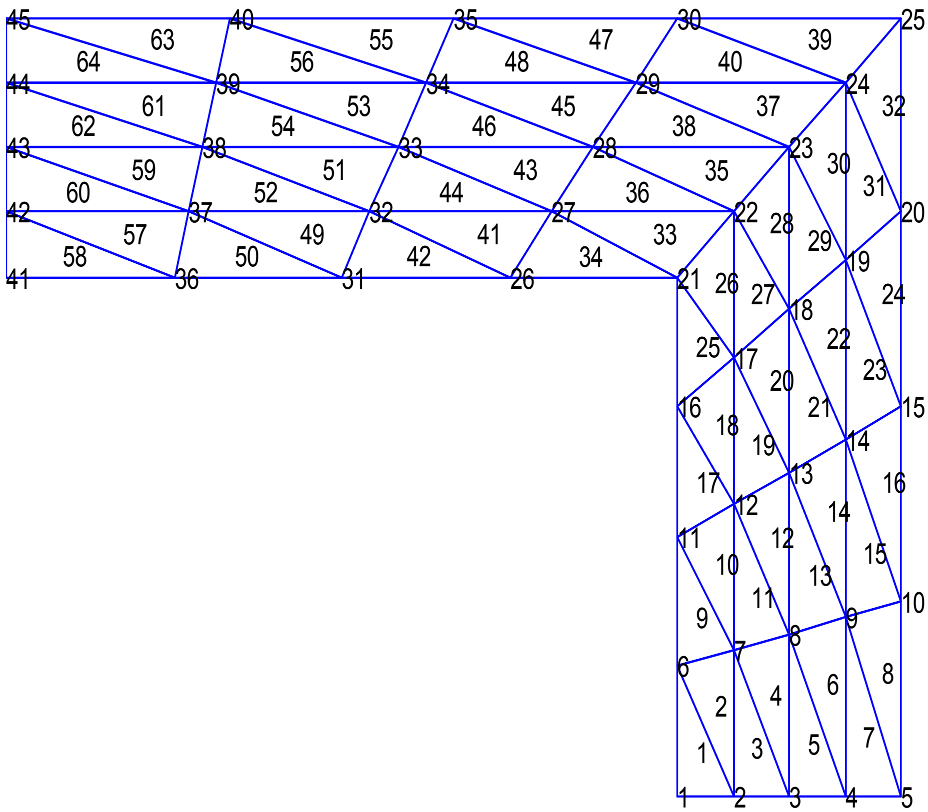
10.13 Brick Chimney
Model (using Symmetry)

- MESHGEN2 was used to create a mesh of 32 triangular elements.
- The resulting file was edited as per the lines of HEAT2D.INP provided in the \Examples sub-directory.

Input File for Program Meshgen

```
Mesh Generation
P10_13
Number of Nodes per Element <3 or 4>
  3
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ)
  1              2             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
  1        4
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        4
  2        4
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
  1      0.3         0
  2      0.4       0
  3      .3        .2
  4      .4        0.3
  5      0         .2
```

```
  6      0         .3
  0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
```



Input File for Program Heat2d

```
TWO DIMENSIONAL HEAT ANALYSIS USING HEAT2D
PROBLEM  10.13 Chimney
NN   NE   NM  NDIM  NEN  NDN
 45  64   1    2     3    1
ND   NL    NMPC
 18   0      0
 Node#         X         Y
    1    3.00000e-001    0.00000e+000
```

2	3.25000e-001	0.00000e+000
3	3.50000e-001	0.00000e+000
4	3.75000e-001	0.00000e+000
5	4.00000e-001	0.00000e+000
6	3.00000e-001	5.00000e-002
7	3.25000e-001	5.62500e-002
8	3.50000e-001	6.25000e-002
9	3.75000e-001	6.87500e-002
10	4.00000e-001	7.50000e-002
11	3.00000e-001	1.00000e-001
12	3.25000e-001	1.12500e-001
13	3.50000e-001	1.25000e-001
14	3.75000e-001	1.37500e-001
15	4.00000e-001	1.50000e-001
16	3.00000e-001	1.50000e-001
17	3.25000e-001	1.68750e-001
18	3.50000e-001	1.87500e-001
19	3.75000e-001	2.06250e-001
20	4.00000e-001	2.25000e-001
21	3.00000e-001	2.00000e-001
22	3.25000e-001	2.25000e-001
23	3.50000e-001	2.50000e-001
24	3.75000e-001	2.75000e-001
25	4.00000e-001	3.00000e-001
26	2.25000e-001	2.00000e-001
27	2.43750e-001	2.25000e-001
28	2.62500e-001	2.50000e-001
29	2.81250e-001	2.75000e-001
30	3.00000e-001	3.00000e-001
31	1.50000e-001	2.00000e-001
32	1.62500e-001	2.25000e-001
33	1.75000e-001	2.50000e-001
34	1.87500e-001	2.75000e-001
35	2.00000e-001	3.00000e-001
36	7.50000e-002	2.00000e-001
37	8.12500e-002	2.25000e-001
38	8.75000e-002	2.50000e-001
39	9.37500e-002	2.75000e-001
40	1.00000e-001	3.00000e-001
41	0.00000e+000	2.00000e-001
42	0.00000e+000	2.25000e-001
43	0.00000e+000	2.50000e-001
44	0.00000e+000	2.75000e-001
45	0.00000e+000	3.00000e-001

Elem#	Node1	Node2	Node3	Mat#	Heat Source
1	1	2	6	1	0
2	7	6	2	1	0
3	2	3	7	1	0
4	8	7	3	1	0
5	3	4	8	1	0
6	9	8	4	1	0
7	4	5	9	1	0
8	10	9	5	1	0
9	6	7	11	1	0
10	12	11	7	1	0
11	7	8	12	1	0
12	13	12	8	1	0

13	8	9	13	1	0
14	14	13	9	1	0
15	9	10	14	1	0
16	15	14	10	1	0
17	11	12	16	1	0
18	17	16	12	1	0
19	12	13	17	1	0
20	18	17	13	1	0
21	13	14	18	1	0
22	19	18	14	1	0
23	14	15	19	1	0
24	20	19	15	1	0
25	16	17	21	1	0
26	22	21	17	1	0
27	17	18	22	1	0
28	23	22	18	1	0
29	18	19	23	1	0
30	24	23	19	1	0
31	19	20	24	1	0
32	25	24	20	1	0
33	21	22	27	1	0
34	27	26	21	1	0
35	22	23	28	1	0
36	28	27	22	1	0
37	23	24	29	1	0
38	29	28	23	1	0
39	24	25	30	1	0
40	30	29	24	1	0
41	26	27	32	1	0
42	32	31	26	1	0
43	27	28	33	1	0
44	33	32	27	1	0
45	28	29	34	1	0
46	34	33	28	1	0
47	29	30	35	1	0
48	35	34	29	1	0
49	31	32	37	1	0
50	37	36	31	1	0
51	32	33	38	1	0
52	38	37	32	1	0
53	33	34	39	1	0
54	39	38	33	1	0
55	34	35	40	1	0
56	40	39	34	1	0
57	36	37	42	1	0
58	42	41	36	1	0
59	37	38	43	1	0
60	43	42	37	1	0
61	38	39	44	1	0
62	44	43	38	1	0
63	39	40	45	1	0
64	45	44	39	1	0

```
DOF#  Displacement (SPECIFIED TEMPERATURE)
1  100
6  100
11  100
16  100
```

```
21  100
26  100
31  100
36  100
41  100
5    30
10   30
15   30
20   30
25   30
30   30
35   30
40   30
45   30
DOF#  Load (NODAL HEAT SOURCE)
MAT#    ThermalConductivity
 1      0.72
No. of edges with Specified Heat flux FOLLOWED BY ...
0
No.of Edges with Convection FOLLOWED BY edge(2 nodes) &  h & Tinf
0
```

Output

```
Output for Input Data from file test.inp
PROBLEM  10.13

 Node#    Temperature
    1         99.9998
    2         82.4876
---------------------------
   43         65.0005
   44         47.5007
   45         30.0005
 -- CONDUCTION HEAT FLOW PER UNIT AREA IN EACH ELEMENT --
ELEMENT#   QX= -K*DT/DX   QY= -K*DT/DY
   1          504.35       0.0033319     Inner wall X-
   8          502.37      -0.0032809  outer wall X-
   9          504.88     4.5541E-005     Inner X-
   16          487.64      0.00043947  outer X-
   17          509.09      0.00031088     Inner X-
   24          389.54       0.0019601  outer X-
   25          542.23       0.0098147     Inner X-
   32          146.18       0.0025944  outer X-
   34         0.00453           508.2     Inner Y-
   39        0.003681          146.18  outer Y-
   42     4.7633E-005          503.59     Inner Y-
   47       0.0015837          433.31  outer Y-
   50    -4.2499E-006          503.95     Inner Y-
   55      0.00017418          501.21  outer Y-
   58       0.0033226          503.99     Inner Y-
   63      -0.0033204          504.08  outer Y-
```

Using

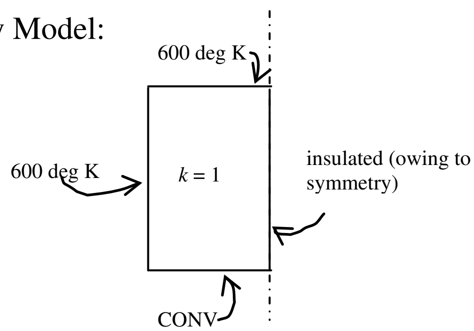
$$\sum(-k\frac{\partial T}{\partial n})\times area_o \equiv q_o \times area_i$$

For ¼ symmetry model:
we have heat flow into chimney = 507*0.2 + 505*.3 = 253 W,
total heat flow out of chimney = 381.4*0.3 + 396.2*.4 = 273 W

On average, total heat flow through chimney = 4 x 263 = 1,052. W per meter length of chimney. Accuracy may be checked by refining the mesh near the corner. ■

10.14 Industrial Furnace

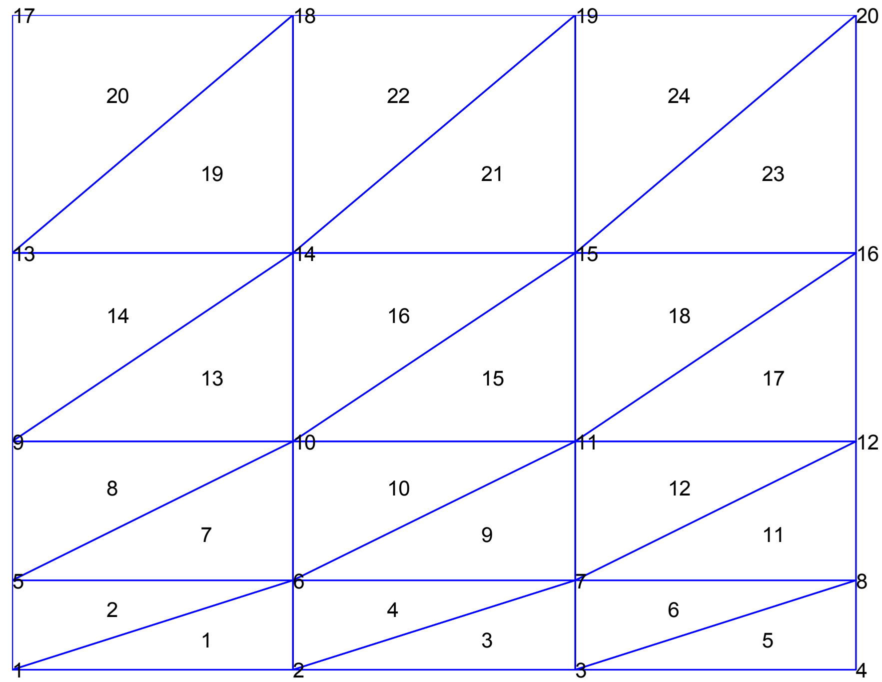
½ Symmetry Model:



Meshgen Input File

```
Mesh Generation
P10_14
Number of Nodes per Element <3 or 4>
  3
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ)
  1             1             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
  1        3
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        4
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
  1      0         0
  2      0.5       0
  3      0        1
  4      .5       1
  0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   1     0           .35
   2     .5          .35
  0
MERGING SIDES (Node1 is the lower number)
```

Pair# Side1Node1 Side1Node2   Side2Node1 Side2node2



Heat2d Input File

```
TWO DIMENSIONAL HEAT ANALYSIS USING HEAT2D
PROBLEM  10.10
NN   NE   NM   NDIM   NEN   NDN
 20   24    1    2      3    1
ND    NL     NMPC
 8    0       0
 Node#         X          Y
    1    0.00000e+000    0.00000e+000
    2    1.66667e-001    0.00000e+000
    3    3.33333e-001    0.00000e+000
    4    5.00000e-001    0.00000e+000
    5    0.00000e+000    1.37500e-001
    6    1.66667e-001    1.37500e-001
    7    3.33333e-001    1.37500e-001
    8    5.00000e-001    1.37500e-001
    9    0.00000e+000    3.50000e-001
   10    1.66667e-001    3.50000e-001
   11    3.33333e-001    3.50000e-001
   12    5.00000e-001    3.50000e-001
   13    0.00000e+000    6.37500e-001
   14    1.66667e-001    6.37500e-001
   15    3.33333e-001    6.37500e-001
   16    5.00000e-001    6.37500e-001
   17    0.00000e+000    1.00000e+000
   18    1.66667e-001    1.00000e+000
   19    3.33333e-001    1.00000e+000
   20    5.00000e-001    1.00000e+000
```

```
 Elem#    Node1   Node2   Node3  Mat#    Elem Heat Source
     1       1       2       6       1    0
     2       6       5       1       1    0
     3       2       3       7       1    0
     4       7       6       2       1    0
     5       3       4       8       1    0
     6       8       7       3       1    0
     7       5       6      10       1    0
     8      10       9       5       1    0
     9       6       7      11       1    0
    10      11      10       6       1    0
    11       7       8      12       1    0
    12      12      11       7       1    0
    13       9      10      14       1    0
    14      14      13       9       1    0
    15      10      11      15       1    0
    16      15      14      10       1    0
    17      11      12      16       1    0
    18      16      15      11       1    0
    19      13      14      18       1    0
    20      18      17      13       1    0
    21      14      15      19       1    0
    22      19      18      14       1    0
    23      15      16      20       1    0
    24      20      19      15       1    0
DOF#  Displacement (SPECIFIED TEMPERATURE)
 1     600
5     600
9     600
13     600
17     600
18     600
19     600
20     600
DOF#  Load (NODAL HEAT SOURCE)
MAT#    ThermalConductivity
 1        1.0
No. of edges with Specified Heat flux FOLLOWED BY two edges & q0 (positive if
out)
No.of Edges with Convection FOLLOWED BY edge(2 nodes) &  h & Tinf
3
 1  2  12.  300.
 2  3  12.  300.
 3  4  12.  300.
```

Output

```
Output for Input Data from file test.inp
PROBLEM  10.14

 Node#    Temperature
    1        599.9936
    2        372.8885
-----------------------
    7        433.1146
    8        420.8970
    9        599.9984
```

```
   20        599.9998
 -- CONDUCTION HEAT FLOW PER UNIT AREA IN EACH ELEMENT --
ELEMENT#   QX= -K*DT/DX   QY= -K*DT/DY
   1          1362.6        -777.35
   3          133.36        -599.66
   5          28.819        -545.74
---------------------------------------
Thus, total heat flow into air stream (accounting for ½-symmetry)is

   2 x {(777.35+599.66+545.74)/3*.5m } = 641.0 W per unit length of furnace ■
```

10.15 2-D Fin

After choosing a suitable value for h, say h = 400 W/m^2 ^{o}C, the 'stiffness' matrix **k** in the program HEAT2D needs to be modified by assembling.

$$+\frac{CA_e}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

where C = -2h/t , for each element. Also, the right side ("Force") needs to be augmented by

$$+CT_\infty \frac{A_e}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

A ¼ symmetry model may then be used. ■

10.16

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}=0$$

Galerkin Approach for Deriving Element Matrices

$$2\pi\int_A \phi\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}\right] rdA = 0$$

for every ϕ satisfying $\phi = 0$ on S_T, where T is specified. The product rule of differentiation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\phi\, r\frac{\partial T}{\partial r}\right)=\frac{1}{r}\frac{\partial\phi}{\partial r}r\frac{\partial T}{\partial r}+\frac{1}{r}\phi\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

Similarly

$$\frac{\partial}{\partial z}\left(\phi\,\frac{\partial T}{\partial z}\right)=\phi\,\frac{\partial^2 T}{\partial z^2}+\frac{\partial\phi}{\partial z}\frac{\partial T}{\partial z}$$

Thus

$$2\pi\int_A\left\{\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\phi\, r\frac{\partial T}{\partial r}\right)-\frac{\partial\phi}{\partial r}\frac{\partial T}{\partial r}\right]+\left[\frac{\partial}{\partial z}\left(\phi\,\frac{\partial T}{\partial z}\right)-\frac{\partial\phi}{\partial z}\frac{\partial T}{\partial z}\right]\right\}rdA=0$$

For every ϕ, $\phi = 0$ on S_T. The divergence theorem:

$$\int_A\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\phi\, r\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial z}\left(\phi\frac{\partial T}{\partial z}\right)\right]rdA=\int_S\left[\phi\frac{\partial T}{\partial r}n_r+\phi\frac{\partial T}{\partial z}n_z\right]dS=\int_S\phi\, q_n dS$$

where

$$q_n=\underline{q}.\underline{n}=-k\left(\frac{\partial T}{\partial r}n_r+\frac{\partial T}{\partial z}n_z\right)$$

is the specified normal heat flux. Thus, introducing k = thermal cond.,

$$2\pi k\int_A\left[\frac{\partial\phi}{\partial r}\frac{\partial T}{\partial r}+\frac{\partial\phi}{\partial z}\frac{\partial T}{\partial z}\right]rdA+2\pi\int_S\phi q_n dS=0$$

Since $\phi = \mathbf{N}\ \boldsymbol{\psi}$, $T = \mathbf{N}\,\mathbf{T^e}$, the above leads to the element matrices:

$$\mathbf{K}_T = 2\pi\, k\,\bar{r}\ A_e\,\mathbf{B}_T^T\ \mathbf{B}_T$$

where $\mathbf{B_T}$ is given in the text, and

$$\mathbf{r}_q = 2\pi \; r_{av} \; q_o \; l_{2-3} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

if q_o is the normal heat flow on edge 2-3. Now, $\mathbf{K}_T$ and $\mathbf{R}_q$ are assembled, specified temp. B.C.'s are handled in the usual manner and the solution is obtained from $\mathbf{K}_T \; \mathbf{T} = \mathbf{R}_q$. ■

10.17 Project

$T(\xi,\eta) = N_1T_1 + N_2T_2 + N_3T_3 + N_4T_4$

$x = \sum_{i=1}^{4} N_i \; x_i, \; y = \sum_{i=1}^{4} N_i \; y_i$

N are given in Eqs. (8.5).

$$\begin{Bmatrix} \partial T/\partial x \\ \partial T/\partial y \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \partial T/\partial \xi \\ \partial T/\partial \eta \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \partial \mathbf{N}/\partial \xi \\ \partial \mathbf{N}/\partial \eta \end{Bmatrix} \mathbf{T^e}$$

$\equiv \mathbf{B}_T \; \mathbf{T}^e$

We have

$$\begin{Bmatrix} \partial T/\partial x \\ \partial T/\partial y \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \partial T/\partial \xi \\ /\partial \eta \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \partial \mathbf{N}/\partial \xi \\ \partial \mathbf{N}/\partial \eta \end{Bmatrix} \mathbf{T^e}$$

$\equiv \mathbf{B}_T \; \mathbf{T}^e$

$$\frac{1}{2}\iint_A \left[k\left(\frac{\partial T}{\partial x}\right)^2 + k\left(\frac{\partial T}{\partial y}\right)^2 \right] dA = \frac{1}{2}\sum_e \mathbf{T}^{eT} \mathbf{k_T} \mathbf{T^e}, \; where$$

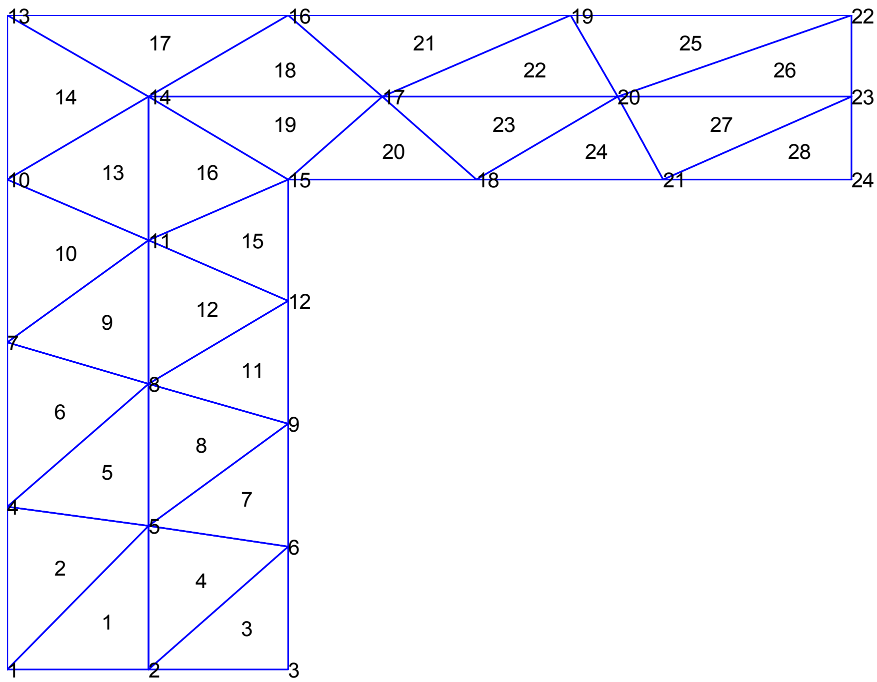
$$\mathbf{k_T} = k_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}_T^T \; \mathbf{B}_T \; \det \mathbf{J} \; d\xi \; d\eta$$

Gaussian quadrature is used to evaluate the integral. Boundary terms are similarly handled. A computer program alongs the lines of HEAT2D may now be readily developed (See also Program QUAD in Chapter 8). ■

10.18 Torsion of L-shaped Beam

Part (a):

Meshgen Input File



Boundary conditions:

Node	Specified Disp
1,2,3,4,6, 7,9,10,12, 13,15,16,18, 19,21,22,23,24	0.0

Interactive Inputs

Torque = 1.

Symmetry Factor = 1.

Input File for Program TORSION

```
TWO DIMENSIONAL HEAT ANALYSIS USING HEAT2D
PROBLEM  10.18 -- Torsion of L-shaped Beam
NN   NE   NM   NDIM   NEN   NDN
 24   28    1    2      3     1
ND    NL     NMPC
 18    0       0
 Node#         X          Y
     1     0.00000e+000    0.00000e+000
     2     1.00000e+001    0.00000e+000
```

```
    3    2.00000e+001    0.00000e+000
    4    0.00000e+000    2.00000e+001
    5    1.00000e+001    1.75000e+001
    6    2.00000e+001    1.50000e+001
    7    0.00000e+000    4.00000e+001
    8    1.00000e+001    3.50000e+001
    9    2.00000e+001    3.00000e+001
   10    0.00000e+000    6.00000e+001
   11    1.00000e+001    5.25000e+001
   12    2.00000e+001    4.50000e+001
   13    0.00000e+000    8.00000e+001
   14    1.00000e+001    7.00000e+001
   15    2.00000e+001    6.00000e+001
   16    2.00000e+001    8.00000e+001
   17    2.66667e+001    7.00000e+001
   18    3.33333e+001    6.00000e+001
   19    4.00000e+001    8.00000e+001
   20    4.33333e+001    7.00000e+001
   21    4.66667e+001    6.00000e+001
   22    6.00000e+001    8.00000e+001
   23    6.00000e+001    7.00000e+001
   24    6.00000e+001    6.00000e+001
 Elem#    Node1   Node2   Node3    Mat#
     1        1       2       5       1
     2        5       4       1       1
     3        2       3       6       1
     4        6       5       2       1
     5        4       5       8       1
     6        8       7       4       1
     7        5       6       9       1
     8        9       8       5       1
     9        7       8      11       1
    10       11      10       7       1
    11        8       9      12       1
    12       12      11       8       1
    13       10      11      14       1
    14       14      13      10       1
    15       11      12      15       1
    16       15      14      11       1
    17       13      14      16       1
    18       17      16      14       1
    19       14      15      17       1
    20       18      17      15       1
    21       16      17      19       1
    22       20      19      17       1
    23       17      18      20       1
    24       21      20      18       1
    25       19      20      22       1
    26       23      22      20       1
    27       20      21      23       1
    28       24      23      21       1
DOF#  Displacement (specified stress function value)
1  0.
2  0.
3  0.
4  0.
6  0.
```

```
7  0.
9  0.
10  0.
12  0.
13  0.
15  0.
16  0.
18  0.
19  0.
21  0.
22  0.
23  0.
24  0.
DOF#  Load
MAT#   ShearModulus(G)
 1        1.
```

Output

```
Output for Input Data from file test.inp
PROBLEM  10.18 -- Torsion of L-shaped Beam

 NODE#   Stress Function Value
    1     3.2030E-005
    2     4.8941E-005
-------------------------
   24     1.0458E-005

TWIST PER UNIT LENGTH =    4.96130E-006

-- SHEARING STRESSES TAUYZ, TAUXZ IN EACH ELEMENT
ELEMENT#    TAUYZ        TAUXZ
1    -8.38995E-012     2.52383E-005
2    -4.41669E-005     1.38570E-011
--------------------------------------
28     1.77940E-011     1.77914E-011
```

$\alpha = 4.9613\text{x}10^{-6}$ T/G / rad/mm, with T in N-mm, G in N/mm^2 or Mpa.

Part (b) requires the following program modification:

$$T = G\alpha_o \sum_e \frac{2}{3} A_e(\psi_1 + \psi_2 + \psi_3) = G\alpha_o \sum_e T_e$$

Thus, $\dfrac{T_e}{\sum T_e} \times 100\%$ is the percentage that each element contributes in resisting the applied torque. The modification to program TORSION is:

```
Percent;    ← is added at the top of the program (main) to call the subroutine

function [] = Percent();
global NN NE NM NDIM NEN NDN
global ND NL NCH NPR NMPC NBW
global X NOC MAT F S PM NU U
global TITLE FILE1 FILE2 FILE3
```

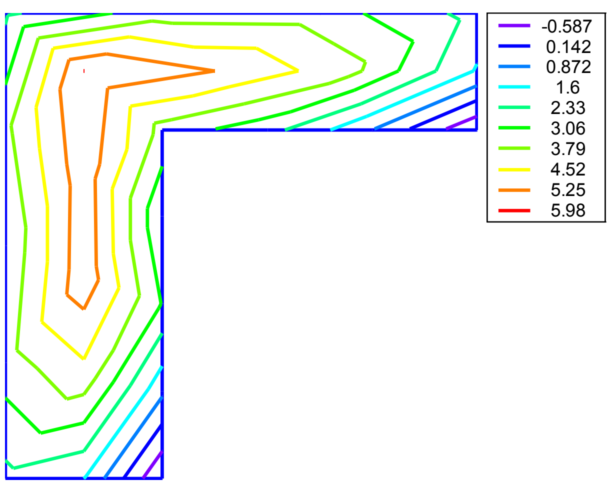
```
global LINP LOUT LOUT2 IPL

SUM=0.;
for I = 1:NE
   I1 = NOC(I, 1);I2 = NOC(I, 2);I3 = NOC(I, 3);
   X32 = X(I3, 1) - X(I2, 1);   X13 = X(I1, 1) - X(I3, 1);
   X21 = X(I2, 1) - X(I1, 1);   Y23 = X(I2, 2) - X(I3, 2);
   Y31 = X(I3, 2) - X(I1, 2);   Y12 = X(I1, 2) - X(I2, 2);
   DETJ = X13 * Y23 - X32 * Y31;
   TE(I) = abs(DETJ) / 3 * (F(I1) + F(I2) + F(I3));
   SUM=SUM + TE(I);
end
LOUT3  = fopen('temp.dat','w');
for I=1:NE
   TE(I)=TE(I)/SUM*100.;
   fprintf(LOUT3,'%14.5E\n',TE(I));
end
fclose(LOUT3)
```

The result is an output file "temp.dat" which contains a dummy at the top followed by the element contributions as:

```
torsion contr.
  2.57641E+000
  2.94447E+000
----------------
  2.43903E+000
  2.53695E-006
```

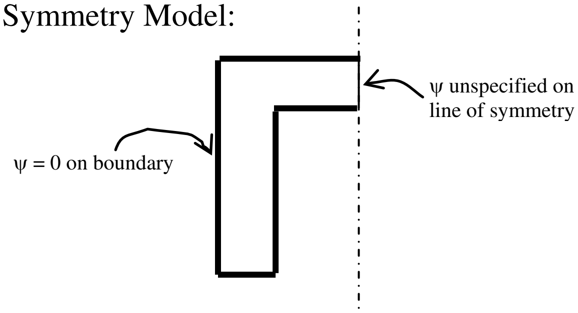
This file & original input file are fed into Program BESTFIT to obtain the nodal values. The nodal file and the original input file again are then fed into Program CONTOURB to obtain a contour plot of the element contributions to resisting twist.



■

10.19

½ Symmetry Model:



T = 5,000 in-lb

$$G = \frac{E}{2(1+\upsilon)} = \frac{30x10^6}{2(1+0.3)} = 11.54E6$$

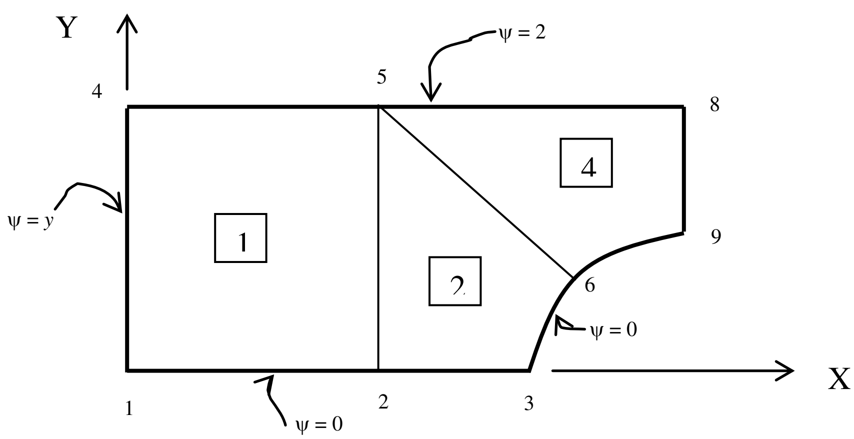
Symmetry Factor = 2

<u>The procedure is identical to P10.18 above.</u>
Using a 40-element model, we obtain , from running TORSION:

$$\alpha = 0.0027058 \text{ rad/in} = 0.155 \text{ deg/in}$$

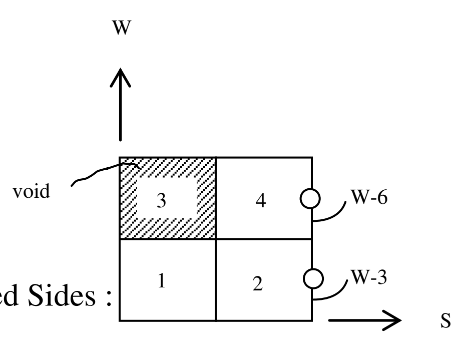
10.20 Potential flow around a cylinder. Governing Equation : $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$

Region with Corner Node #s



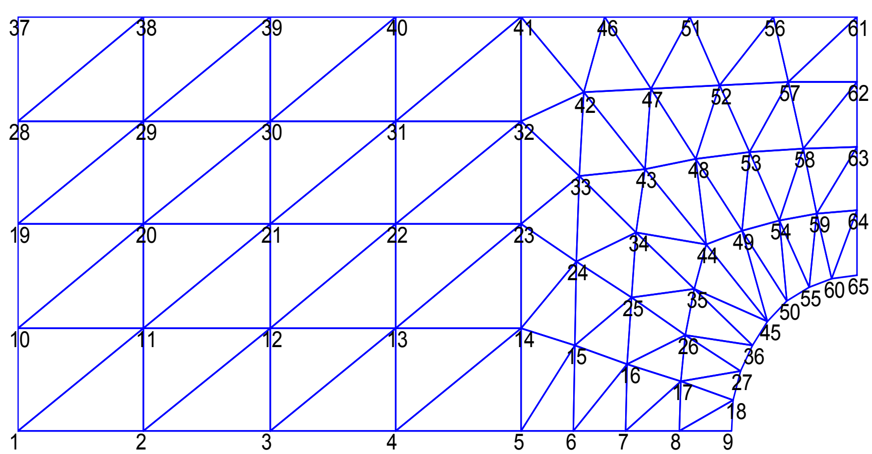
Block

Mid-Side Nodes of Curved Sides :



Meshgen Input File

```
Mesh Generation
P10_18 flow around a cylinder
Number of Nodes per Element <3 or 4>
  3
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ
  2             2             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
  1        4
  2        4
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        4
  2        4
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  3         0
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
  1      0         0
  2      3       0
  3      4.25        0
  4      0       2
  5      3       2
  6      4.46975     .53025
  8      5         2
  9      5         .75
  0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
  3        4.307    .287
  6        4.713    .693
  0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
```



We use the HEAT2D Program to solve this and other scalar field problems (except TORSION, for which a dedicated program is already provided).

Boundary Conditions

Node	Ψ
1,2,3,4,5,6,7,8,9,18,27,36,45,50,55,60,65	0.0
10	0.5
19	1.0
28	1.5
37,38,39,40,41,46,51,56,61	2.0

Input Data for Program Heat2d

```
TWO DIMENSIONAL HEAT ANALYSIS USING HEAT2D
PROBLEM  10.20 -- Flow around a cylinder
NN  NE  NM  NDIM  NEN  NDN
 65  96   1   2     3    1
ND   NL    NMPC
 29   0      0
 Node#        X         Y
    1    0.00000e+000    0.00000e+000
    2    7.50000e-001    0.00000e+000
    3    1.50000e+000    0.00000e+000
    4    2.25000e+000    0.00000e+000
    5    3.00000e+000    0.00000e+000
    6    3.31250e+000    0.00000e+000
    7    3.62500e+000    0.00000e+000
    8    3.93750e+000    0.00000e+000
    9    4.25000e+000    0.00000e+000
   10    0.00000e+000    5.00000e-001
   11    7.50000e-001    5.00000e-001
   12    1.50000e+000    5.00000e-001
   13    2.25000e+000    5.00000e-001
   14    3.00000e+000    5.00000e-001
   15    3.31632e+000    4.12242e-001
   16    3.63264e+000    3.24484e-001
   17    3.94896e+000    2.36727e-001
   18    4.26528e+000    1.48969e-001
   19    0.00000e+000    1.00000e+000
   20    7.50000e-001    1.00000e+000
   21    1.50000e+000    1.00000e+000
   22    2.25000e+000    1.00000e+000
   23    3.00000e+000    1.00000e+000
   24    3.32675e+000    8.21750e-001
   25    3.65350e+000    6.43500e-001
   26    3.98025e+000    4.65250e-001
   27    4.30700e+000    2.87000e-001
   28    0.00000e+000    1.50000e+000
   29    7.50000e-001    1.50000e+000
```

```
   30    1.50000e+000    1.50000e+000
   31    2.25000e+000    1.50000e+000
   32    3.00000e+000    1.50000e+000
   33    3.34379e+000    1.22852e+000
   34    3.68758e+000    9.57047e-001
   35    4.03137e+000    6.85570e-001
   36    4.37516e+000    4.14094e-001
   37    0.00000e+000    2.00000e+000
   38    7.50000e-001    2.00000e+000
   39    1.50000e+000    2.00000e+000
   40    2.25000e+000    2.00000e+000
   41    3.00000e+000    2.00000e+000
   42    3.36744e+000    1.63256e+000
   43    3.73488e+000    1.26513e+000
   44    4.10231e+000    8.97687e-001
   45    4.46975e+000    5.30250e-001
   46    3.50000e+000    2.00000e+000
   47    3.77148e+000    1.65621e+000
   48    4.04295e+000    1.31242e+000
   49    4.31443e+000    9.68633e-001
   50    4.58591e+000    6.24844e-001
   51    4.00000e+000    2.00000e+000
   52    4.17825e+000    1.67325e+000
   53    4.35650e+000    1.34650e+000
   54    4.53475e+000    1.01975e+000
   55    4.71300e+000    6.93000e-001
   56    4.50000e+000    2.00000e+000
   57    4.58776e+000    1.68368e+000
   58    4.67552e+000    1.36736e+000
   59    4.76327e+000    1.05104e+000
   60    4.85103e+000    7.34719e-001
   61    5.00000e+000    2.00000e+000
   62    5.00000e+000    1.68750e+000
   63    5.00000e+000    1.37500e+000
   64    5.00000e+000    1.06250e+000
   65    5.00000e+000    7.50000e-001
```

Edit Meshgen output file to add this column

```
 Elem#     Node1    Node2    Node3   Mat#     Elem Heat  Source
     1        1        2       11       1      0.
     2       11       10        1       1      0.
     3        2        3       12       1      0.
     4       12       11        2       1      0.
     5        3        4       13       1      0.
     6       13       12        3       1      0.
     7        4        5       14       1      0.
     8       14       13        4       1      0.
     9       10       11       20       1      0.
    10       20       19       10       1      0.
    11       11       12       21       1      0.
    12       21       20       11       1      0.
    13       12       13       22       1      0.
    14       22       21       12       1      0.
    15       13       14       23       1      0.
    16       23       22       13       1      0.
    17       19       20       29       1      0.
    18       29       28       19       1      0.
    19       20       21       30       1      0.
    20       30       29       20       1      0.
```

21	21	22	31	1	0.
22	31	30	21	1	0.
23	22	23	32	1	0.
24	32	31	22	1	0.
25	28	29	38	1	0.
26	38	37	28	1	0.
27	29	30	39	1	0.
28	39	38	29	1	0.
29	30	31	40	1	0.
30	40	39	30	1	0.
31	31	32	41	1	0.
32	41	40	31	1	0.
33	5	6	15	1	0.
34	15	14	5	1	0.
35	6	7	16	1	0.
36	16	15	6	1	0.
37	7	8	17	1	0.
38	17	16	7	1	0.
39	8	9	18	1	0.
40	18	17	8	1	0.
41	14	15	24	1	0.
42	24	23	14	1	0.
43	15	16	25	1	0.
44	25	24	15	1	0.
45	16	17	26	1	0.
46	26	25	16	1	0.
47	17	18	27	1	0.
48	27	26	17	1	0.
49	23	24	33	1	0.
50	33	32	23	1	0.
51	24	25	34	1	0.
52	34	33	24	1	0.
53	25	26	35	1	0.
54	35	34	25	1	0.
55	26	27	36	1	0.
56	36	35	26	1	0.
57	32	33	42	1	0.
58	42	41	32	1	0.
59	33	34	43	1	0.
60	43	42	33	1	0.
61	34	35	44	1	0.
62	44	43	34	1	0.
63	35	36	45	1	0.
64	45	44	35	1	0.
65	41	42	46	1	0.
66	47	46	42	1	0.
67	42	43	47	1	0.
68	48	47	43	1	0.
69	43	44	48	1	0.
70	49	48	44	1	0.
71	44	45	49	1	0.
72	50	49	45	1	0.
73	46	47	51	1	0.
74	52	51	47	1	0.
75	47	48	52	1	0.
76	53	52	48	1	0.
77	48	49	53	1	0.

```
    78      54      53      49       1    0.
    79      49      50      54       1    0.
    80      55      54      50       1    0.
    81      51      52      56       1    0.
    82      57      56      52       1    0.
    83      52      53      57       1    0.
    84      58      57      53       1    0.
    85      53      54      58       1    0.
    86      59      58      54       1    0.
    87      54      55      59       1    0.
    88      60      59      55       1    0.
    89      56      57      61       1    0.
    90      62      61      57       1    0.
    91      57      58      62       1    0.
    92      63      62      58       1    0.
    93      58      59      63       1    0.
    94      64      63      59       1    0.
    95      59      60      64       1    0.
    96      65      64      60       1    0.
DOF#  Displacement (SPECIFIED VALUE OF FIELD VARIABLE)
1   0.
2   0.
3   0.
4   0.
5   0.
6   0.
7   0.
8   0.
9   0.
18   0.
27   0.
36   0.
45   0.
50   0.
55   0.
60   0.
65   0.
10   0.5
19   1.0
28   1.5
37   2.
38   2.
39   2.
40   2.
41   2.
46   2.
51   2.
56   2.
61   2.
DOF#  Load
MAT#    ThermalConductivity
 1      1.
No. of edges with Specified Heat flux FOLLOWED BY two edges & q0 (positive if
out)
0
No.of Edges with Convection FOLLOWED BY edge(2 nodes) &  h & Tinf
0
```

Note!

Output

```
Output for Input Data from file test.inp
PROBLEM  10.20 -- Flow around a cylinder

 Node#    Temperature
 -- CONDUCTION HEAT FLOW PER UNIT AREA IN EACH ELEMENT --
ELEMENT#   QX= -K*DT/DX   QY= -K*DT/DY
   1    -1.0152E-005        -0.99797 ←--------  V, -U
   2       0.0013317        -0.99998
   3     9.5836E-008        -0.99327
   4       0.0031364        -0.99797
   5     2.8268E-007         -0.9794
---------------------------------------
   68         0.18098         -1.1462
   69         0.18611         -1.1796
   70         0.41279         -1.1472
   71         0.46746         -1.3107
   72         0.93299         -1.1457
   73     2.1262E-006         -1.1727
   74        0.084706          -1.229
   75         0.08443         -1.2224
   76         0.20249         -1.2667
   77         0.20715         -1.3096
   78         0.39737         -1.3308
   87         0.32937         -1.7412
   88         0.53499         -1.7701
   89    -1.3134E-005         -1.3732
   90        0.031147         -1.4138
   91        0.031334          -1.434
   92        0.064547         -1.4677
   93        0.066949         -1.5697
   94          0.1102         -1.6013
   95         0.12389         -1.8841
   96         0.19668         -1.9172  <----   Max U
```

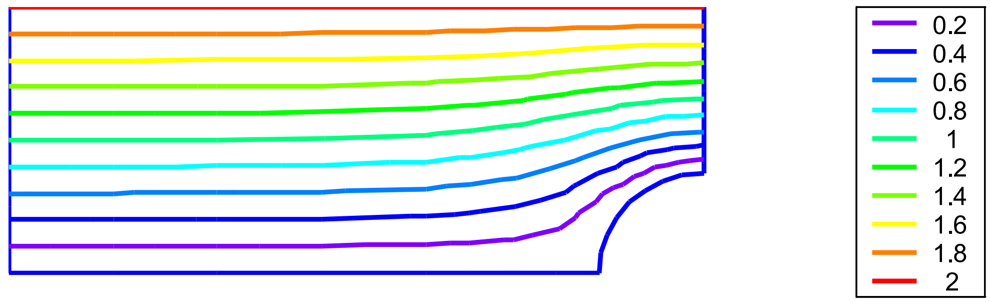
Velocities

The (HEAT2D) output file contains values, in each element, of

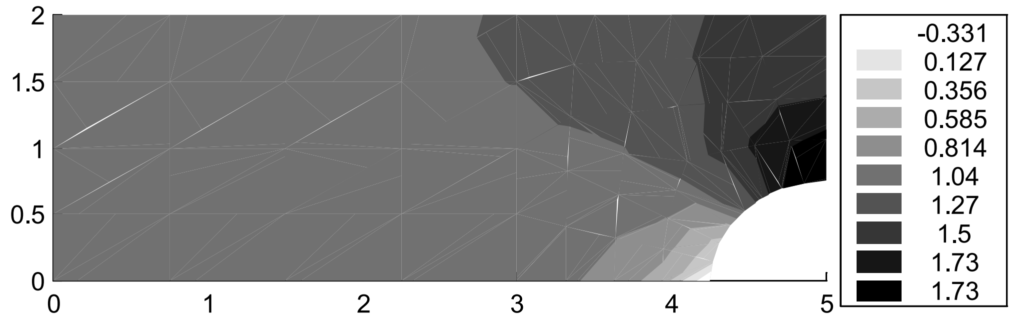
$$\frac{-\partial\psi}{\partial x}, \frac{-\partial\psi}{\partial y}$$

Since $U = \dfrac{+\partial\psi}{\partial y}, V = \dfrac{-\partial\psi}{\partial x}$ these columns of data correspond to element values of V, -U respectively, where U = x – comp., v = y-component of velocity. From the output file, we find max. U = 1.917 m/s.

The contour program plots contours of Ψ, the stream function, giving a good feel for the flow around the cylinder. Also, the flow between two streamlines is analogous to $\Delta\Psi$ of flow in a tube.



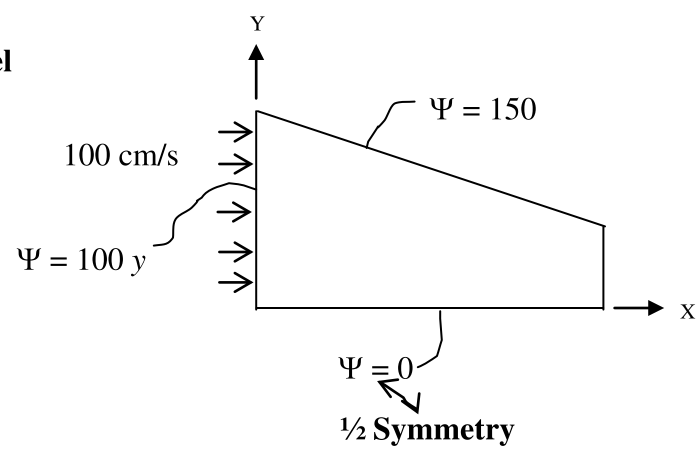
To obtain a contour for , say the x-velocity "U", modify the HEAT2D program to output the velocity <u>in each element</u> into a file, then run BESTFIT followed by ContourA or ContourB. This has been done for this problem to illustrate the U velocity distribution.



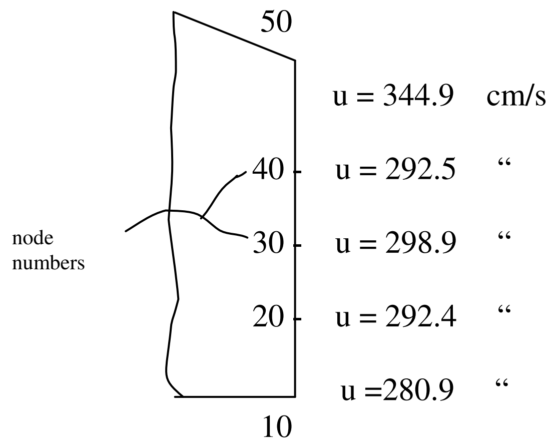
We see that there is a sharp increase in velocity near the neck – this is similar to stress concentration in a plate with a hole subjected to axial load. ■

10.21 Flow in a Venturimeter . $\dfrac{\partial^2\psi}{\partial x^2}+\dfrac{\partial^2\psi}{\partial y^2}=0$

Model



The procedure involving mesh creation, data editing is exactly similar to Problem 10.20 above, and will not be given here. A 50-node model provides the following answer for the u-component of velocity at the neck:



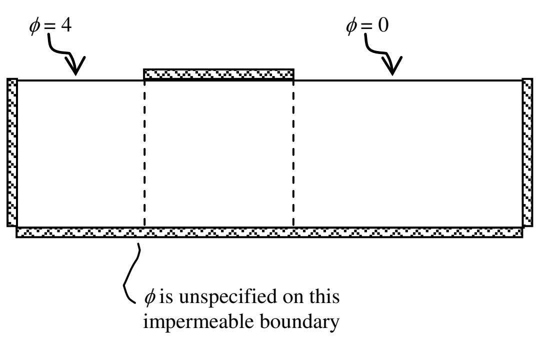
■

10.22 Seepage: $k\dfrac{\partial^2\phi}{\partial x^2} + k\dfrac{\partial^2\phi}{\partial y^2} = 0$

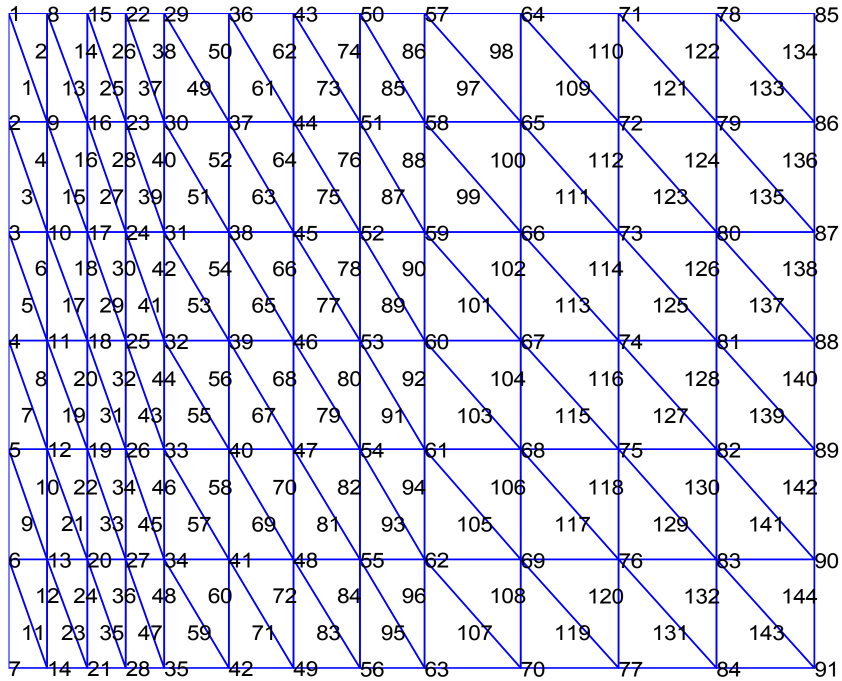
k = Hydraulic Conductivity, ϕ = Hydraulic head

Again, we execute program HEAT2D to solve this seepage flow problem.

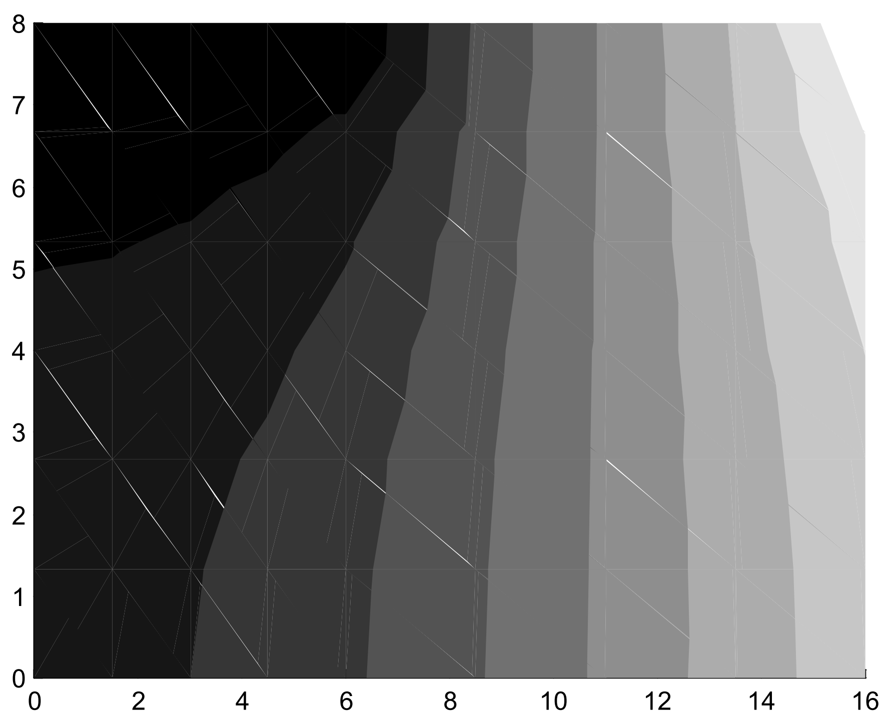
<u>Model</u>



<u>Mesh</u>
Using Program MESHGEN, the following mesh is obtained:



The mesh file and contour file are then input into contour to obtain contours of ϕ , or equi-potential lines:



(Flow is perpendicular to these lines)

The HEAT2D output files provide element values of

$$-k\frac{\partial T}{\partial x} \equiv U(x - velocity)\ and$$

$$k\frac{\partial T}{\partial x} \equiv V(y - velocity)$$

(element nos. obtained from output file)

To obtain the quantity of water seeping under the dam, consider a cross-section consisting of elements 73-84. From the output file, we have

Element	U	-V
73	8.8993	-0.045439
74	9.5344	1.1455
75	8.0044	-0.09163
76	8.8993	1.5864
77	7.2128	-0.10013
78	8.0044	1.3841
79	6.6344	-0.085096
80	7.2128	0.99932
81	6.29	-0.056082
82	6.6344	0.58977
83	6.1762	-0.019502
84	6.29	0.19386

Average = 7.48 m/day (each element has the same edge length)

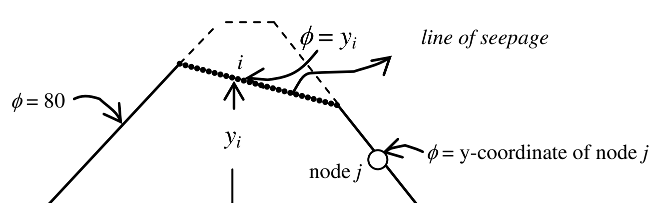
Thus, quantity of seepage = 7.48 x (8x1) = 60.0 m^3/day per unit width of dam

We see from this problem how various scalar field problems are solved using the same programs for analysis and pre- and post- processing. ■

10.23 (Project)

As discussed in the text, an initial choice of the line of seepage has to be made. For example, choosing a = 31.62 feet and a straight-line assumption, we may then define an initial region which has to be meshed. Modeling aspects are given below.

Initial Region with Boundary Conditions



The solution now proceeds by (step1) creating a mesh, (step2) editing the file for solution by Program HEAT2D as in earlier problems. Subsequently, an iterative procedure is necessary, since the condition $\phi_i = y_i$ on the assumed line of seepage (recall we started with a value for a and a straight-line) will not be satisfied. Thus, a correction such as

$$y_i^{new} = y_i + t(\phi_i - y_i)$$

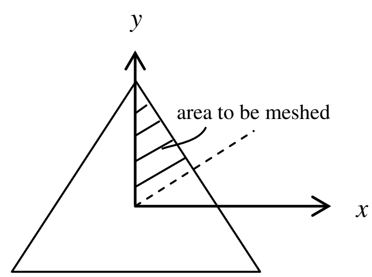
for each node i on the line of seepage where $t \cong 5\%$. ■

10.24 (a) **Triangular Duct**

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} + 1 = 0$$

$$W = 0 \quad on\ Boundary$$

To Find: $C = \dfrac{1}{2W_m},\ where \quad W_m = \dfrac{\int_A W dA}{\int_A dA}$

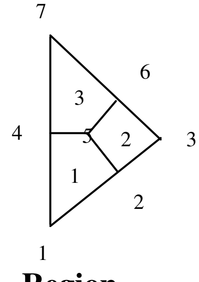


Model

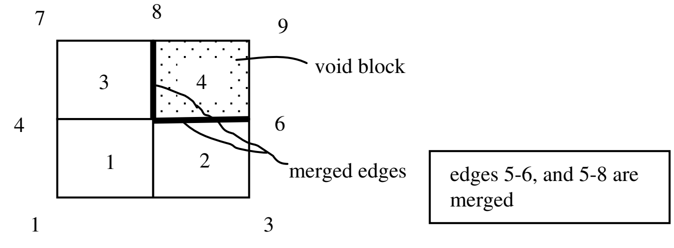
First, normalized coordinates are defined by $X = \dfrac{x}{D_h}, Y = \dfrac{y}{D_h}$

where $D_h = \dfrac{4 \times Area}{P} = \dfrac{4 \times 43.3}{30} = 5.7735$

Program MESHGEN requires a Region and a Block Diagram to be defined:



Region

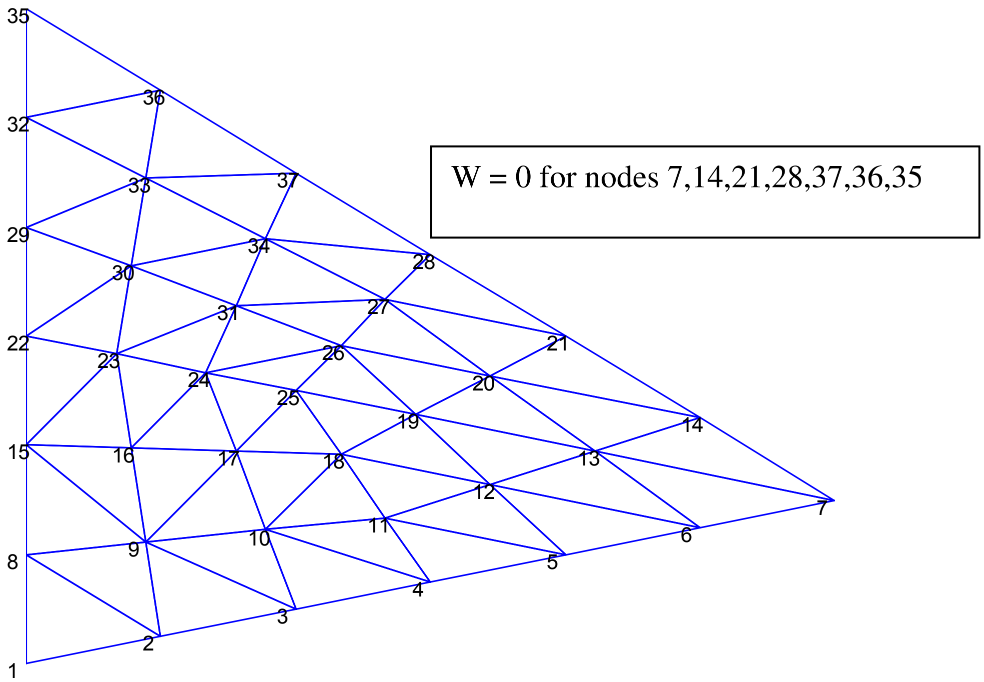


Block

Corner Node	X ,Y Coord.
1	0,0
2	0.2165, 0.125
3	0.433, 0.25
4	0, 0.5
5	0.14433, 0.416667
6,8	0.2165, 0.625
7	0, 1

MESHGEN Input File

```
Mesh Generation
P10_18 flow around a cylinder
Number of Nodes per Element <3 or 4>
  3
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ
  2             2             1
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
  1        3
 2         3
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        3
  2        3
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  4        0
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
      1     0      0
      2   0.2165  0.125
      3   0.433   0.25
      4   0       0.5
      5   0.14433  0.416667
      6   0.2165   0.625
      8   0.2165   0.625
      7   0         1
      0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
  1        5          6              5           8
```



Modification of Program HEAT2D

The evaluation of $W_m = \int_A W dA \Big/ \int_A dA$ reduces to, for CST elements, $W_m = \dfrac{\sum_e \frac{A_e}{3}(W_1 + W_2 + W_3)}{\sum_e A_e}$

where $W_1 + W_2 + W_3$ are the nodal values of W for an element (after 'SOLVE'), $A_e = 0.5*$ ABS(det J). Then, C = $1/2W_m$ is simply printed out.

Also, the constant "1" in the differential eq. $\dfrac{\partial^2 W}{\partial X^2} + \dfrac{\partial^2 W}{\partial Y^2} + 1 = 0$ can be handled by defining the elment heat source to be unity in the data file, as:

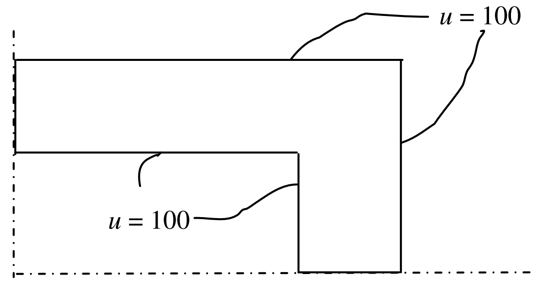
Elem#	Node1	Node2	Node3	Mat#	Elem Heat Source
1	....	....	....	....	1.0
2	....	....	....	....	1.0
54	....	....	....	....	1.0

Solution: C = 13.5

Part (b) may be similarly solved. ■

10.25

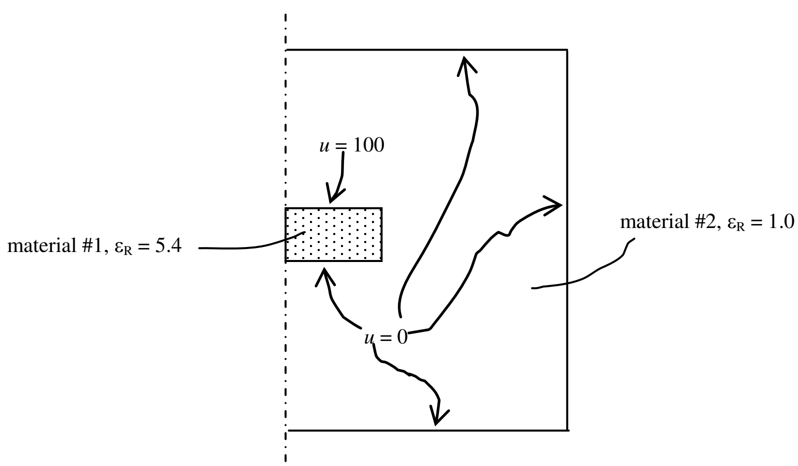
$$\varepsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = -\rho$$



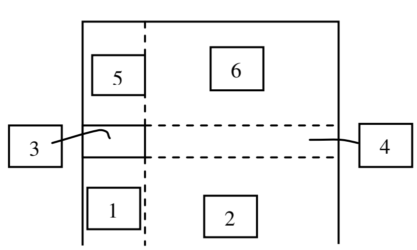
¼ Symmetry Model of Cable

We gave $\varepsilon = \varepsilon_R\ \varepsilon_o = 3 \times 8.854E-12$ and $\rho = 0$. Note that we can simply solve the problem in terms of $\varepsilon_R = 3$, to avoid small numbers. ■

10.26 $$\varepsilon_R\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = 0$$



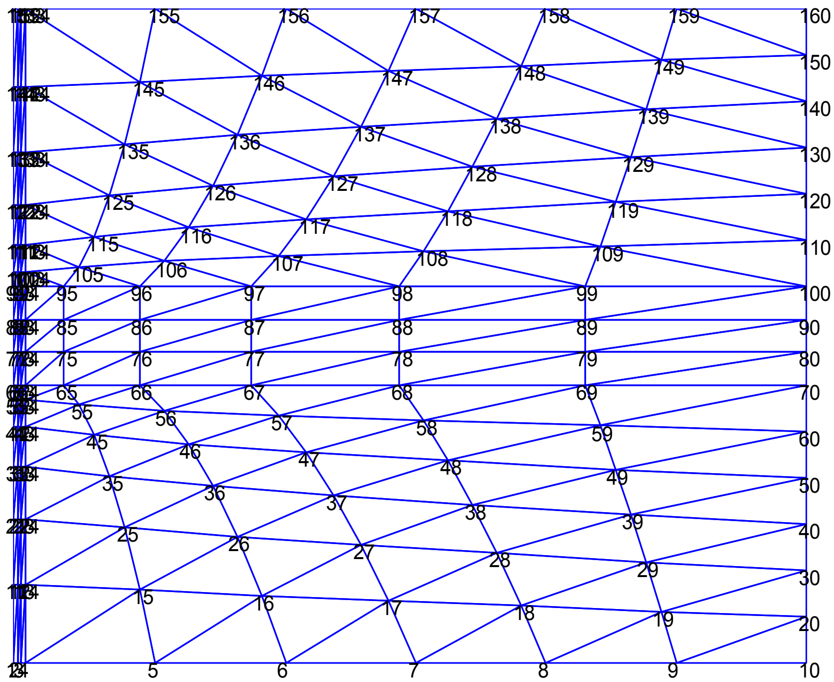
In Program MESHGEN, different material ID #s can be given by defining the different blocks with different material nos. (1 and 2). The Block Diagram used is:



The mesh is graded towards the strips using displaced mid-points. The MESHGEN input file is given below.

MESHGEN Input File

```
Mesh Generation
P10_18 flow around a cylinder
Number of Nodes per Element <3 or 4>
  3
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ
  2             3             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
 1         3
 2         6
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        6
  2        3
  3        6
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  3        2
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
      1     0      -.5
      2    .015    -.5
      3   1.       -.5
      4   0        -.075
      5   .015     -.075
      6   1.       -.075
      7   0         .075
      8   .015      .075
      9   1.        .075
     10   0         .5
     11   .015      .5
     12   1         .5
      0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   4      .3        -.075
   6      .3         .075
   0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   1       0         -.2
   2       .015      -.2
   7       0          .2
   8       .015       .2
   0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
```

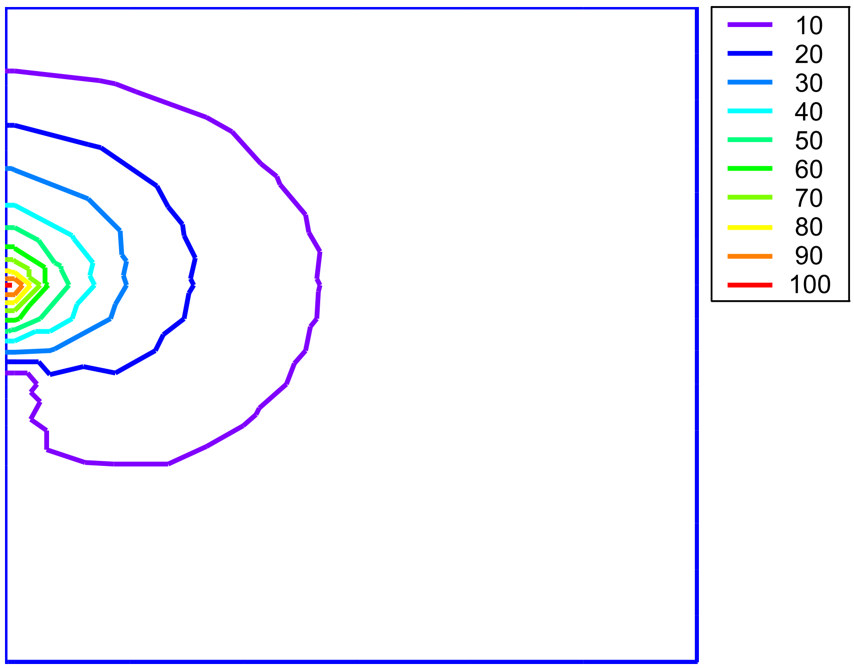


HEAT2D Input File

```
TWO DIMENSIONAL HEAT ANALYSIS USING HEAT2D
PROBLEM  10.20 -- Flow around a cylinder
NN  NE  NM  NDIM  NEN  NDN
160 270  2   2     3    1
ND   NL    NMPC
42   0      0
 Node#         X         Y
    1    0.00000e+000   -5.00000e-001
    2    5.00000e-003   -5.00000e-001
---------------------------------------
  160    1.00000e+000    5.00000e-001
 Elem#     Node1    Node2    Node3  Mat#  Elem Heat Source
     1        1        2       12       1   0.
 -------------------------------------
   270     160     159     150       1   0.
DOF#  Displacement (SPECIFIED VALUE OF FIELD VARIABLE)
1   0
2   0
3   0
4   0
5   0
6   0
7   0
8   0
9   0
10   0
20   0
30   0
```

```
40   0
50   0
60   0
70   0
80   0
90   0
100   0
110   0
120   0
130   0
140   0
150   0
160   0
159   0
158   0
157   0
156   0
155   0
154   0
153   0
152   0
151   0
61   0
62   0
63   0
64   0
91  100
92  100
93  100
94  100
DOF#  Load
MAT#    ThermalConductivity
 1      1.
 2      5.4
No. of edges with Specified Heat flux FOLLOWED BY two edges & q0 (positive if
out)
0
No.of Edges with Convection FOLLOWED BY edge(2 nodes) &  h & Tinf
0
```

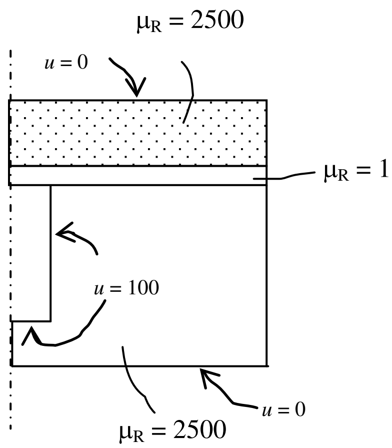
Voltage Field Distribution (Contours)



■

10.27 Magnetic Potential

<u>Model</u>



The solution steps involving meshing, creation of an Heat2d input file and output/contour plotting parallel the solution of earlier problems whose details have been provided. ■

10.28 Acoustic Modes in a Tube Rigid at Both Ends (perfect reflection)

Example 10.6 presents the solution with 6 elements. Solution with more elements simply involves preparing the data file and using Program JACOBI or Program GENEIGEN given in Chapter 11. With 12 elements, each with length 0.5m, we have the following input file based on the matrices in Eq. (10.122) in the text:

Input File for Program Jacobi

```
Stiffness and Mass for Data in File example 10.6, page 346, Text
Num. of DOF    Bandwidth
 13             2
Banded Stiffness Matrix
2.  -2.
4  -2.
4  -2.
4  -2.
4  -2.
4  -2.
4  -2.
4  -2.
4  -2.
4  -2.
4  -2.
4  -2.
2  0.
Banded Mass Matrix
.16667   .083333
.33333   .083333
.33333   .083333
.33333   .083333
.33333   .083333
.33333   .083333
.33333   .083333
.33333   .083333
.33333   .083333
.33333   .083333
.33333   .083333
.33333   .083333
.16667   0.
Starting Vector for Inverse Iteration
1 1 1 1 1 1 1  1  1  1  1  1  1
```

Solution (1st mode is a rigid body mode). Frequencies in cps. $f = c\,\sqrt{\lambda}/(2\pi)$

	2nd Mode	3rd Mode	4th Mode	5th Mode
6 elements	28.9	59.8	94.6	133.7
12 elements	28.7	57.8	88.0	119.6
Theory	28.6	57.2	85.8	114.3

We thus see an improvement with doubling of the elements. The reader may complete the problem as above with 24 and 48 elements, respectively. ■

10.29 Acoustic Modes in a Tube with One End Rigid, the other a Pressure release Condition
The same input file as in Example 10.6, Text, is used, except that a penalty term is introduced to enforce $p = 0$ at node 7. This is done simply by defining K(7,7) = a large value, or equivalently, the banded stiffness component S(7,1) = a large value. Program Jacobi yields

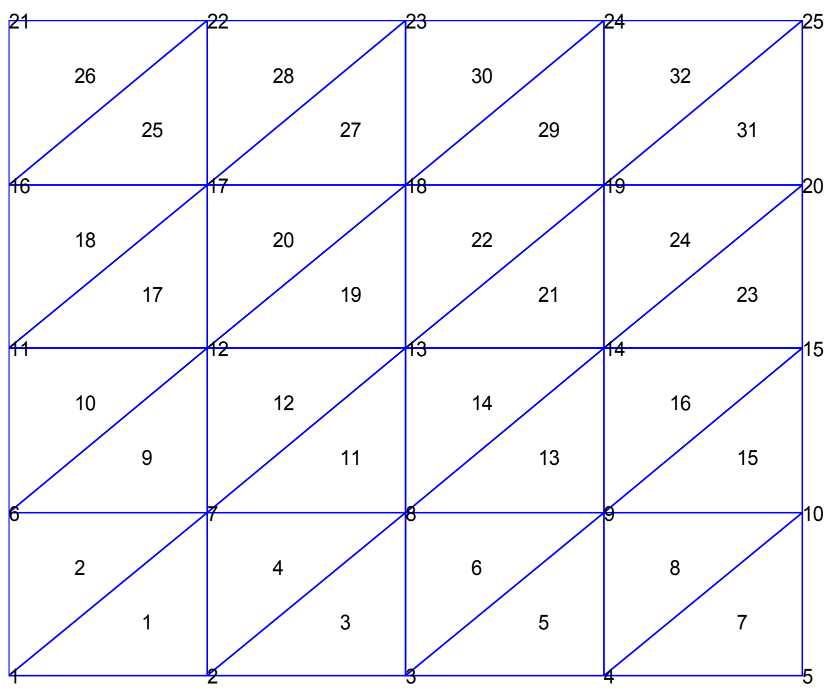
m	0	1	2	3	4	5
6 element, eigenvalue	0.068931	0.649164	1.96876	4.3378	7.922242	11.4068
6 element, frequency in cps	14.33246	43.98355	76.59662	113.6967	153.6517	184.3721
Theory, frequency in cps	14.29167	42.875	71.45833	100.0417	128.625	157.2083

The mode shapes may be plotted from the eigenvectors printed out. ■

10.30 The steps closely parallel those in Chapter 6. ■

10.31 2-D Acoustic Modes

Mesh Used:



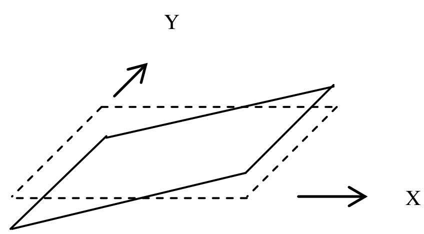
Solotion Approach: The acoustic stiffness and matrices given in the text are implemented into a program analogous to Program CSTKM. In fact, Program CSTKM has been modified. Further, NDN = no. of degrees of freedom per node is defined equal to unity in the input data file. Execution of this program results in an output file containing banded K, M matrices. Program

Jacobi is then run to obtain eigenvalues (λ_i) and eigenvectors. We use $f = c\,\sqrt{\lambda}/(2\pi)$ to get frequency in cps.

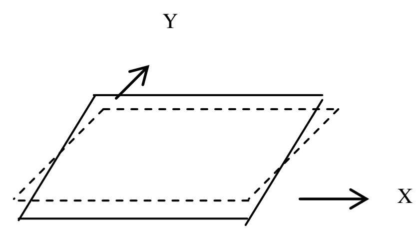
Solution (1st mode is a rigid body mode). Frequencies in cps.

	2nd Mode	3rd Mode	4th Mode
FEA	8.79	17.56	18.84
Theory	$f_{1,0,0} = 8.58$	$f_{0,1,0} = 17.15$	$f_{1,1,0} = 19.17$

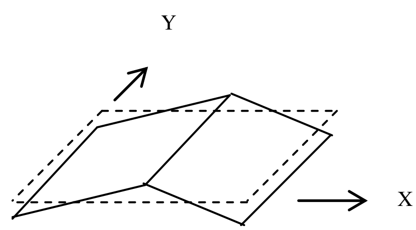
Mode Shapes for the Pressure in the Cavity:



2nd



3rd

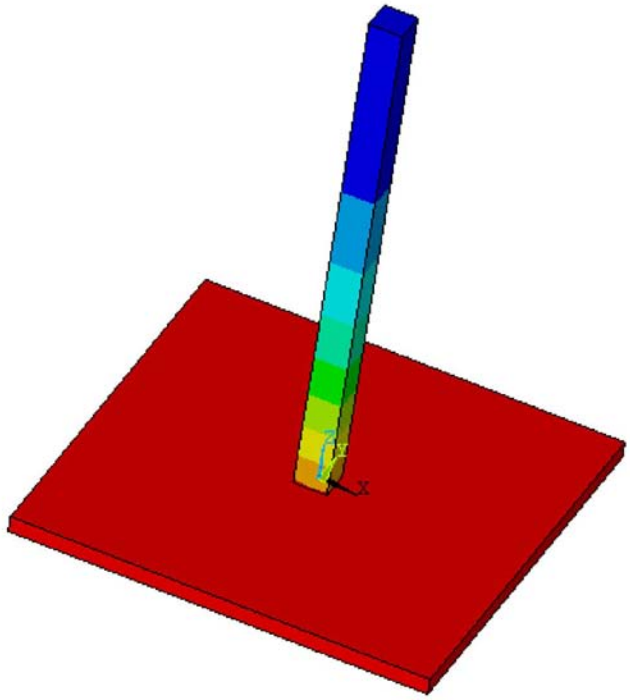


4th

It should be noted that computerized plotting will give better shapes. ■

10.32
Solution has been obtained using ANSYS. The data was entered directly into the program. The contour plot wor w = 2 mm is given below.

w = 2mm

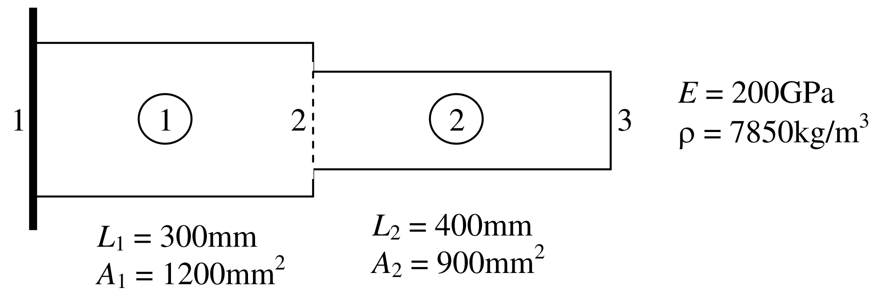


Temperature at base (= chip) is 112 degC and decreases to 64 degC near top of fin ■

10.33
The multiple fin problem is suggested as a project problem using ANSYS or other computer program. ■

CHAPTER 11
DYNAMIC CONSIDERATIONS

11.1



(a) Stiffness matrix

$$\mathbf{K} = E\begin{bmatrix} \frac{A_1}{L_1} & -\frac{A_1}{L_1} & 0 \\ -\frac{A_1}{L_1} & \frac{A_1}{L_1}+\frac{A_2}{L_2} & -\frac{A_2}{L_2} \\ 0 & -\frac{A_2}{L_2} & \frac{A_2}{L_2} \end{bmatrix}$$

$$= 10^8\begin{bmatrix} 8 & -8 & 0 \\ -8 & 12.5 & -4.5 \\ 0 & -4.5 & 4.5 \end{bmatrix}\text{N/m}$$

Mass matrix

$$\mathbf{M} = \frac{\rho}{6}\begin{bmatrix} 2A_1L_1 & A_1L_1 & 0 \\ A_1L_1 & 2(A_1L_1 + A_2L_2) & A_2L_2 \\ 0 & A_2L_2 & 2A_2L_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.942 & 0.471 & 0 \\ 0.471 & 1.884 & 0.471 \\ 0 & 0.471 & 0.942 \end{bmatrix}\text{kg}$$

Since displacement and velocity are zero at node 1, vor hand calculation we strike off terms involving Q_1 and $\dot{Q}_1$ in the potential and kinetic energy expressions. Thus the problem leads to finding λ, Q_2, and Q_3 from

$$10^8\begin{bmatrix} 12.5 & -4.5 \\ -4.5 & 4.5 \end{bmatrix} = \lambda\begin{bmatrix} 1.884 & 0.471 \\ 0.471 & 0.942 \end{bmatrix}$$

(b) We now use the inverse iteration algorithm to solve for λ and Q_2, Q_3.

We first start with a guess vector

$$\mathbf{u}^0 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\mathbf{v}^0 = \mathbf{M}\mathbf{u}^0 = \begin{Bmatrix} 2.355 \\ 1.413 \end{Bmatrix}$$

We then solve for $\mathbf{u}^1$ from $\mathbf{K}\mathbf{u}^1 = \mathbf{v}^0$, and obtain

$$\mathbf{u}^1 = 10^{-9}\begin{Bmatrix} 4.71 \\ 7.85 \end{Bmatrix}$$

$$\mathbf{v}^1 = \mathbf{M}\mathbf{u}^1 = 10^{-9}\begin{Bmatrix} 12.57 \\ 9.613 \end{Bmatrix}$$

λ and **u** are now calculated

$$\lambda = \frac{\left(\mathbf{u}^1\right)^T \mathbf{v}^0}{\left(\mathbf{u}^1\right)^T \mathbf{u}^{1^1}} = 1.6473 \times 10^8$$

$$\mathbf{u} = \begin{Bmatrix} 0.4059 \\ 0.6764 \end{Bmatrix}$$

This **u** used as $\mathbf{u}^0$ and the calculation steps are repeated.
The second iteration gives

$$\lambda = 1.6406 \times 10^8$$

$$\mathbf{u} = \begin{Bmatrix} 0.3921 \\ 0.6942 \end{Bmatrix}$$

The third iteration gives

$$\lambda = 1.6405 \times 10^8$$

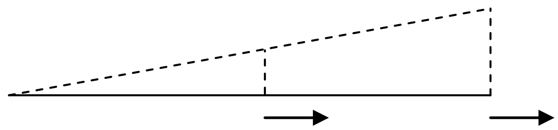
$$\mathbf{u} = \begin{Bmatrix} 0.3905 \\ 0.6962 \end{Bmatrix}$$

The proportionate difference in λ is small and thus convergence has been achieved.

Natural frequency

$$f = \frac{\sqrt{\lambda}}{2\pi} = 2039 \text{ Hz}$$

The mode shape is the pattern of the eigenvector where both nodes 2 and 3 (Q_2, Q_3) are moving to the right.



Inverse iteration gives the lowest eigenvalue. To get other eigenvalues, search is made in the orthogonal space.

(c) The input for program INVITR, JACOBI, or GENEIGEN and output are given here.

Input File for INVITR, JACOBI, or GENEIGEN

```
Stiffness and Mass for Data for Prob 11.1
Num. of DOF    Bandwidth
 2             2
Banded Stiffness Matrix
12.5E8  -4.5E8
4.5E8    0
Banded Mass Matrix
1.884    0.471
0.942    0
Starting Vector for Inverse Iteration
1 1
```

Output File for INVITR

```
Eigenvalues and Eigenvectors for Data in File p111.inp
Eigenvalue Number  1      Iteration Number  4
Eigenvalue =  1.6405E+08   Omega =  1.281E+04   Freq =  2.038E+03 Hz
Eigenvector
 3.903E-01  6.965E-01
Eigenvalue Number  2      Iteration Number  3
Eigenvalue =  1.4131E+09   Omega =  3.759E+04   Freq =  5.983E+03 Hz
Eigenvector
 6.739E-01 -8.535E-01
```

The two eigenvalues and corresponding eigenvectors are obtained.

(d) Using the two eigenvectors above, we note that

$$(\mathbf{u}^1)^T \mathbf{K}\mathbf{u}^2 = 0$$
$$(\mathbf{u}^1)^T \mathbf{M}\mathbf{u}^2 = 0$$
$$(\mathbf{u}^1)^T \mathbf{M}\mathbf{u}^1 = 1$$
$$(\mathbf{u}^1)^T \mathbf{K}\mathbf{u}^1 = \lambda_1$$
$$(\mathbf{u}^2)^T \mathbf{M}\mathbf{u}^2 = 1$$
$$(\mathbf{u}^2)^T \mathbf{K}\mathbf{u}^2 = \lambda_2$$

■

11.2 We need λ and $\mathbf{u}$ such that

$$10^8\begin{bmatrix}12.5 & -4.5\\ -4.5 & 4.5\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \lambda\begin{bmatrix}1.884 & 0.471\\ 0.471 & 0.942\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

$$\Rightarrow\begin{bmatrix}K_{11}-\lambda M_{11} & K_{12}-\lambda M_{12}\\ K_{12}-\lambda M_{12} & K_{22}-\lambda M_{22}\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = 0$$

For a non-trivial solution, we require

$$\det\begin{bmatrix}K_{11}-\lambda M_{11} & K_{12}-\lambda M_{12}\\ K_{12}-\lambda M_{12} & K_{22}-\lambda M_{22}\end{bmatrix} = 0$$

This yields the characteristic polynomial

$$\left(M_{11}M_{22}-M_{12}{}^2\right)\lambda^2-\left(k_{11}M_{22}+K_{22}M_{11}-2K_{12}M_{12}\right)\lambda+K_{11}K_{22}-K_{12}{}^2=0$$

On substitution for K_{11}, … , the characteristic polynomial is

$$1.5529\lambda^2 - 24.492\times10^8\,\lambda + 36\times10^{16} = 0$$

On solving the quadratic equation,

$$\lambda_1 = 1.6405\times10^8,\ \text{and}\ \lambda_2 = 14.131\times10^8.$$

For the eigenvector corresponding to λ_1, we need

$$\left(K_{11}-\lambda_1 M_{11}\right)u_1+\left(K_{12}-\lambda_1 M_{12}\right)u_2 = 0$$
$$\Rightarrow u_1 = 0.56037u_2$$

u_1 and u_2 are chosen such that

$$\mathbf{u}^{\mathrm{T}}\mathbf{M}\mathbf{u} = 1$$

This gives $[u_1,\ u_2]^{\mathrm{T}} = [0.3903,\ 0.6965]^{\mathrm{T}}$

Other eigenvector is calculated similarly. ■

11.3 In the lumped mass technique, mass of each element is calculated and distributed equally to each of its nodes.

Mass of element 1 = $7850\times0.3\times1200\times10^{-6} = 2.826$ kg
Mass of element 2 = $7850\times0.4\times900\times10^{-6} = 2.826$ kg

The lumped mass matrix is

$$\mathbf{M} = \begin{bmatrix} 1.413 & 0 & 0 \\ 0 & \boxed{\begin{matrix} 2.826 & 0 \\ 0 & 1.413 \end{matrix}} \end{bmatrix}$$

Lumped mass matrix is always diagonal. We make use of the 2×2 part of the matrix as discussed in problem 11.1, and create the input file for eigenvalue evaluation.

Input File for INVITR, JACOBI, or GENEIGEN

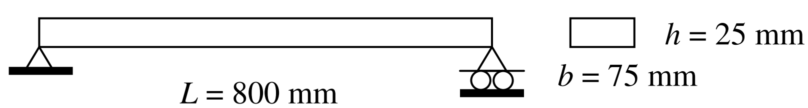
```
Stiffness and Mass for Data for Prob 11.3
Num. of DOF    Bandwidth
 2             2
Banded Stiffness Matrix
12.5E8  -4.5E8
4.5E8    0
Banded Mass Matrix
2.826    0
1.413    0
Starting Vector for Inverse Iteration
1 1
```

Output from program JACOBI

```
Eigenvalues and Eigenvectors for Data in File p113.inp
Eigenvalue Number   1
Eigenvalue =  1.4684E+08   Omega =  1.212E+04   Freq =  1.929E+03 Hz
Eigenvector
 3.606E-01  6.691E-01
Eigenvalue Number   2
Eigenvalue =  6.1395E+08   Omega =  2.478E+04   Freq =  3.944E+03 Hz
Eigenvector
 4.731E-01 -5.099E-01
```

The eigenvalues are 1.468×10^8, and 6.1395×10^8. The consistent mass matrix gave values of 1.6405×10^8, and 14.131×10^8. ■

11.4



$\rho = 7850 \text{ kg/m}^3$

$$I = \frac{bh^3}{12} = \frac{(0.075)(0.025)^3}{12} = 9.7656 \text{ m}^4$$

$L = 0.8$ m

$A = bh = (0.075)(0.025) = 1.875 \times 10^{-3} \text{ m}^2$

(a) We consider the one element model. The input file for BEAMKM is given first. The output from BEAMKM is the input for INVITR, JACOBI, or GENEIGEN.

Input file for BEAMKM

```
<< STIFFNESS MASS USING BEAMKM >>
PROBLEM 11.4a
NN  NE  NM  NDIM  NEN  NDN
 2   1   1   1     2    2
ND   NL    NMPC
 2    0     0
Node#     X
 1        0
 2        0.8
Elem# N1  N2   MAT#   Mom_Inertia    Area
 1     1   2    1      9.7656E-5     1.875E-3
DOF#  Displacement
 1     0
 3     0
DOF#  Load
MAT#   Prop1(E)      Prop2(MassDensity Rho)
 1     200E9         7850
```

Input file for INVITR, JACOBI, or GENEIGEN

```
Stiffness and Mass for Data in File p114.km
Num. of DOF    Bandwidth
 4             4
Banded Stiffness Matrix
 4.578082E+12  1.83105E+08 -4.577625E+08  1.83105E+08
 9.765599E+07 -1.83105E+08  4.8828E+07  0
 4.578082E+12 -1.83105E+08  0  0
 9.765599E+07  0  0  0
Banded Mass Matrix
 4.373571  .4934286  1.513929 -.2915714
 7.177143E-02  .2915714 -5.382857E-02  0
 4.373571 -.4934286  0  0
 7.177143E-02  0  0  0
Starting Vector for Inverse Iteration
1 1 1 1
```

Output from program JACOBI

```
Eigenvalues and Eigenvectors for Data in File p114a.inp
Eigenvalue Number   1
Eigenvalue =  3.8860E+08   Omega =  1.971E+04   Freq =  3.137E+03 Hz
Eigenvector
-1.329E-04 -1.994E+00 -1.329E-04  1.994E+00
Eigenvalue Number   2
Eigenvalue =  8.1439E+09   Omega =  9.024E+04   Freq =  1.436E+04 Hz
Eigenvector
 1.479E-03  5.262E+00 -1.478E-03  5.262E+00
Eigenvalue Number   3
Eigenvalue =  4.6665E+12   Omega =  2.160E+06   Freq =  3.438E+05 Hz
Eigenvector
 7.136E-01 -4.466E+00  7.140E-01  4.458E+00
Eigenvalue Number   4
Eigenvalue =  7.7959E+12   Omega =  2.792E+06   Freq =  4.444E+05 Hz
Eigenvector
-9.219E-01  1.039E+01  9.214E-01  1.039E+01
```

(b) We divide the beam into two elements. The input file for BEAMKM is given first. The output from BEAMKM is the input for INVITR, JACOBI, or GENEIGEN.

Input file for BEAMKM

```
<< STIFFNESS MASS USING BEAMKM >>
PROBLEM 11.4b
NN  NE  NM  NDIM  NEN  NDN
 3   2   1   1     2    2
ND   NL    NMPC
 2    0     0
Node#     X
 1        0
 2        0.4
 3        0.8
Elem# N1  N2   MAT#   Mom_Inertia    Area
 1     1   2    1      9.7656E-5     1.875E-3
 2     2   3    1      9.7656E-5     1.875E-3
DOF#  Displacement
 1     0
 5     0
DOF#  Load
MAT#   Prop1(E)      Prop2(MassDensity Rho)
 1     200E9         7850
```

Input file for INVITR, JACOBI, or GENEIGEN

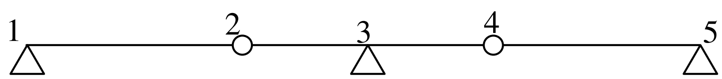
```
Stiffness and Mass for Data in File p114b.km
Num. of DOF    Bandwidth
 6             4
Banded Stiffness Matrix
 7.324566E+13  7.3242E+08 -3.6621E+09  7.3242E+08
 1.95312E+08 -7.3242E+08  9.765599E+07  0
 7.324199E+09  0 -3.6621E+09  7.3242E+08
 3.90624E+08 -7.3242E+08  9.765599E+07  0
 7.324566E+13 -7.3242E+08  0  0
 1.95312E+08  0  0  0
Banded Mass Matrix
 2.186786  .1233571  .7569643 -7.289285E-02
 8.971428E-03  7.289285E-02 -6.728571E-03  0
 4.373571  0  .7569643 -7.289285E-02
 1.794286E-02  7.289285E-02 -6.728571E-03  0
 2.186786 -.1233571  0  0
 8.971428E-03  0  0  0
Starting Vector for Inverse Iteration
1 1 1 1 1 1
```

Output from program GENEIGEN

```
Eigenvalues and Eigenvectors for Data in File p114b.inp
Eigenvalue Number   1
Eigenvalue =  3.1806D+08   Omega =  1.783D+04   Freq =  2.838D+03 Hz
Eigenvector
 6.700D-06  1.631D+00  4.154D-01  5.052D-12  6.700D-06 -1.631D+00
Eigenvalue Number   2
Eigenvalue =  6.2195D+09   Omega =  7.886D+04   Freq =  1.255D+04 Hz
Eigenvector
 6.650D-05  3.990D+00  5.878D-13 -3.990D+00 -6.650D-05  3.990D+00
Eigenvalue Number   3
Eigenvalue =  3.9285D+10   Omega =  1.982D+05   Freq =  3.155D+04 Hz
Eigenvector
 3.113D-04  8.584D+00 -3.755D-01 -5.397D-13  3.113D-04 -8.584D+00
```

```
Eigenvalue Number   4
Eigenvalue =  1.3054D+11   Omega =  3.613D+05   Freq =  5.750D+04 Hz
Eigenvector
 7.390D-04  1.054D+01  4.093D-15  1.055D+01 -7.390D-04  1.054D+01
Eigenvalue Number   5
Eigenvalue =  1.6178D+14   Omega =  1.272D+07   Freq =  2.024D+06 Hz
Eigenvector
 1.051D+00 -1.576D+01  1.618D-01 -1.429D-15  1.051D+00  1.576D+01
Eigenvalue Number   6
Eigenvalue =  1.8674D+14   Omega =  1.367D+07   Freq =  2.175D+06 Hz
Eigenvector
 1.129D+00 -1.976D+01  6.604D-17 -5.650D+00 -1.129D+00 -1.976D+01
```
■

11.5 This problem is similar to Example 11.5 with some changes in data.



$d = 4$ in
$I = \pi d^4/64 = 12.566$ in^4
$A = \pi d^2/4 = 12.566$ in^2

$\rho = 7.324 \times 10^{-4}$ lb.s/in^4
$M_2 = 2400$ lb $= 6.218$ lb.s^2/in
$M_4 = 1200$ lb $= 3.109$ lb.s^2/in

There are 3 bearing supports at 1,3, and 4. There are point masses at nodes 2 and 4. In part (a) we apply boundary conditions $Q_1 = Q_5 = Q_9 = 0$. In part (b) there are no boundary conditions. We introduce stiffness values of 25000 lb/in at K_{11}, K_{55}, K_{99}, interactively in BEAMKM. The masses are also introduced interactively in BEAMKM. The input data sets for BEAMKM and the output from BEAMKM are given these two cases. Eigenvalue and eigenvector evaluation may be carried out using INVITR, JACOBI, or GENEIGEN program.

(a) Input file for BEAMKM

```
<< STIFFNESS MASS USING BEAMKM >>
PROBLEM 11.5a
NN  NE  NM  NDIM  NEN  NDN
 5   4   1   1     2    2
ND   NL    NMPC
 3    0     0
Node#     X
 1        0
 2        30
 3        48
 4        66
 5        96
Elem# N1  N2   MAT#   Mom_Inertia    Area
 1     1   2    1      12.566        12.566
```

```
 2     2   3    1      12.566        12.566
 3     3   4    1      12.566        12.566
 4     4   5    1      12.566        12.566
DOF#  Displacement
 1     0
 5     0
 9     0
DOF#  Load
MAT#   Prop1(E)      Prop2(MassDensity Rho)
 1     30e6          7.324e-4
```

Input file for INVITR, JACOBI, or GENEIGEN

```
Stiffness and Mass for Data in File p115a.km
Num. of DOF    Bandwidth
 10            4
Banded Stiffness Matrix
 1.675467E+12  2513200 -167546.7  2513200
 5.0264E+07 -2513200  2.5132E+07  0
 943225.7  4467911 -775679  6981111
 1.340373E+08 -6981111  4.188667E+07  0
 1.675468E+12  0 -775679  6981111
 1.675467E+08 -6981111  4.188667E+07  0
 943225.7 -4467911 -167546.7  2513200
 1.340373E+08 -2513200  2.5132E+07  0
 1.675467E+12 -2513200  0  0
 5.0264E+07  0  0  0
Banded Mass Matrix
 .1025515  .4338717  3.549859E-02 -.2563787
 2.366573  .2563787 -1.77493  0
 6.382082 -.2776778  2.129916E-02 -9.229634E-02
 2.877752  9.229634E-02 -.3833848  0
 .1230618  0  2.129916E-02 -9.229634E-02
 1.022359  9.229634E-02 -.3833848  0
 3.273082  .2776778  3.549859E-02 -.2563787
 2.877752  .2563787 -1.77493  0
 .1025515 -.4338717  0  0
 2.366573  0  0  0
Starting Vector for Inverse Iteration
1 1 1 1 1 1 1 1 1 1
```

Output from program JACOBI

```
Eigenvalues and Eigenvectors for Data in File p115a.inp
Eigenvalue Number  1      Iteration Number  8
Eigenvalue =  3.6230E+04   Omega =  1.903E+02   Freq =  3.029E+01 Hz
Eigenvector
 1.576E-08  2.198E-02  3.478E-01 -8.950E-03  2.449E-08 -2.178E-02 -2.557E-01
-4.589E-03 -1.008E-08  1.516E-02
Eigenvalue Number  2      Iteration Number  3
Eigenvalue =  1.2959E+05   Omega =  3.600E+02   Freq =  5.729E+01 Hz
Eigenvector
 1.485E-08  1.571E-02  1.834E-01 -1.253E-02  1.759E-07  1.075E-02  4.820E-01
 1.940E-02  2.853E-08 -3.441E-02
```

Mode shapes may be plotted using the eigenvector data.

(b) Input file for BEAMKM

```
<< STIFFNESS MASS USING BEAMKM >>
PROBLEM 11.5b
NN  NE  NM  NDIM  NEN  NDN
```

```
 5   4   1   1     2    2
ND   NL    NMPC
 0    0     0
Node#     X
 1        0
 2        30
 3        48
 4        66
 5        96
Elem# N1  N2   MAT#   Mom_Inertia    Area
 1     1   2    1      12.566        12.566
 2     2   3    1      12.566        12.566
 3     3   4    1      12.566        12.566
 4     4   5    1      12.566        12.566
DOF#  Displacement
DOF#  Load
MAT#   Prop1(E)       Prop2(MassDensity Rho)
 1     30e6           7.324e-4
```

<u>Input file for INVITR, JACOBI, or GENEIGEN</u>

```
Stiffness and Mass for Data in File p115b.km
Num. of DOF    Bandwidth
 10            4
Banded Stiffness Matrix
 192546.7  2513200 -167546.7  2513200
 5.0264E+07 -2513200  2.5132E+07  0
 943225.7  4467911 -775679  6981111
 1.340373E+08 -6981111  4.188667E+07  0
 1576358  0 -775679  6981111
 1.675467E+08 -6981111  4.188667E+07  0
 943225.7 -4467911 -167546.7  2513200
 1.340373E+08 -2513200  2.5132E+07  0
 192546.7 -2513200  0  0
 5.0264E+07  0  0  0
Banded Mass Matrix
 .1025515  .4338717  3.549859E-02 -.2563787
 2.366573  .2563787 -1.77493  0
 6.382082 -.2776778  2.129916E-02 -9.229634E-02
 2.877752  9.229634E-02 -.3833848  0
 .1230618  0  2.129916E-02 -9.229634E-02
 1.022359  9.229634E-02 -.3833848  0
 3.273082  .2776778  3.549859E-02 -.2563787
 2.877752  .2563787 -1.77493  0
 .1025515 -.4338717  0  0
 2.366573  0  0  0
Starting Vector for Inverse Iteration
1 1 1 1 1 1 1 1 1 1
```

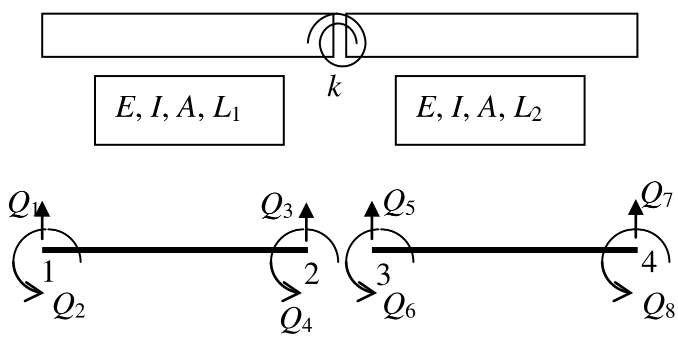
<u>Output from program JACOBI</u>

```
Eigenvalues and Eigenvectors for Data in File p115b.inp
Eigenvalue Number  1      Iteration Number  5
Eigenvalue =  4.7051E+03   Omega =  6.859E+01   Freq =  1.092E+01 Hz
Eigenvector
 1.815E-01  6.967E-03  3.372E-01  1.670E-03  3.257E-01 -2.123E-03  2.705E-01
-4.307E-03  8.950E-02 -6.906E-03
Eigenvalue Number  2      Iteration Number  4
Eigenvalue =  1.9959E+04   Omega =  1.413E+02   Freq =  2.248E+01 Hz
Eigenvector
```

```
-3.993E-01  3.176E-03 -1.910E-01  1.428E-02  1.419E-01  2.046E-02  4.563E-01
 1.179E-02  5.238E-01 -2.667E-03
```

Mode shapes may be plotted using the eigenvector data. ■

11.6 In this problem, we consider two isolated elements. We them impose conditions on the stiffness matrix to equate $Q_3 = Q_5$.



Taking into account the constraint, following energy term gets added.

$$U = \frac{1}{2}C(Q_3 - Q_5)^2 + \frac{1}{2}k(Q_4 - Q_6)^2$$
$$= \frac{1}{2}[Q_3 \quad Q_5]\begin{bmatrix} C & -C \\ -C & C \end{bmatrix}\begin{Bmatrix} Q_3 \\ Q_5 \end{Bmatrix} + \frac{1}{2}[Q_4 \quad Q_6]\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} Q_4 \\ Q_6 \end{Bmatrix}$$

Where the first part is the penalty term. C is a large constant. C is large in comparison to the diagonal terms of the stiffness matrix of the assembled stiffness matrix.

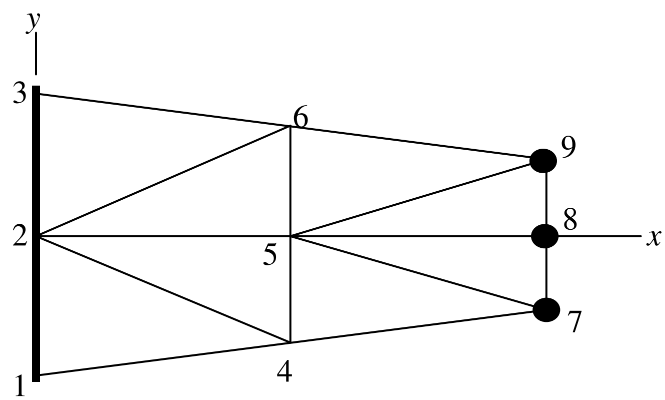
$$C = \frac{12EI}{L_1^3} \times 10^4$$

Above choice is in line with the penalty constant choice suggested in Chapter 3.

After generating the stiffness and mass matrices using BEAMKM, the stiffness terms from the above energy term are added to appropriate locations in the matrix. A better way is to introduce this into the program by modifying the input data set and the program.

After these modifications, the use of INVITR, JACOBI, or GENEIGEN is same as other programs. ■

11.7



A 8 element, 9 node configuration is considered here. The dimensions are all defined in meter units. In the CSTKM run, mass of 25 kg each is interactively input at dof 13, 14, 15, 16, 17, 18. The input data file for CSTKM is given here.

<u>Input Data file for CSTKM</u>

```
<< STIFFNESS MASS USING CSTKM >>
EXAMPLE 11.7
NN NE NM NDIM NEN NDN
9 8 1 2 3 2
ND NL  NMPC
 6  0   0
Node#  Coordinates
1 0     -0.08
2 0     0
3 0     0.08
4 0.2   -0.06
5 0.2   0
6 0.2   0.06
7 0.4   -0.04
8 0.4   0
9 0.4   0.04
Elem#  Nodes  Mat#  Thickness  TempRise
1 1 4 2 1 0.01 0
2 2 4 5 1 0.01 0
3 2 5 6 1 0.01 0
4 2 6 3 1 0.01 0
5 4 7 5 1 0.01 0
6 5 7 8 1 0.01 0
7 5 8 9 1 0.01 0
8 5 9 6 1 0.01 0
DOF#  Displacement
1 0
2 0
3 0
4 0
5 0
6 0
DOF#  Load
MAT#  E    Nu  Alpha   MassDensity (Rho)
```

```
1     200e9 .3  12E-6   7850
B1  i  B2 j  B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

The file generated by CSTKM is not given here. Banded stiffness and mass matrices are generated by CSTKM. The generated file is used with INVITR, JACOBI, or GENEIGEN.

For finer meshes, MESHGEN may be used in generating the data file. However, care must be taken in editing the file for the data to be added. ■

11.8 From Eq. 11.29, we note that

$$\mathbf{m}^e = \rho t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{N}^T \mathbf{N} \det \mathbf{J} d\xi d\eta$$

Following the integration steps used in the element stiffness calculations in program QUAD, the element mass matrix can be introduced into the program. The formation of the global mass matrix also follows the same steps as the stiffness matrix. The global mass matrix also has the same bandwidth.

Program CSTKM may be used as a guideline in modifying the QUAD program to develop the program QUADKM. First copy the QUAD program with name QUADKM and make the changes. The equation solving and stress calculation aspects are not necessary for QUADKM program. Make the output data similar to the output from CSTKM. ■

11.9 The idea in this project is to develop program FRAME2DKM using element mass matrix defined in Eq. 11.32. Copy the program FRAM2D into FRAME2DKM and use BEAMKM as a reference to make necessary changes. The input data file for this program must be similar to the FRAME2D file with mass and area properties introduced as in BEAMKM.INP. ■

11.10 This problem needs to use the FRAME2DKM program. The development steps for this program are stated in the solution steps of problem 11.9. This problem also needs the step of reading in lumped masses. ■

11.11 Follow the steps given in Example 11.6 in solving this problem. ■

11.12

$$\mathbf{A} = \begin{bmatrix} 6 & 1 & 2 & 0 & 0 \\ 1 & 3 & 1 & 2 & 0 \\ 2 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

We show here one step in the tridiagonalization process as discussed in the chapter. We use the vector $\mathbf{a} = [1\ \ 2\ \ 0\ \ 0]^T$ formed by excluding the first element in the first column.

$$|\mathbf{a}| = \sqrt{1^2 + 2^2} = 2.2361$$

We denote $\mathbf{e}_1 = [1\ \ 0\ \ 0\ \ 0]^T$. $\mathbf{w}_1$ is the unit vector along $\mathbf{a} - |\mathbf{a}|\,\mathbf{e}_1$. $\mathbf{w}_1$ is given by

$$\mathbf{w}_1 = [-0.526\ \ 0.851\ \ 0\ \ 0]^T$$

$\mathbf{H}_1$ is given by

$$\mathbf{H}_1 = \mathbf{I} - 2\mathbf{w}_1\mathbf{w}^T = \begin{bmatrix} 0.447 & 0.894 & 0 & 0 \\ 0.894 & -0447 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We then get

$$\mathbf{H}_1 \begin{bmatrix} 3 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \mathbf{H}_1 = \begin{bmatrix} 3 & 1 & 1.789 & 0 \\ 1 & 2 & 1.342 & 0 \\ 1.789 & 1.342 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The operations at the first step on the entire matrix gives the 5×5 matrix

$$\begin{bmatrix} 6 & 2.236 & 0 & 0 & 0 \\ 2.236 & 3 & 1 & 1.789 & 0 \\ 0 & 1 & 2 & 1.342 & 0 \\ 0 & 1.789 & 1.342 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

The process is repeated for two more iterations to tridiagonalize the matrix. On performing this we obtain the tridiagonal matrix:

$$\begin{bmatrix} 6 & 2.236 & 0 & 0 & 0 \\ 2.236 & 3.0000 & 2.0494 & 0 & 0 \\ 0 & 2.0494 & 3.1429 & 1.1206 & 0 \\ 0 & 0 & 1.1206 & 1.6805 & 1.1509 \\ 0 & 0 & 0 & 1.1509 & 2.1766 \end{bmatrix}$$

■

11.13 We show one step in the Jacobi process. The details of the steps are given in Chapter 11.

$$\mathbf{K} = \begin{bmatrix} 4 & 3 & 2 & 0 \\ 3 & 3 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} 3 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

At the first step, we simultaneously make K_{13}, K_{31}, M_{13}, M_{31} zero. Following the notation in the Jacobi procedure in the chapter,

$$A = K_{11} M_{13} - M_{11} K_{13} = (4)(2) - (3)(2) = 2$$
$$B = K_{33} M_{13} - M_{33} K_{13} = (2)(2) - (3)(2) = -2$$
$$C = K_{11} M_{33} - M_{11} K_{33} = (4)(3) - (3)(2) = 6$$

$$\alpha = \frac{-0.5C + \text{sgn}(C)\sqrt{0.25C^2 + AB}}{A} = -0.382$$

$$\beta = -\frac{A\alpha}{B} = -0.382$$

We form the transformation matrix **P** for the first iteration as

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

On performing $\mathbf{P}^T\mathbf{KP}$, $\mathbf{P}^T\mathbf{MP}$, we get

$$\mathbf{P}^T\mathbf{KP} = \begin{bmatrix} 2.764 & 2.618 & 0 & -0.382 \\ 2.618 & 3 & -0.146 & 0 \\ 0 & -0.146 & 1.056 & 1 \\ -0.382 & 0 & 1 & 4 \end{bmatrix}$$

$$\mathbf{P}^T\mathbf{MP} = \begin{bmatrix} 1.91 & 0.618 & 0 & -0.382 \\ 0.618 & 2 & 0.618 & 0 \\ 0 & 0.618 & 1.91 & 1 \\ -0.382 & 0 & 1 & 2 \end{bmatrix}$$

Note that the value at location 1,4 has changed to a non-zero value. We continue with the iteration to zero the location 1,2, since 1,4 is zero. Next we zero location 2,3 and so on. We come back to zero location 1,4 at the next sweep.

Following diagonal values are obtained by printing the values from program JACOBI.

Stiffness = [0.4262   17.923   1.1447 7.0855]^T
Mass = [3.3123   6.3556   1.7875 3.6216]^T ■

11.14 The solution of this problem is straightforward. Follow the steps given in problems 11.4 and 11.5. ■

11.15 For the geometry shown, the velocity of the center of gravity of the mass and the angular velocity of the rigid body are easily established.

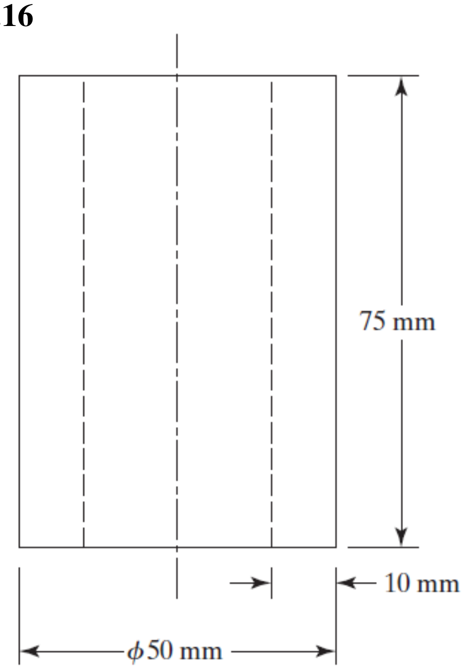
$$v = \dot{Q}_1 + a\dot{Q}_2$$
$$\omega = \dot{Q}_2$$

The kinetic energy is

$$T = \frac{1}{2}M\left(\dot{Q}_1 + a\dot{Q}_2\right)^2 + \frac{1}{2}I_C\dot{Q}_2^{\,2}$$
$$= \frac{1}{2}\left[\dot{Q}_1 \quad \dot{Q}_2\right]\begin{bmatrix} M & Ma \\ Ma & Ma^2 + I_c \end{bmatrix}\begin{Bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \end{Bmatrix}$$

The mass matrix is the 2 × 2 matrix above. ■

11.16



Material steel $E = 2006$ Pa
$v = 0.3$
$p = 7850$ kg/m^3

An element mass matrix as given in Eqn11.26 is to be introduced. See CSTKM program for the CST mass matrix implementation.
The ExcelVB implementation is as follows:

```
Private Sub ElemKM(N, RBAR)
     '--- Element Stiffness
     For I = 1 To 6
       For J = 1 To 6
         C = 0
         For K = 1 To 4
           C = C + Abs(DJ) * B(K, I) * DB(K, J) * PI * RBAR
         Next K
         SE(I, J) = C
       Next J
     Next I
   '--- Element Mass  EM()
   RHO = PM(MAT(N), 4)
   CM = PI * RHO * 0.5 * Abs(DJ) / 10
   For I = 1 To 6: For J = 1 To 6: EM(I, J) = 0: Next J: Next I
   '--- Non-zero elements of mass matrix are defined
   r1 = X(NOC(N, 1), 1): r2 = X(NOC(N, 2), 1): r3 = X(NOC(N, 3), 1)
   RBAR = (r1 + r2 + r3) / 3
   EM(1, 1) = CM * (4 / 3 * r1 + 2 * RBAR): EM(1, 3) = CM * (2 * RBAR - r3 / 3)
   EM(1, 5) = CM * (2 * RBAR - r2 / 3): EM(3, 1) = EM(1, 3): EM(5, 1) = EM(1, 5)
   EM(2, 2) = CM * (4 / 3 * r1 + 2 * RBAR): EM(2, 4) = CM * (2 * RBAR - r3 / 3)
   EM(2, 6) = CM * (2 * RBAR - r2 / 3): EM(2, 4) = EM(2, 4): EM(6, 2) = EM(2, 6)
   EM(3, 3) = CM * (4 / 3 * r2 + 2 * RBAR): EM(3, 5) = CM * (2 * RBAR - r1 / 3)
   EM(5, 3) = EM(3, 5)
   EM(4, 4) = CM * (4 / 3 * r2 + 2 * RBAR): EM(4, 6) = CM * (2 * RBAR - r1 / 3)
   EM(6, 4) = EM(4, 6)
   EM(5, 5) = CM * (4 / 3 * r3 + 2 * RBAR): EM(6, 6) = CM * (4 / 3 * r3 + 2 * RBAR)
End Sub
```

Four element input to the program and output are now given.

<< AXISYMMETRIC STRESS ANALYSIS USING TRIANGULAR ELEMENT >>
PROBLEM 11.16

NN	NE	NM	NDIM	NEN	NDN
6	4	1	2	3	2

ND	NL	NMPC
0	0	0

Node#	X	Y
1	15	0
2	25	0
3	15	37.5
4	25	37.5
5	15	75
6	25	75

Elem#	N1	N2	N3	Mat#	Temp
1	1	2	3	1	0
2	3	2	4	1	0

3	3	4	5	1	0
4	5	4	6	1	0
DOF#	**Load**				
DOF#	**Displ.**				
MAT#	**E**	**Nu**	**Alpha**	**Rho**	
1	2.00E+05	0.3	1.20E-05	7.85E-06	

Results from Program AxisymKM
PROBLEM 11.16
NumDOF BandWidth
12 6
Banded Stiffness Matrix

52191142.1	9867823.1	-56227368.5	-4430451.2	-3790293.7	-5437371.9
20749279.7	-7853981.6	-16614191.9	-5638756.0	-4135087.8	0
72157089.1	-1208304.9	3548867.4	13089969.4	4761444.1	-4027682.9
21501113.8	11076127.9	0	-9062286.5	-4886921.9	0
116581515.9	11076127.9	-123323143.6	-11076127.9	-3790293.7	-5437371.9
44519321.5	-13089969.4	-36249146.0	-5638756.0	-4135087.8	0
152964542.5	13089969.4	3548867.4	13089969.4	4761444.1	-4027682.9
46022989.8	11076127.9	0	-9062286.5	-4886921.9	0
64390373.9	1208304.9	-67095775.1	-6645676.8	0	0
23770041.9	-5235987.8	-19634954.1	0	0	0
80807453.4	14298274.3	0	0	0	0
24521876.0	0	0	0	0	0

Banded Mass Matrix

0.026203	0	0.014643	0	0.013101	0
0.026203	0	0.014643	0	0.013101	0
0.067819	0	0.030827	0	0.017725	0
0.067819	0	0.014643	0	0.017725	0
0.081691	0	0.030827	0	0.013101	0
0.081691	0	0.030827	0	0.013101	0
0.103270	0	0.030827	0	0.017725	0
0.103270	0	0.014643	0	0.017725	0
0.055488	0	0.016184	0	0	0
0.055488	0	0.016184	0	0	0
0.035451	0	0	0	0	0
0.035451	0	0	0	0	0

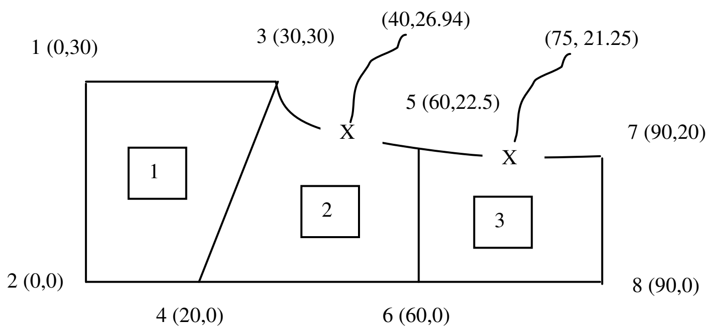
Starting Vector for Inverse Iteration
1 1 1 1 1 1 1 1 1 1 1 1

CHAPTER 12

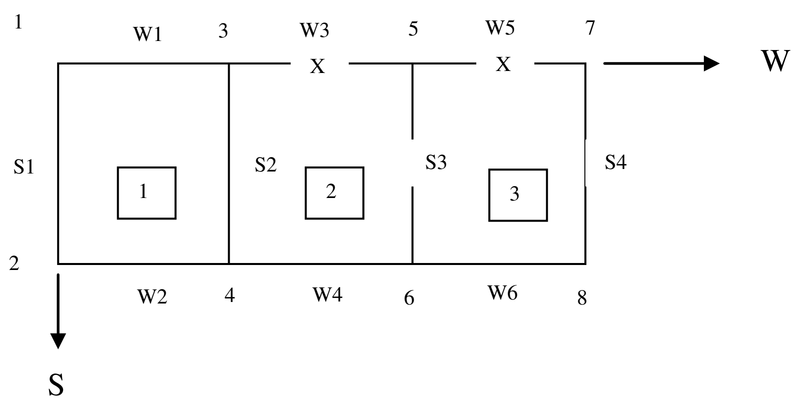
PREPROCESSING AND POSTPROCESSING
(See also solutions to Chapters 6,8 and 10 for details on pre/post processing)

12.1

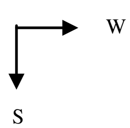
(a)

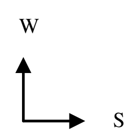


Region



Block

Note: 

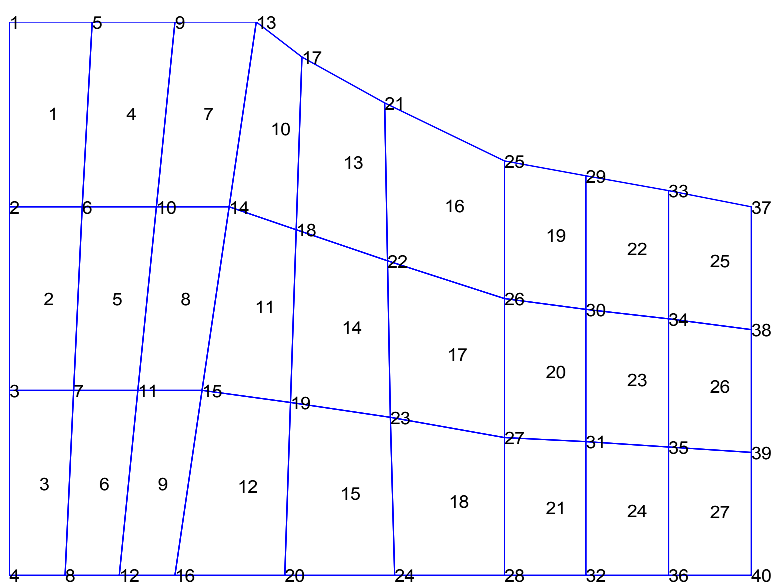
leads to smaller bandwidth than 

MESHGEN Input File

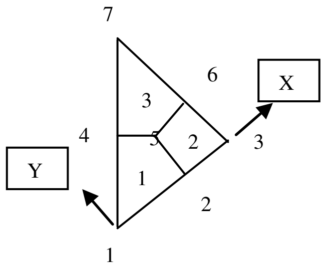
```
Mesh Generation
P12_1(a)
Number of Nodes per Element <3 or 4>
  4
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ
  1             3             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
```

```
 1          3
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1         3
  2         3
  3         3
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
      1     0      30
      2    0       0
      3    30      30
      4    20      0
      5    60      22.5
      6    60      0
      7    90      20
      8    90      0
      0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   3       40         26.94
   5       75         21.25
   0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
```

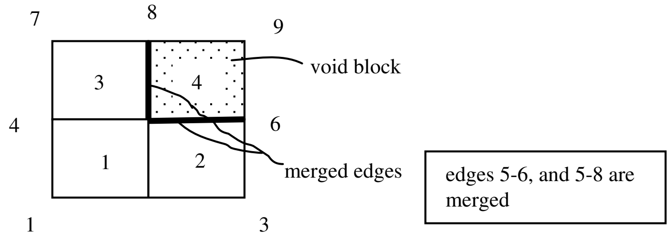
Plot2d (with meshgen output file):



(b)



Region



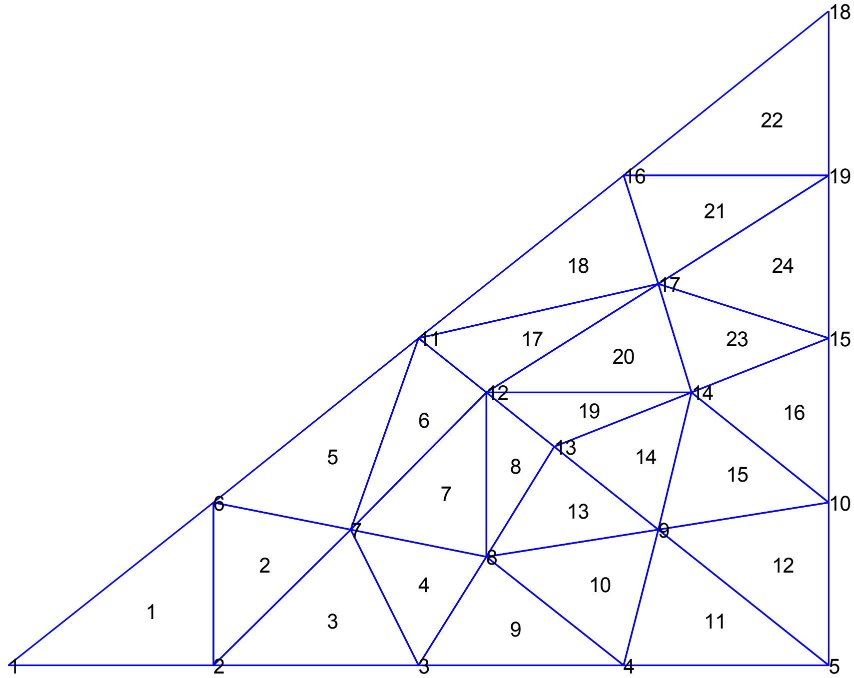
Block

Corner Node	X ,Y Coord.
1	0, 0
2	25, 0
3	50, 0
4	25, 25
5	16.7, 16.7
6,8	50, 25
7	50, 50

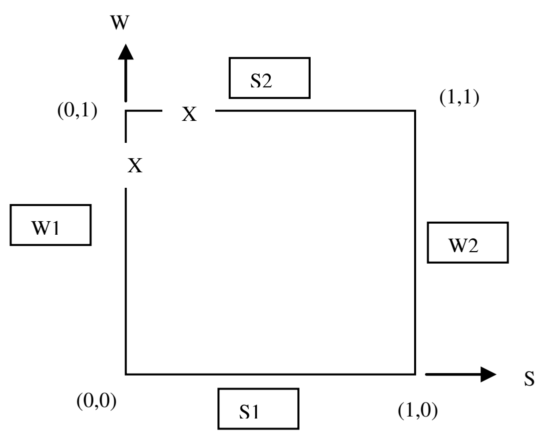
MESHGEN Input File

```
Mesh Generation
P12_1(b)
Number of Nodes per Element <3 or 4>
  3
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ
  2              2             1
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
 1         2
 2         2
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        2
  2        2
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  4        0
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
      1    0   0
      2   25   0
      3   50   0
      4   25   25
      5   33.3  16.7
      6   50   25
      8   50   25
      7   50   50
      0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   0
```

```
MERGING SIDES (Node1 is the lower number)
Pair#   Side1Node1   Side1Node2    Side2Node1   Side2node2
  1         5           6              5            8
```



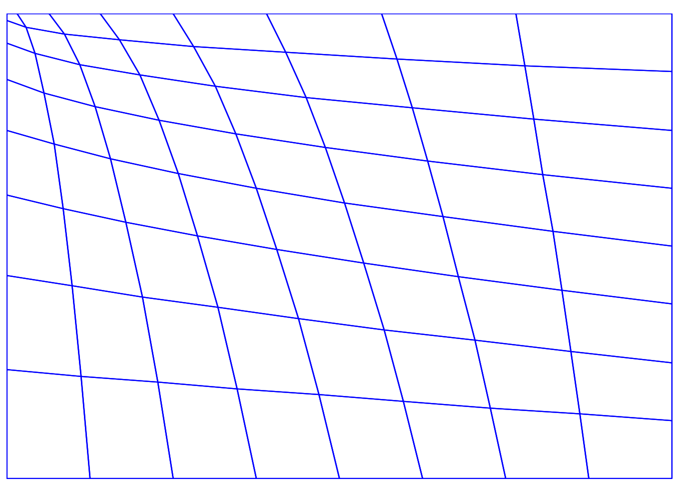
12.2 Implementation of a graded mesh using MESHGEN program is done by defining shifted mid-side nodes. For instance, consider a square shaped region where it is desired to have small elements towards the left top corner. We may then define



MESHGEN Input File

```
Mesh Generation
```

```
Example of (Mesh) Grading
Number of Nodes per Element <3 or 4>
  4
BLOCK DATA
#S-Spans(NS)  #W-Spans(NW)  #PairsOfEdgesMergedNSJ
  1             1             0
SPAN DATA
S-Span#  Num-Divisions  (for each S-Span/ Single division = 1)
 1         8
W-Span#  Num-Divisions  (for each W-Span/ Single division = 1)
  1        8
BLOCK MATERIAL DATA (for Material Number other than 1)
Block#   Material  (Void => 0   Block# = 0 completes this data)
  0
BLOCK CORNER DATA
Corner#  X-Coord   Y-Coord (Corner# = 0 completes this data)
      1    0   0
      2    1   0
      3    0   1
      4    1   1
      0
MID POINT DATA FOR CURVED OR GRADED SIDES
S-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   2       .25       1
   0
W-Side#  X-Coord   Y-Coord (Side# = 0 completes this data)
   1        0        .75
   0
MERGING SIDES (Node1 is the lower number)
Pair#  Side1Node1  Side1Node2   Side2Node1  Side2node2
```



■

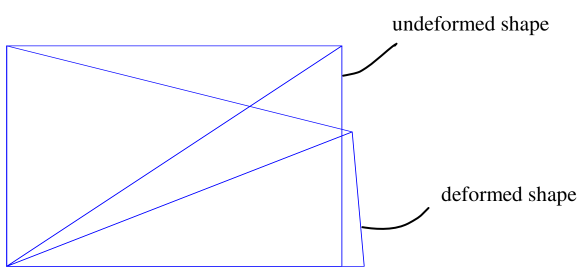
12.3 Several examples of isotherms have been given in detail in Chapter 10. The temperature for each node is to be stored in a file. CONTOUR needs the coordinate and connectivity data file and the nodal temperature data file. Isotherms are constant temperature line. ■

12.4 Plotting of Deformed Shape

Plots of the deformed shape, with maximum deflection scaled to about 10% of the physical geometry may be obtained using the Matlab code given below. The code below is attached to Program CST at the end.

```
function [] = Deformed();
global NN NE NM NDIM NEN NDN
global ND NL NCH NPR NMPC NBW
global X NOC F AREA MAT TH DT S
LOUT1  = fopen('deformed.dat','w');
%Determine size limits
UMAX = F(1);  VMAX = F(2); UMIN = F(1); VMIN = F(1);
for I = 2:NN
if  (UMAX < F(2*I-1))
  UMAX = F(2*I-1);
end
if  (VMAX < F(2*I))
   VMAX = F(2*I);
end
if  (UMIN < F(2*I-1))
   UMIN = F(2*I-1);
end
if  (VMIN< F(2*I))
   VMIN = F(2*I);
end
end
XMAX=max(X(:,1));XMIN=min(X(:,1));YMAX=max(X(:,2));YMIN=min(X(:,2));
SC = 0.1 * (XMAX -XMIN)/abs(UMAX - UMIN);
SCY = 0.1 * (YMAX -YMIN)/abs(VMAX - VMIN);
SC = min(SC,SCY);
for N1=1:NN
   X1 = X(N1,1) + SC*F(2*N1-1);   Y1 = X(N1,2) + SC*F(2*N1);
   fprintf(LOUT1,' %4d %15.4E %15.4E\n',N1,X1,Y1);
end
   fclose(LOUT1);
```

<u>Example</u>



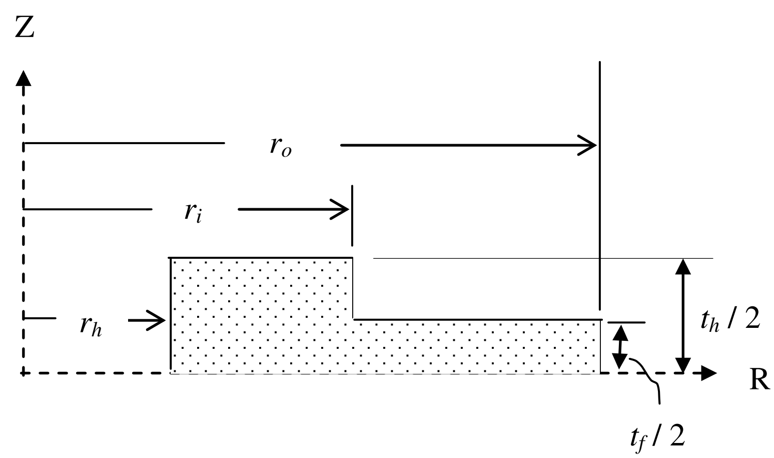
■

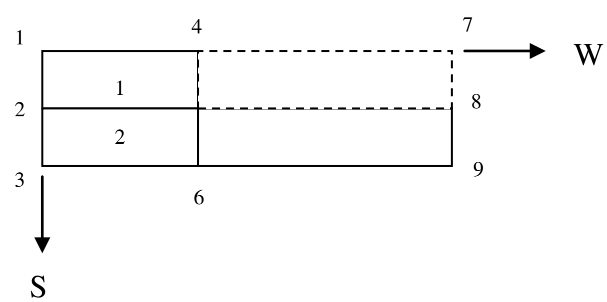
12. 5

The eigenvector **U** has normalized components, which are displacements in the DOF directions. Of interest are those components that correspond to translational dof's rather than rotational dof's. Plotting the mode shape is thus analogous to plotting deformed shape explained above, except for one additional computation : It is necessary to get coordinates of additional points in the case of beams. This is done by using the (Hermite) shape functions to interpolate geometry from the known nodal values. ■

12. 6 (Project)

In this problem, at the MESHGEN stage, one may define the 'region' using a parametric representation. That is, the coordinates of the corner nodes in the block representation can be defined in terms of the parameters r_i, r_o, r_h, t_h and t_f. Block (3) is defined as void. Number of divisions can also be automatically decided by making a decision on the size of the element to be used. Running of other programs is same as discussed in earlier chapters.

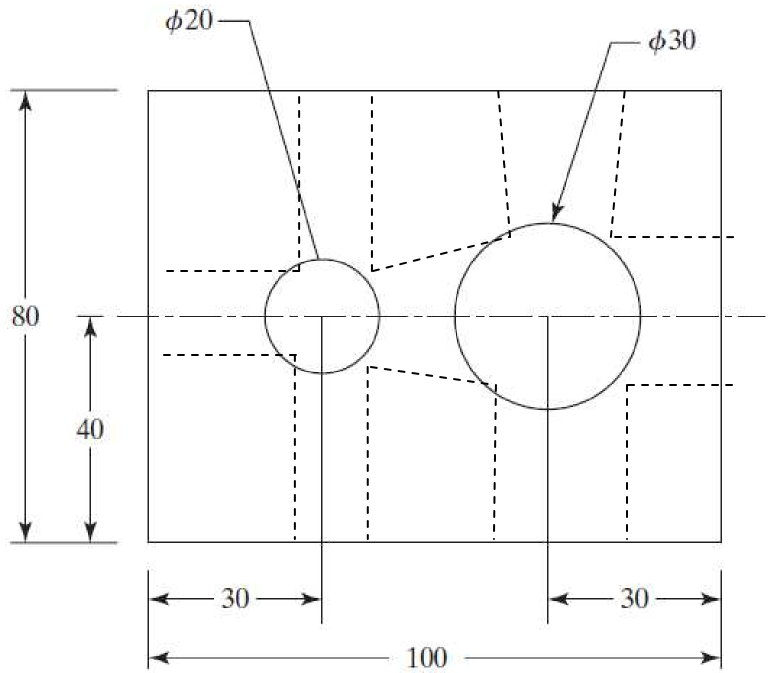




■

12.7 Save the shearing stress values say τ_{xz} into a file. These values are element values. First BESTFIT is run using MESHGEN-data and the element shear stress data. The output file is one that gives nodal values. Now CONTOUR can be run using MESHGEN-data and the nodal value data to plot the contours. Similar procedure can be used to plot τ_{yz} . Several examples have been given in Chapter 10 in detail, related to use of Bestfit/Contour programs. ■

12.8



All dimensions in mm

Meshgen input file (ExcelVB)

MESH GENERATION
PROBLEM 12.8
Number of Nodes per Element <3 or 4>
4
BLOCK DATA NS=#S-Spans
NS NW NSJ NW=#W-Spans
3 5 0 NSJ=#PairsOfEdgesMerged
SPAN DATA
S-Span# #Div (for each S-Span/ Single division = 1)
1 1
2 2
3 1
W-Span# #Div (for each W-Span/ Single division = 1)
TempRise (NCH=2 El Char: Th Temp)
1 1
2 2
3 1
4 2
5 1

BLOCK MATERIAL DATA
Block# Material (Void => 0 Block# = 0 completes this data)

5 0
11 0
0

BLOCK CORNER DATA
Corner# X-Coord Y-Coord (Corner# = 0 completes this data)

1 0 80
2 0 47.07
3 0 32.93
4 0 0
5 22.93 80
6 22.93 47.07
7 22.93 32.93
8 22.93 0
9 37.07 80
10 37.07 47.07
11 37.07 32.93
12 37.07 0
13 59.39 80
14 59.39 47.07
15 59.39 32.93
16 59.39 0
17 80.61 80
18 80.61 47.07
19 80.61 32.93
20 80.61 0
21 100 80
22 100 47.07
23 100 32.93
24 100 0
0

MID POINT DATA FOR CURVED OR GRADED SIDES

S-Side# X-Coord Y-Coord (Sider# = 0 completes this data)

5 20 40
8 40 40
11 55 40
14 85 40
0

W-Side# X-Coord Y-Coord (Sider# = 0 completes this data)

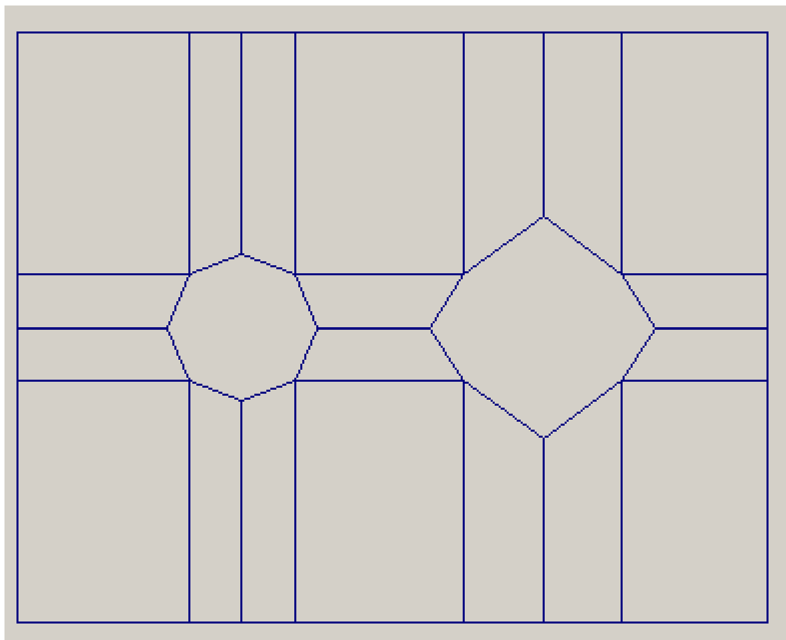
6 30 50

7	30	30
14	70	55
15	70	25
0		

MERGING SIDES (Node1 is the lower number)

Pair#	Sid1Nod1	Sid1Nod2	Sid2Nod1	Sid2Nod2

Plot2D of output



■